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PLURICANONICAL MAPPINGS OF SURFACES WITH $K^2 = 1,2,\ q=p_g=0$

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PLURICANONICAL MAPPINGS OF SURFACES WITH $K^2 = 1,2$, $q=p_g=0$ (*)

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I. Introduction.

This lecture is a continuation of Dolgacev's ones on surfaces with $q=p_g=0$, and considers those minimal models of such surfaces for which $K^2=2$ (numerical Campedelli surfaces) and those for which $K^2=1$ (numerical Godeaux surfaces): they are of general type by classification of surfaces.

The Main Theorem of [1] (to which we will refer as [CM]) asserted among other things that for a minimal surface of general type, Φ_{mK} denoting the rational map associated to the complete linear system $|mK_S|$, Φ_{mK} is birational, for $m \geq 3$, with these exceptions

(*) This seminar is an exposition of joint work of E. Bombieri-F. Catanese.

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a) $K^2=1$, $p_g=2$, $m=4,3$
b) $K^2=2$, $p_g=3$, $m=3$ and, possibly,
c) $K^2=1$. $p_g=0$, $m=4,3$, $K^2=2$, $p_g=0$, $m=3$.

It was later shown that the exceptions of c) don't really occur: the case $K^2=1$, $p_g=0$ $m=4$ was proven by Bombieri (unpublished) and subsequently by us along a simpler line of proof ([4]), the case $K^2=1$, $p_g=0$, $m=3$ by Miyaoka [6] and subsequently by Kulikov and us (along a different line of proof, unpublished), the case $K^2=2, p_g=0$, $m=3$, by Peters ([10]) in the particular case of a Campedelli double plane, in the general one by us ([3]) and later, independently by X. Benveniste (unpublished).

The main goal of this lecture is by one side to prove these results in the simplest fashion and by the other one to exhibit the application of some new lemmas (of [3]) which allow one to handle reducible curves in nearly the same way than non singular ones. We will give our proof for the first case, for the second we will give the main steps (in which our differs from Miyaoka's proof): for numerical Campedelli surfaces, finally, we remark that the proof appearing here is a combination of our with an argument of Benveniste's proof.

II. Some auxiliary results.

Lemma 1. On a surface S of general type with $K^2 \leq 2$, $q=0$ there is only a finite number of irreducible curves C with $K \cdot C \leq 1$.

Proof. Observe that if $C^2 < 0$ C is isolated in its class of numerical equivalence, hence in this case it suffices to show that the number of such classes is finite. Here we use the index theorem ([9] page 128) to the effect that on the subspace of numerical classes orthogonal to K the intersection form is negative definite: if $K\cdot C=0$ $C^2=-2$ (as $C^2<0$ and $KC+C^2=$ $=2\,p(C)-2\geq -2$) and the number of such classes is finite (moreover such curves are numerically independent, see [5] pag. 177, [C.M.] pag. 174-5). If $K\cdot C=1$, then $(K-(K^2)C)$ is orthogonal to K, hence $0\geq (K-(K^2)C)^2=K^2((K^2)C^2-1)$ and so $C^2\leq 1/_{K^2}$; however $2p(C)-2=1+C^2\geq -2$ implies C^2 odd, $C^2\geq -3$, so $C^2<0$ unless only if it is $K^2=1$, $C^2=1$, K homologous to C.

Note that $(K-(K^2)C)$ belongs to a numerical class orthogonal to K with selfintersection bounded from below by -14, hence can belong only to a finite number of classes, and the same then occurs for C; finally if C is homologous to K $h^\circ(\mathcal{O}_S(C))=1$ (compare [6] or Dolgacev's lecture), and, the surface being regular, there is a finite number of such curves.

Q.E.D.

We refer to [3] for the proof of the following lemmas A,B,B'.

<u>Lemma</u> A. Let C be a positive divisor on a smooth surface S, $\mathcal{L}$ an invertible sheaf on C with $h^\circ(C,\mathcal{L})\geq 1$: then either

i) there exists a section s not vanishing identically on any component of C, and $\deg_C\mathcal{L}\geq 0$, equality holding iff $\mathcal{L}=\mathcal{O}_C$

or

ii) there exists a section σ, $C_1,C_2>0$ such that $C=C_1+C_2$, $\sigma|_{C_1}\equiv 0$ but $\sigma|_{C'}\not\equiv 0$ if $C_1<C'\leq C$, and $C_1\cdot C_2\leq \deg_{C_2}(\mathcal{L}\otimes\mathcal{O}_{C_2})$.

