SUBCANONICAL GRADED RINGS WHICH ARE NOT COHEN-MACAULAY

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This article is dedicated to Rob Lazarsfeld on the occasion of his 60-th birthday.

Abstract. We answer a question by Jonathan Wahl, giving examples of regular surfaces (so that the canonical ring is Gorenstein) with the following properties:
1) the canonical divisor $K_S \equiv rL$ is a positive multiple of an ample divisor $L$
2) the graded ring $\mathcal{R} := \mathcal{R}(X, L)$ associated to $L$ is not Cohen-Macaulay.

In the appendix Wahl shows how these examples lead to the existence of Cohen-Macaulay singularities with $K_X \mathbb{Q}$-Cartier which are not $\mathbb{Q}$-Gorenstein, since their index one cover is not Cohen-Macaulay.

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1. Introduction

The situation that we shall consider in this paper is the following: $L$ is an ample divisor on a complex projective manifold $X$ of complex dimension $n$, and we assume that $L$ is subcanonical, i.e., there exists an integer $h$ such that we have the linear equivalence $K_X \equiv hL$, where $h \neq 0$.

Date: February 18, 2014.

AMS Classification: 14M05, 14J29, 13H10, 32S20.

The present work took place in the realm of the DFG Forschergruppe 790 “Classification of algebraic surfaces and compact complex manifolds”.

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There are then two cases: $h < 0$, and $X$ is a Fano manifold, or $h > 0$ and $X$ is a manifold with ample canonical divisor (in particular $X$ is of general type).

Assume that $X$ is a Fano manifold, and that $-K_X = rL$, with $r > 0$: then, by Kodaira vanishing

$$H^j(mL) := H^j(\mathcal{O}_X(mL)) = 0 \forall m \in \mathbb{Z}, \forall 1 \leq j \leq n - 1.$$

For $m < 0$ this follows from Kodaira vanishing (and holds for $j \geq 1$), while for $m \geq 0$ Serre duality gives $h^j(mL) = h^{n-j}(K - mL) = h^{n-j}((-r - m)L) = 0$.

At the other extreme, if $K_X$ is ample, and $K_X \equiv rL$, (thus $r > 0$) by the same argument we get vanishing outside of the interval

$$0 \leq m \leq r.$$

To $L$ we associate as usual the finitely generated graded $\mathbb{C}$-algebra

$$\mathcal{R}(X, L) := \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mL))$$

Therefore in the Fano case, the divisor $L$ is arithmetically Cohen-Macaulay (see [Hart77]) and the above graded ring is a Gorenstein ring.

The question is whether also in the case where $K_X$ is ample one may hope for such a good property.

The above graded ring is integral over the canonical ring $\mathcal{A} := \mathcal{R}(X, K_X)$, which is a Gorenstein ring if and only if we have pluriregularity, i.e., vanishing

$$H^j(\mathcal{O}_X) = 0 \forall 1 \leq j \leq n - 1.$$

Jonathan Wahl asked the following question (which makes only sense for $n \geq 2$):

**Question 1.** (J. Wahl) *Are there examples of subcanonical pluriregular varieties $X$ such that the graded ring $\mathcal{R}(X, L)$ is not Cohen-Macaulay?*

We shall show that the answer is positive, also in the case of regular subcanonical surfaces with $K_X$ ample, where by the assumption we have the vanishing

$$H^1(mL) = 0 \forall m \leq 0, \text{ or } r \leq m$$

and the question boils down to requiring the vanishing also for $1 \leq m \leq r - 1$.

The following theorem answers the question by J. Wahl:

**Theorem 2.** For each $r = n - 3$, where $n \geq 7$ is relatively prime to 30, and for each $m$, $1 \leq m \leq r - 1$ there are Beauville type surfaces $S$ with $q(S) = 0$ ($q(S) := \dim H^1(S, \mathcal{O}_S)$) s.t. $K_S = rL$, and $H^1(mL) \neq 0$. 
Subcanonical Rings

We get therefore examples of the following situation: \( A := \mathcal{R}(S, K_S) \)
is a Gorenstein graded ring, and a subring of the ring \( \mathcal{R} := \mathcal{R}(S, L) \), which is not arithmetically Cohen-Macaulay; hence we have constructed examples of non Cohen-Macaulay singularities (\( \text{Spec} (\mathcal{R}) \)) with \( K_Y \) Cartier which are cyclic quasi-étale covers of a Gorenstein singularity (\( \text{Spec} (A) \)).

