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SURFACES WITH $K^2 = 6$, $\chi = 4$, AND WITH TORSION (**)

§ 0. Introduction.

The purpose of the present paper is twofold: firstly, as suggested by the title, we prove the following

MAIN THEOREM. The minimal surfaces of general type with $K^2 = 6$, $\chi = 4$, and torsion, belong to one irreducible family, and have fundamental group $\cong \mathbb{Z}/2$.

Their moduli space $\mathcal{M}$ is reduced and rational. More specifically, all such surfaces S are obtained as follows: there is a normal symmetrical cubic surface Σ in $\mathbb{P}^3$ (i.e., Σ is defined by $\det(\alpha_{ij}(y)) = 0$, where $(\alpha_{ij}(y))$ is a symmetrical 3×3 matrix of linear forms) and there exists a double cover $\psi : S \to \Sigma$ branched on the singular points of Σ and on the section of Σ with a quartic surface P not passing through the singular points of Σ.

The canonical system $|K|$ of S is not composed of a pencil, has no fixed part on the canonical model X of S, but it has there a base locus B which is a 0-dimensional subscheme of length 4.

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Secondly, working out this specific example, we illustrate a further generalization of a systematic method introduced in [C-D] and [Ca 4] in order to study all the regular surfaces of general type with given numerical invariants K^2 and χ.

In fact, using the existence of a divisor η such that 2η is linearly equivalent to zero, we enlarge the role of the pluricanonical systems $|mK|$ by considering also the twisted pluricanonical systems $|mK+\eta|$, and apply again the general machinery of free resolutions and ring structure, used in [Ca 1], [Ca 4], [Ca 5].

The method of construction of these surfaces can be applied more generally to all the symmetrical normal surfaces Σ in $\mathbb{P}^3$ (cf. [Ca 1]) in order to produce infinitely many families of surfaces with torsion $\mathbb{Z}/2$; we have focused our attention to this particular class of surfaces for a couple of reasons, one is because when the degree of Σ is high the surfaces thus obtained are not stable by deformations, a second one is because we are more interested on the (more difficult) problem of classification rather than on the construction aspect.

Also, we believe that the remarkable behaviour of the canonical map singles out our family of surfaces (previous examples where the canonical system has precisely one base point have been investigated in detail by Horikawa in [Hor 1], [Hor 2]).

The proof of the main theorem will be given through a series of propositions, theorems, remarks and corollaries, and the paper is organized as follows.

We notice first (see § 1) that a surface with $K^2=6$, $\chi=4$ has a torsion group T of order at most 3. The case when the torsion group T is $\mathbb{Z}/3$ is shown not to occur, since S should appear then as a quotient by a free action of $\mathbb{Z}/3$ of a surface Y with $K^2=2p_g-4$: by the above mentioned result of Horikawa ([Hor 2]) we show that such an action must have a fixed point.

The rest of paragraph 1 is devoted to a very careful analysis of the linear systems $|K+\eta|$ and $|K|$: by a strategy involving several "reductiones ad absurdum" we show that in our specific case we can dispense with assumptions of generality (as in [En], [Ca 4]), because indeed both systems are not composed of a pencil, and $|K+\eta|$ is base point free. Since then the twisted canonical system $|K+\eta|$ has dimension bigger than $|K|$ we can apply the standard Clifford-type lemmas to show that $|K+\eta|$ gives a morphism ψ which yields the above mentioned double cover $\psi: S\to\Sigma$.

Paragraph 2 is devoted to the description of the canonical ring of the unramified double cover $f: Y \to S$ naturally associated to the divisor η: this establishes the structure theorem for S (here it is proven in particular that Σ is a symmetrical cubic surface).

Finally, paragraph 3 contains the results about the moduli space $\mathcal{M}$ of the family thus constructed.

These last results appear already in the thesis of the first author ([Ba]), and indeed we refer to [Ba] for some technical details whose proof we omit; in particular, using the Kodaira-Spencer-Kuranishi theory of deformations, we prove that $\mathcal{M}$ is smooth on an open set ($\mathcal{M}$ is irreducible!), whereas we only prove that $\mathcal{M}$ is birational to a quotient of an affine space $\mathbb{C}^{30}$ by the symmetric group $\mathcal{S}_4$, referring the reader to [Ba] for the computation of the field of invariant rational functions, which proves the rationality of $\mathcal{M}$.

