ON THE RATIONALITY OF CERTAIN MODULI SPACES RELATED TO CURVES OF GENUS 4

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INTRODUCTION

Let $M_g$ be the coarse moduli space for complete smooth curves of genus $g$, let $R_g$ be the "Prym moduli space" of unramified (connected) double covers of curves of genus $g$; a general problem is: what can be said about the birational structure of $M_g, R_g$? From the point of view of birational geometry we can also talk about $M_{g,n}$, the moduli space of curves of genus $g$ together with an ordered $n$-tuple of points, though this moduli functor is not representable in general (cf. [10]).

Our main results are:

Theorem A: $R_4$ is a rational variety.
Theorem B: $M_{4,1}$ is a rational variety.
Theorem C: $M_4$ admits a covering of degree 24 by a rational variety.

To put these results into perspective, we notice that, while the rationality of $M_4, R_4$ is classical and well-known, the rationality of $M_2$ has been proved by Igusa (cf. [8], also [17]).

For higher values of the genus $g$, the situation is as follows:

i) $M_g$ is known to be unirational for $g \leq 10$, ([16], [1]), $g = 12$ ([14]), uniruled for $g = 11$ ([9]), whereas, for $g$ odd $\geq 25$ $M_g$ is variety of general type ([7]), and D. Mumford and J. Harris announced a similar result also for $g$ even $\geq 40$

ii) the unirationality of $R_g$ for $g = 5,6$ has been proven only recently ([4], [6]).

If the base field is of characteristic $\neq 2$, $R_g$ is a covering of $M_g$ of degree $2^{2g} - 1$, so that theorems A and C produce two rational coverings of

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$M_3$ and $M_4$ of degrees, respectively, 255 and 24.

I should finally remark that it is conjectured that $M_3$, $M_4$ are rational varieties, but, to my knowledge, this is still unsolved.

Our method of proof does not use classical invariant theory: our strategy consists in constructing, using the geometry of curves of genus 4, some rational Galois covers by some rational variety, and then computing explicitly the subfield of invariants.

These covers are constructed with elementary arguments in the case of theorems B), C)
For theorem A) I use a classical result of Wirtinger ([18]), which can also be found in [5], and about which I was told by S. Recillas, (cf. [13]), to whom I am indebted for noticing a mistake in an earlier proof of Theorem 1.5.

Our notation is as follows:

$k$ is an algebraically closed field of char. #2
$X$ is a complete smooth curve of genus $g$ defined over $k$
$\text{Pic}(X)$ is the group of divisors on $X$ modulo linear equivalence, here denoted by $\equiv$.
$\eta$ is a divisor in $\text{Pic}_2(X) - \{0\}$, i.e. $2\eta \equiv 0, \eta \neq 0$.
$K$ is a canonical divisor on any Gorenstein variety $Y$, i.e. $\mathcal{O}_Y(K_Y) \cong \omega_Y$.
If $D$ is a divisor, $|D|$ is the linear system of effective divisors $D' \equiv D$.
$S_n$ is the symmetric group in $n$ letters, and
$V_n$ is the standard (permutation) representation on $k^n$.

Given any coherent sheaf $F$ on a complete variety $Y$, we denote by $h^i(F)$ the dimension of $H^i(Y,F)$ as a $k$-vector space.
If $U$ is a $k$-vector space, we denote by $U^*$ its dual space.
R.R. is an abbreviation for the Riemann-Roch theorem.
§1. GEOMETRY OF CURVES OF GENUS 4.

Let $X$ be a non-hyperelliptic curve of genus 4. Then the linear system $\mid K_X \mid$ gives an embedding of $X$ in $\mathbb{P}^3$ such that the image of $X$ is the complete intersection of a quadric $Q$ and of a cubic $G$.

The quadric $Q$ is uniquely determined, and, since it is normal, as well as $G$, there are only two possibilities:

i) $Q$ is smooth
ii) $Q$ is a quadric cone.

Case ii) occurs if and only if there exists a half canonical divisor $\nu$ (i.e. $2\nu = K_X$) such that $h^0(\mathcal{O}_X(\nu)) = 2$: we say then that $X$ has a vanishing thetanull.

It is well-known that $M_4$ is an irreducible variety of dimension 9 and that curves with a vanishing thetanull form an 8-dimensional subvariety, hyperelliptic curves form a 7-dimensional subvariety.

Definition 1.1. Let $\eta \in \text{Pic}_2(X) - \{0\}$. We shall say that the pair $(X, \eta)$ is bielliptic if there exist an elliptic curve $E$, a double covering $f: X \rightarrow E$, and a divisor $\eta' \in \text{Pic}_2(E) - \{0\}$ such that $\eta \sim f^*(\eta')$.

Definition 1.2. A normal cubic surface $G$ in $\mathbb{P}^3$ is said to be symmetric if its equation can be written as the determinant of a symmetric $3 \times 3$ matrix of linear forms (cf. [2]). A symmetrization of $G$ is the datum of such a matrix $(a_{ij}(y)) = (a)$, where $y = (y_0, y_1, y_2, y_3)$ are coordinates in $\mathbb{P}^3$, up to the action of $\text{PGL}(3)$ (such that, for $g \in \text{GL}(3)$, $(a) \mapsto g(a)g$).

How many symmetric cubics with a symmetrization are there in $\mathbb{P}^3$, up to the action of $\text{PGL}(4)$?

The answer is: as many as there are pencils of conics in $\mathbb{P}^2$, up to the action of $\text{PGL}(3)$.

In fact, let $U$ be the space $\text{Sym}^2(k^3)$ of symmetric $3 \times 3$ matrices; then $\mathbb{P}(U)$ is the space of conics in $\mathbb{P}^2$, and $\mathbb{P}(U)$ contains the cubic determinantal hypersurface $\Delta = \{\det(a_{ij}) = 0\}$: $\Delta$ is the dual variety of the Veronese surface $W^*$ in $\mathbb{P}(U^*)$, and its singular locus is the Veronese surface $W$ in $\mathbb{P}(U)$.

Now, the datum of a symmetrization amounts to giving a $\mathbb{P}^3 \subset \mathbb{P}(U)$ such that $\mathbb{P}^3 \cap \Delta$ is a normal cubic. But giving a $\mathbb{P}^3 \subset \mathbb{P}(U)$ is equivalent to giving a $\mathbb{P}^1$ in $\mathbb{P}(U^*)$, i.e. a pencil of conics.

Notice that the number of base points in the pencil of conics is the cardinality of $\mathbb{P}^3 \cap W$, the number of degenerate conics in the pencil is the cardinality of $\mathbb{P}^1 \cap \Delta^*$. 
The following is the list of pencils of conics (up to projective equivalence):

i) pencils of reducible conics: $\lambda x_1^2 + \mu x_1 x_2 = 0$, or $\lambda x_1 x_2 + \mu x_2 x_3 = 0$

ii) pencil with 4 base points: $\lambda x_1 x_2 + \mu x_3 (x_1 + x_2 + x_3) = 0$

iii) pencil with 3 base points: $\lambda x_1 x_2 + \mu x_3 (x_1 - x_2) = 0$

iv) pencil with 2 base points, 2 degenerate conics: $\lambda x_1 x_2 + \mu (x_1 x_3 - x_2) = 0$

v) pencil with 2 base points, one reducible conic: $\lambda x_1 x_2 + \mu (x_1 x_3 - x_2^2) = 0$

vi) pencil with 1 base point: $\lambda x_1^2 + \mu (x_1 x_3 - x_2^2) = 0$.