Lemma B. If Γ is an irreducible Gorenstein curve and $|\omega_\Gamma| \neq \emptyset$, then $|\omega_\Gamma|$ has no base points.

More generally a reduced point p of a curve C on a smoth surface is not a base point of $|\omega_C|$ if either

i) p is simple on C and belongs to a component Γ with $p(\Gamma) \geq 1$

or

ii) p is singular and for every decomposition $C=C_1+C_2$ $(C_i > 0)$ one has $C_1 \cdot C_2 > (C_1 \cdot C_2)_p$ = intersection multiplicity of C_1, C_2 at p.

Remark. If C is given by two elliptic curves meeting transversally at a point p, p is a base point of $|\omega_C|$, and in fact condition ii) is violated; however if C is given by more than three lines in the projective plane all meeting in the same point p, condition ii) is violated but p is not a base point. On a numerical Godeaux surface if ε is a torsion class $\neq -\varepsilon$, (D_ε denoting the unique curve of $|K+\varepsilon|$), $C=D_\varepsilon+D_{-\varepsilon}$ has $b=D_\varepsilon \cap D_{-\varepsilon}$ as a base point of $|\omega_C|$, and infact $D_\varepsilon \cdot D_{-\varepsilon}=1=$ $=(D_\varepsilon \cdot D_{-\varepsilon})_b$ (compare Dolgacev's lecture and the following of this).

Lemma B'. If p is a reduced singular point of a curve C lying on a smooth surface, denote by $\mathfrak{M}_p$ the maximal ideal of p in C, and let $\pi:\tilde{C} \longrightarrow C$ be a normalization of C at p. Then $\mathrm{Hom}(\mathfrak{M}_p, \mathcal{O}_{C,p})$ can be embedded in the ring A of regular functions of $\tilde{C}$ at $\pi^{-1}(p)$.

Lemma 2 (X. Benveniste). Let S be a numerical Campedelli surface and m a positive integer: then the family $\mathfrak{F}_m$ of irreducible curves E such that $K\cdot E \leq m$ and $|E| = \{E\}$ is a finite one.

Proof. If $E \in \mathfrak{F}_m$, $h^2(E)=h^0(K-E) \leq h^0(K)=0$, hence R.R. gives $1+\frac{1}{2}(E^2-KE) = \chi(\mathcal{O}(E)) \leq 1$, so $(\#)$ $KE \geq E^2$. Associate to $E \in \mathfrak{F}_m$ the following numerical class ξ_E orthogonal to K: $\xi_E = 2E - (K\cdot E)K$.

Here, as in lemma 1, we use the index theorem to infer $\xi_E^2 \leq 0$. But $\xi_E^2 = 2E\cdot\xi_E = 4E^2 - 2(K\cdot E)^2$, and this, together with the already used inequality $E^2 \geq -KE-2$, gives the result that $\xi_E^2 \geq -2(K\cdot E)^2 - 8 - 4K\cdot E \geq -(8+4m+2m^2)$; in turn this implies that ξ_E may belong only to a finite set of numerical classes.

Suppose now $\xi_{E_1} \sim \xi_{E_2}$: if we prove that then either

$E_1 \sim E_2$ or $E_1 \sim K+E_2$ we are done (the surface being regular each class can be given by at most $2m$ such curves where m is the order of the torsion subgroup T of $Pic(S)$).

$\xi_{E_1} \sim \xi_{E_2}$ can be read as $2(E_1-E_2) \sim (KE_1-KE_2)K$ and we can

assume $r = KE_1 - KE_2 > 0$; more over r is an even number, because $r(K\cdot E_i) = 2(E_1-E_2)\cdot E_i$ (r odd would imply $K\cdot E_1, K\cdot E_2$ to be even, which is obviously absurd).

This equality, in turn, when $i=1$, can be read as $2E_1^2 = 2E_1E_2 + rK\cdot E_1$ and the fact that $E_1\cdot E_2 \geq 0$

$$(\#)\quad E_1^2 \le KE_1 \text{ soon imply } \begin{cases} E_1\cdot E_2=0 \\ r = 2 \\ E_1^2=K\cdot E_1 \end{cases} \text{, finally } 2(E_1-E_2)\sim 2K \text{ i.e. } E_1\sim E_2+K .$$

Q.E.D.

