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(C.I.M.E.)

ON A CLASS OF SURFACES OF GENERAL TYPE

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I. Introduction.

This lecture contains an exposition, without many details and proofs, (they will appear in a future paper), of a joint research of E.Bombieri-F.Catanese, dealing whith surfaces having the following numerical invariants: $K^2=2$, $p_g=q=1$ (of course they are of general type).

The interest about the existence of these surfaces was motivated by thefollowing remark: if S is a minimal surface of general type, and one considers the numerical characters K^2, $\chi = \chi(\mathcal{O}_S) = 1-q+p_g$, then the following inequalities hold

(*) Research made when this author was a member of G.N.S.A.G.A. of C.N.R..

(Noether's inequality) $p_g \leq \frac{1}{2} K^2 + 2 \Rightarrow \chi \leq \frac{1}{2} K^2 + 3$

(Bogomolov-Miyaoka's inequality) $K^2 \leq 9\chi$.

(a consequence of Castelnuovo's criterion) $\chi \geq 1$.

Moreover, if S is irregular, S admits unramified covers $\hat{S}$ of any order m ; as these two invariants for $\hat{S}$ are those of S multiplied by m , and Noether's inequality must hold for then, it turns out that $K_S^2 \geq 2\chi(\mathcal{O}_S)$. In the (χ, K^2) plane then, the minimal irregular surfaces of general type lie in a convex ragion whose lower vertex corresponds to surfaces with $K^2=2$, $\chi=1$.

Finally in this case one must have $q=p_g=1$, by the results of [C.M.] (pag. 212), and one is conducted to check if these minimal values of K^2, χ are really attained, trying to construct such surfaces.

We have proved firstly an existence and unicity theorem about these surfaces, namely that there exist double ramified covers of the double symmetric product of an elliptic curve which have these numerical invariants, and moreover that all such surfaces arise in this way.

It has been then possible to show that their canonical models all belong to a family with non singular base space, hence by the results of [11] they are all deformation of each other (with non singular base), and are all diffeomorphic; their fundamental group is proven to be abelian exploiting this remark and a suitable degeneration of the branch locus: finally it is possible to prove that the constructed family coincides with the Kuranishi family (when the canonical models are nonsingular).

II. Geometry of the double symmetric product of an elliptic curve.

Let E be an elliptic curve; $E^2 = E \times E$: $E^{(2)}$, the double symmetric product of E is the quotient of E^2 by the involution taking (x,y) E^2 to (y,x).

The quotient map $\pi: E^2 \longrightarrow E^{(2)}$ has the diagonal Δ of E^2 as ramification locus, and $E^{(2)}$ is non singular (compare [1]). Fixing a base point in E as the zero element of a group law on E, the map $p': E^2 \longrightarrow E$ s.t. $p'((x,y)) = x+y$ induces a mapping $p: E^{(2)} \longrightarrow E$ (note that a different choice of the base point alters p only up to a translation on E).

The following hold:

Proposition 1. $p: E^{(2)} \longrightarrow E$ makes $E^{(2)}$ into a $\mathbb{P}^1$-bundle over E.

Proposition 1 bis. p possesses a 1-dimensional family of cross sections, two of which intersect transversally in only one point.

Proof. For $a \in E$ consider the section $\sigma_a : E \longrightarrow E^{(2)}$ s.t. $\sigma_a(z) = (a, z-a)$. Denoting by Γ_a the image of σ_a, Γ_a and $\Gamma_{a'}$ meet transversally in (a,a').

Q.E.D.

Denoting by γ the homology class of a section and by f that of a fibre, we know that the second homology group of $E^{(2)}$ is freely generated by γ, f and $\gamma \cdot f = 1$, $\gamma^2 = 1$, $f^2 = 0$ (the first assertion is by the Leray-Hirsch theorem).

If D is a divisor, denote by $\deg D$ the intersection num ber of D with a fibre; one has

Proposition 3. The following sequence is exact

$$0 \longrightarrow \mathrm{Pic}(E) \xrightarrow{P_*} \mathrm{Pic}(E^{(2)}) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0 .$$

The proof of the preceding assertion is by diagram chasing, while, by the fact that every morphism of P^n into an abelian variety is constant, one gets.

