The Geometry and Topology of Moduli Spaces of Cubic Threefolds

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Throughout the talk let
\[ X \subset \mathbb{P}^4 \]
be a (smooth) cubic threefold (over the complex numbers \( \mathbb{C} \)).

**Theorem (Clemens-Griffiths, 1972)**

A smooth cubic threefold \( X \) is unirational, but not rational.

The main tool for proving this is the intermediate Jacobian

\[ \text{IJ}(X) = H^{2,1}(X)^* / H_3(X, \mathbb{Z}). \]

This is a 5-dimensional principally polarized abelian variety.
The claim then follows from two facts:

**Fact (1)**

*If there is a birational map $X \to \mathbb{P}^3$, then $IJ(X) = J(C)$ where $J(C)$ is the Jacobian of a genus 5 curve of compact type. In particular, the singular locus of the theta-divisor of $IJ(X)$ has dimension at least 1.*

**Fact (2)**

*The theta divisor of the Jacobian of a smooth cubic threefold has a unique singularity, namely a triple point at an odd 2-torsion point. Mumford has moreover shown that the cubic $X$ can be recovered as the projectivized tangent cone of this singularity (thus giving a proof of the Torelli theorem for cubic threefolds).*
The moduli space of (smooth) cubic threefolds is the GIT quotient

\[ \mathcal{M} = \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)))^{\text{smooth}} / \text{SL}(5, \mathbb{C}). \]

Here we use the obvious \( \text{SL}(5, \mathbb{C}) \)-linearization. This quotient is well defined as every smooth cubic threefold is GIT stable.

\[ \dim \mathcal{M} = \binom{4+3}{3} - 1 - 24 = 35 - 25 = 10. \]

We shall later also consider the GIT compactification

\[ \overline{\mathcal{M}} := \overline{\mathcal{M}}^{\text{GIT}} = \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)))/\text{SL}(5, \mathbb{C}) = \]

\[ = \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)))^{\text{ss}} / \text{SL}(5, \mathbb{C}). \]
1. Period maps

There are two types of period maps associated to cubic threefolds. We have already seen a *Griffiths period map* of moduli spaces

\[ \mathcal{I}_J : \mathcal{M} \to \mathcal{A}_5 \]

where \( \mathcal{M} \) is the GIT moduli space of smooth cubic threefolds and \( \mathcal{A}_5 \) is the moduli space of 5-dimensional principally polarized abelian varieties.

Note that

\[ \dim \mathcal{M} = 10, \quad \dim \mathcal{A}_5 = 15. \]
The second period map is due to Allcock, Carlson and Toledo. Given a cubic threefold

\[ X = \{ f_3(x_0, \ldots, x_4) = 0 \} \subset \mathbb{P}^4 \]

one can associate to this a cubic fourfold

\[ Z = \{ z^3 - f_3(x_0, \ldots, x_4) = 0 \} \subset \mathbb{P}^5. \]

By construction we have a 3 : 1 map

\[ Z \to \mathbb{P}^4 \]

which is branched over the cubic threefold \( X \subset \mathbb{P}^4 \).
The *Fano variety of lines*

\[ F(Z) = \{ \ell \subset \mathbb{P}^5 \mid \ell \subset Z \} \subset \text{Gr}(1, 5) \]

is a smooth fourfold. By a result of Beauville and Donagi this is an *irreducible holomorphic symplectic manifold* (IHSM), also known as *hyperkähler manifold* (HK). It is a deformation equivalent to a Hilbert square \( S^{[2]} \) of a \( K3 \) surface.

To any smooth cubic fourfold \( Z \), respectively its Fano variety \( F(Z) \), one can thus associate (after choosing a marking) a period point

\[ \omega(Z) = \omega(F(Z)) \in \Omega \]

where \( \Omega \) is the period domain of (suitably polarized) IHSM of \( K3^{[2]} \)-type. This is a homogeneous domain of type IV and has dimension 20.
By construction the Fano varieties $F(Z)$ of cubic fourfolds coming from cubic threefolds $X$ come with a $\mathbb{Z}/3\mathbb{Z}$-action (which defines a non-symplectic automorphism of $F(Z)$).