In the Appendix, J. Wahl uses these to construct Cohen-Macaulay singularities with \( K_X \) \( \mathbb{Q} \)-Cartier whose index one cover is not Cohen-Macaulay.

In fact, we can consider three graded rings, two of which are subrings of the third, and which are cones associated to line bundles on the surface \( S \):

- \( Y := \text{Spec} (\mathcal{R}) \), the cone associated to \( L \), which is not Cohen-Macaulay, while \( K_Y \) is Cartier;
- \( Z := \text{Spec} (A) \), the cone associated to \( K_S \), which is Gorenstein;
- \( X := \text{Spec} (B) \), the cone associated to \( K_S + L \) (for instance), which is Cohen-Macaulay with \( K_X \) \( \mathbb{Q} \)-Cartier, but whose index 1 (or canonical) cover \( Y = \text{Spec} (\mathcal{R}) \) is not Cohen-Macaulay.

2. The special case of even surfaces

Recall: a smooth projective surface \( S \) is said to be even if there is a divisor \( L \) such that \( K_S \equiv 2L \).

This is a topological condition, it means that the second Stiefel Whitney class \( w_2(S) = 0 \), or, equivalently, the intersection form

\[ H^2(S, \mathbb{Z}) \rightarrow \mathbb{Z} \]

is even (takes only even values).

In particular, an even surface is a minimal surface.

In particular, if \( S \) is of general type and even, the self intersection

\[ K_S^2 = 4L^2 = 8k \]

for some integer \( k \geq 1 \).

The first numerical case is therefore the case \( K_S^2 = 8 \).

**Proposition 3.** Assume that \( S \) is an even surface of general type with \( K_S^2 = 8 \) and \( p_g(S) = h^0(K_S) = 0 \). Then, if \( K_S \equiv 2L \), then \( H^1(L) = 0 \).

**Proof:** We assume that \( S \) is even, \( K \equiv 2L \), and \( p_g = 0 \).

Since the intersection form is even, and \( K^2 \leq 9 \) by the Bogomolov-Miyaoka - Yau inequality, we obtain that \( L^2 = 2 \).

The Riemann Roch theorem tells us: \( \chi(L) = 1 + \frac{1}{2}L(L - K) = 1 + \frac{1}{2}L(-L) = 0 \).

On the other hand, by Serre duality \( \chi(L) = 2h^0(L) - h^1(L) \), so if \( H^1(L) \) is different from zero, then \( H^0(L) \neq 0 \), contradicting \( p_g = 0 \). \( \square \)
Our construction for \( n = 5 \) shall show in particular that the ‘Beauville surface’, constructed by Beauville in [Bea78] is an even surface with \( K^2_S = 8, q(S) = p_g(S) = 0 \), but with \( H^1(L) = 0 \).

3. Canonical linearization on Fermat curves

Fix a positive integer \( n \geq 5 \), and let \( C \) be the degree \( n \) Fermat curve

\[ C := \{(x, y, z) \in \mathbb{P}^2 | f(x, y, z) := x^n + y^n + z^n = 0\}. \]

Let as usual \( \mu_n \) be the group of \( n \)-roots of unity. The group \( G := \mu^2_n = \mu_n^3/\mu_n \) acts on \( C \), and we obtain a natural linearization of \( \mathcal{O}_C(1) \) by letting

\[ (\zeta, \eta) \in \mu^2_n \text{ act as follows:} \]

\[ z \mapsto z, x \mapsto \zeta x, y \mapsto \eta y. \]

In other words, \( H^0(\mathcal{O}_C(1)) \) splits as a direct sum of one dimensional eigenspaces (respectively generated by \( x, y, z \)) corresponding to the characters \( (1, 0), (0, 1), (0, 0) \in (\mathbb{Z}/n)^2 \cong \text{Hom}(G, \mathbb{C}^*) \).

