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The direct image of the relative dualizing sheaf needs not be semiample

L'image directe du faisceau dualisant relatif n'est pas nécessairement semi-ample

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A B S T R A C T

We provide details for the proof of Fujita’s second theorem and prove that for a Kähler fibre space \( f : X \to B \) over a smooth projective curve \( B \), the direct image of the relative dualizing sheaf \( V := f_*\omega_{X/B} \) is the direct sum of an ample and a unitary flat bundle. We also show that \( V \) needs not be semiample, which is our main result.

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résumé
Nous donnons des détails sur la démonstration du second théorème de Fujita et nous montrons que l'image directe du fibré canonique relatif \( V := f_*\omega_{X/B} \) d'une fibration \( f : X \to B \) sur une courbe \( B \) est la somme directe d’un fibré vectoriel ample et d’un fibré vectoriel unitairement plat si l’espace total \( X \) est une variété kählérienne compacte. Nous montrons en outre que \( V \) n’est en général pas semi-ample, ce qui constitue notre résultat principal.

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1. Introduction

An important progress in classification theory was stimulated by a theorem of Fujita, who showed [3] that if \( X \) is a compact Kähler manifold and \( f : X \to B \) is a fibration onto a smooth projective curve \( B \) (i.e., \( f \) has connected fibres), then the direct image of the relative dualizing sheaf \( V := f_*\omega_{X/B} \) is a numerically semipositive vector bundle on \( B \) (over a curve, this is equivalent to saying that the bundle is nef). In this note, which is an abridged version of the article [1], we study further properties of \( V \), related to semipositivity.

Recall that a vector bundle \( V \) on a curve is numerically semipositive if and only if every quotient bundle \( Q \) of \( V \) has degree \( \deg(Q) \geq 0 \), and \( V \) is ample if and only if every quotient bundle \( Q \) of \( V \) has degree \( \deg(Q) > 0 \) ([9], Theorem 2.4, cf. [1], Prop. 7, see also [15]). In the note [4], Fujita announced the following stronger result (in fact, a flat unitary bundle is numerically positive, cf. [1], Thm. 9):

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Theorem 1.1 (Fujita’s second theorem). Let \( f : X \to B \) be a fibration of a compact Kähler manifold \( X \) over a projective curve \( B \), and consider the direct image sheaf \( V := f_* \omega_X|_B \). Then \( V \) splits as a direct sum \( V = A \oplus Q \), where \( A \) is an ample vector bundle and \( Q \) is a unitary flat bundle.\(^1\)

Fujita sketched the proof, but referred to a forthcoming article concerning the positivity of the so-called local exponents which however did not appear since. A first purpose of this article is to outline in Section 2 the missing details for the proof of the second theorem of Fujita, which are fully given in \([1]\). It is important to have in mind Fujita’s second theorem in order to understand the question posed by Fujita in 1982 (\([10]\), Problem 5): Is the direct image \( V := f_* \omega_X|_B \) semi-ample? In our particular case, where \( V = A \oplus Q \), with \( A \) ample and \( Q \) unitary flat, it simply means that the representation of the fundamental group \( \rho : \pi_1(B) \to U(\mathbb{R}, \mathbb{C}) \) associated with the flat bundle \( Q \) has finite image \([1]\), Thm. 9). The second aim of this article is to outline the proof of \([1]\), Thm. 3, stating that this question has a negative answer:

Theorem 1.2. There exists a surface \( X \) endowed with a fibration \( f : X \to B \) onto a curve \( B \) of genus \( \geq 3 \), and with fibres of genus \( 6 \), such that \( V := f_* \omega_X|_B \) splits as a direct sum \( V = A \oplus Q_1 \oplus Q_2 \), where the summands \( Q_i \) \((i = 1, 2)\) are flat unitary rank-2 bundles having infinite monodromy group and where \( A \) is ample. In particular, \( V \) is not semi-ample.

