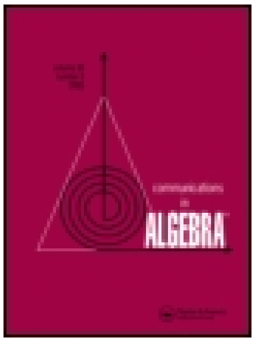

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ON SEVERI'S PROOF OF THE DOUBLE POINT FORMULA

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§0. Introduction

In his paper [6] of 1902 Severi considers the following situation: let $M \longrightarrow \mathbb{P}^{2k}(\mathbb{C})$ be an irreducible variety of dimension k with "generic" singularities, i.e., only a finite number of transversal double points, $P_1, \cdots, P_d$ ("transversal" means that locally, at each P_i, M consists of two smooth branches intersecting transversally).

Severi then gives a formula expressing d in terms of certain projective characters of M:

$$2d = n(n-1) - \sum_{i=1}^{k} \omega_i .$$

Here n is the degree of M, and ω_i is the i^{th} ceto

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of M, which can be conveniently defined as follows: if $V \longrightarrow \mathbb{P}^N$ is a variety of dimension m, the m^{th} ceto $\omega_m(V)$ is the number of m-dimensional linear subspaces tangent to V at smooth points which meet a general 2m-codimensional linear subspace, while $\omega_i(V)$ is the i^{th} ceto of the intersection of V with a general (m-i)-codimensional linear subspace. During the past few years there has been a renewed interest in enumerative geometry and Severi's double point formula has been generalized to a greater extent (see [2], and especially [3] for wider historical and bibliographical references): we believe however it may be interesting to give in this note an account of Severi's elementary proof, clarifying it and filling in some details skipped in his paper [6].

We note that this proof works word by word in the case of any algebraically closed field of char. 0.

Notations

$G(r,N)$ is the Grassmanian of r-dimensional linear subspaces of $\mathbb{P}^N$.

O is a general point in $\mathbb{P}^{2k}$, $\sigma_O \subset G(k,2k)$ is the Schubert cell of the subspaces containing O,

$\Sigma_O \subset G(1,2k)$ is defined analogously.

$f: M' \longrightarrow M \longrightarrow \mathbb{P}^{2k}$ is the normalization of M (so that f is an immersion).

$\mu: M' \longrightarrow G(k,2k)$ is the Gauss map of f.

P_i', P_i'' are the two distinct points in M' whose image is P_i.

φ_0, φ_1 are two general linear forms on $\mathbb{P}^{2k}$.

S is the (2k-2)-dimensional linear subspace defined by $\varphi_0 = \varphi_1 = 0$ and H is the hyperplane spanned by O,S.

Δ_M is the diagonal in $M \times M$.

Steps of Proof.

1) The projection of $M \cap H$ from O to S is a birational immersion and its image $\widetilde{M}$ has generic double points.

2) If $\widetilde{d}$ is the number of double points of $\widetilde{M}$, $2\widetilde{d} = 2d + \omega_k$.

Theorem: Steps 1), 2) imply that $2d = n(n-1) - \sum_{i=1}^{k} \omega_i$.

Proof: By induction on k. For k = 1 we have a plane curve with d nodes and of degree n: hence the equation is a particular case of the first Plücker formula. We remark then that by step 1) $n(M) = n(\widetilde{M})$, $\omega_i(M) = \omega_i(\widetilde{M})$ so that the inductive assumption plus step 2) imply the desired result.

§I. The basic construction ($k > 1$)

Consider in $(M \times M - \Delta_M) \times G(1,2k)$ the graph of the map which takes (P,Q) to the line joining them, and its closure Λ in $(M \times M) \times G(1,2k)$: Λ is irreducible of dimension $2k$.

Denote by $p: \Lambda \longrightarrow G(1,2k)$ the canonical projection; set $Z = p(\Lambda)$.

Lemma 1: $p: \Lambda \longrightarrow Z$ is generically a 2-1 map if M is not contained in a linear subspace of dimension $k+1$ (note that if $d \geq 1$, M is not contained in any hyperplane): in particular, then, Z is irreducible of dimension $2k$.

Proof: In our hypotheses you can find $k+3$ points of M spanning a subspace of dimension $k+2$, so that a general subspace L of dimension k will be such that $L \cap M$ is a finite set with the property that any $k+3$ points in it are linearly independent (compare [O], chapter II, iii), "Special linear systems I"), hence a fortiori any line through two of them won't contain a third one: this however immediately implies our assertion.

Fix a general point O and denote by $\hat{\tilde{Y}} = p^{-1}(\Sigma_O \cap Z)$, by $\hat{\tilde{Y}}'$ its inverse image in $M' \times M' \times G(1,2k)$, by Y, Y' the respective projections on M, M', by $\tilde{Y}, \tilde{Y}'$ the projections on $M \times M$, $M' \times M'$.

Proposition 2: $\tilde{\gamma}'$ is smooth; $\tilde{\gamma}'$ is the graph of a birational involution τ on γ' such that $\tau(P_i') = P_i''$.

