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ON A PROBLEM OF CHISINI

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§0. Introduction. The objects of this note are generic multiple planes, defined according to the following

Definition 1. A multiple plane is a pair (S, f) where S is a compact smooth connected complex surface and f is a finite holomorphic map $f: S \to \mathbf{P}^2 = \mathbf{P}^2_{\mathbf{C}}$. (S, f) is said to be generic if the following properties are satisfied:

(1i) the ramification divisor R of f is smooth and reduced

(1ii) $f(R) = B$ has only nodes and ordinary cusps as singularities

(1iii)* $f_{|R}: R \to B$ has degree 1.

Moreover, two multiple planes (S, f), (S', f'), are said to be isomorphic if there is an isomorphism $\phi: S \to S'$, and a projectivity $g: \mathbf{P}^2 \to \mathbf{P}^2$ such that $f' \circ \phi = g \circ f$, and strictly isomorphic if furthermore $g =$ identity.

Obviously, a necessary condition in order that two multiple planes be isomorphic is then that the two branch curves B, B' be projectively equivalent. Without loss of generality, therefore, we shall consider pairs of generic multiple planes (S, f), (S', f') such that $B = B'$, and we will investigate the problem of deciding whether they are strictly isomorphic. i.e., there does exist an isomorphism $\phi: S \to S'$ such that $f' \circ \phi = f$. Such a problem was considered by Chisini (cf. [C]) who conjectured that two generic multiple planes with the same branch curve would be strictly isomorphic "under some suitable conditions of generality."

The problem has a negative answer in general (contrary to the statement of the main theorem of [L]), as it is shown by a very nice example of Chisini himself in [C] (a previous example given by B. Segre in [S] yields a nongeneric triple plane).

Our result consists in giving a necessary and sufficient condition for strict isomorphism: at the end of the paper we shall discuss Chisini's example to illustrate our theorem. To explain our condition, we need a technical definition.

Definition 2. A marked curve $(C, p_1, \ldots, p_\gamma)$ consists of a (compact connected complex) curve C, together with an ordered set of γ points of C.

A marked line bundle $\mathscr{L} = (L, h_1, \ldots, h_\gamma)$ on $(C, p_1, \ldots, p_\gamma)$ consists of the datum of a holomorphic line bundle L on C, and of isomorphisms h_i $(i = 1, \ldots, \gamma)$ of the fibre L_{p_i} of L over p_i with $\mathbf{C}$.

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*As pointed out by the referee, if $f: S \to \mathbf{P}^2$ is generic, and $h: S' \to S$ is finite and etâle, $f \circ h: S' \to \mathbf{P}^2$ satisfies (1i), (1ii), but not (1iii).

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It is easy to guess how all the standard notions (isomorphism, tensor products, . . .) extend from the case of line bundles to the case of marked line bundles.

Let's go back and consider two generic multiple planes (S, f), (S', f') with the same branch curve B. Since $f_{|R} : R \to B, f'_{|R'} : R' \to B$ give the desingularization of B, there is a natural isomorphism $\phi' : R \to R'$, thus to the invertible sheaves $\mathcal{O}_R(R), \mathcal{O}_{R'}(R')$ are naturally associated two line bundles $L, \hat{L}$, on R (the normal bundles). One easily sees that $L^{\otimes 2}$ is isomorphic to $(\hat{L})^{\otimes 2}$; moreover, if $p_1, \ldots, p_\gamma$ are the points of R mapping to the cuspidal points of B, we show that the datum of f and f' determines (non canonically) a pair of marked line bundles $\mathcal{L}$ and $\hat{\mathcal{L}}$ on the marked curve $(R, p_1, \ldots, p_\gamma)$ in such a way that $\eta = \eta(S, S') = \mathcal{L}^{-1} \otimes \hat{\mathcal{L}}$ is a marked line bundle with $\eta^{\otimes 2}$ trivial (η is now canonically associated to f, f'). We can now formulate our main theorem.

THEOREM. *The multiple planes (S, f), (S', f') admit a strict isomorphism $\phi : S \to S'$ (i.e., with $f' \circ \phi = f$) if and only if the marked line bundle $\eta(S, S')$ is trivial.*

It seems necessary to digress on results stated about the problem in the existing literature.