2. Fujita’s second theorem

Let \( B \) be a smooth complex projective curve. A holomorphic vector bundle over it is identified with its sheaf of holomorphic sections. Assume now that \( f : X \to B \) is a fibration of a compact Kähler manifold \( X \) over \( B \), and consider the invertible sheaf \( \omega := \omega_X|_B \). By Hironaka’s theorem, there is a sequence of blow ups with smooth centres \( \pi : \hat{X} \to X \) such that \( \hat{f} := f \circ \pi : \hat{X} \to B \) has the property that all singular fibres \( F \) are such that \( F = \sum m_iF_i \), and \( F_{\text{red}} = \sum F_i \) is a normal crossing divisor. Since \( \pi_* \omega_X(K_X) = \omega_X(K_X) \), we obtain \( \hat{f}_* \omega_{\hat{X}} = \hat{f}_* \omega_X(K_{\hat{X}} - \hat{f}^*K_B) = f_* \omega_X(K_X - f^*K_B) \). We therefore shall assume that all the reduced fibres of \( f \) are normal crossing divisors. By \([12]\), there exists a cyclic Galois covering of \( B, B' \to B = B'/G \), such that the normalization \( X' \) of the fibre product \( B' \times_B X \) admits a resolution \( X' \to X'' \) such that the resulting fibration \( f' : X' \to B' \) has all the fibres which are reduced and normal crossing divisors. It is proved in \([1]\), Prop. 13, that the sheaf \( V' := f'_* \omega_{X'}|_B \) is a subsheaf of the sheaf \( u^*(V) \), where \( V := f_* \omega_X|_B \), and the cokernel \( u^*(V)/V' \) is concentrated on the set of points corresponding to singular fibres of \( f' \). In particular, since \( V \) and \( V' \) are semipositive by Fujita’s first theorem, if \( V' \) satisfies the property that for each degree 0 quotient bundle \( Q' \) of \( V' \) then there is a splitting \( V' = E' \oplus Q' \) for the projection \( p : V' \to Q' \) and \( Q' \) is unitary flat, then \( V' \) splits as the direct sum \( V' = A \oplus Q \), where \( A \) is an ample vector bundle and \( Q \) flat unitary bundle, and the same conclusion holds also for \( V \) (cf. \([1]\), Prop. 13).

Theorem 2.1. (See Fujita, \([4]\).) Let \( f : X \to B \) be a fibration of a compact Kähler manifold \( X \) over a projective curve \( B \), and consider the direct image sheaf \( V := f_* \omega_X|_B \). Then \( V \) splits as a direct sum \( V = A \oplus Q \), where \( A \) is an ample vector bundle and \( Q \) is a unitary flat bundle.

Proof. By the above discussion it suffices to prove the theorem in the semistable case. Let \( n \) be the dimension of \( X \). Let \( V^* \) denote the restriction of \( V \) to the noncritical locus \( B^* \) of \( f \) and let \( \mathcal{H}^* = (\mathcal{H}^*, \nabla, F) \) denote the variation of polarized Hodge structures underlying the local system \( R^{n-1}f_*\mathcal{C} \) such that \( V^* = F^{n-1}(\mathcal{H}^*) \). Let \( \mathcal{D}\mathcal{H} \) be the canonical extension of \( \mathcal{H}^* \) to \( B \), characterized in the semistable case by the nipotence of the residue matrices of \( \nabla \) at the singular points. By the results of Schmid \([17]\), the Hodge filtration extends to a holomorphic filtration of \( \mathcal{D}\mathcal{H} \), also denoted by \( F \), and it is proved in \([11]\) (cf. also \([14]\) ) that \( V = F^{n-1}(\mathcal{D}H) \). The restriction to \( V^* \) of the polarization on \( \mathcal{H}^* \) induces the structure of a Hermitian vector bundle on \( V^* \). By \([19]\), Prop. 4.4, for each singular point \( s \in S := B \setminus B^* \), there exists a basis of \( V \) given by elements \( \sigma_j \) such that their norm in the flat metric outside the punctures grows at most logarithmically (cf. \([8]\)). Hence, for each quotient bundle \( Q \) of \( V^* \), denoting the restriction of \( Q \) to \( B^* \), the determinant \( \det(Q) \) admits a metric \( h \) with growth at most logarithmic at the punctures \( s \in S \). By \([11]\), Lemma 5, and \([16]\), Prop. 3.4, the degree \( \deg(\det(Q)) \) of \( Q \) is hence given by the integral of the first Chern form \( c_1(\det(Q), h) = \theta_h \) of the singular metric. One has (see \([6]\), Lecture 2):