Similarly, for \( m \leq n - 1 \), the monomial \( x^ay^b z^{m-a-b} \in H^0(\mathcal{O}_C(m)) \) generates the unique eigenspace for the character \( (a, b) \) (we identify here \( \mathbb{Z}/n \cong \{0, 1, \ldots, n-1\} \) and we obviously require \( a + b \leq m \)).

However, any two linearizations differ (see [Mum70]) by a character of the group.

**Definition 4.** Assume that \( n \) is not divisible by 3.

We call the canonical linearization on \( H^0(\mathcal{O}_C(1)) \) the one obtained from the natural one by twisting with the character \( (n - 3)^{-1}(1, 1) \). Thus \( x \) corresponds to the character \( v_1 := (1, 0) + (n - 3)^{-1}(1, 1) = (-3)^{-1}(-2, 1), y \) corresponds to the character \( v_2 := (0, 1) + (n - 3)^{-1}(1, 1) = (-3)^{-1}(1, -2), z \) corresponds to the character \( v_3 := (-3)^{-1}(1, 1) \).

**Remark 5.** (I) Observe that \( v_1, v_2 \) are a basis of \((\mathbb{Z}/n)^2\) as soon as \( n \) is not divisible by 3.

Indeed, \( v_1 + v_2 = \frac{1}{3}(1, 1) = 3^{-1}(1, 1) \), hence

\[ (1, 0) = v_1 + 3^{-1}(1, 1) = 2v_1 + v_2, \quad (0, 1) = 2v_2 + v_1. \]

(II) Observe that the above linearization induces a linearization on all multiples of \( L \), and, in the case where \( m = (n - 3) \), we obtain the natural linearization on the canonical divisor of \( C \), \( \mathcal{O}_C(n - 3) \cong \Omega^1_C \).

Since, if we take affine coordinates where \( z = 1 \), and we let \( f \) the equation of \( C \), we have

\[ H^0(\Omega^1_C) = \{P(x, y) \frac{dx}{fy} = -P(x, y) \frac{dy}{fx}\} \]

and the monomial \( P = x^a y^b \) corresponds under this isomorphism to the character \((a + 1, b + 1)\).
(III) In particular, Serre duality

\[ H^0(\mathcal{O}_C(m)) \times H^1(\Omega^1_C(-m)) \to H^1(\Omega^1_C) \cong \mathbb{C}, \]

where \( \mathbb{C} \) is the trivial \( G \)-representation, is \( G \)-invariant.

From the previous discussion follows also

**Lemma 6.** The monomial \( x^ay^bz^c \in H^0(\mathcal{O}_C(m)) \) (here \( a, b, c \geq 0, a + b + c = m \)) corresponds to the character \( \chi \) equal to:

\[ (a, b) + (-3)^{-1}(m, m) = (a - c)v_1 + (b - c)v_2. \]

**Proof.** \( v_1 + v_2 = \frac{1}{3}(1, 1) \), hence \( (a, b) + (-3)^{-1}(m, m) = av_1 + bv_2 + (-m + a + b)(3)^{-1}(1, 1) = (a - c)v_1 + (b - c)v_2. \)

\( \square \)

### 4. Abelian Beauville Surfaces and their subcanonical divisors

We recall now the construction (see also [Cat00], or [BCG05]) of a Beauville surface with Abelian group \( G \cong (\mathbb{Z}/n)^2 \), where \( n \) is not divisible by 2 and by 3.

**Definition 7.**

(1) Let \( \Sigma \subset G \) be the union of the three respective subgroups generated by \((1, 0), (0, 1), (1, 1)\).