III. Birationality of Φ_{4K} for a numerical Godeaux surface.

Let S be a numerical Godeaux surface (a minimal model, as we always assume). We recall that for any S of general type, and $m \geq 2$ $P_m=h^0(\mathcal{O}(mK)) = \frac{1}{2}m(m-1)K^2+\chi(\mathcal{O})$ ([5] pag. 184, [C.M.] pag. 185), so that in this case $P_2=2$, $P_3=4$, $P_4=7$.

Take U to be the Zariski open set whose complement is given by the union of the curves D such that $K\cdot D \leq 1$, with the locus of base points of $|2K|$ and of singular points of curves $C \in |2K|$.

Remark indeed that Φ_{4K} restricted to U is a regular map, then we claim

Theorem 1. $\Phi_{4K}|_U$ is an injective morphism.

Proof. Suppose that x,y are two points of U such that $\Phi_{4K}(x) = \Phi_{4K}(y)$; by our choice of U we may choose a curve $D \in |2K|$ s.t. $y \notin D$, and the unique curve $C\in|2K|$ s.t. $x \in C$: now $C+D$ is a curve of $|4K|$ passing through x, hence $y \in C$, and x,y are simple points of C.

Given a sheaf $\mathcal{F}$, denote by $\mathcal{F}(-x)=\mathcal{F}\mathfrak{m}_x$ (where $\mathfrak{m}_x$ is the ideal sheaf of the point x), and by $\mathbb{K}_x$ the sheaf supported at x with stalk $\mathbb{K}$. We obtain that $H^0(C,\mathcal{O}_C(K))=0$ by the exact sequence $0\longrightarrow\mathcal{O}_S(-K)\longrightarrow\mathcal{O}_S(K)\longrightarrow\mathcal{O}_C(K)\longrightarrow 0$ together with the vanishing of $H^1(\mathcal{O}_S(-K))$ (this is by the vanishing theorem of Mumford, [8], asserting that if $\mathcal{L}$ is an invertible sheaf such that for large n $\mathcal{L}^n$ is spanned by global sections and has three algebraically independent sections, then $H^1(\mathcal{L}^{-1})=0$: it can be applied to $\mathcal{O}_S(mK)$, $m\geq 1$, and for later use we observe that by duality also gives $h^i(\mathcal{O}_S(mK))=0$ for $i\geq 1$, $m\geq 2$).

We will derive a contradiction by showing that $\mathcal{O}_C(K)$ is isomorphic to $\mathcal{O}_C(x+y)$.

Consider for this the following exact sequences:

$$0\longrightarrow\mathcal{O}_S(4K-x-y)\longrightarrow\mathcal{O}_S(4K)\longrightarrow\mathbb{K}_x\oplus\mathbb{K}_y\longrightarrow 0$$

$$0\longrightarrow\mathcal{O}_S(2K)\longrightarrow\mathcal{O}_S(4K-x-y)\longrightarrow\mathcal{O}_C(4K-x-y)\longrightarrow 0\ .$$

Then from their cohomology sequences one gets $1=h^1(\mathcal{O}_S(4K-x-y))=h^1(\mathcal{O}_C(4K-x-y))$, and by Serre duality on the curve C $h^0(C,\mathcal{L})=1$, where $\mathcal{L}=\omega_C\otimes\mathcal{O}_C(4K-x-y)^{-1}=\mathcal{O}_C(x+y-K)$.

We are ready to apply lemma A, after observing that $\deg_C\mathcal{L}=2-K\cdot C=0$; then if i) occurs $\mathcal{L}\cong\mathcal{O}_C$, what we wanted to show. Case ii) cannot occur: infact one would have $C=C_1+C_2$, $C_1\cdot C_2\leq\deg\mathcal{O}_{C_2}\otimes\mathcal{L}=-KC_2+$ (number of points of $C_2\cap\{x,y\}$).