Our interest about the problem was aroused by the paper [L], where appears the nice idea* that, R, R' being ample divisors, the complements $X = S - R$, $X' = S' - R'$, are affine varieties: therefore, once one has an isomorphism $\tilde{\phi} : U \to U'$ of some tubular neighbourhoods U (resp.: U') of R (resp.: R'), this isomorphism can be extended to a global isomorphism $\phi : S \to S'$.

The error in [L] lies in the fact that it is not always possible to construct such a local isomorphism $\tilde{\phi}$: as a matter of fact, Lanteri refers to the first part of [C], where it is wrongly asserted that such a local isomorphism $\tilde{\phi}$ always exists.

However, in the second part of [C], Chisini proves the unicity of the multiple plane (i.e., the existence of ϕ) under a very strong assumption about the possibility of having a good degeneration of the multiple plane (and, furthermore, if the degree of f is at least 5).

These assumptions boil down to a nice van Kampen presentation of $\pi_1(\mathbf{P}^2 - B)$.

These global methods have been recently taken up again in a series of papers by Moishezon, who e.g. (cf. [M] cor. 3), proves unicity in the special case when $f : S \to \mathbf{P}^2$ is a generic projection of a smooth surface in $\mathbf{P}^3$ (in this case f can degenerate in a slightly worse way than the one allowed by Chisini).

In this special case unicity seems to depend upon the well-known result that the alternating group $\mathfrak{A}_n$ is simple for $n \geqslant 5$. In fact Moishezon proves that $\pi_1(\mathbf{P}^2 - B) = \mathcal{B}'_n$, the quotient of the braid group $\mathcal{B}_n$ by its cyclic centre; therefore, in this case, the global monodromy homomorphism μ associated to a generic multiple plane must factor through the canonical surjection of $\mathcal{B}'_n$ onto the symmetric group $\mathfrak{S}_n$.

*Pointed out by A. Andreotti many years ago, according to the referee.

Chisini's problem deserves to be better understood: in particular the connection between the local condition of triviality of $\eta(S, S')$ and the global structure of $\pi_1(\mathbf{P}^2 - B)$ should be investigated.

A final remark is that everything holds verbatim for generic finite morphisms $f: S \to Y$ where Y is any algebraic surface other than $\mathbf{P}^2$, provided the ramification divisor R of f is ample.

§1. Auxiliary results ("Locally around the cusps"). Let B be a plane curve and let q be an ordinary cusp: then there are local holomorphic coordinates around q, say (x, y), such that, if $W_\epsilon = W$ is a ball in $\mathbf{C}^2$ with centre the origin (i.e., q) and radius ϵ,

$$W_\epsilon \cap B = \{(x, y) \in W_\epsilon \mid y^2 - x^3 = 0\}. \tag{1.1}$$

It is well known that the fundamental group $\pi_1(W_\epsilon - B)$ is isomorphic to the abstract group

$$\Pi = \langle \xi, \eta; \xi\eta\xi = \eta\xi\eta \rangle, \qquad \text{(cf. e.g., [La] p. 76)} \tag{1.2}$$

i.e., Π is the group with generators ξ, η (corresponding to the two generators of $\pi_1((W_\epsilon - B) \cap \{x = \epsilon^2\}))$ with relation $\xi\eta\xi = \eta\xi\eta$.

A normal irreducible finite covering $f: Y \to W$, unramified outside B, is determined by a monodromy homomorphism $\mu: \Pi \to \mathfrak{S}_d$, where $\mu(\Pi)$ is a transitive subgroup of $\mathfrak{S}_d$.

It is easy to see that above the origin q there lies only one point p of Y (in fact, if $\epsilon' < \epsilon$, $\pi_1(W_{\epsilon'} - B) \cong \pi_1(W_\epsilon - B)$) and that p is the only possible singular point of Y.