Some Notation

S : a complete smooth algebraic surface over $\mathbb{C}$, minimal and of general type

T : the torsion group of S, i.e. the torsion part of $H_1(S, \mathbb{Z})$ and of $H^2(S, \mathbb{Z})$

$D \equiv C$: the divisors D and C are linearly equivalent

$D \sim C$: D and C are numerically equivalent

$|D|$: the linear system of effective divisors linearly equivalent to D

K : a canonical divisor on S, i.e. $\mathcal{O}_S(K) \cong \Omega_S^2$

p_g : $\dim_{\mathbb{C}} H^0(\mathcal{O}_S(K))$

q : $\dim_{\mathbb{C}} H^1(\mathcal{O}_S) = \dim_{\mathbb{C}} H^1(\mathcal{O}_S(K))$

$\chi(\mathcal{O}_S) = 1 - q + p_g$

When $q = 0$, the group $\mathrm{Pic}(S)$ of divisors modulo linear equivalence coincides with $\ker : H^2(S, \mathbb{Z}) \to H^2(S, \mathcal{O}_S)$, hence T is naturally identified to a subgroup of $\mathrm{Pic}(S)$

η : a divisor whose class belongs to T, in this paper a divisor of 2-torsion (i.e., such that $\eta \not\equiv 0$, $2\eta \equiv 0$)

ϕ, ψ : the rational maps associated to the respective linear systems $|K|$, $|K + \eta|$

Σ : the image of the rational map ψ

ω_X : the dualizing sheaf of a Gorenstein variety, for X a divisor in S $\omega_X = \mathcal{O}_X(K+X)$

Θ_S : the sheaf of holomorphic vector fields on S.

§ 1. The canonical and twisted canonical maps.

LEMMA 1.1. Let S be a minimal surface of general type with $K^2 = 6$, $\chi = 4$. Then $q = 0$, the order of the torsion group T is at most 2 and $H^1(\mathcal{O}_S(\eta)) = 0$ $\forall$ divisor η whose class is torsion.

Proof. (cf. [Bo], thm. 14, [Re]). If $Y \xrightarrow{f} S$ is an unramified cover of degree m, $\chi(\mathcal{O}_Y) = 4m$, $K_Y^2 = 6m$.

Since $K^2 \geqslant 2p_g - 4$ by Noether's inequality ([Bo], thm. 9), $6m \geqslant 8m - 6$ and $m \leqslant 3$. Hence $q = 0$, and T has order at most 3; the same argument implies that if $m \geqslant 2$, Y has no unramified covers, hence $q(Y) = 0$, in particular $H^1(f_* \mathcal{O}_Y) = 0$, and $H^1(\mathcal{O}_S(\eta))$ vanishes being a direct summand of the previous cohomology group.

It remains to exclude the case: T has order 3. Then, if $f\colon Y \to S$ is the $\mathbb{Z}/3$ Galois cover associated to T, we have $K_Y^2 = 18$, $p_g(Y) = 11$ and (cf. [Hor 2, I]) the canonical map φ of Y yields a double cover of a ruled surface $\mathbb{F}$.

The action of $\mathbb{Z}/3$ on Y induces an action on $\mathbb{F}$. But it is easy to verify that any action of $\mathbb{Z}/3$ on $\mathbb{F}$ has a fixed point p: since $\varphi^{-1}(p)$ consists either of at most two points, or of a fundamental cycle (cf. [Bo]), $\mathbb{Z}/3$ cannot act freely on Y, a contradiction.

Q.E.D.

REMARK 1.2. From now on, and throughout the paper, we shall assume that S is as in lemma 1.1 and that furthermore the torsion group T is of order exactly: we shall denote by η a divisor whose class generates T.

PROPOSITION 1.3. $|K + \eta|$ is not composed of a pencil (i.e., the image $\Sigma = \psi(S)$ is a surface).

Proof. Let Λ be the fixed part of the linear system $|K + \eta|$, so that $|K + \eta| = \Lambda + |N|$, where N is the movable part of the system.

If $|K + \eta|$ is composed of a pencil, we have $N \sim dF$, where F is a fibre of the pencil, $d = \deg(\Sigma) \geqslant 3$.

We have ([Bo], prop. 1) $KF \geqslant 1$, but $d\,KF \leqslant K^2 = 6$, hence $KF \leqslant 2$.

Then either $K \cdot F = 1$ or $K \cdot F = 2$, $d = 3$, $K\Lambda = 0$.

If $K \cdot F = 1$, since F has genus at least 2, $(K+F) \cdot F \geqslant 2$ and $F^2 \geqslant 1$. In this case $K \cdot F = (dF + \Lambda) F \geqslant dF^2 \geqslant 2$, a contradiction. On the other hand, $K \cdot F = 2$ and $K \sim 3F + \Lambda$ implies $2 = 3F^2 + F\Lambda$, hence $F^2 = 0$ and $F\Lambda = 2$.

Since $|F|$ is a base point free pencil, and Λ is a sum of (-2) curves, i.e. irreducible curves E with $E^2 = -2$, $KE = 0$, we can write $\Lambda = \Lambda_1 + \Lambda_2$, where $F \cdot \Lambda_2 = 0$, $F\Lambda_1 = 2$, $\Lambda_1 \cdot \Lambda_2 \geqslant 0$.