But by our choice of U, if x, or $y\in C_2$, then $KC_2\geq 2$, so in any case $C_1\cdot C_2$ should be non positive, and this contradicts the following result of Bombieri about connectedness of divisors homologous to pluricanonical divisors ([C.M.] pag. 181) : if $D\sim mK$, $m\geq 1$ and $D=D_1+D_2$, $D_i>0$, then $D_1\cdot D_2\geq 2$, except when $K^2=1$, $m=2$ but then $D_1\sim D_2\sim K$.

Q.E.D.

IV. Birationality of Φ_{3K} for a numerical Campedelli surface.

By the just quoted formula here $P_2=3$, $P_3=7$.

From now on we suppose that x,y are two points such that $\Phi_{3K}(x) = \Phi_{3K}(y)$: as $P_2=3$ there exists a curve $C \in |2K|$ containing them both.

Lemma 3. $h^1(\mathcal{O}_C(3K-x-y)) = 2$.

Proof. The cohomology sequences of

$$0 \longrightarrow \mathcal{O}_S(3K-x-y) \longrightarrow \mathcal{O}_S(3K) \longrightarrow \mathbb{K}_x \oplus \mathbb{K}_y \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_S(\underset{\substack{\parallel\!\!| \\ K}}{3K-C}) \longrightarrow \mathcal{O}_S(3K-x-y) \longrightarrow \mathcal{O}_C(3K-x-y) \longrightarrow 0$$

give $h^1(\mathcal{O}_C(3K-x-y)) = h^1(\mathcal{O}_S(3K-x-y)) + h^2(\mathcal{O}(K)) = 1+1$, as $h^1(\mathcal{O}_S(3K)) = 0$ and $\Phi_{3K}(x) = \Phi_{3K}(y)$.

Q.E.D.

Proposition 4. For general x,y , and $C \in |2K-x-y|$, x and y are simple points of the curve C which is hyperelliptic having $h_C = \mathcal{O}_C(x+y)$ as its hyperelliptic invertible sheaf.

Proof. The second part of the statement is an easy consequence of the first part, lemma 3 plus Serre duality on C , the first will be proven in two steps.

Step I: x belongs to only one irreducible component of C , the same holds for y .

In fact if, say, x belongs to two components Γ_1, Γ_2 of C , by lemma 1 $K \cdot \Gamma_i \geq 2$, hence is equal to 2, and by lemma 2

$h^\circ(\mathcal{O}_S(\Gamma_i)) \geq 2$. Write $C = \Gamma_1 + \Gamma_2 + F$ $(K \cdot F = 0)$ and consider that $|2K| \supset |\Gamma_1| + |\Gamma_2| + F$, $\dim|2K| = 2$: from this we deduce that $\Gamma_1 \equiv \Gamma_2$. In fact if Γ_1' is an irreducible curve $\in |\Gamma_1|$ there exist by the previous remark Γ_2', $\Gamma_2'' \in |\Gamma_2|$ such that $\Gamma_1 + \Gamma_2'' = \Gamma_1' + \Gamma_2'$, and so $\Gamma_2'' = \Gamma_1' + H_1'$, $\Gamma_2' = \Gamma_1 + H_1$, $H_1' \equiv H_1$, $H_1 \cdot K = 0$; but this implies $H_1' = H_1$ and so H_1 would be a fixed part of $|\Gamma_2|$: then H_1 must be 0 and $\Gamma_1 \equiv \Gamma_2$.

If it were $\Gamma_1 \neq \Gamma_2$ x would be a base point of $|\Gamma_1|$, which cannot hold for general x (the numerical class of Γ_1 can range in a finite set); while if it were $\Gamma_1 = \Gamma_2$ one could take, by what has just been said, a curve $\Gamma_1' \in |\Gamma_1|$ not passing through x, and then consider $C' = \Gamma_1 + \Gamma_1' + F$.

<u>Step</u> II: x and y are simple points of C.

For this we can use lemma 3 and Grothendieck duality to infer that $\dim_{\mathbb{K}} \operatorname{Hom}(\mathfrak{m}_x \mathfrak{m}_y, \mathcal{O}_C) = 2$. Taking $\tilde{C}$ a normalization of C at x,y we observe that by step I $\tilde{C}$ is connected exactly as C, hence by Ramanujam's result ([11], lemma 3) $h^\circ(\mathcal{O}_{\tilde{C}}) = 1$. We can apply lemma B': x,y both singular would imply $h^\circ(\mathcal{O}_{\tilde{C}}) \geq$ ≥ 2, a contradiction, while if however x is simple, y singular, one gets $h^\circ(\mathcal{O}_{\tilde{C}}(x)) = 2$ so x belongs to a rational curve; S being of general type, this cannot occur for general x.