Proposition 4. If D is a positive divisor on $E^{(2)}$, $D^2 \geq 0$.

We come now to the canonical bundle of $E^{(2)}$: considering the map π and the triviality of K_{E^2}, we obtain the equality $-2K = \hat{\Delta}$, where $\hat{\Delta} = \pi_* \Delta$.

For $z \in E$ denote by $F_z = p^{-1}(z)$; using prop. 3 and the structure of Pic (E) we can show that for a suitable choice of the base point in E $K = -2\Gamma + F_o$.

As for positive divisor D, $\deg(D) \geq 0$, one can sharpen prop. 4 to

Proposition 5. If B is an irreducible curve with $B^2=0$,

either i) B is a fibre

or ii) B is homologous to $-nK$, $n \geq 1$.

There are on $E^{(2)}$ two rational pencils of curves, which will be of particular use in the sequel, that we are just going to describe.

Consider on E^2 the following family of curves, $\{C_a\}_{a \in E}$, given by $C_a = \{(x,x+a)\ x \in E\}$: these curves are isomorphic to E and are "translated" of $\Delta = C_o$.

Observe that $C_a \cap C_b = \emptyset$ if $a \neq b$, and the symmetry involution takes C_a to C_{-a} (which is a different curve if

$2a \neq 0$). Denote by a_i, $i=1,2,3$, one of the three points such that $a \neq 0$ $2a=0$: the symmetry operation operates trivially on Δ, not on $C_{\hat{a}_i}$.

If $\hat{C}_i = \pi(C_{\hat{a}_i})$ $\pi : C_{\hat{a}_i} \longrightarrow \hat{C}_i$ is a double unramified cover, so $\hat{C}_i$ is an elliptic curve not isomorphic to E, $\pi_*(C_{\hat{a}_i}) = 2\hat{C}_i$ while for the other values of a

$$\pi : C_a \longrightarrow p(C_a) \text{ is an isomorphism.}$$

We can summarize this in the following

<u>Proposition</u> 6. There exists on $E^{(2)}$ a rational pencil of curves linearly equivalent to $-2K$, whose elements are all isomorphic to E, but three particular members which are twice a non singular elliptic curve not isomorphic to E.

<u>Proof</u>. The family is given by $\{\pi_*(C_a)\}_{a \in E}$.
As $\pi_*(C_a) = \pi_*(C_{-a})$ the family can be parametrized by $\mathbb{P}^1$: then recall that $-2K \equiv \hat{\Delta} \equiv \pi_*(C_o)$.

Q.E.D.

By an entirely equal argument we have

<u>Proposition</u> 7. $|2\Gamma|$ contains the rational pencil given by

$$\{\Gamma_a + \Gamma_{-a}\}_{a \in E}.$$

<u>Proposition</u> 8. On $E^{(2)}$ the three curves $\hat{C}_i$ are the only curves homologous to $-K$.

<u>Sketch of proof</u>. If C is such a curve, $\hat{\Delta} \cdot C = 0$, so

$\hat{\Delta} \cap C = \emptyset$: by an easy numerical argument one gets that $\pi^{-1}(C)$ is a connected curve $\tilde{C}$. However, $\tilde{C}$ being homologous to Δ and disjoint from it, it must be the graph of a translation on E (cf. [10]), so one of the curves C_a: finally the condition $\pi^*\pi_*(\tilde{C})=2\tilde{C}$ implies $\tilde{C}=C_{\hat{a}_i}$ for some $i \in \{1,2,3\}$.

Corollary 9. $|-3K| \neq \emptyset$, $h^{\circ}(-nK) = h^1(-nK)$, for $n \geq 0$.

Proof. Consider that $\hat{C}_i \equiv 2\Gamma - F_{\hat{a}_i}$, so that the sum of those three curves is $\equiv -3K$.
The other assertion comes from R.R. plus duality.

Q.E.D.

Proposition 10.

$$h^{\circ}(\mathcal{O}(-rK)) = \begin{cases} m+1 & \text{if } r=2m \\ m & \text{if } r=2m+1 \end{cases}$$

Proof. The proof of the first assertion uses induction on m in the cohomology sequence of

$$0 \longrightarrow \mathcal{O}(m(-2K)) \longrightarrow \mathcal{O}((m+1)(-2K)) \longrightarrow \mathcal{O}_{\hat{\Delta}} \longrightarrow 0 \quad ,$$

to obtain $h^{\circ}(\mathcal{O}(-2mK)) \leq m+1$, and exploits the fact that $|-2mK| \supset m|-2K|$ to deduce that $h^{\circ}(\mathcal{O}(-2mK)) \geq m+1$.