We let $\bar{f}: \bar{Y} \to W$ be the standard covering of W given by the normalized equation of third degree:

$$\begin{cases} \bar{Y} = \{(x, y, z) \mid (x, y) \in W, z^3 - 3xz + 2y = 0\}, \\ \bar{f}(x, y, z) = (x, y). \end{cases} \tag{1.3}$$

$\bar{Y}$ is smooth, x and z being coordinates, the ramification divisor R is smooth, $R = \{(x,z) \in \bar{Y} \mid x - z^2 = 0\}$, and if Γ is the curve in $\bar{Y}$ with $\Gamma = \{(x,z) \in \bar{Y} \mid 4x - z^2 = 0\}$, $f^*(B) = 2R + \Gamma$. $\bar{Y}$ corresponds to the homomorphism $\mu: \Pi \to \mathfrak{S}_3$ such that, setting

$$\alpha = \mu(\xi), \qquad \beta = \mu(\eta), \tag{1.4}$$

one has $\alpha = (1,2)$, $\beta = (2,3)$.

Remark 1.5. There are plenty of transitive homomorphisms $\mu: \Pi \to \mathfrak{S}_d$, e.g., one can take in $\mathfrak{S}_6$ $\alpha = (1,2,3,4)$, $\beta = (2,5,4,6)$, which satisfy $\alpha\beta\alpha = \beta\alpha\beta$. However, with some restrictions upon α and β, one is left only with the previous homomorphism μ onto $\mathfrak{S}_3$, and the related one into $\mathfrak{S}_6 = \mathfrak{S}(\mathfrak{S}_3)$, μ' (such that, for $g \in \Pi$, $h \in \mathfrak{S}_3$, $\mu'(g)(h) = \mu(g) \cdot h$).

LEMMA 1.6. *Let α, $\beta \in \mathfrak{S}_d$ be such that $\alpha\beta\alpha = \beta\alpha\beta$, and let Γ be the subgroup generated by α and β.*

(i) *Γ is abelian* iff $\alpha = \beta$.

(ii) *if the cycle decomposition of α, resp. β, consists of a product of transpositions, and Γ is transitive, nonabelian, then $\Gamma \cong \mathfrak{S}_3$ and after renumbering, one has either $d = 3$, $\alpha = (1,2)$, $\beta = (2,3)$, or $d = 6$, with $\mathfrak{S}_3$ acting on itself by left translations.*

Proof. First of all, $\beta = \alpha\beta\alpha\beta^{-1}\alpha^{-1} = (\alpha\beta)\alpha(\alpha\beta)^{-1}$, hence α and β are conjugate permutations, in particular (i) follows immediately.

We can assume that α permutes 1 with 2: we shall later consider the case when exactly one of them is left fixed by β. Assuming $\beta(1) = 1$, $\beta(2) = 2$, we get $\alpha\beta\alpha(1) = 1$, $\beta\alpha\beta(1) = 2$, and we have a contradiction. If neither 1 nor 2 are left fixed by β, there are elements $A, B \in \{1, \dots, d\}$ such that β permutes 2 with B, 1 with A, and, by our assumptions in (ii), the set $\{1,2,A,B\}$ has 4 elements.

If one of the two elements A, B is left fixed by α, say that $\alpha(A) = A$, then we have $\beta\alpha\beta(1) = 1$, $\alpha\beta\alpha(1) = \alpha(B) \neq 1$, a contradiction; on the other hand, if α permutes A with B, we get $\alpha\beta\alpha(1) = A$, $\beta\alpha\beta(1) = 2$, again a contradiction.

If, instead, $\alpha(A) = A'$, $\alpha(B) = B'$, where the six elements $1,2,A,A',B,B'$ are distinct, one has $\alpha\beta\alpha(1) = B'$, $\beta\alpha\beta(1) = \beta(A')$, hence $B' = \beta(A')$, and we are in the case where $d = 6$ and, as it is easily verified, $\mathfrak{S}_3$ acts on itself by left translations. Finally, if $\beta(1) = 1$, and $\beta(2) \neq 2$, we can assume $\beta(2) = 3$ and we conclude since $\alpha(3) = \alpha\beta\alpha(1) = \beta\alpha\beta(1) = 3$, hence $\{1,2,3\}$ is a Γ-orbit and $\alpha = (1,2)$, $\beta = (2,3)$. Q.E.D.