Q.E.D.

From now on we suppose $C \in |2K - x - y|$ to satisfy the requirements of proposition 4.

Lemma 5. $\omega_C \cong h_C^{\otimes 6}$

Proof. If $s_1,\dots s_6$ are six distinct general sections of $H^\circ(C,h_C)$ and $\mathrm{div}(s_i) = a_i+b_i$, one has that a section of $H^\circ(\omega_C)$ vanishing at $a_1,\dots a_6$ vanishes at $b_1,\dots b_6$ too.

Q.E.D.

We can pass now to the proof of

Theorem 2. For a numerical Campedelli surface S $\Phi = \Phi_{3K}$ is a birational map.

Proof. Consider Φ : S —— $\mathbb{P}^6$: $V = \Phi(S)$ is not contained in any hyperplane so that $d = \deg V \geq 5$.
V is not a curve, otherwise the general element of $|3K|$ would be decomposable in more than d elements, while we know that the curves D with $K \cdot D \leq 1$ are a finite number.

By Theorem 5.1 of [2] (also [7] Th. A) we know that $|3K|$ has no base points, hence if $m=\deg \Phi$, $dm = (3K)^2 = 18$.

We must then prove that it is impossible to have either $\deg \Phi = 2$ or $\deg \Phi = 3$.

Case I. $\deg \Phi = 2$

There is defined on S a birational involution σ such that $y=\sigma(x)$ if $\Phi(x) = \Phi(y)$; S being a minimal model σ is an automorphism, hence $\sigma^*(\mathcal{O}(K)) = \mathcal{O}(K)$.

Remark that $\sigma(C) = C$: in fact if a is a general point of C, and b is the point of C s.t $\mathcal{O}_C(a+b) \cong h_C$, $\Phi(a) = \Phi(b)$ (by lemma 5).

The exact sequence $0 \longrightarrow H^{0}(\mathcal{O}(-K)) \longrightarrow H^{0}(\mathcal{O}(K)) \longrightarrow$ $\longrightarrow H^{0}(\mathcal{O}_C(K)) \longrightarrow 0$ implies $h^{0}(\mathcal{O}_C(K)) = 0$, which is impossible by virtue of the following.

<u>Lemma</u> 6. $\mathcal{O}_C(K) \cong h_C^{\otimes 2}$.

<u>Proof of the lemma</u>. By lemma 5 $\mathcal{O}_C(3K) \cong h_C^{\otimes 6}$ so it suffices to prove that, for instance, $\mathcal{O}_C(8K) \cong h_C^{\otimes 16}$. But $|4K|$ has no base points, and we may pick up $s \in H^{0}(\mathcal{O}_S(4K))$ with 16 simple zeros on C, $a_1, \ldots a_{16}$. Then $\sigma^{*}s$ has 16 simple zeros too, $\sigma(a_1), \ldots \sigma(a_{16})$, therefore $\sigma^{*}(s) \cdot s|_C$ is a section of $\mathcal{O}_C(8K)$ whose divisor is linearly equivalent to $h_C^{\otimes 16}$.

Q.E.D.

<u>Case</u> II: $\deg \Phi = 3$.

Let a be a general point of C, and $\bar{a}$ be such that $a+\bar{a} \in |h_C| : \Phi(a) = \Phi(\bar{a})$, and the third point a' with the same image under Φ cannot lie on C. Set $N = \Phi(C)$: then $\Phi^{-1}(C) = C \cup N'$ and N' is a rational curve (there is a birational map from $N' = \text{locus}\{a'\}$ and $|h_C| \cong \mathbb{P}^1$). So S contains a continuous family of rational curves, which is though absurd.

Q.E.D.

V. <u>Birationality of Φ_{3K} for a numerical Godeaux surface</u>.

Denote by $\Phi = \Phi_{3K_S}$ and by $V = \Phi(S)$, $d = \deg V$.

<u>Lemma</u> 7. V is not a curve.