For the second one we apply Ramanujam's theorem A ([9] [C.M.]): $|-2mK|$ being composed of the rational pencil $|-2K|$, $h^1(\mathcal{O}(2mK))=m-1$.

Then Serre duality and corollary 9 are enough to fulfill the proof.

Q.E.D.

If τ is an automorphism of E , $(x,y) \longrightarrow (\tau(x),\tau(y))$ defines an automorphis $K(\tau)$ of $E^{(2)}$; any automorphism g of

$E^{(2)}$ moreover has the effect of permuting the fibres of p and so induces an automorphism $H(g)$ of E. A simple calculation shows that $H(K(\tau))(x) = \tau(x)+\tau(0)$ so that Ker $H\circ K$ is given by the four translations of period two. Then we observe that, as $(a,b) = \Gamma_a \cap \Gamma_b$ and $|\Gamma_a| = \{\Gamma_a\}$, if g^* is the identity on $\mathrm{Pic}(E^{(2)})$, g must be the identity ; so if $g \in$ Ker H, $g^*(F_a)=F_a$, $g^*(K)=K$, and $g^*(2\Gamma)=g^*(-K+F_o)=2\Gamma$.

Hence $g^*(\Gamma) = \Gamma - F_o + F_{\hat{a}_i}$, and Ker H has order 4.

Then, noticing that K is injective, Ker $H\circ K$=Ker H, and $H\circ K$ is onto, one gets.

<u>Proposition</u> 11. H is an isomorphism of $\mathrm{Aut}(E)$ onto $\mathrm{Aut}(E^{(2)})$.

III. Existence of surfaces with $K^2 = 2$, $p_g=q=1$.

Given a line bundle L on $E^{(2)}$, suppose we are given with a cover $\{U_i\} = \mathcal{U}$, a cocycle $(r_{ij}) \in H^1(\mathcal{U}, \mathcal{O}^*)$ defining L, and a positive divisor D defined by a section $S=\{s_i\}$ of $L^{\otimes 2}$: then one may take in L the surface S' defined by taking the square roots of S, that is S' is defined in $U_i \times \mathbb{C}$ (with coordinates (x,z_i)) by the equation $s_i(x)=z_i^2$. If $D=\mathrm{div}(S)$ is smooth, then S' is smooth too, and S' is normal if D has no multiple components.

Supposing that D exists, smooth, one obtains, using the signature formula of 4 and other computations, that, L being $\equiv$ $\equiv a\Gamma + ((b-1)F_o + F_x)$, S' has the desired numerical invariants iff $a=3$, $b=1$.

So we are left with the task of showing that the generic element of $|D|$ is non singular if $D \equiv 6\Gamma - 2F_x \equiv -3K+F_{-2x}$; in view of the result of prop. 11 one can restrict himself to consider only the divisor $D \equiv -3K+F_o$.

<u>Proposition</u> 12. $h^1(\mathcal{O}(D))=0$, $|D|$ has dimension 6 and its generic element is irreducible non singular.

<u>Sketch of proof</u>. as $|D| \supset |-2K| + \hat{C}_i + F_{-\hat{a}_i}$ $(i=1,2,3)$, $|D| \ni \hat{C}_1+\hat{C}_2+\hat{C}_3+F_o$ it is easily seen that $|D|$ has neither fixed part nor base points, and Bertini's theorem applies. The first two assertions follow from R.R., Serre duality and Ramanujam's theorem $(h^1(\mathcal{O}(D)) = h^1(-4K+F_o))$.

By the results of [5] , when D has no multiple components, S' has at most rational double points as its singularities iff D has no singular points of multiplicity greater than three, and any triple point has no infinitely near singularities of multiplicity greater than two; moreover if this conditions are satisfied , a minimal desingularization $g:S \to S'$ of S' has the desired numerical invariants.