Let $\bar{Z}$ be the smooth cover of $W_\epsilon = W$ branched on B given by the ordered triples of roots of the normalized equation of third degree, i.e.,

$$\bar{Z} = \{(z_1, z_2, z_3) \mid z_1 + z_2 + z_3 = 0, x = -(z_1z_2 + z_1z_3 + z_2z_3)/3,$$
$$y = -\tfrac{1}{2} z_1z_2z_3, \text{ are such that } (x, y) \in W\}$$
$$\text{with} \quad f(z_1, z_2, z_3) = (x, y). \quad (1.7)$$

We can now rephrase the previous lemma as follows

LEMMA 1.8. *Let W, B be as above, and let $f\colon Y \to W$ be a normal irreducible finite cover with ramification divisor R, and with branch curve B (i.e., $B = f(R)$). Assume R to be reduced: then one of the following holds*

(i) $\deg f = 2$, *and R is isomorphic to B (in particular, R is singular)*

(ii) $\deg f = 6$, *Y is isomorphic to $\bar{Z}$, and R is singular*

(iii) $\deg f = 3$, *R is smooth, and there exists only one biholomorphism $g\colon Y \to \bar{Y}$ such that $\bar{f} \circ g = f$.*

Proof. If R is reduced, one can easily see that the monodromy of each of the two generators ξ, η of $\Pi \cong \pi_1(W - B)$ is given by a product of commuting transpositions, hence lemma 1.6 applies (Γ is transitive by the irreducibility of Y).

Since there are only 3 choices for the monodromy, and moreover Y is normal, Y is either isomorphic to $\bar{Z}$, or to $\bar{Y}$, or to the cyclic cover $\bar{T}$ of degree 2, i.e.,

$$\bar{T} = \{(t, x, y) \mid (x, y) \in W, t^2 = (y^2 - x^3)\}.$$

To prove the last assertion, it suffices to show that the covering $\bar{f}: \bar{Y} \to W$ has no automorphism.

But this is in fact more generally true for the unramified covering $\bar{Y} - R \to W - B$, since, the monodromy homomorphism $\mu: \Pi \to \mathfrak{S}_3$ being given by $\mu(\xi) = \alpha = (1,2)$, $\mu(\eta) = \beta = (2,3)$ the 3 associate subgroups of the covering are all distinct (they map onto the 3 cyclic subgroups of order 2 in $\mathfrak{S}_3$). Q.E.D.

§2. Isomorphism of generic multiple planes. In this section, (S, f) being a generic multiple plane with branch curve B, we shall denote by $q_1, \dots, q_\gamma$ the cuspidal points of B, by $c_1, \dots, c_\nu$ the nodes of B. By assumption (1iii), there is exactly one point $p_i \in R$ mapping to q_i (for $i = 1, \dots, \gamma$), and there are exactly two points $a_j, b_j \in R$ mapping to c_j ($j = 1, \dots, \nu$).

For each cusp q_i we choose, by virtue of lemma 1.8, respective neighbourhoods W_i of q_i, Y_i of p_i, such that

there are holomorphic coordinates (x_i, y_i) on W_i giving an isomorphism τ_i of W_i onto W mapping $W_i \cap B$ to the curve $\{(x, y) \mid y^2 - x^3 = 0\} \cap W$ (2.1)

there exists an isomorphism (a unique one, by 1.8) of the covering $f_{|Y_i}: Y_i \to W_i$ with the standard one $\tau_i^{-1} \circ \bar{f}: \bar{Y} \to W_i$. Hence on Y_i there are coordinates x_i and z_i such that (2.2)

$$-y_i^2 + x_i^3 = (x_i - z_i^2)^2 (x_i - \tfrac{1}{4} z_i^2)$$

$$\left(\text{since } y_i = \tfrac{1}{2}(3x_i z_i - z_i^3)\right). \quad (2.3)$$

We consider now (cf. def. 2) the marked curve $(R, p_1, \dots, p_\gamma)$ and remark that the choice of coordinates (x_i, y_i) on W_i defines a marking $\mathcal{N}$ of the normal bundle N_R of R in S, since on Y_i there is a unique choice of coordinates (x_i, z_i) (cf. 2.2, 2.3), and then the fibre of N_R at p_i is naturally identified to the complex line spanned by $\partial/\partial x_i$.

Remark 2.4. It is clear that if there exists a strict isomorphism $\phi: S \to S'$ of the generic multiple planes $(S, f), (S', f')$, then $\phi^1 = \phi_{|R}: R \to R'$ induces an isomorphism of the marked normal bundles $\mathcal{N}, \mathcal{N}'$.