Proof. Otherwise d would be ≥ 3 and the moving part of $|3K|$ would be decomposable in more than d elements, which, by lemma 1, have intersection with K at least 2: then one would have $3K\cdot K \geq 2d \geq 6$.

Lemma 8. $|3K|$ has no fixed part.
See [6] , pag. 103 for the proof.

However $|3K|$ can have base points, and they are characterized by the following proposition, in which we denote, as in the following, by $T = \mathrm{Tors}\,(\mathrm{Pic}(S))$ and by D_ε, if $\varepsilon \neq 0$, $\varepsilon \in T$, the unique curve in $|K+\varepsilon|$.

Proposition 9. If b is a base point of $|3K|$ there exists $\varepsilon \in T$, $\varepsilon \neq -\varepsilon$, such that $C=D_\varepsilon + D_{-\varepsilon}$ is the unique curve in $|2K|$ passing through b : moreover D_ε and $D_{-\varepsilon}$ have b as the unique point of transversal intersection. Conversely, if $\varepsilon \neq -\varepsilon$, $D_\varepsilon \cap D_{-\varepsilon}$ gives a base point of $|3K|$.

Proof. We recall first Miles Reid's lemma (see Dolgacev's lecture) which asserts that if $\varepsilon \neq \tau$ are non zero torsion classes, D_ε and D_τ have no common component, hence intersect transversally in only one point.

Take $C \in |2K|$ such that $b \in C$: then $\mathcal{O}_C(3K) \cong \omega_C$ and b is a base point of $|\omega_C|$ (because $h^1(\mathcal{O}(K))=0$, so $|3K|\big|_C = |\omega_C|$).
$2p(C)-2 = (C+K)\cdot C=6 \Rightarrow p(C)=4$ and by lemma B C must be reducible; moreover if Γ is a component of C containing b, $K\cdot\Gamma \geq 1$ ($K\cdot\Gamma=0 \Rightarrow \Gamma$ is rational non singular, hence that every section of $\mathcal{O}(3K)$, as it vanishes at b , vanishes on the whole of Γ ; this would contradict lemma 8), so that b belongs to at most two components. Then pick up an irreducible component

$\Gamma_\circ$ of C such that, if $C=\Gamma_\circ+C_\circ$, b belongs to one and only one component of $C_\circ$. Consider the exact sequence:

$$0 \longrightarrow \mathcal{O}(K+\Gamma_\circ) \longrightarrow \mathcal{O}(3K) \longrightarrow \mathcal{O}_{C_\circ}(3K) \longrightarrow 0 .$$

By Serre's duality $h^i(\mathcal{O}(K+\Gamma_\circ))=h^{2-i}(\mathcal{O}(-\Gamma_\circ))=0$ for $i=0,1$ ($i=1$ is a consequence of the exact sequence $0 \longrightarrow \mathcal{O}(-\Gamma_\circ) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{\Gamma_\circ} \longrightarrow 0$ and the irreducibility of $\Gamma_\circ$) and one obtains that b is a base point for $|\mathcal{O}_{C_\circ}(3K)|$.

Then one has the exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow H^1(\mathfrak{M}_b\mathcal{O}_{C_\circ}(3K)) \text{ —— } H^1(\mathcal{O}_{C^\circ}(3K)) \text{ —— } 0$$

and, dualizing,

$$(*) \quad 0 \longleftarrow \mathbb{K} \longleftarrow \mathrm{Hom}(\mathfrak{M}_b, \mathcal{O}_{C^\circ}(-\Gamma_\circ)) \longleftarrow H^\circ(\mathcal{O}_{C^\circ}(-\Gamma_\circ)) \longleftarrow 0$$

First, b must be a simple point of $C_\circ$, otherwise lemma B' implies $h^\circ(\mathcal{O}_{\tilde{C}_\circ}(-\Gamma_\circ)) \geq 1$, and using Lemma A plus the already quoted connectedness theorem for pluricanonical divisor, $\mathcal{O}_{\tilde{C}_\circ}(-\Gamma_\circ)$ has degree ≤ -1, hence there exists a decomposition of $C_\circ = C_1+C_2$ such that $-\Gamma_\circ\cdot C_2 \geq C_1C_2$: however then $0 \geq (\Gamma_\circ+C_1)\cdot C_2$, and $C=(\Gamma_\circ+C_1)+C_2$ is not numerically connected. The same reasoning gives the vanishing of $h^\circ(\mathcal{O}_{C_\circ}(-\Gamma_\circ))$, and b being simple, (*) amounts to $h^\circ(\mathcal{O}_{C_\circ}(-\Gamma_\circ+b))=1$. Again lemma A gives either $\mathcal{O}_{C_\circ}(\Gamma_\circ)\cong\mathcal{O}_{C_\circ}(b)$ or $C_\circ=C_1+C_2$ such that $-\Gamma_\circ C_2+1 \geq C_1C_2$ if $b \in C_2$