We first prove two auxiliary lemmas:

Lemma A: Let P be a smooth point of M, and O a point in the space T_P tangent to M at P. Then $P \in \gamma$ and γ is smooth at P if σ_O is transversal to $\mu(M')$ at $\mu(P)$; in this case also $(d\tau)_P = -$ Identity.

Proof: We can take affine coordinates $x = (x_1,\cdots,x_k)$, $y = (y_1,\cdots,y_k)$ so that P corresponds to the origin, $y = 0$ is T_P, O is the point at infinity on the x_1-axis: $(x,f(x))$, where $f \in (\mathfrak{m}_x^2)^k$, will be then a parametric equation of M in a neighborhood U of P. Take coordinates (x,z) in $U \times U$: then the line through $(x,f(x))$, $(z,f(z))$ contains $O \Leftrightarrow x_2-z_2 = 0,\cdots,x_k-z_k = 0$, and $f_h(x) - f_h(z) = 0$ for $h = 1,\cdots,k \Leftrightarrow x_2-z_2 = 0$, $x_k-z_k = 0$

$$f_h(x_1,x_2,\cdots,x_k) - f_h(z_1,x_2,\cdots,x_k) = 0 \quad (h = 1,\cdots,k).$$

We can write $f_h(x_1,x_2,\cdots,x_k) \equiv \sum_{\nu=0}^{2} f_h^{\nu}(x_2,\cdots,x_k)\cdot x_1^{\nu} \pmod{\mathfrak{m}_x^3}$, hence

$$f_h(x_1,x_2,\cdots,x_k) - f_h(z_1,x_2,\cdots,x_k) \equiv (x_1-z_1)\left[f_h^1+(x_1+z_1)f_h^2\right] \pmod{\mathfrak{m}_x^3};$$

if at the origin $\det(f_h^2\, \partial f_h^1/\partial x_r) \neq 0$ $\left(\begin{matrix} h = 1,\cdots,k \\ r = 2,\cdots,k \end{matrix}\right)$

then it is easily seen by the implicit function theorem that $\tilde{\gamma}$ is smooth and x_1, z_1 can be both chosen as a local coordinate (hence γ is smooth too). Moreover, $f_h^2(0)(dx_1+dz_1) = 0 \ \forall\ h \Longrightarrow dx_1 = -dz_1$ at P. We are going then to check the non-vanishing of our determinant. In fact, the subspaces near T_P have parametric equations $(w, Aw+b)$, σ_0 consists locally of the submanifold defined by $Ae_1 = 0$ (the first column of A must vanish), the Gauss max μ takes x to $A = (\partial f_h/\partial x_r)$, $b = f$: hence transversality to σ_0 at $\mu(P)$ means that $\partial f_h/\partial x_1 \colon U \longrightarrow \mathbb{C}^k$ has invertible differential at the origin.

But $\partial f_h/\partial x_1 \equiv f_h^1 + 2x_1 f_h^2 \pmod{\mathfrak{m}_x^2}$ so its Jacobian matrix is at the origin

$$(2f_h^2\, \partial f_h^1/\partial x_r)\left(\begin{matrix} h = 1,\cdots,k \\ r = 2,\cdots,k \end{matrix}\right).$$

We remark finally that the tangent to γ at P passes through O.

Lemma B: Let P be a double point of M: then if O does not belong to $\mu(P')$, $\mu(P'')$, $\tilde{\gamma}'$ is smooth at (P',P''), γ' is smooth at P',P'', τ is biregular at P',P''.

Proof: We can take affine coordinates $x = (x_1,\cdots,x_k)$, $y = (y_1,\cdots,y_k)$ such that P is the origin, $\mu(P')$ is $y = 0$, $\mu(P")$ is $x = 0$. The two branches have parametric equations $(x,f(x))$, respectively $(g(y),y)$, ($f \in (\mathfrak{m}_x^2)^k$, $g \in (\mathfrak{m}_y^2)^k$). We can also suppose O to be the point at infinity on the line through P and $((1,0,\cdots,0),(1,0,\cdots,0))$, so that in a neighborhood of P two points of M can be collinear with O only if they lie in different branches.

$\tilde{Y}'$ is then defined locally by the following equations:

$$\begin{aligned} F_h &= y_h - f_h(x) = 0 \\ G_h &= x_h - g_h(x) = 0 \end{aligned} \qquad h = 2,\cdots,k$$

and

$$G_1 = x_1 - g_1(y) - y_1 + f_1(x) = 0.$$

Clearly at the origin (corresponding to $(P',P")$), $\partial G_h/\partial x_k = \delta_{hk}$, $\partial F_h/\partial x_k = 0$, $\partial F_h/\partial y_k = \delta_{hk}$, $\partial G_h/\partial y_k = -\delta_{1h}\delta_{1k}$ so that by the implicit function theorem $\tilde{Y}'$ is smooth and both x_1, y_1 can be taken as a local coordinate.

Proof of Prop. 2: Observe that Λ is smooth outside $\Delta_M \times G(1,2k)$: by theorem 2 of [1] (page 290), then $\tilde{Y}$ is smooth there, and (Λ being there a graph) $\tilde{Y} - \Delta_M$ is smooth.