We observe that the condition is fulfilled in our linear system if D is irreducible (by the genus formula), and it is possible to describe explicitly which are the reducible curves in $|-3K+F_o|$, and which the "bad" ones: we omit this point for the sake of brevity, as well as the verification of

<u>Proposition</u> 13:

i) S is a minimal surface

ii) S' is the canonical model of S (see [C.M.] for its definition and properties)

iii) if $f:S' \longrightarrow E^{(2)}$ is the projection induced from L, $h = p \circ f \circ g : S \longrightarrow E$ is the Albanese mapping of S.

We have moreover

Proposition 14. Let S, S_o be two surfaces obtained in the above described way: if $\Psi: S \longrightarrow S_o$ is an isomorphism, $E \cong Alb(S) \cong Alb(S_o) \cong E_o$ and under this identification of E and E_o Ψ is induced by an automorphism ψ of $E^{(2)}$ taking L to L_o, D to D_o.

Sketch of proof. By proposition 13 you may identify E, E_o and suppose that Ψ commutes with the respective Albanese maps of S, S_o.

Ψ induces an isomorphism ψ' of the canonical models S', moreover the fibres $\hat{F}$ of the Albanese map are curves of genus two, and $f(x)=f(y)$ iff $\mathcal{O}_{\hat{F}}(x+y)$ is the unique hyperelliptic bundle of $\hat{F}$: this remark enables us to define ψ on $E^{(2)}$.

IV. Surfaces with $K^2=2$, $\chi=1$, $q=1$ are double covers of the symmetric product of their Albanese variety.

Let $\vartheta : S \longrightarrow E = Alb(S)$ be the Albanese map of S, and, for $u \in E$, set $G_u = \vartheta^{-1}(\{u\})$, $K_u \equiv K + F_u - F_o$.

We refer to [C.M.] for the proof of the following facts

i) $\forall\, u \in E$ $h^o(S, \mathcal{O}(K_u))=1$ (denote then by C_u the unique curve in $|K_u|$).

ii) C_u is generically irreducible $p(C_u)=3$

iii) $h^o(\mathcal{O}(K+K_v)) = 3$ $\forall\, v \in E$.

Fixing $v \in E$, $\forall u \in E$ we have the divisor $C_u + C_{v-u} \in \in |K+K_v|$.

<u>Proposition</u> 15. The mapping $u \longrightarrow C_u + C_{v-u}$ defines an holomorphic mapping ψ_v of E into $|K+K_v| \cong \mathbb{P}^2$.

<u>Proposition</u> 16. The image Δ_v of ψ_v is an irreducible rational curve.

<u>Proof</u>. $C_u + C_{v-u} = C_{v-u} + C_{v-(v-u)}$, so ψ_v is invariant by the involution which takes $u \longrightarrow v-u$, and whose quotient is isomorphic to $\mathbb{P}^1$ (denote now by $\varphi_v : \mathbb{P}^1 \longrightarrow |K+K_v|$ the induced mapping). Q.E.D.

<u>Proposition</u> 17. For general v Δ_v is non singular.

<u>Sketch of proof</u>. An uniformizing parameter on the universal cover of E induces a derivation D on each vector bundle, and, given a section σ, we use the classical notation σ' for $D(\sigma)$.

Take a cover $\{U_i\}$ of S on which $\mathcal{O}(K_u)$ is trivialized for u near $\hat{u}$: then, σ_u being the section of $\mathcal{O}(K_u)$ defining C_u, $\sigma_u = \{\sigma_{iu}\}$, where σ_{iu} depends holomorphically on u.

The condition that ψ_v is not of maximal rank at $u=\hat{u}$ can be read as $\varphi_i = \dfrac{\sigma'_{i\hat{u}}}{\sigma_{i\hat{u}}} = \dfrac{\sigma'_{i(v-\hat{u})}}{\sigma_{i(v-\hat{u})}} + \lambda$, for λ a suitable constant independent of i.

Now, for general v, if $\hat{u} \neq v-\hat{u}$, this equality implies that φ_i is a regular function on U_i.

But then, if $f_{ij}(u)$ is a cocycle defining $\mathcal{O}(K_u)$, one gets after a simple computation that $\frac{f'_{ij}(\hat{u})}{f_{ij}(\hat{u})} = \varphi_i - \varphi_j$, hence it is a coboundary in $H^1(S,\mathcal{O})$. A similar conclusion can be drawn in general if φ_v is not of maximal rank at any point.