Correspondingly we get the following symmetrizations:

i) \[
\begin{pmatrix}
0 & 0 & y_1 \\
y_0 & y_2 & 0 \\
y_1 & y_2 & y_3
\end{pmatrix},
\begin{pmatrix}
y_0 & 0 & y_3 \\
y_0 & y_1 & 0 \\
y_3 & 0 & y_2
\end{pmatrix}
\] $G = \{y_0y_1^2 = 0\}$, respectively

$G = \{y_0y_1y_2 - y_1y_3^2 = 0\}$, so $G$ is reducible, and this case must be excluded,

ii) \[
\begin{pmatrix}
y_0 & 0 & y_2 \\
y_0 & y_1 & y_3 \\
y_2 & y_3 & (-y_2 - y_3)
\end{pmatrix}
\] $G = \{y_0y_1(y_2 + y_3) + y_0y_3^2 + y_1y_2^2 = 0\}$

Here $G$ has 4 singular points, and is also projectively equivalent to the 4-nodal cubic of Cayley of equation $\sigma_3(y) = \frac{3}{\sum_{i=0} y_0y_1y_2y_3}{y_i} = 0$.

iii) \[
\begin{pmatrix}
y_0 & 0 & y_2 \\
y_0 & y_1 & y_2 \\
y_2 & y_2 & y_3
\end{pmatrix}
\] $G = \{y_0y_1y_3 - y_0y_2^2 - y_1y_2^2 = 0\}$.

$G$ has three singular points, two nodes and a singularity of type $A_3$ at $\{y_0 = y_1 = y_2 = 0\}$.

iv) \[
\begin{pmatrix}
y_0 & 0 & y_2 \\
y_0 & y_1 & y_3 \\
y_2 & y_3 & 0
\end{pmatrix}
\] $G = \{y_0y_3^2 + y_1y_2^2 = 0\}$.

The line $y_3 = y_2 = 0$ is singular, so this case must be excluded.
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v) \[
\begin{pmatrix}
\gamma_0 & 0 & \gamma_1 \\
0 & \gamma_1 & \gamma_2 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{pmatrix}
\]
\[G = \{y_0y_1y_3 - y_0y_2^2 - y_1^3 = 0\}.
\]

$G$ has two singular points, one node at \((y_1 = y_2 = y_3 = 0)\), and a singular point of type $A_5$ at \((y_0 = y_1 = y_2 = 0)\).

vi) \[
\begin{pmatrix}
0 & \gamma_0 & \gamma_1 \\
\gamma_0 & \gamma_1 & \gamma_2 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{pmatrix}
\]
\[G = \{(y_0^2 + y_1^3 - 2y_0y_1y_2)\}
\]

The line $y_0 = y_1 = 0$ is singular: moreover the plane \((y_0 = 0)\) is in the tangent cone at every point of the singular line so that this cubic is not projectively equivalent to the one in iv).

Theorem 1.3. Every symmetric cubic $G$ has only one symmetrization. Moreover, $G$ has exactly one finite irreducible double cover ramified exactly at the singular points of $G$.

Proof. The first statement follows from the above list. Now, let $Y \to G$ be a double cover, and notice that $G$ has only singularities of type $A_n$, $n = 1,3,5$. The fibre product $Z' = Y \times_{Y_0} \tilde{G}$, where $\tilde{G}$ is a minimal desingularization of $G$, is an irreducible finite cover of $\tilde{G}$.

Therefore, if $Z$ is the normalization of $Z'$, there exists a reduced effective divisor $E$ with support in the exceptional divisor of $p:\tilde{G} \to G$, and a divisor $L$ on $\tilde{G}$ such that $2L = E$, and $f:Z \to \tilde{G}$ is the double cover of $\tilde{G}$ in $\mathcal{O}_{\tilde{G}}(L)$ branched over $E$. Hence $Z$ has only nodes as singularities, and $\omega_Z = f^*(H) + f^*(L)$, where $H$ is a hyperplane section of $\tilde{G}$. $Z$ is then a rational variety, therefore $f_*\mathcal{O}_Y$ is a Cohen-Macaulay sheaf on $\mathbb{P}^3$, with support on $G$ (cf. [2], prop. 2.18). It follows also, by the Riemann-Roch theorem, that $h^0(\mathcal{O}_G(H-L)) = 3$. Applying theorem 2.19 of [2] we prove that a double cover as above gives a symmetrization.

Conversely, consider the sheaf $F$ cokernel of $0 \to O^3 \xrightarrow{(a_i(y))} \mathbb{P}^3 \to \mathcal{O}_{\mathbb{P}^3}(1)^3 \to F \to 0$, and define $Y$ to be $\text{Spec}(\mathcal{O}_G \oplus F)$, with algebra structure given as in [2], cor. 2.17.

Q.E.D.

We observe now that if we write $X = Q \cap G$, where $G$ is symmetric, we are
giving, as $X$ is smooth, an unramified double cover $\tilde{X}$ of $X$, induced from the double cover $Y \rightarrow G$, or, equivalently a divisor $\eta \in \text{Pic}_2(X)$. Now $\eta$ is not trivial if $h^0(X, O_X(K+\eta)) = 2$, and this follows from the exact sequence

$$0 \rightarrow H^0(G(-H+L)) \rightarrow H^0(G(H+L)) \rightarrow H^0(O_X(K+\eta)) \rightarrow 0.$$ 

(1.4)

We are going now to prove a converse to this statement,

**Theorem 1.5. (Wirtinger-Coble-Recillas).**

Let $X$ be a curve of genus 4, not hyperelliptic and with no vanishing theta null. Then giving a divisor $\eta \in \text{Pic}_2(X) - \{0\}$, such that $(X,\eta)$ is not bielliptic, is equivalent to writing $X$ as the complete intersection of a smooth quadric $Q$ and a symmetric cubic $G$.

**Proof.** We have already proven that giving $G$ symmetric containing $X$ determines an $\eta \in \text{Pic}_2(X) - \{0\}$.

Conversely, consider the rational mapping $\phi: X \rightarrow \mathbb{P}^2$ given by the linear system $|K_X + \eta|$. We break up the proof in several steps.

(1.6) $|K + \eta|$ has no base points if $X$ is not hyperelliptic.

**Proof.** Let $p \in X$: since $H^1(\Omega_X(K+\eta)) = 0$ $p$ is not a base point if and only if $H^1(\Omega_X(K+\eta-p)) = 0$. By Roch's duality, this is equivalent to $|p-\eta| \neq \emptyset$. But if $q \in |p-\eta|$, then $2p = 2q$, with $q \neq p$, and $X$ is hyperelliptic. Q.E.D.

Then $\phi$ is a morphism. Denote by $C = \phi(X)$, so that $\deg C \cdot \deg \phi = 6$

(1.7) $\deg C \geq 3$ if $X$ is not hyperelliptic.