For general O, σ_0 has ω_k transversal intersections with $\mu(M')$ (by [1]) at $\mu(Q_1),\cdots,\mu(Q_{\omega_k})$, where $Q_j \neq P_i$, hence we can apply Lemma A, and putting together with the above result and Lemma B we obtain that $\tilde{Y}'$ is smooth. Moreover Lemma 1 guarantees that the projection from $\tilde{Y}'$ to Y' is birational, and we observe for later use that the Q_i are the only fixed points of τ.

Remark: Though $\tilde{Y}'$ is smooth, Y' needs not to be so (a trisecant through O contributes three double points of Y).

§II. Proof of the main steps.

Step 1): For H general $M \cap H$ is smooth of dimension k-1 so you can find $O \in H$ such that the projection of $M \cap O$ with center O is a birational immersion and its image $\tilde{M}$ has "generic" singularities. (This is well known, see e.g. [4]).

Step $1+\frac{1}{2}$): For general O, Y, you can find H a general hyperplane containing O such that

i) H intersects Y transversally in deg Y distinct smooth points $R_1,\cdots,R_{\deg Y}$.

ii) H contains no Q_i (so you can suppose deg $\gamma = 2m$ and $\tau(R_i) = R_{m+i}$ for $i \leq m$).

iii) H,O satisfy the requirements of step 1).

To check i), ii), one needs to consider that only ω_k tangents of γ pass through O, so that the hyperplanes containing O and not satisfying i), ii) form a (2k-2)-dimensional subvariety.

Now for S a general (2k-2)-dimensional linear subspace of H the projection $\pi\colon H-\{O\} \longrightarrow S$ is such that $\widetilde{M} = \pi(M \cap H)$ has $\tilde{d}$ double points corresponding to the pairs $\{R_i, \tau(R_i)\}$ in $H \cap \gamma$, hence $m = \tilde{d}$.

<u>Step 2</u>): Take φ_0 a linear form vanishing on H, φ_1 a general one, S defined by $\varphi_0 = \varphi_1 = 0$. We will have $\gamma \cap S = \emptyset$ and (φ_0, φ_1) defines a morphism $g\colon \mathbb{P}^{2k}-S \longrightarrow \mathbb{P}^1$, and naturally

$$(g \cdot f, g \cdot f)\colon (M'-f^{-1}(S)) \times (M'-f^{-1}(S)) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1;$$

denote by ψ its restriction to $\widetilde{\gamma}'$, by $D = \psi(\widetilde{\gamma}')$, by Δ the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$.

Because $O, P, \tau(P)$ are collinear, it is clear that $g(P) = g(\tau(P))$ iff either $P = \tau(P)$ or $P \in H$, hence $\psi^{-1}(\Delta)$ consists of the pairs $(R_i, \tau(R_i)), (P_j', P_j''), (P_j'', P_j')$, (Q_h, Q_h).

Assertion: For general φ^1, Ψ is transversal to Δ, $\Psi(Q_i,Q_i) \neq \Psi(Q_j,Q_j)$ for $i \neq j$ ($\Longleftrightarrow g(Q_i) \neq g(Q_j)$).

As a first consequence of this we get that $\Psi^{-1}((Q_i,Q_i)) = (Q_i,Q_i)$ so that Ψ is birational, hence D has bidegree $(2\tilde{d},2\tilde{d})$: in fact, if $(\mu_0,\mu_1) \in \mathbb{P}^1$, the intersections of D with $(\mu_0,\mu_1)\times\mathbb{P}^1$ correspond then to the points of γ in the hyperplane $\mu_1\varphi_0-\mu_0\varphi_1 = 0$, and are then $2\tilde{d} = \deg \gamma$, and moreover D is clearly symmetric in $\mathbb{P}^1\times\mathbb{P}^1$. As a second consequence $D\cdot\Delta = 2\tilde{d} + 2d + \omega_k$: but, having computed the bidegree of D, we know also that $D\cdot\Delta = 4\tilde{d}$, hence we infer that $2\tilde{d} = 2d + \omega_k$.

Proof of the assertion: By Lemma A,B, we must prove that g has maximal rank at each Q_j, at each R_j and P_h $g\circ f$ and $g\circ f\circ\tau$ have not the same differential, the Q_j's have all distinct images.

If P is any of these points, take a nonzero tangent vector t_P of γ' at P and pick a hyperplane H' in $\mathbb{P}^{2k}$ not containing f(P) for any of these points: then on $\mathbb{P}^{2k}-H'$ choose affine coordinates $(z_1,\cdots,z_{2k})$ such that

$$g = (\varphi_0,\varphi_1) = (z_1,\ a_0 + \sum_{i=1}^{2k} a_i z_i).$$

You can then identify $f(P), df(t_p)$ to a point and an applied vector at it in this affine space, and if $f(P) = f(\tau P)$, $df(t_p) \neq d(f\cdot\tau)(t_p)$: what we want then is that g must separate a finite number of points (including the vertices of the applied vectors), for which $z_1 \neq 0$, and this can clearly be achieved for general a_i's.

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