Simple-connectedness of Fano log pairs with semi-log canonical singularities

Wenfei Liu

Xiamen University

Joint work with Osamu Fujino

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- Do $n$-dimensional Fano manifolds form a bounded family?
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  Rational connectedness ($\Rightarrow$ simple connectedness)
  by KMM’92 and Campana’92
A log pair \((X, \Delta)\) consists of

- an equi-dimensional demi-normal scheme \(X\), and
- an effective \(\mathbb{R}\)-divisor \(\Delta\) on \(X\) such that
  1. \(\Delta\) does not contain any irreducible component of the non-normal locus of \(X\)
  2. \(K_X + \Delta\) is \(\mathbb{R}\)-Cartier.

A projective log pair \((X, \Delta)\) is Fano if 
\[-(K_X + \Delta)\] is ample.

Using resolution of singularities, one can define various types of singularities naturally appearing in the minimal model program:

- Kawamata log terminal (klt) for \(\epsilon > 0\)
- \(\epsilon\)-Kawamata log terminal (\(\epsilon\)-klt)
- log canonical (lc)
- semi-log-canonical (slc)

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Theorem (Birkar)

Given a natural number \( n > 0 \) and a positive real number \( \epsilon > 0 \), Fano log pairs of dimension \( n \) with \( \epsilon \)-klt singularities form a bounded family.

Note: "\( \epsilon \)-klt" cannot be relaxed to "klt".
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Theorem (Qi Zhang 06, Hacon–McKernan 04)

Let $(X, \Delta)$ be a (connected) Fano log pair. Then

1. $X$ is rationally connected if $(X, \Delta)$ is klt;
2. $X$ is rationally chain connected if $(X, \Delta)$ is lc.
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Corollary

Log canonical Fano pairs are simply connected.

Note: "Fano" cannot be replaced by "weak Fano".
Main results: Fano log pairs with slc singularities

However, "lc" can be indeed replaced by "slc":

Theorem A (Fujino–L. 17)
Let \((X, \Delta)\) be a (connected) Fano log pair with slc singularities. Then \(X\) is rationally chain connected and simply connected.

Note: on non-normal varieties, rational connectedness does not imply finiteness of fundamental groups.

To prove Theorem A, we need to prove a more general

Theorem B (Fujino–L. 17)
Let \((X, \Delta)\) be a (connected) Fano log pair with slc singularities. Then any union of slc strata is rationally chain connected and simply connected.
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**Definition**

Let \((X, \Delta)\) be a log pair with semi-log canonical singularities. Let \(\nu : \bar{X} \to X\) be the normalization and write \(\nu^*(K_X + \Delta) = K_{\bar{X}} + \bar{\Delta}\).

- An *slc center* of \((X, \Delta)\) means the image of an lc center of \((\bar{X}, \bar{\Delta})\), and \(\mathbb{N}_{\text{klt}}(X, \Delta)\) is the union of all slc centers.
- An *slc stratum* means either an slc center of \((X, \Delta)\) or a component of \(X\).

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### Theorem (Hacon–McKernan 04)

For an lc Fano pair \((X, \Delta)\), \(\pi_1(\mathbb{N}_{\text{klt}}(X, \Delta)) \to \pi_1(X)\) is surjective.
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Let $X$ be a demi-normal scheme and $\nu: \bar{X} \to X$. Then we have a push-out diagram in the category of algebraic spaces:

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\begin{array}{ccc}
\bar{D}^\nu & \longrightarrow & \bar{D} \\
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- $D \subset X$ and $\bar{D} \subset \bar{X}$ are the conductors.
- $\bar{D}^\nu \to \bar{D}$ and $\bar{D}^\nu \to \bar{D}$ are normalizations.
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Two issues:

1. Normalization can change the topology dramatically.
2. $D$ can be not slc even if $X$ is.
Quasi-log canonical pairs

Way out: go to the larger category of quasi-log canonical pairs.