**Proof.** If $C$ is a smooth conic, then, let $D$ be the inverse image of a general point in $C$: we have $h^0(\Omega_X(D)) \geq 2$, and $D$ has degree 3. Let $D' = K - D$: by R.R. $h^0(\Omega_X(D')) = 2$, and $D' \neq D$, since $2D = K + \eta$, $\eta \neq 0$. Since $|D|$ has no base points, by the "base point free pencil trick" (cf. [11]), $H^0(O_X(K)) \cong H^0(\Omega_X(D')) \otimes H^0(\Omega_X(D))$. Since $X$ is not hyperelliptic $H^0(O_X(K)) \otimes_2 H^0(\Omega_X(2K))$ is surjective. Since $2D' = 2D$, it follows that then $H^0(\Omega_X(2D)) \otimes_2 H^0(\Omega_X(2K))$ is surjective: this is anyhow absurd since $|2K|$ gives a birational morphism. Q.E.D.

(1.8) If $C$ is a singular cubic, then its normalization $\tilde{C}$ is $\mathbb{P}^1$, and,
since $\phi$ factors through $\tilde{C}$, $X$ is hyperelliptic.

(1.9) $C$ is a smooth cubic if and only if $(X,\eta)$ is bielliptic.

Proof. If $C$ is a smooth cubic, then $\phi_*\mathcal{O}_X = \mathcal{O}_C \oplus \mathcal{O}_C(-E)$, $K_X = \phi^*(E)$, where $E$ is an effective divisor on $C$ of degree 3. Let $H$ be a hyperplane divisor on $C$. Then $\eta = \phi^*(H-E)$, but $\phi^*:\text{Pic}(C) \to \text{Pic}(X)$ is injective ([12], pag. 332), hence $(X,\eta)$ is bielliptic, with $\eta' = H-E$.

Conversely, $K_X + \eta = f^*(E+\eta')$, where $f_*\mathcal{O}_X \cong \mathcal{O}_C \oplus \mathcal{O}_C(-E)$. But, by the Leray spectral sequence for the map $f$, $H^0(X,\mathcal{O}_X(K_X+\eta)) \cong f^*(H^0(C,\mathcal{O}_C(E+\eta')))$, therefore $\phi$ factors through $f$ and an embedding of $C$ as a plane cubic. Q.E.D.

Remark 1.10. An easy computation shows that bielliptic pairs form a six-dimensional subvariety of $\mathbb{R}_4$.

We are at the last step of the proof: $\phi:X \to C$ is a birational morphism, therefore can be factored through a finite sequence of blow-ups.

We are therefore in the following situation: we are given a surface $S$ obtained from $\mathbb{P}^2(\phi:S \to \mathbb{P}^2)$ by a finite sequence of blow ups of the (possibly infinitely near) singular points of $C$, of multiplicities $r_1 \geq r_2 \geq \ldots \geq r_k$.

Let $E_1, \ldots, E_k$ be the total transforms of the exceptional curves of each blow up, $H$ the total transform of a line in $\mathbb{P}^2$.

Then, on $S$, we have

\begin{equation}
H^2 = 1, \quad H \cdot E_i = 0, \quad E_i^2 = -1, \quad E_i \cdot E_j = 0 \quad \text{for } i \neq j,
\end{equation}

\begin{equation}
K_S = -3H + \sum_{i=1}^{k} E_i, \quad X = 6H - \sum_{i=1}^{k} r_i E_i.
\end{equation}

\begin{equation}
\sum_{i=1}^{k} (r_i-1)E_i; \quad \text{by the adjunction formula } \mathcal{O}_X(K_X) = \mathcal{O}_X(3H-\Delta),
\end{equation}

and, on $X$, $\Delta = 3H - K_X = 2H + \eta$.

Therefore

i) $H^0(\mathcal{O}_X(2H-\Delta)) = 0$, hence $H^0(\mathcal{O}_S(2H-\Delta)) = 0$

ii) $\Delta \cdot K_X = 12 = \sum_{i=1}^{k} r_i(r_i-1)$.

Since $H^1(S,K_S) = H^1(S,\mathcal{O}_S) = 0$, we have an isomorphism of $H^0(\mathcal{O}_S(3H-\Delta)) \to H^0(\mathcal{O}_X(K_X))$ given by restriction.
Let $H'$ be the inverse image of a line not passing through the singular points of $C$. Since $H'(3H-A) = 3$, the exact sequence

$$0 = H^0(\mathcal{O}_S(2H-A)) \to H^0(\mathcal{O}_S(3H-A)) \to H^0(\mathcal{O}_{H'}(3H-A))$$

says that the rational map $\psi: S \to \mathbb{P}^3$ given by the linear system $|3H-A|$ embeds $H'$ as a twisted cubic.

Therefore, if $G = \psi(\mathbb{P}^2)$, $G$ contains a $2$-parameter family of twisted cubics, two of which intersect in only one point, so that $G$ is not a smooth quadric.

If $X$ has no vanishing thetanull, $G$ must be a cubic surface.

Let $F$ be the fixed part of the linear system $|3H-A|$, so that $|3H-A| = F + |M|$, $M^2 = \deg(G)$, $M \cdot F \geq 0$. Since $\psi$ embeds a general line in $\mathbb{P}^2$, $F$ is a sum of exceptional curves, each of which can be written either in the form $E_i$, or in the form $E_i - E_j$, $i > j$.

Then $M^2 = (F+M)^2 - (F+M) \cdot F = (3H-A)^2 - (3H-A) \cdot F$.

Now $(3H-A) \cdot F = -\Delta F = -\sum \frac{k}{i} (r_i-1)E_i \cdot F$, and for each component of $F$, $-\Delta F = 0$ or

$$1 \quad (-\Delta E_i = r_i - 1 \geq 1, \quad -\Delta (E_i - E_j) = r_i - r_j \geq 0 \text{ since } i > j).$$

Hence $M^2 \geq (3H-A)^2 = 9 - \sum \frac{k}{i} (r_i-1)^2 = 9 - \sum \frac{k}{i} r_i (r_i-1) - \sum \frac{k}{i} (r_i-1) = 3 + \sum \frac{k}{i} (r_i-1)$.

If $\deg(G) = M^2 = 3$, then $r_1 = 2$; if $\deg(G) = 2$, then the only other possibility is $r_1 = 3$, $r_2 = 2$.

We can assume from now on $r_1 = 2$. By the exact sequence

$$0 \to \mathcal{O}_S(4H-2A-X) \to \mathcal{O}_S(4H-2A) \to \mathcal{O}_X \to 0$$

since $H^1(\mathcal{O}_S(4H-2A-X)) = H^1(\mathcal{O}_S(-2H)) = 0$, we conclude that

$$|4H-2A| \text{ has dimension } 0.$$
therefore, if we denote still by \( \psi \) the rational map \( \psi : \mathbb{G} \to \mathbb{P}^2 \) given by \( |H - L| \), \( \psi \) embeds \( \mathbb{G}' \) as a smooth plane cubic. Since through any two general points \( x, y \) of \( \mathbb{G} \) there passes a plane section \( \mathbb{G}' \) as above, \( \psi \) is birational, and the inverse map \( \psi : \mathbb{P}^2 \to \mathbb{G} \) is given by a system of plane cubics. Since \( \psi \) is a morphism, clearly \( |K + \eta| \) gives a birational morphism and \( (X, \eta) \), by (1.9), is not bi-elliptic.