PROPOSITION 2.5. *Let $(S, f), (S', f')$ be generic multiple planes with the same branch curve B. Then the marked line bundle $\eta = \mathcal{N}^{-1} \otimes (\phi')^*(\mathcal{N}')$ is of*

2-torsion (*i.e.*, $\eta^{\otimes 2}$ *is trivial*). *Moreover* η *is trivial if and only if there does exist an isomorphism* $\tilde{\phi}: U \to U'$ *between respective neighbourhoods of* R, R' *such that* $f' \circ \tilde{\phi} = f$.

Proof. In order to treat in a uniform way all the multiple planes with branch curve B, we shall consider a suitable "neighbourhood" $\tilde{V}$ of the normalization of B at the nodes. To construct $\tilde{V}$, we shall take local coordinates (u_j, v_j) around c_j and a neighbourhood T_j s.t.

$$T_j = \{(u_j, v_j) \mid |u_j|, |v_j| < \epsilon\}, \qquad T_j \cap B = \{(u_j, v_j) \in T_j \mid u_j v_j = 0\}. \quad (2.6)$$

We set $\hat{T}_j$ the closed polydisc of radius $\epsilon/2$, $\hat{T}_j = \{(u_j, v_j) \mid |u_j|, |v_j| \leqslant \epsilon/2\}$, $\hat{W}_i = \tau_i^{-1}(\overline{W}_{\epsilon/2})$, and $B^{\#} = B - (\bigcup_{j=1}^{\nu} \hat{T}_j) - (\bigcup_{i=1}^{\gamma} \hat{W}_i)$. Furthermore we choose a small tubular neighbourhood $V^{\#}$ of $B^{\#}$, set $V = V^{\#} \cup (\bigcup_j T_j) \cup (\bigcup_i W_i)$, and construct a smooth manifold $\tilde{V}$ with an immersion $\rho: \tilde{V} \to V$ by simply replacing in V each T_j by 2 copies of T_j (we are obviously assuming all the T_j's, W_i's to be disjoint), labelled by the two branches of B at c_j, and glueing them to $V^{\#}$ by the obvious identification of points in $V^{\#} \cap T_j$.

Definition 2.7. We shall say that the datum of $\rho: \tilde{V} \to V$, of the W_i's and of isomorphisms $\tau_i: W_i \to W$ for $i = 1, \dots, \gamma$ is a *monk's belt* for the generic multiple plane (S, f) if, d being the degree of f,

(1) $f^{-1}(W_i)$ has $(d-2)$ connected components (one of them being Y_i, the remaining ones mapping isomorphically onto W_i)

(2) $f^{-1}(V - (\bigcup_i W_i))$, has $(d-1)$ connected components of which only one, denoted here by $\hat{U}$, intersects R.

We shall moreover say that $U = \hat{U} \cup (\bigcup_{i=1}^{\gamma} Y_i)$ is the *balanced* neighbourhood of R associated to the monk's belt.

Now, let us choose a common monk's belt for (S, f), (S', f'), for which we shall use the notations introduced before, denoting by U, (resp. U') the associated balanced neighbourhood of R (resp. R').

Notice that $f: U \to V$ factors through $g: U \to \tilde{V}$ and $\rho: \tilde{V} \to V$ (respectively: $f' = \rho \circ g'$).

Before proceeding to an explicit computation with covers and 1-cocycles, let's explain geometrically why the normal bundles of R and R' differ only by a 2-torsion bundle. We have in fact

$$f^*(B) = 2R + \Gamma, \quad (2.8)$$

where Γ is reduced, intersects R transversally at the points a_j, b_j, intersects R at the points p_i with intersection multiplicity equal to 2 (being smooth there).

Let b be the degree of B, and H be the divisor on R which is the pull-back of a line in $\mathbb{P}^2$.

We have then, by (2.8), a linear equivalence of divisors on R, namely

$$2N \equiv bH - \sum_{i=1}^{\nu} (a_j + b_j) - 2 \sum_{i=1}^{\gamma} p_i \quad (2.9)$$

where N is a divisor associated to the normal bundle of R in S: the upshot is that the right hand side depends only on $f_{|R}$, the normalization map for B.