Denoting by $\lambda_{ij}(u)$ a cocycle for $\mathcal{O}_E(u-0)$ (relative to a covering $\{V_i\}$ compatible with $\{U_i\}$), we have that $\frac{f'_{ij}(\hat{u})}{f_{ij}(\hat{u})} = \vartheta^* \left(\frac{\lambda'_{ij}(\hat{u})}{\lambda_{ij}(\hat{u})} \right)$: $\vartheta^* H^1(E,\mathcal{O}) \longrightarrow H^1(S,\mathcal{O})$ is however an isomorphism (compare [3]), so $\frac{\lambda'_{ij}(\hat{u})}{\lambda_{ij}(\hat{u})}$ is a coboundary. By the homogeneity of E under translation, it is possible to write for all u $\frac{\lambda'_{ij}(u)}{\lambda_{ij}(u)} = \varphi_i(u) - \varphi_j(u)$, with $\varphi_i(u) \in \Gamma(V_i,\mathcal{O})$.

Now, integrating on any path from 0 to u, one gets $\frac{\lambda_{ij}(u)}{\lambda_{ij}(0)} = \frac{\exp(\int_0^u \varphi_i(t)dt)}{\exp(\int_0^u \varphi_j(t)dt}$, and so 0 and u should be two linearly equivalent points, which is clearly absurd.

Q.E.D.

The fact that $C_u \cdot C_v = 2$ and the irrationality of the pencil $\{C_u\}$ implies that this pencil has no base points hence

Proposition 18. $\forall\, v$ $|K+K_v|$ has no base points and Δ_v is an irreducible conic.

Proof. It suffices to show that $\deg \Delta_v = 2$ (Δ_v being irreducible), and yet only that $\deg \Delta_v \geq 2$ (Δ_v being generally non singular). Take then $C_u, C_{u'}$ general, meeting in two points

x,y : C_u+C_{v-u}, $C_{u'}+C_{v-u'}$ represent two points in the plane $|K+K_v|$ which lie in the intersection of Δ_v with the line $|K+K_v-x|$.

Q.E.D.

Proposition 19. For general $y \in S$, there exists a unique point $(u,u') \in E^{(2)}$ (with $u \neq u'$), s.t. $y \in C_u \cap C_{u'}$.

Proposition 20. The correspondence of prop. 19 extend to a holomorphic mapping $f' : S \longrightarrow E^{(2)}$. Moreover f' factors as $S \xrightarrow{g} S' \xrightarrow{f} E^{(2)}$, where

a) S' is the canonical model of S, g the canonical mapping

b) f is a finite map of degree two

c) there exists a line bundle L on $E^{(2)}$ and a section $\sigma \in H^0(L^{\otimes 2})$ such that S' is isomorphic to the surface of the square roots of σ in L .

V. Some results on the structure of these surfaces and on their deformations.

Sharpening the result of prop. 20 one can show that it is possible to choose $L \equiv 3\Gamma - F_0$ (essentially by using $\mathrm{Aut}(E^{(2)})$).

We will sketch very rapidly the construction of a family containing all the canonical models S' of our surfaces: take $\mathcal{E} \longrightarrow H$ the local universal family of elliptic curves over the Siegel upper halfplane, and form its symmetric fibre product $\mathcal{E}^{(2)} \longrightarrow H$. All the invertible sheaves $\mathcal{O}_{E^{(2)}}(3\Gamma - F_0)$ fit together to form an invertible sheaf $\mathcal{L}$ on $\mathcal{E}^{(2)}$ and one can choo se sections $s_o,\dots s_6 \in \Gamma(\mathcal{E}^{(2)}, \mathcal{L}^{\otimes 2})$ such that for all $t_o\dots t_6$, $s_t = \sum t_i s_i$ defines a relative divisor $\mathcal{D}_t$ on $\mathcal{E}^{(2)} \longrightarrow H$ (we use the terminology of [8]). Then on $\mathcal{E}^{(2)} \times \mathbb{P}^6 \longrightarrow H \times \mathbb{P}^6$ is defined an invertible sheaf that we still denote by $\mathcal{L}$ and a relative divisor $\mathcal{D}$ linearly equivalent to $2\mathcal{L}$. Define $\mathcal{S} \hookrightarrow \mathcal{L}$ to be the variety defined in $\mathcal{L}$ by the square roots of a section defining $\mathcal{D}$: then we have a family $\mathcal{S} \longrightarrow H \times \mathbb{P}^6$ and an open dense subset V of $H \times \mathbb{P}^6$ such that the fibres over points of V are all the canonical models of the surfaces we are considering.