Q.E.D.

Just for completeness, we indicate, for the three types of symmetric cubics, which are the systems of plane cubics giving the rational map \( \psi \).

In case ii) we consider the six points of intersection of four independent lines in \( \mathbb{P}^2 \), and we blow then up to get \( S \simeq \tilde{\mathbb{G}} \), with \( A = \sum_{i=1}^{6} E_i \), and \( D \in |4H - 2\Delta| \) given by the of the proper transforms of the four lines (cf. e.g. [3]).

In case iii): take three lines \( L_1, L_2, L_3 \) in general position in \( \mathbb{P}^2 \) and blow up \( \mathbb{P}^2 \) at the three points \( L_1 \cap L_2 \), at a fourth point \( P_4 \in L_3 \), and then at the 2 infinitely near points \( P_{4+i} \) lying over \( L_1 \cap L_3 = P_i \) (i=1,2) in the direction of \( L_i \). Let \( P_3 = L_1 \cap L_2 \).

Here you obtain \( S \) where \( D \in |4H - 2\Delta| \) is given by the proper transform of

\[ 2L_3 + L_1 + L_2 \]

together with \( E_1 - E_5 \), \( E_2 - E_6 \), and \( S \simeq \tilde{\mathbb{G}} \).

The double cover \( Z \) of \( S \) is smooth, being branched on the proper transforms of \( L_1, L_2, \) and \( (E_1 - E_5) (E_2 - E_6) \), i.e. on a smooth divisor consisting of four (-2) rational curves, while the finite cover \( Y \) has just a node as singularity, lying over the \( A_3 \) singular point of \( \mathbb{G} \).

Since we believe that case v) is the least known, we explain how to obtain the mapping \( \psi \).

Choose \( w_0, w_1, w_2 \) a basis of \( H^0(\mathbb{G}(H-L)) \) such that (cf. [2], cor. 2.17) the following relations hold:

\[
\begin{align*}
  y_0w_0 + y_1w_2 &= 0 \\
  y_1w_1 + y_2w_2 &= 0 \\
  y_1w_0 + y_2w_1 + y_3w_2 &= 0 \\
\end{align*}
\]

We can solve these as linear equations in \( y_0, \ldots, y_3 \) and express them as homogeneous polynomials in \( (w_0, w_1, w_2) \).

We get \( y_0 = w_3^3, \ y_1 = -w_0w_2^2, \ y_2 = w_0w_1w_2, \ y_3 = w_0(w_0w_2-w_1^2) \), and this is an expression of \( \psi \) in appropriate coordinates on \( \mathbb{P}^2 \) and \( \mathbb{P}^3 \).

The system of cubics has 2 base points, namely \( \{w_2 = w_0 = 0\} = P, \) and \( \{w_2 = w_1 = 0\} = P' \), and a general cubic of the system is smooth at \( P, P' \):
but to obtain a system free of base points one has to blow up three times over \( P \) at the points where the line \( \{w_0 = 0\} \) (whose proper transform will be denoted by \( L_0 \)) passes and three times over \( P' \) at the points where the conic \( \{w_0w_2 - w_1^2 = 0\} \) passes through.

Denote by \( L_2 \) the proper transform of the line \( \{w_0 = 0\} \).

We get thus \( E_1, E_2, E_3, E'_1, E'_2, E'_3 \) on \( S \), and we notice that \( L_2^2 = L_0^2 = -2 \), \( L_2 \) intersects transversally in exactly one point \( (E_1 - E_2) \), \( (E'_2 - E'_3) \); \( L_0 \) intersects \( E_3 \) transversally in exactly one point.

The total transform of the quartic \( \{w_0w_2^3 = 0\} \) is thus
\[
L_0 + (E_1 + E_2 + E_3) + 3(E_1' + E_2' + E_1') + 3L_2 \quad \text{i.e.} \quad 3L_2 + L_0 + 2\Delta + (E_1' - E_2') + 2(E_2' - E_3') + 2(E_1' - E_2') \n\]

The normal double cover \( Z \) of \( S = \widetilde{\pi} \) is thus ramified on
\[
L_2^2 + L_0^2 + (E_1 - E_2) + (E_2 - E_3), \quad \text{hence} \quad Z \text{ is smooth, and the finite cover } Y \text{ of } G \text{ has just a singular point of type } A_2 \text{ lying over the singular point of } G \text{ of type } A_5.
\]

The meaning of theorem 1.5 in terms of \( R_4 \) is the following

Theorem 1.18. \( R_4 \) is an irreducible variety, birational to the quotient
\[
P(\text{Sym}^2(V_4))/S_4, \quad \text{where } V_4 \text{ is the standard representation of}
\]

Proof. Since \( R_4 \) is a finite cover of \( M_4 \), it is pure dimensional.

Let \( A \) be the open set of \( R_4 \) corresponding to pairs \((X,\eta)\) such that:

i) \( X \) is not hyperelliptic

ii) \( X \) has no vanishing thetanull

iii) \( (X,\eta) \) is not bielliptic.

By remark 1.10 and the considerations made at the beginning of the paragraph \( A \) is dense.

Let \( Q \) be a fixed smooth quadric in \( \mathbb{P}^3 \), and let \( B \) be the open set in the space of symmetric 3×3 matrices of linear forms such that, if \((a_{ij}(y)) \in B\), \( G = \text{det}(a_{ij}(y)) \) is a normal cubic and \( X = \pi Q \) is a smooth curve of degree 6.

In view of theorem 1.5, there is a morphism of \( B \) onto \( A \) which is a quotient by the previously described action of \( \text{GL}(3) \) on \( B \). Hence \( R_4 \) is irreducible (actually this was known already).

Moreover, let \( B' \) be the open subset of \( B \) such that \( G \) is a 4-nodal cubic (case ii)), and \( A' \) its image in \( R_4 \): \( A' \) is again dense, being non-empty.

Assume that \((X,\eta)\) corresponds to giving generators \( Q, G \) of the ideal of \( X \) in \( \mathbb{P}^3 \) such that \( G \) is a symmetric cubic, and analogously \((X',\eta')\) corresponds to \((Q',G')\); if \( f : X \to X' \) is an isomorphism such that \( f^*(\eta') = \eta \), then
$f$ is induced by a projectivity $g : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ such that $Q = g^*(Q')$, $G = g^*(G')$, by Theorem 1.5, and, conversely, such a projectivity induces an isomorphism of the pair $(X, \eta)$ with the pair $(X', \eta')$.

Since all 4-nodal cubics are projectively equivalent, we can fix the 4-nodal cubic to be $G_0$, the cubic of equation $\sigma_3(y) = \sum_{i=1}^{4} \frac{y_1^2 y_2 y_3 y_4}{y_i} = 0$.