\[
\theta_{V^*} = \theta_{\mathcal{H}^*}|_{V^*} + \sigma \Theta^\sigma = \sigma \Theta^\sigma,
\]

with \( \sigma \) denoting the second fundamental form. Griffiths proves (\([5]\), cf. \([6]\), Corollary 5) that the curvature of the dual \( (V^*)^* \) is semi-negative, since its local expression is of the form \( \partial h^*(z) \partial \overline{z} \wedge dz \), where \( h^*(z) \) is a semipositive definite Hermitian matrix (cf. \([1]\), Section 2, for a discussion on the various notions of curvature positivity). In particular, the curvature \( \theta_{V^*} \) of \( V^* \) is semipositive. The dual of the principle ‘curvature decreases in Hermitian subbundles’ \([7]\) implies that the curvature of \( Q^* \) is also semipositive. Therefore we can conclude that, since \( \deg(Q) = 0 \), the quotient \( Q^* \) carries a flat connection. Moreover, using the Hermitian splitting, we can view \( Q^* \) as a subbundle of \( V^* \). Since the local monodromy of \( Q^* \) at the

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\(^1\) We remark that, while unitary flatness of a bundle implies numerical semipositivity, flatness alone does not, as shown by the following result (\([11]\), Thm. 4): Let \( f : X \to B \) be a Kodaira fibration, i.e., \( X \) is a surface and all the fibres of \( f \) are smooth curves not all isomorphic to each other. Then the direct image sheaf \( V := f_* \omega_X|_B \) has strictly positive degree hence \( \mathcal{H} := R^1 f_*\mathcal{C} \otimes \mathcal{O}_B \) is a flat bundle which is not numerically semipositive.
singular points \( s \in S \) is unipotent (the fibration \( f \) being semistable) and moreover unitary, the local monodromy at each \( s \in S \) is trivial. Hence we conclude that \( Q^* \) has a flat extension to \( B \) which we denote by \( \hat{Q} \). This extension is tautologically the canonical extension of \( Q^* \) and hence we can view \( \hat{Q} \) as a subbundle of \( DF \). Since \( Q^* \subseteq F^{n-1}(H^*) \), we have the inclusion \( \hat{Q} \subseteq V = F^{n-1}(DH) \subseteq DF \), and we obtain a homomorphism \( \psi : \hat{Q} \to Q \) composing the inclusion \( \hat{Q} \to V \) with the surjection \( V \to Q \). From the fact that \( \psi \) is an isomorphism over \( B^* \), we infer that \( \psi \) is an isomorphism: since \( \det(\psi) \) is not identically zero, and is a section of a degree zero line bundle. Hence we conclude that the composition of \( \psi^{-1} \) with the inclusion \( \hat{Q} \to V \) gives then the desired splitting of the surjection \( V \to Q \).  

3. A counterexample to Fujita’s question

Consider the fibration of projective curves \( \varphi : Y \to P_{[y_0:y_1]} =: P \) defined by the minimal resolution of singularities of \( \Sigma \to P \), where \( \Sigma \) is the singular \( \mu_7 \)-Galois cover of \( P^1_{[y_0:y_1]} \times P \) (\( \mu_7 \) denoting the cyclic group of order 7), given by the equation:

\[
\psi^2 = \rho_{10}(y_1 - y_0)(x_0 y_1 - x_1 y_0)^4 x_0^3.
\]