We choose open covers $\{U_\alpha\}$ of U, (resp.: $\{U'_\alpha\}$ for U'), V_α for $\tilde{V}$, such that:

for $\alpha = i \leqslant \gamma$ $V_\alpha = W_i$, $U_\alpha = Y_i$, $(U'_\alpha = Y'_i)$, for $\alpha > \gamma$ $g(U_\alpha) = V_\alpha$ and there are coordinates (u_α, w_α) on U_α, (v_α, w_α) on V_α, such that $g(u_\alpha, w_\alpha) = (u_\alpha^2, w_\alpha)$ (hence $u_\alpha = 0$ is the local equation for R on U_α). (2.10)

Similarly there are coordinates (u'_α, w_α) on U'_α, and we have

$$(u'_\alpha)^2 = v_\alpha = u_\alpha^2. \tag{2.11}$$

We can now prove the first assertion: in fact, via the isomorphism $\phi_i : Y_i \to Y'_i$ such that $x'_i = x_i$, $z'_i = z_i$, we have that the chosen trivializations of N_R on the cover $\{U_\alpha\}$, and of N'_R on the cover $\{U'_\alpha\}$ induce the same markings. Therefore the triviality of $\eta^{\otimes 2}$ follows directly from (2.11). If $\tilde{\phi} : U \to U'$ exists, then η is trivial (cf. 2.6). Conversely, since we have the exact sequence

$$1 \longrightarrow H^1(R, \{\pm 1\}) \longrightarrow \mathrm{Pic}^0(R) \xrightarrow{\otimes 2} \mathrm{Pic}^0(R) \longrightarrow 1,$$

the marked line bundle η is trivial if and only if there do exists numbers $\epsilon_\alpha \in \{\pm 1\}$, with $\epsilon_\alpha = 1$ for $\alpha \leqslant \gamma$, such that on R, after identifying R' with R, we have

$$u'_\beta / u'_\alpha = \epsilon_\beta / \epsilon_\alpha \cdot u_\beta / u_\alpha$$
$$\text{(where we set, for } \alpha \leqslant \gamma,\ u_\alpha = x_\alpha - z_\alpha^2\text{)}. \tag{2.12}$$

We have now that if η is trivial the isomorphisms $\phi_\alpha : U_\alpha \to U'_\alpha$ given by

$$\begin{cases} x'_i = x_i, \quad z'_i = z_i & \text{for } \alpha = i \leqslant \gamma \\ u'_\alpha = \epsilon_\alpha u_\alpha & \text{for } \alpha > \gamma \end{cases} \tag{2.13}$$

patch together, by (2.12), to give the desired isomorphism $\tilde{\phi} : U \to U'$. Q.E.D.

THEOREM. *Two generic multiple planes $(S, f), (S', f')$ with the same branch curve B are strictly isomorphic if and only if $\eta(S, S')$ is a trivial marked line bundle.*

Proof. In view of proposition 2.7 η is trivial iff there exists $\tilde{\phi} : U \to U'$ which is an isomorphism of respective neighbourhoods of R, R', with $f' \circ \tilde{\phi} = f$ (on U).

Set $X = S - R$, $K = S - U$, and define similarly X', K'. Now R, R' are ample divisors (e.g., by [L] thm. 3.1 in the case of surfaces, and, more generally, for every finite morphism to $\mathbf{P}^n$ by [E] thm. 1) therefore X is an affine variety in $\mathbf{C}^n$ (resp. $X' \subset \mathbf{C}^{n'}$). $\tilde{\phi}$ determines n' holomorphic functions on the complement in X of the compact set K: since X is Stein, these functions extend to the whole of X by Hartogs' theorem (cf. [Hö]), and patch with $\tilde{\phi}$ to give a holomorphic map of S to S' (in fact X maps into X' by analytic continuation since X' is the locus of zeros of polynomials on $\mathbf{C}^{n'}$). Similarly we can extend $(\tilde{\phi})^{-1}$ to a holomorphic map $\hat{\phi}: S' \to S$, and the equalities $\phi \circ \hat{\phi} = id_{S'}$, $\hat{\phi} \circ \phi = id_S$, $f' \circ \phi = f$ hold again by analytic continuation. Q.E.D.