Consider now the open set $\Lambda$ in $\mathbb{P}(\text{Sym}^2(V_4))$ corresponding to the quadrics $Q$ in $\mathbb{P}(V_4)$ such that $Q \cap G_0$ is a smooth sextic curve $X$. We get thus a morphism $f$ of $\Lambda$ into $\mathbb{P}^4$, with $f(\Lambda) = \Lambda'$, such that $Q$, $Q'$ map to the same pair $(X, \eta)$ if and only if there exists $g \in \text{PGL}(4)$ such that $g(G_0) = G_0$, $g(Q) = Q'$. We conclude the proof since it is well-known that $S_4$ is the group of projective automorphisms of $G_0$. 

Q.E.D.

We want to find now a dominant rational map of $\mathbb{P}^{10}$ to $M_{4,1}$. To do this, recall that a curve of genus 4 $X$ which is not hyperelliptic and has no vanishing thetanull is a smooth divisor of bidegree $(3,3)$ on $Q = \mathbb{P}^1 \times \mathbb{P}^1$.

Fix three points $\omega, 0, 1$ in $\mathbb{P}^1$ and let $p \in Q$ be the point $(\omega, \omega)$, $M$ be the (unordered) set of five points $\{(\omega, \omega), (\omega, 0), (\omega, 1), (0, \omega), (1, \omega)\}$.

Given a general $[C', p'] \in M_{4,1}$ we can assume to have chosen coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $p = (\omega, \omega)$, and that the two lines $\{\omega\} \times \mathbb{P}^1$, $\mathbb{P}^1 \times \{\omega\}$ intersect $C'$ in three distinct points. Let $I_M$ be the ideal sheaf of $M$ on $Q$. Therefore if we take the linear system $|G| = |I_M(3,3)|$ we obtain a rational dominant map of $|G|$ onto $M_{4,1}$ just by sending $C \in |G|$ to the pair $[C, (\omega, \omega)]$.

Assume now that two pairs $C, C' \in |G|$ are isomorphic: then there exists an automorphism $g$ of $\mathbb{P}^1 \times \mathbb{P}^1$ which leaves $(\omega, \omega)$ fixed and such that $g(C) = C'$, since all the automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ are induced, via the Segre embedding, by automorphisms of $\mathbb{P}^3$.

But now $g$ leaves the set $A = (\mathbb{P}^1 \times (\omega)) \cup ((\omega) \times \mathbb{P}^1)$ invariant, and, since $M = A \cap C = A \cap C'$, $g(M) = M$.

Let us choose affine coordinates $(x, y)$ on $\mathbb{P}^1 \times \mathbb{P}^1 - \Lambda$: then $g$ belongs to the group generated by the involution $g_3$ such that $g_3(x, y) = (y, x)$, and by the two involutions $g_1, g_2$ such that $g_1(x, y) = (1-x, y)$, $g_2(x, y) = (x, 1-y)$.

Let $r = g_3 g_1$: then $r$ has period 4; if we set $s = g_3$, then $s^2 = 1$, $r^4 = 1$, $sr^3 = rs = g_2$, $r^2 = g_1 g_2$, and our group is the dihedral group $D_4$.

We can thus reformulate our discussion with the following

**Theorem 1.19.** $M_{4,1}$ is the quotient of $\mathbb{P}^{10}$ by a suitable action of the dihedral group $D_4$. 
For the geometrical construction underlying theorem C, consider again a non-hyperelliptic curve \( C \subset \mathbb{P}^1 \times \mathbb{P}^1 = Q \). In this picture we have in \( Q \) a family of lines of the form \( \mathbb{P}^1 \times \{a\}, \ a \in \mathbb{P}^1 \), and another of the form \( \{b\} \times \mathbb{P}^1, \ b \in \mathbb{P}^1 \), which we visualize as being orthogonal to the first.

**Definition 1.20.** A rectangle \( R \) in \( Q \) is the union of four distinct lines in \( Q \), of the form \( R = (\mathbb{P}^1 \times \{b\}) \cup (\mathbb{P}^1 \times \{b'\}) \cup (\{a\} \times \mathbb{P}^1) \cup (\{a'\} \times \mathbb{P}^1) \). Its vertices are the four points \((a,b), (a,b'), (a',b), (a'b')\) and if they all belong to \( C \) we shall say that \( R \) is inscribed into \( C \).

**Theorem 1.21.** A general curve \( C \) of genus 4 admits 6 inscribed rectangles (lying in the unique quadric \( Q \) containing the canonical image of \( C \)).

**Proof.** Consider \( C^4 \) and the four projections \( f_i : C^4 \to C \) (\( i = 1, \ldots, 4 \)) on the four factors of the product. Let moreover \( p, p' : Q \to \mathbb{P}^1 \) be the two natural projections: they define two divisors of degree 3 on \( C \), which we denote, respectively, by \( D \) and \( D' \).

Let \( D_i \) be the divisor on \( C^4 \) such that \( D_i = f_i^*(D) \) (resp. \( D'_i = f_i^*(D') \)); let moreover \( \Delta_{ij} \subset C^4 \) be \( \{(y_1, y_2, y_3, y_4) \mid y_i = y_j\} \).

Consider in \( C^4 \) the subvariety \( W = \{(y_1, y_2, y_3, y_4) \mid p(y_1) = p(y_2), p(y_3) = p(y_4), p'(y_1) = p'(y_4), p'(y_2) = p'(y_3)\} \).

Given an inscribed rectangle \( R \) and a vertex \( x \) of \( R \) one determines a unique point \( y = (y_1, y_2, y_3, y_4) \) in \( W \) with \( y_1 = x \), and such that \( y \in W - \bigcup_{i<j} \Delta_{ij} \).

Conversely, if \( y \in W - \Delta_{12} - \Delta_{34} - \Delta_{14} - \Delta_{23} \), then also \( y_1 \neq y_3 \) since otherwise \( p'(y_1) = p'(y_3) = p'(y_2) \), and, since \( p(y_1) = p(y_2) \), one would have \( y_1 = y_2 \); analogously one has \( y_2 \neq y_4 \).

Therefore the points of \( W - \Delta_{12} - \Delta_{34} - \Delta_{14} - \Delta_{23} \) are in a bijection with the pairs \( (R, x) \) where \( R \) is a rectangle inscribed into \( C \), \( x \) is a vertex of \( R \). Now the above mentioned set is the complete intersection of four divisors.

In fact, consider in \( C^2 \) the divisor \( B = \{(y_1, y_2) \mid p(y_1) = p(y_2)\} \). \( B = \Delta + \Gamma \), where \( \Delta \) is the diagonal of \( C \times C \), and \( \Gamma \) is smooth away from \( \Delta \) since \( p \) is a covering of degree equal to three.

\( B \) is the pull back of the diagonal in \( \mathbb{P}^1 \times \mathbb{P}^1 \) under the morphism \( p \times p : C^2 \to (\mathbb{P}^1)^2 \), therefore its class as a divisor on \( C^2 \), using our previous notations \( (f_i : C^2 \to C, \ i = 1, 2, \) being the two projections), is just \( D_1 + D_2 \). Since \( C \) has genus four \( \Delta^2 = -6 \), moreover \( B \cdot \Delta = 6 \), so that \( \Gamma \cdot \Delta = 12 \).
Consider the monodromy of \( p: C \to \mathbb{P}^1 \): if \( C \) is general, then \( p \) has only ordinary ramification, i.e.

a) \( \Gamma \) and \( \Delta \) intersect transversally

b) the monodromy of \( p \) is generated by transpositions

Since \( C \) is connected b) implies that the monodromy is the full symmetric group, hence, in general, \( \Gamma \) is smooth, irreducible, transversal to \( \Delta \) in the points corresponding to the ramification points of \( p \).