(2) Let \( \psi : G \to G \) a homomorphism such that, setting \( \Sigma^* := \Sigma \setminus \{(0, 0)\}, \psi(\Sigma^*) \cap \Sigma^* = \emptyset \) (equivalently, \( \psi(\Sigma) \cap \Sigma = \{(0, 0)\} \)).

(3) Let \( C \) be the degree \( n \) Fermat curve and let \( S = (C \times C)/(\text{Id} \times \psi)(G) \), i.e., the quotient of \( C \times C \) by the action of \( G \) such that \( g(P_1, P_2) = (g(P_1), \psi(g)(P_2)) \).

**Remark 8.**

(i) By property (2) \( G \) acts freely and \( S \) is a projective smooth surface with ample canonical divisor.

(ii) The line bundle \( \mathcal{O}_{C \times C}(1, 1) \) is \( G \times G \) linearized, in particular it is \( G \cong (\text{Id} \times \psi)(G) \)-linearized, therefore it descends to \( S \), and we get a divisor \( L \) on \( S \) such that the pull back of \( \mathcal{O}_S(L) \) is the above \( G \)-linearized bundle.

(iii) By the previous remarks, we have a linear equivalence

\[ K_S \equiv (n - 3)L. \]

### 5. Cohomology of multiples of the subcanonical divisor \( L \)

We consider now an integer \( m \) with

\[ 1 \leq m \leq n - 4, \]

and we shall determine the space \( H^1(\mathcal{O}_S(mL)) \).

Observe first of all that \( H^1(\mathcal{O}_S(mL)) \cong H^1(\mathcal{O}_{C \times C}(m, m))^G \).
By the Künneth formula
\[ H^1(\mathcal{O}_{\mathcal{C} \times \mathcal{C}}(m,m)) \cong \]
\[ \cong [H^0(\mathcal{O}_{\mathcal{C}}(m)) \otimes H^1(\mathcal{O}_{\mathcal{C}}(m))] \bigoplus [H^1(\mathcal{O}_{\mathcal{C}}(m)) \otimes H^0(\mathcal{O}_{\mathcal{C}}(m))]. \]

We want to decompose the right hand side as a representation of $G \cong (\text{Id} \times \psi)(G)$.

Explicitly, $H^0(\mathcal{O}_{\mathcal{C}}(m)) = \oplus \chi V_\chi$, where if we write the character $\chi = (a, b) + (-3)^{-1}(m, m)$ (as we saw) then $V_\chi$ has dimension equal to one and corresponds to the monomial $x^a y^b z^{m-a-b}$, where $a, b \geq 0$, $a + b \leq m$.

By Serre duality, $H^1(\mathcal{O}_{\mathcal{C}}(m)) = \oplus \chi V_{-\chi'}$, where if we write as above $\chi' = (a', b') + (-3)^{-1}(m', m')$, then $V_{-\chi'}$ is the dual of $V_\chi$, corresponding to the monomial $x^{a'} y^{b'} z^{m'-a'-b'}$, where $m' = n - 3 - m$, so $1 \leq m' \leq n - 4$ also, and where $a', b' \geq 0$, $a' + b' \leq m'$.

Now, the homomorphism $\psi : G \to G$ induces a dual homomorphism $\phi := \psi^\vee : G^\vee \to G^\vee$, therefore we can finally write $H^1(\mathcal{O}_{\mathcal{C} \times \mathcal{C}}(m,m))$ as a representation of $G \cong (\text{Id} \times \psi)(G)$:
\[ H^1(\mathcal{O}_{\mathcal{C} \times \mathcal{C}}(m,m)) = \bigoplus_{\chi, \chi'} [(V_\chi \otimes V_{-\phi(\chi')}) \oplus (V_{-\chi'} \otimes V_{\phi(\chi')}]]. \]

We have proven therefore the

**Lemma 9.** $H^1(\mathcal{O}_{S}(mL)) \neq 0$ if and only if there are characters $\chi = (a-c)v_1 + (b-c)v_2$ and $\chi' = (a'-c')v_1 + (b'-c')v_2$ with $a, b \geq 0$, $a + b \leq m$, $a', b' \geq 0$, $a' + b' \leq m' = n - 3 - m$ such that
\[ \chi = \phi(\chi') \text{ or } \chi' = \phi(\chi). \]

**Proof of theorem**

We take now $\phi$ to be given by a diagonal matrix in the basis $v_1, v_2$, i.e., such that
\[ \phi(v_j) = \lambda_j v_j, \ j = 1, 2, \ \lambda_j \in (\mathbb{Z}/n)^*. \]

For further use we also set $\lambda := \lambda_1, \mu := \lambda_2$.