$$-\Gamma_\circ C_2 \geq C_1C_2 \quad \text{if} \quad b \notin C_2 .$$

This last is impossible, the other two possibilities imply $C=C'+C''$ where $C'\sim K\sim C''$, again by the connectedness theorem , and $\mathcal{O}_{C'}(C'')=\mathcal{O}_{C'}(b)$ by lemma A again.

Finally the exact sequence

$$0 \to H^0(\mathcal{O}(C''-C')) \to H^0(\mathcal{O}(C'')) \to H^0(\mathcal{O}_{C'}(C'')) \to 0$$

implies that $C' \neq C''$, so $C'=D_\varepsilon$ for a suitable $\varepsilon \neq -\varepsilon$. Conversely, if $b=D_\varepsilon \cap D_{-\varepsilon}$ we claim that b is a base point for $|3K|$. In fact $h^1(\mathcal{O}(3K-D_\varepsilon)) = h^1(-D_{-\varepsilon})=0$ (the D_τ's are $\sim K$ hence numerically connected), so that $|3K||_{D_\varepsilon} = = |\mathcal{O}_{D_\varepsilon}(3K)|$: then, as $\mathcal{O}_{D_\varepsilon}(3K) \cong \omega_{D_\varepsilon} \otimes \mathcal{O}_{D_\varepsilon}(b)$, $h^1(\mathcal{O}_{D_\varepsilon}(3K)) = = h^0(\mathcal{O}_{D_\varepsilon}(-b)) = 0$, so b is a base point of $|3K||_{D_\varepsilon}$, and therefore of $|3K|$.

Q.E.D.

<u>Lemma</u> 10. If there exists $\varepsilon \in T$ s.t. $\varepsilon \neq -\varepsilon$, then $|3K|$ is spanned by the two linear subsystems $D_\varepsilon + |2K-\varepsilon|$, $D_{-\varepsilon} + |2K+\varepsilon|$.

<u>Proof</u>. Note that by R.R. $\forall$ non zero torsion class τ $h^0(\mathcal{O}(2K+\tau))=2$, while $P_3=4$, hence it suffices to show that these two subsystems have no common element. This is clear, however, since if one should have $D_\varepsilon + M_{-\varepsilon} = D_{-\varepsilon} + M_\varepsilon$, $M_\varepsilon - D_\varepsilon$ would be a positive divisor $\equiv K$ (in fact D_ε and $D_{-\varepsilon}$ have no common component), contradicting $p_g=0$.

Q.E.D.

<u>Proposition</u> 11. Two general curves of $|3K|$ are simple at a base point b of $|3K|$, and have there a transversal intersection.

<u>Proof</u>. If a general curve of $|3K|$ would be singular at $b=D_\varepsilon \cap D_{-\varepsilon}$, b would then be a double base point of $|3K||_D = = b+|\omega_{D_\varepsilon}|$, so b would be a base point of the canonical system

of D_ε ; so if Γ is the component of D to which b belongs $p(\Gamma)=0$ and, by Lemma A, $D_\varepsilon = C_1+C_2$, where $C_1 \geq \Gamma$, $C_1 \cdot C_2 \leq$ $\leq C_2(K+D_\varepsilon)=2K\cdot C_2=0$ (as C_2 is made out of curves E s.t. $K\cdot E=0$): this however contradicts the numerical connectedness of D_ε. This reasoning tells also that a general curve is not tangent at b neither to D_ε, nor to $D_{-\varepsilon}$. If b is not a base point of $|2K-\varepsilon|$, or is not a base point of $|2K+\varepsilon|$, we are through: but in the contrary case, by lemma 10, b would be a singular base point of $|3K|$, which we have just shown to be impossible.

Q.E.D.