Considering the projection \( p' \) instead of \( p \), we define analogously \( \Gamma' \subset C^2 \).

Let \( \Gamma_{ij} = (f_i \times f_j)^*(\Gamma) \), \( \Gamma'_{hk} = (f_h \times f_k)^*(\Gamma') \).

Then we claim that \( W - \Delta_{12} - \Delta_{34} - \Delta_{14} - \Delta_{23} = \Gamma_{12} \cap \Gamma_{34} \cap \Gamma'_{14} \cap \Gamma'_{23} \), and the intersection is transversal, for \( C \) general.

In fact, if \((y_1, y_2) \in \Delta_{12} \cap \Gamma_{12}\), then \( y_1 = y_2 \) and \( y_1 \) is a ramification point of \( p \): since \( y = (y_1, \ldots, y_4) \in W \) it follows that \( y_3 = y_4 \), hence \( y_3 \) is a second ramification point of \( p \), and \( p'(y_1) = p'(y_3) \).

It is easy to see that curves \( C \) of type \((3,3)\) in \( Q \) such that the above situation can hold form a proper subvariety in the linear system \( |O_Q(3,3)| \) gives a transversal intersection for \( C \) general, we consider the variety \( \Lambda \subset |O_Q(3,3)| \times Q^4 \) defined by

\[
\Lambda = \{ (C, y_1, y_2, y_3, y_4) \mid y_i \in C, \quad i=1,\ldots,4, \quad p(y_1) = p(y_2), \quad p(y_3) = p(y_4), \\
p'(y_2) = p'(y_3), \quad p'(y_1) = p'(y_4) \}
\]

\( \Lambda \) is of dimension 15 and smooth at the general point, hence our assertion is proven if the projection of \( \Lambda \) on \( |O_Q(3,3)| \) is surjective: but if this were not the case, for \( C \) general, \( \Gamma_{12} \cap \Gamma_{34} \cap \Gamma'_{14} \cap \Gamma'_{23} \) would be empty.

Finally we compute:

\[
\Gamma_{12} \cap \Gamma_{34} \cap \Gamma'_{14} \cap \Gamma'_{23} = (D_1 + D_2 - \Delta_{12}) \cdot (D_3 + D_4 - \Delta_{34}) \cdot (D'_1 + D'_4 - \Delta_{14}) \cdot \\
(D'_2 + D'_3 - \Delta_{23}) = 2 \cdot 3^4 - 2 \cdot 3^3 \cdot 4 + 2 \cdot 3^2 \cdot 6 - 2 \cdot 3 \cdot 4 - 6 = 3^3 \cdot 2 - 30 = 24; \quad \text{in fact} \quad \Delta_{12} \cdot \Delta_{34} \cdot \Delta_{14} \cdot \Delta_{23} \quad \text{equals the self-intersection of} \quad \Delta \quad \text{in} \quad C \times C.
\]

Q.E.D.

Theorem C is now a straightforward consequence of theorem 1.21.

Namely, consider in \( Q = \mathbb{P}^1 \times \mathbb{P}^1 \) the following set of six points:

\[
M' = \{ (\infty, \infty), \ (0,0), \ (0,\infty), \ (\infty, 0), \ (1,\infty), \ (\infty,1) \}.
\]

Let \( |C'| \) be the linear system \( |I_M(3,3)| \): we can choose affine coordinates \((x,y)\) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) \{-\infty\} \times \mathbb{P}^1 \times \{-\infty\}. \) Then \( |C'| \) is the projective space associated with the vector space \( U \) spanned by the monomials

\[
\begin{align*}
x, \quad x^2, \quad x^2y, \quad x^3y(1-y) \\
y, \quad y^2, \quad y^2x, \quad y^3x(1-x) \quad \text{and} \\
xy, \quad x^2y \quad \text{as monomials.}
\end{align*}
\]
These monomials are permuted by the action on $|G'|$ induced by the automorphism $s: Q \to Q$ such that $s(x, y) = (y, x)$. It is then obvious that $|G'|/s = \mathbb{P}(U)/s$ is birational to $\mathbb{P}^9$. We conclude then this paragraph with

**Theorem C.** $\mathcal{M}_4$ has covering of degree 24 by a rational variety. More precisely, the rational map of $|G'| \to \mathcal{M}_4$ is a covering of degree 48 which factors through the action of $s$ on $|G'|$, and a general point of $|G'|/s$ corresponds to the datum of a triple $(C, R, p)$ where $C$ is a curve of genus 4, $R$ is a rectangle inscribed into $C$, $p$ is a vertex of $R$.

**Proof.** To a curve $C \in |G'|$ we associate the triple

$$(C, (\mathbb{P}^1 \times \{0, \infty\} \cup \{(0, \infty) \times \mathbb{P}^1\}, \{0, \infty\}).$$

Assume now that $C, C'$ give isomorphic triples: then there exists $g \in \text{Aut}(Q)$ such that $g(\infty, \infty) = (\infty, \infty)$, $g(C) = C'$, and for the rectangle $R = (\mathbb{P}^1 \times \{0, \infty\}) \cup \{(0, \infty) \times \mathbb{P}^1\}$ one has $g(R) = R$. In particular, $g(M') = M'$, so that necessarily $g$ is either the identity or the involution $s$. The fact the degree of the rational map of $|G'|$ onto $\mathcal{M}_4$ is 48 follows immediately from theorem 1.21.

Q.E.D.
52. RATIONALITY OF THE INVARIANT SUBFIELDS.

Before turning to prove the rationality of $R_4$, we first state a more general auxiliary result.

Let $V$ be the standard permutation representation of the symmetric group $S_n$, $V^n_m$ the direct sum of $m$ copies of $V$. Then the field of rational functions on $V^n_m$, $k(V^n_m)$, can be written as $k(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{mn})$ and a permutation $\tau$ acts on $x_{ij}$ by sending it to $x_{i\tau(j)}$. Consider the following invariant rational functions, where $\sigma_i$ denotes the $i$-th elementary symmetric function, and a variable with a cap has to be omitted:

$$\sigma_i^{(2)}(x_{ij}, \ldots, x_{in}) \quad (i = 1, \ldots, n)$$

$$\sigma_i^{(h)} = \sum_{j=1}^{n} x_{hj} x_{lj}^{(1)} \quad (h = 2, \ldots, m)$$

$$\sigma_i^{(h)} = \sum_{j=1}^{n} x_{hj} \sigma_{i-1}(x_{11}, \ldots, x_{lj}, \ldots, x_{1n}) \quad (h = 2, \ldots, m; i = 2, \ldots, m).$$

Lemma 2.2. The invariant subfield $k(V^n_m)_n$ is a rational field: more precisely the $nm$ functions given by 2.1 form a basis of the purely transcendental extension over $k$.