As a result the period points of these Fano varieties are special, more precisely they lie in a 10-dimensional ball

$$\omega(F(Z)) \in \mathbb{B} := \mathbb{B}^{10} \subset \Omega.$$

**Theorem (Allcock, Carlson, Toledo)**

*The above construction defines an open embedding*

$$P : \mathcal{M} \hookrightarrow \mathbb{B}/\Gamma$$

*where $\Gamma$ is a suitable arithmetic group.*
It is natural to ask whether the maps

\[ \mathcal{I}J : \mathcal{M} \to A_5, \quad P : \mathcal{M} \to \mathbb{B}/\Gamma \]

can be extended to suitable compactifications of these spaces? For this we consider:

- \( \overline{\mathcal{M}} = \overline{\mathcal{M}}^{\text{GIT}} \) : GIT compactification
- \( \widehat{\mathcal{M}} \) : partial Kirwan blow-up of \( \overline{\mathcal{M}} \)
- \( \widetilde{\mathcal{M}} \) : wonderful blow-up of \( \overline{\mathcal{M}} \)
- \( (\mathbb{B}/\Gamma)^* \) : Baily-Borel compatification
- \( A_5^{\text{Sat}} \) : Satake compactification
- \( A_5^{\text{Vor}} \) : Second Voronoi compactification
We can summarize this in the diagram

The resolutions $q$ is due to ACT and Looijenga, Swierstra
The resolution of $IJ^{\text{Sat}}$ is due to C-ML
The fact that $IJ^{\text{Sat}}$ can be factored through the Voronoi compactification is due to GC-MHL.
2. The problem

For the rest of this talk I will concentrate on the diagram

\[ \begin{array}{ccc}
\mathcal{M}^K & \to & (\mathbb{B}/\Gamma)^* \\
\downarrow & & \downarrow \\
\hat{\mathcal{M}} & \to & \\
\mathcal{M} & & \\
\end{array} \]

where

- \( \mathcal{M}^K \) : full Kirwan blow-up

**Question**

*What is the (intersection) topology of these spaces?*
3. GIT Analysis

We will need a good understanding of the semi-stable and polystable points.

Theorem (Allcock, Yokoyama)

Let $\overline{M} := \overline{M}^{GIT}$ be the GIT compactification of the moduli space of smooth cubic threefolds. The following holds:

1. A cubic threefold is GIT stable if and only if it has at worst isolated $A_1, \ldots, A_4$-singularities.
2. A cubic threefold is GIT semi-stable if and only if it has at worst isolated $A_1, \ldots, A_5$ or $D_4$-singularities or is the chordal cubic (the secant variety of a rational normal curve.).
3. The GIT boundary (i.e., the complement of the stable locus in $\overline{M}$) consists of a rational curve $T$ and an isolated point $\Delta$. 
The polystable (non-stable) points are described by

**Proposition**

*Under a suitable identification $T \cong \mathbb{P}^1$ the following holds:*

1. $t \in \mathbb{P}^1 \setminus \{0, 1\} \iff X_t$ has $2A_5$ singularities
2. $0 \iff X_0$ has $2A_5 + A_1$ singularities
3. $1 \iff X_1$ is a chordal cubic ($\Xi$)

*Moreover for the isolated point $\Delta$ we have*

- $\Delta \iff X_\Delta$ has $3D_4$ singularities.
We return to the Allcock-Carlson-Toledo period map

\[
\begin{array}{ccc}
\widehat{\mathcal{M}} & \xrightarrow{p} & \mathcal{M} \\
\downarrow & & \downarrow \mathcal{P} \\
\mathcal{M} & \xrightarrow{q} & (\mathbb{B}/\Gamma)^* \\
\end{array}
\]

Here

- The map \( p : \widehat{\mathcal{M}} \to \mathcal{M} \) is a blow-up in the point \( \Xi \) corresponding to the chordal cubic (we shall later relate this to the Kirwan blow-up).

- The exceptional locus of \( p \) is mapped to a Heegner divisor in \((\mathbb{B}/\Gamma)^*\), namely the so-called hyperelliptic locus (which is itself a 9-dimensional ball quotient).

- The strict transform of the curve \( T \cong \mathbb{P}^1 \) is again mapped to a \( \mathbb{P}^1 \). The point corresponding to \( 0 \in T \) (or \( 2A_5 + A_1 \) cubics) is mapped to a cusp in \((\mathbb{B}/\Gamma)^*\).