In the next section we shall show Chisini's example producing several multiple planes with the same branch curve, and will compute explicitly that η is a 2-torsion bundle which is nontrivial even as an unmarked bundle. Unfortunately we don't have yet an example where the nontriviality of η depends only upon the marking.

§3. Chisini's example. In this section S will be $\mathbf{P}^2$ and S' a certain ruled surface Σ: we shall show that η is nontrivial also as an unmarked bundle.

Let $f: \mathbf{P}^2 \to \mathbf{P}^2$ correspond to a generic projection of the Veronese surface, i.e., taking homogeneous coordinates (x_0, x_1, x_2) on the domain of f, and (y_0, y_1, y_2) on the range, we assume that

$$f(x_0, x_1, x_2) = \left(Q^0(x), Q^1(x), Q^2(x) \right), \quad \text{with} \\ y_k = Q^k(x) = \sum_{i,j} Q^k_{i,j} x_i x_j \qquad \left(Q^k_{i,j} = Q^k_{j,i} \right). \tag{3.1}$$

The ramification divisor is given by the cubic curve (that we can assume to be smooth)

$$R = \{(x_0, x_1, x_2) \mid \det_{j,k}(\textstyle\sum_i Q^k_{i,j} x_i) = 0\} \text{ and the normal bundle of } R \text{ corresponds to the sheaf } \mathcal{O}_R(3). \tag{3.2}$$

To determine $f(R) = B$ it is easier to consider the projective plane $(\mathbf{P}^2)^*$ dual to the $\mathbf{P}^2$ with y-coordinates, and to consider on it homogeneous coordinates $(\lambda_0, \lambda_1, \lambda_2)$ (dual to (y_0, y_1, y_2)).

Now the pull-back of the line $\sum_k \lambda_k y_k = 0$ is singular if and only if on the one hand the line is tangent to B, on the other hand if the conic $\sum_k \lambda_k Q^k(x)$ is singular. Therefore, by biduality, B is the dual curve of the (smooth) cubic curve

$$B^* = \left\{ (\lambda_0, \lambda_1, \lambda_2) \mid \det_{i,j}\left(\sum_k \lambda_k Q^k_{i,j} \right) = 0 \right\}. \tag{3.3}$$

In general, given a curve B, with nodes and cusps only, which is the dual curve of a smooth curve B^*, there is a natural multiple plane (Σ, ψ) attached to it, as follows: $\Sigma \subset (\mathbf{P}^2)^* \times \mathbf{P}^2$, ψ being given by projection on the second factor

$$\Sigma = \left\{ ((\lambda_0, \lambda_1, \lambda_2), (y_0, y_1, y_2)) \mid \sum_k \lambda_k y_k = 0, (\lambda_0, \lambda_1, \lambda_2) \in B^* \right\} \tag{3.4}$$

If $p: B^* \times \mathbf{P}^2 \to B^*$ is given by projection on the first factor, Σ is the divisor in $B^* \times \mathbf{P}^2$ of a section of $p^*(\mathcal{O}_{B^*}(1) \otimes \psi^*(\mathcal{O}_{\mathbf{P}^2}(1))$. It is clear that $(y_0, y_1, y_2) \notin B$ iff $\sum_k y_k \lambda_k = 0$ is not tangent to B^*, i.e., iff $\psi^{-1}((y_0, y_1, y_2))$ has exactly $\deg(B^*)$ points, hence B is the branch curve of ψ: it is easy to see that ψ is generic.

Furthermore if B^* has degree d, and $g(\lambda)$ is a homogeneous equation for B^*, the ramification curve of ψ is the graph of the morphism $(\partial g/\partial\lambda_0, \partial g/\partial\lambda_1, \partial g/\partial\lambda_2) : B^* \to B \subset \mathbf{P}^2$, which we shall identify thus with the plane curve B^*, and in particular

$$\psi^*(\mathcal{O}_{\mathbf{P}^2}(1)) \otimes \mathcal{O}_{B^*} = \mathcal{O}_{B^*}(d-1). \tag{3.5}$$