Denote by n the order of the torsion group T of S : by theorem 14 of [C.M.] (pag. 214-5) $n \leq 6$; moreover $n=6 \Longrightarrow$

$\mathrm{Tors}(\mathrm{Pic}(S)) \cong \mathbb{Z}/_{2\mathbb{Z}} + \mathbb{Z}/_{3\mathbb{Z}}$, hence there would exist a double unramified cover $p:\tilde{S} \longrightarrow S$, with $\chi(\mathcal{O}_{\tilde{S}})=2$, $K^2_{\tilde{S}}=2$, $q(\tilde{S})=0$ ([C.M.] lemma 14, pag. 212), but then $\tilde{T}=\mathrm{Tors}(\mathrm{Pic}(\tilde{S}))$ should be either 0 or $\mathbb{Z}/_{2\mathbb{Z}}$ ([C.M.] th. 15, pag. 215).

Hence $n \leq 5$, and $\mathbb{Z}/_{2\mathbb{Z}} + \mathbb{Z}/_{2\mathbb{Z}}$ cannot be the torsion group, by Miles Reid's lemma (compare Dolgacev's lecture).

Combining these with the previous results, we obtain.

Corollary 12. There are no infinitely near base points for $|3K|$, and the number b of them is

0 if $T \cong 0, \mathbb{Z}/_{2\mathbb{Z}}$

1 if $T \cong \mathbb{Z}/_{3\mathbb{Z}}$, $\mathbb{Z}/_{4\mathbb{Z}}$

2 if $T \cong \mathbb{Z}/_{5\mathbb{Z}}$.

<u>Theorem</u> 3. $\bar{\Phi}$ is birational.

We refer the reader to [6], pag. 107-108 for the proof of this last part. We only remark that one has to prove that m=deg $\bar{\Phi}$ cannot be more than 1, and so one must show that (as $9 = md + b$, and $d \geq 2$) the following cases cannot hold

i) $d = 2 \quad m = 4 \quad b = 1$

ii) $d = 3 \quad m = 3 \quad b = 0$

iii) $d = 4 \quad m = 2 \quad b = 1$

For case i) it suffices to consider that V, a quadric, contains a pencil of reducible hyperplane sections, and by taking inverse images one contradicts lemma 1.

Case ii) is managed showing that V cannot have a double line (by a similar argument to the preceding one), hence it is a normal cubic, and then that there exists a pencil of quadrics cutting on V the images of curves in $|2K|$: however this gives rise to a numerical contradiction.

Finally case iii) makes direct use of the existence of the divisors D_{ε} homologous to K (guaranteed by corollary 12).

REFERENCES

[1] ≅ [C.M.] Bombieri, E. Canonical models of Surfaces of General Type, Publ. Math. I.H.E.S. 42 , pp. 171-219.

[2] Bombieri, E. The Pluricanonical Map of a Complex Surface, Several complex Variables I, Maryland 1970 Springer Lecture Notes 155, pp. 35-87.

[3] Bombieri,E.-Catanese,F. The tricanonical map of a surface with $K^2=2$, $p_g=0$, Preprint Pisa (to appear in a volume dedicated to C.P. Ramanujam).

[4] Bombieri,E-Catanese,F. Birationality of the quadricanonical map for a numerical Godeaux surface (to appear in B.U.M.I).

[5] Kodaira,K. Pluricanonical systems on algebraic surfaces of general type, J. Math. Soc. Japan, 20 (1968) pp. 170-192.

[6] Miyaoka,Y. Tricanonical maps of numerical Godeaux Surfaces Inventiones Math. 34 (1976), pp. 99-111.

[7] Miyaoka,Y. On numerical Campedelli Surfaces, Complex Analysis and Algebraic Geometry, Cambridge Univ. Press 1977, pp. 113-118.

[8] Mumford,D. The canonical ring of an algebraic surface, Annals of Math., 76(1962), pp. 612-615.

[9] Mumford,D. Lectures on Curves on an algebraic surface, Annals of Math. Studies, 59 (1966).

[10] Peters,C.A.M. On two types of surfaces of general type with vanishing geometric genus, Inventiones Math. 32, (1976), pp. 33-47.

[11] Ramanujam, C.P. Remarks on the Kodaira vanishing theorem, J. Indian Math. Soc. 36 (1972), pp.41-51.