By using the result of Tyurina about local resolution of singularities of such families ([11]) we deduce that our surfaces are all diffeomorphic.

Proposition 21. $\pi_1(S) \xrightarrow{h_*} \pi_1(E)$.

Sketch of proof. By the above remark we may consider the particular surface for which $D = F_o + \hat{C}_1 + \hat{C}_2 + \hat{C}_3$.
Then we observe that by Van Kampen's theorem $\pi_1(S) \cong \pi_1(S')$; call $\hat{F}_o = f^{-1}(F_o)$, and observe that $S' - \hat{F}_o$ —— $E - \{0\}$ is a differentiable fibre bundle with fibre a smooth curve of genus two. Take U to be a small disk around 0 in E : $(p \circ f)^{-1}(U) = S'|_U$ is contractible to $\hat{F}_o$, which is simply connected, and we can apply again Van Kampen's theorem to the open sets $S'|_U$, $S' - \hat{F}_o$ in S', U and $E - \{0\}$ in E , together with the exact homotopy sequence of a bundle (where $u \in U - \{0\}$, $\hat{F}_u$ is the fibre of S' over u), to obtain the following diagram, commutative, exact in the columns and the rows , and which gives easily our result.

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
1 & \longrightarrow & \pi_1(U-\{0\}) & \longrightarrow & \pi_1(E-\{0\}) & \longrightarrow & \pi_1(E) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & \nearrow & \\
1 & \longrightarrow & \pi_1(S'_U-\hat{F}_o) & \longrightarrow & \pi_1(S'-\hat{F}_o) & \longrightarrow & \pi_1(S') & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \pi_1(\hat{F}_u) & \cong & \pi_1(\hat{F}_u) & & & &
\end{array}
$$

Q.E.D.

We end this talk by showing

Proposition 22. If θ_S denotes the tangent bundle of S, then $h^1(\theta_S)=7$.

Sketch of the proof (in the simpler case when D is smooth). Using the Hirzebruch Riemann-Roch theorem one sees that it is equivalent to show that $h^\circ(\Omega^1_S \otimes \Omega^2_S)=1$. One exploits first one fact: you have an involution on S determined by $f':S \longrightarrow X=E^{(2)}$, so $H^\circ(\Omega^1_S \otimes \Omega^2_S) = H^+ \times H^-$, where H^+ is the subspace of invariant, H^- of anti invariant sections of $\Omega^1_S \otimes \Omega^2_S$, and by one side the sections of H^+ correspond to sections of $\Omega^1_X \times \Omega^2_X \times L, \cong \Omega^1_X \otimes \mathcal{O}(\Gamma)$, by the other, sections of H^- vanish on the ramification locus R, hence they come from sections of $\Omega^1_S \otimes \mathcal{O}(K_S-R) \cong \Omega^1_S \times f^*(K_X)$. Secondly one exploits the unique section of Ω^1_X (which defines a trivial subbundle T of Ω^1_X) and the fact that $\Omega^1_X \otimes \mathcal{O}(\Gamma)$ $\Omega^1_X \cong \mathcal{O}(-\Gamma+F_o)$ has no non zero sections, to infer that every section of $\Omega^1_X \otimes \mathcal{O}(\Gamma)$ comes from a section of the subbundle $T \otimes \mathcal{O}(\Gamma) \cong$ $\cong \mathcal{O}(\Gamma)$, which, though, possesses only one section. The same method can be applied to show the vanishing of $H^{\circ}(\Omega^1_S \otimes f^*(K_X))$.

Q.E.D.

Finally, when D is non singular, one can take the constructed family and show the injectivity of the Kodaira-Spencer map: then locally our family is the Kuranishi family (compare [6] , [7]).

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