- The point \( \Delta \) corresponding to \( 3D_4 \) cubics is mapped to the second cusp in \((\mathbb{B}/\Gamma)^*\).
The curve $T$

Let

$$F_{A,B} = Ax_2^3 + x_0x_3^2 + x_1^2x_4 - x_0x_2x_4 + Bx_1x_2x_3 \quad (A, B) \neq (0, 0).$$

We define $C := A/B^2$. Then

$$\{F_{A,B} = 0\} \sim \{F_{A',B'} = 0\} \iff C = C'.$$

For $C = 0$ we obtain the (unique) cubic with $2A_5 + A_1$ singularities and for $C = 1$ we obtain the chordal cubic

$$F_{1,-2} = \det \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}.$$
We also note that the connected component of the stabilizer of these points in $\text{SL}(5, \mathbb{C})$ is given by

$$R = \text{Stab}^0_{F_A, B} = \begin{cases} 
\mathbb{C}^* & \text{if } C \neq 1 \\
\text{SL}(2, \mathbb{C}) & \text{if } C=1.
\end{cases}$$

In the latter case this is the automorphism group of $\mathbb{P}^1$ which is embedded into $\mathbb{P}^4$ via the quadruple Veronese embedding.
The $3D_4$ case

There is a unique cubic threefold with $3D_4$ singularities, given by

$$F_{3D_4} := x_0x_1x_2 + x_3^3 + x_4^3.$$ 

In this case

$$R = \text{Stab}^0_{F_{3D_4}} = (\mathbb{C}^*)^2$$

which acts by

$$(\lambda, \mu) : (x_0 : x_1 : x_2) \mapsto (\lambda x_0 : \mu x_1 : (\lambda \mu)^{-1} x_2).$$
Perfect equivariant stratification

We consider the following general set-up

- $G \actson X$: reductive group $G$ acting on a variety $X$ with a $G$-linearized ample line bundle $\mathcal{L}$
- $K \subset G$: maximal compact subgroup
- $T \subset K$: maximal torus
- $t_+$: positive Weyl chamber

In our case

- $G = \text{SL}(5, \mathbb{C})$ acting on $\mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3))) = \mathbb{P}^3$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^4}(1)$
- $K = \text{SU}(5)$ with root system $A_4$
- $T = S^4$
Following Kirwan’s approach to the computation of the (intersection) cohomology of GIT quotients one can define a stratification of $X$ by measuring how unstable points are (Kempf-Ness-Hesselink stratification).

This stratification is indexed by a finite set

$$0 \in \mathcal{B} \subset \mathfrak{t}_+$$

leading to a decomposition

$$X = S_0 \sqcup \left( \bigsqcup_{0 \neq \beta \in \mathcal{B}} S_\beta \right).$$

Here

$$S_0 = X^{ss}$$

is the locus of semi-stable points. The strata $S_\beta$ are locally closed and the stratification is $G$-equivariant.
The first relevant formula for the computation of the cohomology is

\[ P^G_t(X^{ss}) = P^G_t(X) - \sum_{0 \neq \beta \in B} t^{2d(\beta)} P^G_t(S_\beta) \]

Here

- \( P_t(X) = \sum_{i \geq 0} b_i(X) t^i \): Poincaré polynomial
- \( P^G_t(X) \): Poincaré polynomial of \( G \)-equivariant cohomology
- \( d(\beta) = \text{codim}_\mathbb{C} S_\beta \)

**Remark**

Due to the fact that intersection cohomology satisfies Poincaré duality we will later only need to determine very few of the Betti numbers \( b^G_i(S_\beta) \).
Aside: $G$-equivariant cohomology

We consider a topological group $G$. A *classifying space* $BG$ of $G$ is the base of a left principal $G$-bundle $EG \to BG$ where the total space $EG$ is contractible. The space $BG$ is determined up to homotopy equivalence, in particular $H^\bullet(BG)$ is well defined.