Given a smooth variety X, let's denote by Θ_X its tangent sheaf: if Y is a smooth subvariety of X let's denote by $N_{Y/X}$ the normal sheaf to Y in X. We claim that

$$N_{B^* \mid \Sigma} \cong \mathcal{O}_{B^*}(2d-3). \tag{3.6}$$

In fact, by (3.5), $\Theta_{B^*}(3(d-1)) = \det((\Theta_{B^* \times \mathbf{P}^2}) \otimes \mathcal{O}_{B^*}) = \det(\Theta_\Sigma \otimes \mathcal{O}_{B^*}) \otimes (N_{\Sigma \mid B^* \times \mathbf{P}^2} \otimes \mathcal{O}_{B^*}) = (\Theta_{B^*} \otimes N_{B^* \mid \Sigma}) \otimes \mathcal{O}_{B^*}(d)$.

Let's return to our specific case. The symmetric matrix $Q = \sum_k \lambda_k Q^k_{i,j}$ determines a nontrivial line bundle of 2-torsion on B^*, such that the associated invertible sheaf η is the cokernel of the following exact sequence on $\mathbf{P}^2$

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-2)^3 \xrightarrow{(Q)} \mathcal{O}_{\mathbf{P}^2}(-1)^3 \longrightarrow \eta \longrightarrow 0. \tag{3.7}$$

and (cf. [Tu], [Ca]) there are 3 natural sections of $\eta(1)$, (ξ_0, ξ_1, ξ_2), without common zeros, such that

$$\begin{cases} (\xi_i \xi_j) \text{ gives, for } \lambda \in B^*, \text{ the adjoint matrix of } \left(\sum_k \lambda_k Q^k_{i,j}\right) \\ \sum_j \sum_k \lambda_k Q^k_{i,j} \xi_j = 0 \qquad (\text{for } \lambda \in B^*) \end{cases} \tag{3.8}$$

Since $\eta^3(-3) \otimes \mathcal{O}_{B^*}(3) \cong \eta$, we have shown that $\eta = \eta(S, \Sigma)$ if we prove that

$$\xi = (\xi_0, \xi_1, \xi_2) = f^{-1}\psi(\lambda), \qquad \text{where we consider} \tag{3.9}$$

the following composition of birational maps, $B^* \xrightarrow{\psi} B \xrightarrow{f^{-1}} R$.

In fact, then, the sheaf $\mathcal{O}_R(3)$ corresponds to the sheaf $\eta^3(3)$ on B^* under the isomorphism $f^{-1}\psi : B^* \to R$, and we are done. So, let's prove 3.9 and, to this purpose, set $y = \psi(\lambda)$. Since y represents the tangent line to B^* at λ, by 3.8, we have

$$y_k = \sum_{i,j} Q^k_{i,j} \xi_i \xi_j . \tag{3.10}$$

(3.10) tells us that $y = f(\xi)$, moreover (3.8) tells us that $\sum_j Q^k_{i,j} \xi_j$ is not an invertible matrix, therefore ξ belongs to R, and $f(\xi) = \psi(\lambda)$, Q.E.D.

Our previous considerations allow us to improve upon the Chisini counterexample. Fix in fact the smooth cubic curve B^* and thus also the multiple plane (Σ, ψ): now B^* has 3 nontrivial distinct line bundles of 2-torsion, each one occurring (cf. e.g., [Ca], thm. 2.28) as a cokernel of an exact sequence like (3.7) and therefore giving rise to another generic multiple plane of degree four. We have therefore (keeping track of translations of order 2 in B^*)

PROPOSITION 3.11. *Given the dual curve B of a smooth cubic curve B^*, there do exist* 4 *generic multiple planes with B as branch curve: three of them have degree* 4 *and are isomorphic but not strictly isomorphic, the other has degree* 3.

We end the paper with a curious remark: a generic multiple plane determines $f_{|\Gamma}: \Gamma \to B$, hence an unramified $(d-2)$ covering $\tilde{\Gamma} \to R$, where $\tilde{\Gamma}$ is the normalization of Γ. In turn this corresponds to a (nontrivial) line bundle of $(d-2)$ torsion $\mathscr{P}$ on R: given $(S,f),(S',f')$ with the same B how are $\mathscr{P},\mathscr{P}'$ related? By our result $\mathscr{P}^{-1} \otimes \mathscr{P}'$ is determined by $\eta(S,S')$.

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