Given $n$ relatively prime to 30 and $1 \leq m \leq n - 4$, we want to find $\lambda_1$ and $\mu$ such that the equations
\[ (a-c) = \lambda(a'-c') \]
\[ (b-c) = \mu(b'-c') \]

have solutions with $a, b, c \geq 0$, $a + b + c = m$, and $a', b', c' \geq 0$, $a' + b' + c' = m'$.

The first idea is simply to take $b = c$ and $b' = c'$, so that $\mu$ can be taken arbitrarily.
For the first equation some care is needed, since we want that $\lambda$ be a unit: for this it suffices that $(a-c), (a'-c')$ are both units, for instance they could be chosen to be equal to one of the three numbers 1, 2, 3, according to the congruence class of $m$, respectively $m'$, modulo 3.

With this proviso we have to verify that we have a free action on the product.

**Lemma 10.** If $n \geq 7$, given $\lambda$ a unit, there exists a unit $\mu$ such that $\psi = \phi^\vee$ satisfies the condition $\psi(\Sigma) \cap \Sigma = \{(0,0)\}$.

**Proof.** Since $(1,0) = 2v_1 + v_2$ and $(0,1) = v_1 + 2v_2$, the matrix of $\phi$ in the standard basis is the matrix

$$\phi = \frac{1}{3} \begin{pmatrix} 4\lambda - \mu & 2(\lambda - \mu) \\ 2(\mu - \lambda) & 4\mu - \lambda \end{pmatrix}$$

while the matrix of $\psi$ is the matrix

$$\psi = \frac{1}{3} \begin{pmatrix} A := 4\lambda - \mu & B := 2(\mu - \lambda) \\ C := 2(\lambda - \mu) & D := 4\mu - \lambda \end{pmatrix}$$

The conditions for a free action boil down to:

$$A, B, C, D, A + B, C + D$$

are units in $\mathbb{Z}/n$, and moreover $A \neq B, C \neq D, A + B \neq C + D$.

These are in turn equivalent to the condition that

$$\lambda, \mu, \lambda - 4\mu, \lambda - \mu, \mu - 4\lambda, \lambda + 2\mu, 2\lambda + \mu \in (\mathbb{Z}/n)^*.$$ 

Given $\lambda \in (\mathbb{Z}/n)^*$, consider its direct sum decomposition given by the Chinese remainder theorem and the primary factorization of $n$. For each prime $p$ dividing $n$, the residue classes modulo $p$ which are excluded by the above condition are at most five values inside $(\mathbb{Z}/p)^*$, hence we are done if $(\mathbb{Z}/p)^*$ has at least six elements.

Now, since $n$ is relatively prime to 30, each prime number dividing it is greater or equal to $p = 7$.

\[\square\]

**Proposition 11.** Consider the Beauville surface $S$ constructed in [Bea78], corresponding to the case $n = 5$.

Then $S$ is an even surface and $K_S \equiv 2L$, where $H^1(L) = 0$.

**Proof.** We observe that $L$ is unique, because the torsion group of $S$ is of exponent 5 (see [BC04]).

The existence of $L$ follows exactly as in the proof of the main theorem, where the condition $n \geq 7$ was not used. That $H^1(L) = 0$ follows directly from proposition 3.

\[\square\]
Acknowledgements. I would like to thank Jonathan Wahl for asking the above question. In the appendix below he describes a construction based on our main result.