Let \( P^* = P \setminus \{0, 1, \infty\} \) and let \( \psi : Y^* \to P^* \) denote the restriction of \( \varphi \) to \( \varphi^{-1}(P^*) =: Y^* \). The group \( \mu_7 \) acts fibrewise on the family and \( V := \varphi_*((\omega_{P/Y})^\vee) \) as well as \( H^* = R^1 \varphi_* C_Y \otimes O_P \) splits according to the eigenspaces for the characters \( \chi_j : \mu_7 \to \mathbb{C}^*, \sigma \mapsto \varepsilon^{2\pi i j} (j = 0, 1, \ldots, 6) \) (we shall denote by \( V_j \), resp. \( H_j^* \), the \( \chi_j \)-eigenspace of \( V \), resp. \( H^* \)). The fibres \( H_j^*(x) \) of \( H_j^* \) over a point \( x \in P^* \) are the vector spaces \( H^1(C_x, C) \), which have dimension 2, and we have \( H_j(x) = H^0(C_x, \Omega^1_{C_x}) \otimes \mathbb{C} \) for \( x \in P^* \). It is proven in [1] that in the case \( j = 1 \) there is a basis of \( H^0(C_x, \Omega^1_{C_x}) \) given by \( \eta_1 \) and \( \eta_2 \), where (in affine coordinates):

\[
\eta_1 = \frac{x - y}{(y_1 - y_0)(x_0 y_1 - x_1 y_0)}, \quad \eta_2 = \frac{y - x}{(y_1 - y_0)(x_0 y_1 - x_1 y_0)}.
\]

This implies that for any \( x \in P^* \) there is an equality \( V_1(x) = H_1^*(x) \) which implies an equality of rank-2 vector bundles \( H_1^*(V) = V_1^{[P]} \) (cf. [2]). The Gauß–Manin connection \( \nabla_1 \) on \( H_1^*(V) \) is a flat connection whose local horizontal sections are integrals of the form \( \kappa(x) = \int_c \kappa(x) \), where \( \kappa(x) = \eta_1(x) \). Hence, by the Riemann scheme of \( \Sigma \), the cyclic group \( \mathbb{Z}_7 \) acts on \( H_1^*(V) \) with \( \mathbb{Z}_7 \)-linear and hence irreducible. Therefore the monodromy group of \( \nabla_1 \) is irreducible. Moreover, by the Riemann scheme of \( D(\frac{2}{7}, \frac{3}{7}, \frac{5}{7}) \) (computed as in [13], p. 164) the local monodromy of \( \nabla_1 \) at the punctures \( 0, 1 \in P \) is a homology of order 7 and hence is of order 7 in the associated projective linear group. Hence, by the results of Schwarz [18], the monodromy of \( \nabla_1 \) is infinite. Consider now a ramified covering \( \varphi : B \to P \), locally at each branch point 0, 1, \( \infty \) of type \( x \to x^\prime \), and let \( \psi : B^* := \psi^{-1}(P^*) \to P^* \) denote the restriction of \( \psi \) to \( \psi^{-1}(P^*) \). Let \( f : X \to B \) be the minimal resolution of the fibre product \( B \times_P Y \). Again, the cyclic group \( \mu_7 \) acts fibrewise on \( X \) and it follows fibre-by-fibre that the restriction of \( \psi^* \) to \( B^* \) coincides with the pullback of the flat bundle \( \psi^*(\nabla_1) \). The fibration \( f \) has only three singular fibres, but around them the local monodromy of \( f_* \omega_{X/B} \) is trivial, because the local monodromy of \( \nabla_1 \) at 0, 1, \( \infty \) is of order 7. Therefore the vector bundle \( f_* \omega_{X/B} \) extends to a vector bundle \( Q_2 \subseteq f_* \omega_{X/B} \) on \( B \) carrying a flat connection. But since the monodromy of \( \nabla_1 \) is infinite, the monodromy of the flat connection on \( Q_2 \) is also infinite. Hence \( Q_1 \) is a flat (and unitary) summand in \( f_* \omega_{X/B} \) with infinite monodromy. The same arguments can be carried out for the character \( \chi_2 \), leading to another flat summand \( Q_2 \) in \( f_* \omega_{X/B} \) having also infinite monodromy, and hence leading to the proof of Theorem 1.2.

References