Proof. $\sigma', \sigma^{(2)}, \ldots, \sigma^{(m)}$ determine a morphism $\psi: V^n_m \rightarrow (\mathbb{A}^n)^m$ and to prove that $\psi$ induces a birational map of $V^n_m/S_n$ onto the affine space $(\mathbb{A}^n)^m$ it is enough to prove that on a Zariski open set of $V^n_m$ $\psi(x) = \psi(y)$ if and only if there exists $\tau \in S_n$ such that $\tau(x) = y$.

The "if" part being obvious, let's assume that $\psi(x) = \psi(y)$, then, in particular, $\sigma'(x) = \sigma'(y)$.

By virtue of the fundamental theorem on symmetric functions, we can assume, acting on $y$ by a suitable $\tau \in S_n$, that $x_{1j} = y_{1j}$ for $j = 1, \ldots, n$.

Let us set for convenience $z_j = x_{1j}$ ($j = 1, \ldots, n$). Then the variables $x_{hj}$, $y_{hj}$ ($h = 2, \ldots, m; j = 1, \ldots, n$) are solutions, by 2.1, of the same system of $n(m-1)$ linear non-homogeneous equations, hence they are equal if the determinant of the system is non-zero.

The system being given by the matrix

$$\begin{pmatrix}
z_1, & \ldots, & z_n \\
\sigma_1(z_2, \ldots, z_n), & \ldots, & \sigma_1(z_1, \ldots, z_{n-1}) \\
\vdots & & \vdots \\
\sigma_{n-1}(z_2, \ldots, z_n), & \ldots, & \sigma_{n-1}(z_1, \ldots, z_{n-1})
\end{pmatrix}$$

is non-zero.
it suffices to verify that the determinant of the matrix \((2.3)_n\) is not identically zero. We prove this by induction on \(n\), since for \(n = 2\) we get
\[
\det \begin{pmatrix} z_1 & z_2 \\ z_2 & z_1 \end{pmatrix} = z_1^2 - z_2^2.
\]
For bigger \(n\), the determinant of \((2.3)_n\) modulo \(z_n\), is given, up to sign, by the product of \(z_1 \cdots z_{n-1} = \sigma_{n-1}(z_1, \ldots, z_{n-1})\) times the determinant of \((2.3)_{n-1}\).

\[\text{Q.E.D.}\]

**Theorem A.** \(R_4\) is a rational variety.

**Proof.** In view of theorem 1.18 we have to show the rationality of \(\mathbb{P}(\text{Sym}^2(V_4))/S_4\). We use here the fact that \(S_4\) has a normal subgroup \(G \simeq (\mathbb{Z}/2)^2\) given by the double cycles in \(S_4\); the quotient \(S_4/G\) is isomorphic to \(S_3\) and in this way any representation of \(S_3\) induces canonically a representation of \(S_4\) that we shall denote by the same symbol.

Since the action of \(S_4\) on \(\text{Sym}^2(V_4)\) is linear, it is clearly sufficient to prove the rationality of the quotient \(\text{Sym}^2(V_4)/S_4\).

We subdivide the proof in four steps, noticing that we have the following chain of inclusions
\[(2.4) \quad k(\text{Sym}^2(V_4)) \supset k(\text{Sym}^2(V_4))^G \supset k(\text{Sym}^2(V_4))^{S_4} = \left(\frac{k(\text{Sym}^2(V_4))^G}{S_3}\right).
\]

Let \(W_4\) be the irreducible \(S_4\)-submodule of \(V_4\) generated by \(x_1 - x_2, x_2 - x_3, x_3 - x_4\): \(V_4 = 1 \oplus W_4\), \(1\) being the trivial one dimensional representation spanned by \(\sigma_1(x_1, \ldots, x_4)\).

**Step I.** \(\text{Sym}^2(V_4) \simeq 1 \oplus W_4^2 \oplus V_3^2\)

**Proof.** \(\text{Sym}^2(V_4) \simeq V_4^t \oplus V_3^t \oplus W_4^t\) where \(V_4^t\) is spanned by \(x_1^2, x_2^2, x_3^2, x_4^2\), \(V_3^t\) is spanned by \(y_1 = x_1 x_2 + x_3 x_4, y_2 = x_1 x_3 + x_2 x_4, y_3 = x_1 x_4 + x_2 x_3\), \(W_4^t\) is spanned by \(w_1 = x_1 x_2 - x_3 x_4, w_2 = x_1 x_3 - x_2 x_4, w_3 = x_1 x_4 - x_2 x_3\).

\(V_4^t\) is clearly isomorphic to \(V_4\); also, since \(G\) acts trivially on \(V_3^t\), \(V_3^t\) is induced by a representation of \(S_3\).

\(V_3^t\) has as basis three vectors corresponding to the three non-trivial double cycles of \(S_4\), and the action of \(S_4\) on the basis is given by conjugation in \(S_4\) (\(G\) acts trivially being abelian).

Observing that the transposition \((1,4)\) permutes \(y_1\) with \(y_2\) and leaves \(y_3\) fixed, \((1,2)\) leaves \(y_1\) fixed and permutes \(y_2\) with \(y_3\), we conclude that \(V_3^t\) is isomorphic to \(V_3\).
On \( W_4' \) we have the following actions:

\[
\begin{align*}
(12)(34) \text{ acts by} & \quad \begin{cases} 
  w_1 \mapsto w_1 \\
  w_2 \mapsto -w_2 \\
  w_3 \mapsto -w_3 
\end{cases}, & 
(12) \text{ by} \quad \begin{cases} 
  w_1 \mapsto w_1 \\
  w_2 \mapsto -w_3 \\
  w_3 \mapsto -w_2 
\end{cases} \\
(123) \text{ by} \quad \begin{cases} 
  w_1 \mapsto w_3 \\
  w_2 \mapsto w_1 \\
  w_3 \mapsto -w_2 
\end{cases}, & 
(1234) \text{ by} \quad \begin{cases} 
  w_1 \mapsto -w_3 \\
  w_2 \mapsto w_2 \\
  w_3 \mapsto w_1 
\end{cases}
\end{align*}
\]

Let \( \chi' \) be the character of \( W_4' \): the character of \( W_4 \) equals the character \( \chi \) of \( V_4 \) minus 1, hence we conclude that \( \chi - 1 = \chi' \) by computing explicitly the table of characters.

<table>
<thead>
<tr>
<th>Conjugacy classes</th>
<th>id</th>
<th>(12)(34)</th>
<th>(12)</th>
<th>(123)</th>
<th>(1234)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi' )</td>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi )</td>
<td>4</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

If the characteristic of \( k \) is different from 2,3, this implies that \( W_4' \cong W_4 \); in characteristic 3, this is also true, because both representations are irreducible: in fact (cf. [15], pag. 155) their modular characters are indecomposable. \( \Box \)

To compute \( k(\text{Sym}^2(V_4))^G \), in view of step I, suffices

**Step II.** \( k(W_4^2)^G = L \), if \( w_1', w_2', w_3', w_1, w_2, w_3 \) are coordinates on \( W_4 \oplus W_4 \), is generated by

\[
(2.5) \quad w_1^2, w_2^2, w_1 w_2 w_3, w_i w_j ^i (i=1,2,3)
\]

**Proof.** The six given functions are \( G \)-invariant, and \( k(W_4^2) = L(w_1, w_2) \), so we have an extension of degree 4 and \( L \) is the whole subfield of \( G \)-invariants. \( \Box \)

**Step III.** Let \( F \) be the subfield of \( L \) generated by \( w_1^2, w_2^2, w_3^2, w_i w_j ^i \). Then \( L \)
is a quadratic extension of $F$ given by $F(t)$, where $t = w_1w_2w_3$. $F = k(V_3^2)$ as a representation of $S_3 = S_4/G$.