References


6. Appendix by Jonathan Wahl: A non-$\mathbb{Q}$-Gorenstein Cohen-Macaulay cone $X$ with $K_X$ $\mathbb{Q}$-Cartier

A germ $(X, 0)$ of an isolated normal complex singularity of dimension $n \geq 2$ is called $\mathbb{Q}$-Gorenstein if

1. $(X, 0)$ is Cohen-Macaulay
2. The dualizing sheaf $K_X$ is $\mathbb{Q}$-Cartier (i.e., the invertible sheaf $\omega_{X-\{0\}}$ has finite order $r$)
3. The corresponding cyclic index one (or canonical) cover $(Y, 0) \rightarrow (X, 0)$ is Cohen-Macaulay, hence Gorenstein.

Alternatively, $(X, 0)$ is the quotient of a Gorenstein singularity by a cyclic group acting freely off the singular point. Some early definitions did not require the third condition, which is of course automatic for $n = 2$.

If $(X, 0)$ is $\mathbb{Q}$-Gorenstein, a one-parameter deformation $(\mathcal{X}, 0) \rightarrow (\mathcal{C}, 0)$ is called $\mathbb{Q}$-Gorenstein if it is the quotient of a deformation of the index one cover of $(X, 0)$; this is exactly the condition that $(\mathcal{X}, 0)$
is itself $\mathbb{Q}$-Gorenstein. These notions were introduced by Kollár and Shepherd-Barron [2], who made extensive use of the author’s explicit smoothings of certain cyclic quotient surface singularities in [3] (5.9); these deformations were patently $\mathbb{Q}$-Gorenstein, and it was important to name this property.

Recently, the author and others considered rational surface singularities admitting a rational homology disk smoothing (i.e., with Milnor number 0). The three-dimensional total space of the smoothing had a rational singularity with $K_{\mathbb{Q}}$-Cartier, but it was not initially clear whether the smoothings were $\mathbb{Q}$-Gorenstein. (This was later established [5] by proving the stronger result that the total spaces were log-terminal.) In fact, one needs to be careful because of the examples of A. Singh:

**Example.** [4]: There is a three-dimensional isolated rational (hence Cohen-Macaulay) complex singularity $(X, 0)$ with $K_X$ $\mathbb{Q}$-Cartier which however is not $\mathbb{Q}$-Gorenstein.

The purpose of this note is to use F. Catanese’s result to provide other examples; they are not rational, but are cones over a smooth projective variety, which could for instance be assumed to be projectively normal with ideal generated by quadrics.

**Proposition 12.** Let $S$ be a surface as in Theorem 2 of Catanese’s paper, with $h^1(S, \mathcal{O}_S) = 0$, $L$ ample, $K_S = rL$ (some $r > 1$), and $h^1(mL) \neq 0$ for some $m > 0$. Let $t$ be greater than $r$ and relatively prime to it. Then

1. The cone $R = \mathcal{R}(S, tL) := \oplus_{m \geq 0} H^0(S, \mathcal{O}_S(mtL))$ is Cohen-Macaulay.
2. The dualizing sheaf of $R$ is torsion, of order $t$.
3. The index one cover is $\mathcal{R}(S, L) := \oplus_{m \geq 0} H^0(S, \mathcal{O}_S(mL))$, and is not Cohen-Macaulay.

In particular, $R$ is not $\mathbb{Q}$-Gorenstein.

**Proof.** The Cohen-Macaulayness for $R$ follows because $h^1(itL) = 0$, all $i$, thanks to Kodaira Vanishing. Let $\pi : V \to S$ be the geometric line bundle corresponding to $-tL$; then $H^0(V, \mathcal{O}_V) \equiv R$. Since $K_V \equiv \pi^*(K_S + tL)$, one has that $jK_R \equiv \oplus_{n \in \mathbb{Z}} H^0(S, j(K_S + tL) + nL)$; since $tK_S = r(tL)$ with $r$ and $t$ relatively prime, $K_R$ has order $t$. Making a cyclic $t$-fold cover and normalizing gives that $\mathcal{R}(S, L)$ is the index one cover, which as Catanese has noted is not Cohen-Macaulay.

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