Let $G \curvearrowright X$. Then one sets

$$X_G := X \times_G EG := (X \times EG)/G \to BG$$

which is a locally trivial fibration with fibre $X$ and structure group $G$. The *$G$-equivariant cohomology* is then defined by

$$H^\bullet_G(X) := H^\bullet(X_G).$$
The situation in our case is the following:

\[ X = \mathbb{P}^{34}, \quad G = \text{SL}(5, \mathbb{C}) \]

\[ P_t^G(\mathbb{P}^{34}) = P_t(\mathbb{P}^{34}) P_t(\text{BSL}(5, \mathbb{C})). \]

The relevant cohomology is well known:

\[ P_t(\mathbb{P}^n) = \frac{1}{1 - t^2} \mod t^{2n+1} \]

and

\[ P_t(\text{SL}(n, \mathbb{C})) = \prod_{\ell=2}^{2n} \frac{1}{1 - t^{2\ell}} \]

**Remark**

It turns out that in the end only one stratum \( S_\beta \) will enter the final computation. The general point of this stratum is a cubic with a \( D_5 \) singularity and we only need \( H^0(S_\beta) = \mathbb{C} \).
The Kirwan blow-up

We start with the set of all connected positive dimensional stabilizers of polystable points (up to conjugacy):

\[ \mathcal{R} := \{ R \subseteq G \mid R = \text{Stab}_F^0, \dim R > 0, F \text{ polystable}\}/ \sim \]

In our situation this set is

\[ \mathcal{R} := \{ \mathbb{C}^*, (\mathbb{C}^*)^2, \text{SL}(2, \mathbb{C})\} \]

corresponding to cubics with $2A_5(+A_1)$, $3D_4$ singularities and the chordal cubic respectively.
Now fix $R \in \mathcal{R}$ of maximal dimension and consider its fixed locus

$$Z_R^{ss} := \{ x \in X^{ss} | R \text{ fixes } x \} \subset X^{ss}.$$ 

The set $GZ_R^{ss}$ is smooth and closed in $X^{ss}$. Blowing up along this set gives

$$\pi_1 : X_1^{ss} \to X^{ss}.$$ 

The group $G$ acts on $X_1^{ss}$. We choose a polarization

$$\mathcal{L}_1 := \pi_1^* \mathcal{L} \otimes d \otimes \mathcal{O}_{X_1^{ss}}(-E_1).$$

Here $E_1$ is the exceptional divisor and $d \gg 0$ (the precise choice will not matter). This admits a $G$-linearization.
We now replace \((X^{ss}, \mathcal{L})\) by \((X^{ss}_1, \mathcal{L}_1)\) and repeat the process. This results in a series of blowups

\[
\tilde{X}^K := X_{r}^{ss} \xrightarrow{\pi_r} X_{r-1}^{ss} \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_2} X_1^{ss} \xrightarrow{\pi_1} X^{ss}
\]

where all points in \(\tilde{X}^K\) are stable. Then

\[
\mathcal{M}^K := \tilde{X}^K \sslash G
\]

is the *Kirwan blow-up* of \(\mathcal{M} = X^{ss} \sslash G\). This has only finite quotient singularities.

**Remark**

In our situation we first blow up the chordal cubic. This gives the partial Kirwan blow-up \(\hat{\mathcal{M}}\). Then we blow up in the \(2A_5(\geq A_1)\) locus. The blow-up of the \(3D_4\) locus is independent as the \(3D_4\) locus does not intersect the other blown up loci.
The Kirwan blow-up has only finite quotient singularities and hence its singular cohomology and the intersection cohomology (with rational coefficients) agree. One obtains

\[ P_t(\mathcal{M}^K) = P_t^G(\tilde{X}^K) = P_t^G(X^{ss}) + \sum_{R \in \mathcal{R}} A_R(t). \]

The term \( P_t^G(X^{ss}) \) is known from our previous step.