**Proof.** Clearly $t \notin F$, $t^2 = (w_1^2)(w_2^2)(w_3^2)$. Also the action of $S_4$ on $V_3$ differs from the one of $S_4$ on $W_3$ only up to sign, i.e., as it is easy to verify, $S_4$ acts by permuting the basis given by $y_1, y_2, y_3$, and if $\tau(y_i) = y_j$, then $\tau(w_i) = \pm w_j$, hence $\tau(w_i^2) = w_j^2$, $\tau(w_i w_j) = w_j w_i$.

**Step IV.** Let $M$ be the field generated by $w_i^2, w_i w_j, y_i$ $(i=1,2,3)$. $M$ is a purely transcendental extension of $k$, $M \cong k(V_3^2)$, $k(Sym^2 V_4)^{S_4} = M^{S_3}(t,\sigma)$, where $\sigma = \sigma_1(x_1, \ldots, x_4)$, $t = w_1 w_2 w_3$.

**Proof.** $k(Sym^2 V_4)^{S_4} = (M(t,\sigma))^{S_3}$, but $\sigma$ is an invariant for $S_4$ from the very beginning, while $t$ is an $S_3$ invariant by the formulas written in step I. That $M$ is isomorphic to $k(V_3^2)$ follows by step III.

End of the proof. $k(P(Sym^2 V_4))^{S_4} = S_3^2(t)$. But, by lemma 2.1, $S_3$ is a rational field with basis of transcendency $\sigma_1^1, \sigma_2^1, \sigma_3^1, \sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_1^3, \sigma_2^3, \sigma_3^3$. We conclude observing that $\sigma_1^4 = \sigma_1(w_1^2, w_2^2, w_3^2)$, hence $t^2 = \sigma_3^1$. Q.E.D.

**Theorem B.** $M_{4,1}$ is a rational variety.

**Proof.** By theorem 1.19 and the arguments preceding it we have a linear representation $\rho: D_4 \rightarrow Aut(U)$ where $U$ is 11-dimensional, and we know that $M_{4,1}$ is birational to $P(U)/D_4$.

$U$ has a basis given by the polynomials $1, x, x^2, y, y^2, xy, x^2 y, x^2 y^2, xy^3(1-x), x^3 y(1-y)$, and, if $r, s$ are the generators of $D_4$, such that $s^2 = r^4 = 1, sr^3 = rs$,

$$(2.6) \quad s(x,y) = (y,x), \quad r(x,y) = (y,1-x).$$

To decompose $U$ as a direct sum of irreducibles, since $D_4$ has order 8 and we assume $\text{char}(k) \neq 2$, we compute the character $\chi$ of $\rho$.

For $s$ we observe that $\rho(s)$ permutes the elements of the basis, leaving $1, xy, x^2 y^2$ fixed: hence $\chi(s) = 3$.

For $sr$, $sr(x,y) = (1-x, y)$ and choosing for $U$ the new basis $x, (1-x), x(1-x)$, $yx, y(1-x), yx(1-x), y^2 (1-x), y^2 x(1-x), y^3 x(1-x), x^3 y(1-y)$, we see that the
trace of $\rho(sr)$ is 3 since $sr$ permutes the first 10 elements of the basis, leaving 4 of them fixed, while $sr(x^3y(1-y)) = (1-x)^3(1-y)y = -x^3y(1-y) + \text{terms}$ of lower degree in $x$.

For $\chi(r)$, $x^2, y^2, xy, xy(1-x), x^3y(1-y)$ are easily seen to give a zero contribution (by degree considerations), while 1 is invariant, $xy \mapsto y-xy$, $x^2y^2 \mapsto y^2 - 2xy^2 + x^2y^2$, therefore $\chi(r) = 1$.

Since $r^2(x,y) = (1-x,1-y)$, the trace of $\rho(r^2)$ is easily seen (in the first given basis) to be equal to $1 - 1 + 1 - 1 + 1 - 1 - 1 - 1 - 1 = -1$.

We now put together the character $\chi$ of $\rho$ and the characters of the irreducible representations of $D_4$.

(2.7) Conjugacy classes

<table>
<thead>
<tr>
<th>Characters</th>
<th>$I$</th>
<th>${r, r^3}$</th>
<th>${s, sr^2}$</th>
<th>$sr, sr^3$</th>
<th>$r^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_2$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_3$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_4$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi'$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>$\chi$</td>
<td>11</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>-1</td>
</tr>
</tbody>
</table>

Since $\chi'$, and the $\psi_i$'s are an orthogonal basis for the space of class functions, by computing scalar products we obtain that $\chi = 3\psi_1 + \psi_3 + \psi_4 + 3\chi'$. Now $\psi_1$ is the trivial representation, hence we conclude:

**Step I.** $k(\mathbb{P}(U)/D_4)$ is a purely transcendental extension of degree 2 of the invariant subfield $k(V)^{D_4}$, where $V$ is the representation with character $3\chi' + \psi_3 + \psi_4$.

**Step II.** The cyclic subgroup generated by $r$ is normal, hence $k(V)^{D_4} = (k(V)^r)^{\mathbb{Z}/2}$.

Now, if $i$ is a square root of $-1$, then the representation $\lambda$ corresponding to $\chi'$ is given by

\[
\lambda(r) = \begin{pmatrix} i & 0 \\
0 & -i \end{pmatrix}, \quad \lambda(s) = \begin{pmatrix} 0 & 1 \\
1 & 0 \end{pmatrix}.
\]
Therefore we can choose coordinates $x_j, y_j$ ($j=1,2,3$), $z_1, z_2$ on $V$ such that $r$ acts by $x_j \mapsto ix_j$, $y_j \mapsto iy_j$, $z_h \mapsto -z_h$ while $s$ acts by permuting $x_j$ with $y_j$, while $s(z_1) = z_1$, $s(z_2) = -z_2$.

**Step III.** $k(V)^r$ is a purely transcendental extension of $k$, $K$, generated by $x_1^4, x_1/x_2, x_1/x_3, x_1 y_1, y_1/y_2, y_1/y_3, z_1(x_1^2+y_1^2), z_2(x_1^2+y_1^2)$.

Proof. $K \subset k(V)^r$, and clearly $k(V) = K(x_1)$, but $x_1^4 \in K$, so we have equality. Unfortunately in this way the action of $s$ is not linear any more: to avoid this we replace first in the basis $x_1^4$ by $u = x_1^2/y_1^2 = x_4^2/x_1 y_1^2$.

Then $s(u) = 1/u$: finally we replace $u$ by $(u-1)/u+1 = w$ so that $s(w) = -w$.

End of the proof. In this way we have a linear action of $\mathbb{Z}/2$ on an 8-dimensional vector space, and with 4 eigenvalues equal to $(+1)$, 4 equal to $(-1)$. The quotient is obviously rational. Q.E.D.
REFERENCES