**Definition**

Let $X$ be a scheme and $\omega$ an $\mathbb{R}$-Cartier divisor on $X$. Let $f : Z \to X$ be a proper morphism from a globally embedded simple normal crossing pair $(Z, \Delta_Z)$. If

- $\Delta_Z = \Delta_Z^{\leq 1}$, and $f^*\omega \sim_\mathbb{R} K_Z + \Delta_Z$,
- the natural map $\mathcal{O}_X \to f_*(\lceil - (\Delta_Z^{\leq 1}) \rceil)$ is an isomorphism

then $[X, \omega]$ is called a *quasi-log canonical pair* (qlc pair, for short).
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Important examples:

- An slc log pair $(X, \Delta)$ with quasi-canonical class $\omega = K_X + \Delta$
- Any union $Y$ of slc strata with $\omega = (K_X + \Delta)|_Y$
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One can define qlc centers, qlc strata, Nqklt locus for a qlc pair, which are compatible with slc centers, slc strata, and Nklt locus.
Let $[X, \omega]$ be a qlc pair, and $Y$ any union of qlc strata.

1. (Adjunction) $[Y, \omega|_Y]$ is a quasi-log canonical pair. Moreover, the qlc strata of $[Y, \omega|_Y]$ are exactly those of $[X, \omega]$ that are contained in $Y$.

2. (Vanishing) Assume that $X$ is proper. Let $L$ be a Cartier divisor on $X$ such that $L - \omega$ is nef and log big with respect to $[X, \omega]$. Then

$$H^i(I_Y \otimes L) = 0 \text{ for any } i > 0.$$
Adjunction and Vanishing for qlc pairs

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**Corollary**

Let $(X, \Delta)$ be a Fano log pair with slc singularities. Then

1. any union of slc strata is connected;
2. there is a unique minimal slc stratum, which is normal.
Key ingredient: subadjunction formula for slc strata

- \( W \): an slc stratum of an slc log pair \((X, \Delta)\)
- \( E \): the union of all slc strata that are strictly contained in \( W \)

Let \( \nu: \bar{W} \to W \) be the normalization.

**Lemma**

There is an effective \( \mathbb{Q} \)-divisor \( B_{\bar{W}} \) on \( \bar{W} \) such that

1. \( K_{\bar{W}} + B_{\bar{W}} \sim_{\mathbb{R}} (K_X + \Delta)|_{\bar{W}}, \) and
2. \( \text{Nklt}(\bar{W}, B_{\bar{W}}) \subset \nu^{-1}(E). \)
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If $(X, \Delta)$ is Fano, then $\mathrm{Nklt}(\bar{\mathcal{W}}, B_{\bar{\mathcal{W}}})$ is connected.
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**Corollary**

Let \((X, \Delta)\) be a Fano log pair with slc singularities. Then any union of slc strata of \((X, \Delta)\) is rationally chain connected.

Lemma (Franciosi–Pardini–Rollenske 14)

Let \( \nu: \tilde{X} \to X \) be a holomorphic map of compact complex analytic spaces. Assume \( A \subset X \) is a connected closed analytic subspace such that

- \( \tilde{A} = \pi^{-1}(A) \) is connected, and
- the map \( \pi: \tilde{X} \setminus \tilde{A} \to X \setminus A \) is an isomorphism.

Then \( \pi_1(X) \cong \pi_1(A) \ast_{\pi_1(\tilde{A})} \pi_1(\tilde{X}) \).
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In particular, if $\pi_1(\tilde{A}) \to \pi_1(\tilde{X})$ is surjective then so is $\pi_1(A) \to \pi_1(X)$. 
\begin{itemize}
    \item $(X, \Delta)$: a Fano log pair with slc singularities
    \item $W$: any union of slc strata of $(X, \Delta)$
    \item $W_0$: the minimal slc stratum of $(X, \Delta)$
\end{itemize}
(\(X, \Delta\)): a Fano log pair with slc singularities

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Define a filtration of reduced subschemes of \(W\):

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W = W^{(0)} \supset W^{(1)} \supset \cdots \supset W^{(k)} = W_0.
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where \(W^{(i+1)}\) is the union of slc strata that are strictly contained in an irreducible component of \(W^{(i)}\) for \(0 \leq i \leq k - 1\).
(X, Δ): a Fano log pair with slc singularities
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\[ \{1\} = \pi₁(W^{(k)}) \twoheadrightarrow \pi₁(W^{(k-1)}) \twoheadrightarrow \cdots \twoheadrightarrow \pi₁(W^{(1)}) \twoheadrightarrow \pi₁(W^{(0)}). \]
Thank You!