The calculation of the correction terms \( A_R(t) \) requires a careful study of the geometry of the loci of the blow-ups in the Kirwan desingularization process.
The cohomology of the GIT quotient

The final step in the calculation of the intersection cohomology of the GIT quotient \( \overline{M}^{\text{GIT}} \) is an application of the decomposition theorem due to Beilinson, Bernstein, Deligne, and Gabber to the map

\[ M^K \to \overline{M}^{\text{GIT}}. \]

This results in

\[ IP_t(\overline{M}^{\text{GIT}}) = P_t(M^K) - \sum_{R \in R} B_R(t). \]

To calculate the correction terms \( B_R(t) \) one has (among other things) to compute the normal spaces \( N_x \) of a general point \( x \in GZ^s_R \) and the intersection cohomology of \( \mathbb{P}(N_x) // R \).
Summary

We have the following steps

\begin{align*}
(1) \quad P_t^G(X^{ss}) &= P_t^G(X) - \sum_{0 \neq \beta \in \mathcal{B}} t^{2d(\beta)} P_t^G(S_\beta) \\
\end{align*}

$P_t^G(X)$ and the contributions $P_t^G(S_\beta)$ are easy to control.

\begin{align*}
(2) \quad &\quad P_t(\mathcal{M}^K) = P_t^G(\tilde{\mathcal{X}}^K) = P_t^G(X^{ss}) + \sum_{R \in \mathcal{R}} A_R(t). \\
\end{align*}

$P_t^G(X^{ss})$ is known from from (1) and the $A_R(t)$ need an analysis of the geometry of the fixed loci of the stabilizers $R$.

\begin{align*}
(3) \quad &\quad IP_t(\mathcal{M}^{GIT}) = P_t(\mathcal{M}^K) - \sum_{R \in \mathcal{R}} B_R(t). \\
\end{align*}

$P_t(\mathcal{M}^K)$ is known from (2) and the $B_R(t)$ require a detailed analysis of the topology of the exceptional loci of the blow-ups.
The ball quotient

To compute the cohomology of the ball quotient we make again use of the decomposition theorem.

**Theorem**

Let \( f : X \to Y \) be a morphism or projective varieties which contracts a divisor \( E \). Assume that \( X \) has only finite quotient singularities. Then

\[
H^i(X) = \begin{cases} 
  IH^i(Y) & \text{if } j=2n, 2n-1 \\
  IH^i(Y) + H^i(E) & \text{if } j=n, \ldots, 2n-2 \\
  IH^i(Y) + H^{2n-j}(E) & \text{if } j=2, \ldots, n-1 \\
  IH^i(Y) & \text{if } j=0,1 
\end{cases}
\]

where all cohomology groups are with rational coefficients.
We apply this to the morphism

\[ f : M^K \to (B/\Gamma)^*. \]

This morphism contracts two divisors, namely the divisor \( D_{2A_5} \) belonging to cubics with \( 2A_5(+A_1) \) singularities and the divisor \( D_{3D_4} \) which is the blow-up of the \( 3D_4 \) cubic. Their cohomology is known from previous computations.
The final result

Combining all the steps described we finally obtain the

**Theorem**

The Betti numbers of the Kirwan blow-up $\mathcal{M}^K$, and the intersection Betti numbers of $\overline{\mathcal{M}}^{\text{GIT}}$, $\hat{\mathcal{M}}$, and $(\mathbb{B}/\Gamma)^*$ are as follows:

<table>
<thead>
<tr>
<th>$j$</th>
<th>$0$</th>
<th>$2$</th>
<th>$4$</th>
<th>$6$</th>
<th>$8$</th>
<th>$10$</th>
<th>$12$</th>
<th>$14$</th>
<th>$16$</th>
<th>$18$</th>
<th>$20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dim H^j(\mathcal{M}^K)$</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>10</td>
<td>13</td>
<td>15</td>
<td>13</td>
<td>10</td>
<td>6</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>$\dim IH^j(\mathcal{M}^{\text{GIT}})$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\dim IH^j(\hat{\mathcal{M}})$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>9</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$\dim IH^j((\mathbb{B}/\Gamma)^*)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

while all the odd degree (intersection) cohomology vanishes. All cohomology groups are taken with rational coefficients.
Further remark and outlook

Remark
Deligne and Mostow considered various moduli spaces of configurations of points on $\mathbb{P}^1$ which are ball quotients. Their topology was studied by Kirwan, Lee and Weintraub. These examples all have dimension $\leq 9$.

Question

- Can one compute the intersection cohomology of the ball quotient $(\mathbb{B}/\Gamma)^*$ by representation theoretic methods?
- Can one understand the cohomology of $(\mathbb{B}/\Gamma)^*$ in terms of Shimura subvarieties?