We have thus $0 = K \cdot \Lambda_1 = (3F + \Lambda_1 + \Lambda_2)\Lambda_1 \geqslant 6 + \Lambda_1^2$; since $\Lambda_1 \cdot F = 2$, Λ_1 consists of at most 2 irreducible components, but $\Lambda_1^2 \leqslant -6$ implies $\Lambda_1 = 2E$, where E is a (-2) curve.

In particular we should also have $E \cdot \Lambda_2 = 1$, $\Lambda_2^2 = -2$.

If F is a smooth fibre, and p is the point of intersection of E with F, $\mathcal{O}_F(K+\eta) = \mathcal{O}_F(2p)$, and $\mathcal{O}_F(K)$ is the canonical bundle of the genus 2 curve F; thus $4p$ belongs to $|\mathcal{O}_F(2K)|$, hence p is a Weierstrass point of F, and $\mathcal{O}_F(K) \cong \mathcal{O}_F(K+\eta)$. Therefore $\mathcal{O}_F(\eta) \cong \mathcal{O}_F$ and, if $f\colon Y \to S$ is the double cover associated to η, f splits over F.

In other words, $\psi \circ f\colon Y \to \Sigma \cong \mathbb{P}^1$ has reducible fibres and if $\tilde{\psi}\colon Y \to \Gamma$ is its Stein factorization, Y is the normalization of $S \times_{\mathbb{P}^1} \Gamma$ and the covering involution $\tilde{i}$ on Y is induced by the covering involution i on Γ. But $g\colon \Gamma \to \mathbb{P}^1$ is a double cover branched at some point $q \in \mathbb{P}^1$: hence $\tilde{i}$ has a fixed point on Y unless the fibre $\psi^{-1}(q)$ is multiple with multiplicity 2. But there cannot be multiple fibres (for several reasons, first because $E \cdot F = 1$, second because multiple fibres cannot occur for genus 2 pencils).

Q.E.D.

PROPOSITION 1.4. $|K + \eta|$ has no fixed part.

Proof. As in 1.3, let Λ be the fixed part of $|K+\eta|$ and write $|K+\eta| = \Lambda + |N|$.

We have $6 - K\Lambda = K \cdot N = \Lambda N + N^2$; by 2-connectedness ([Bo], lemma 1) of the divisors $\sim K$, $\Lambda N \geqslant 2$ if $\Lambda \neq 0$, but $N^2 \geqslant \deg(\psi)\deg(\Sigma)$ in view of 1.3.

But either $\deg(\psi) \geqslant 2$, $\deg(\Sigma) \geqslant 2$, or $\deg(\Sigma) \geqslant 5$, and the chain of inequalities $4 \geqslant 6 - K\Lambda - \Lambda N \geqslant N^2 \geqslant 4$ can hold if and only if $K\Lambda = 0$, $\Lambda N = 2$, $N^2 = 4$, $\Lambda^2 = -2$, $|N|$ has no base points, $\deg(\psi) = \deg(\Sigma) = 2$.

Since Σ is irreducible, either it is smooth, or it is a quadric cone.

If Σ is smooth, we take ψ^{-1} of one generic line for each of the two rulings of $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$ to write $N = N_1 + N_2$, where clearly $N_1 \cdot N_2 = 2$ (ψ is of degree 2, $|N|$ has no base points); hence $N_1^2 = N_2^2 = 0$, and by the genus formula $K \cdot N_i$ is even, but $KN_1 + KN_2 = KN = K^2 - \Lambda K = 6$. We can therefore assume $KN_1 = 2$, $N_1\Lambda = 0$, $KN_2 = 4$, $N_2\Lambda = 2$, and $|N_1|$ is a base point free pencil of genus 2 curves.

Now $\mathcal{O}_{N_1}(K+\eta) = \mathcal{O}_{N_1}(N_2)$ is a line bundle of degree 2 with two independent sections, so $\mathcal{O}_{N_1}(K+\eta) \cong \omega_{N_1} = \mathcal{O}_{N_1}(K)$, thus $\mathcal{O}_{N_1}(\eta) \cong \mathcal{O}_{N_1}$, and the same argument used at the end of the proof of prop. 1.3 applies in order to derive a contradiction.

So, let's consider the case when Σ is a quadric one: taking the inverse image of a generic line passing through the vertex of Σ we can write $N = 2N' + A$ (N' is the inverse image of a line, and clearly then $NN' = 2$, since $|N|$ has no base points). We have $2N'^2 + AN' = 2$, and N'^2, AN' are $\geqslant 0$, hence either $N'^2 = 1$, $N'A = 0$, or $N'^2 = 0$, $N'A = 2$. The latter case gives rise to the same contradiction as before, since $\Lambda N' \leqslant 1$, and KN' equals $\Lambda N' + AN'$ but must be an even number, hence $\Lambda N' = 0$ and again $\mathcal{O}_{N'}(K + \eta)$ must coincide with the canonical bundle of the genus 2 curve N', implying again a contradiction. On the other hand in the former case $KN' = \Lambda N' + 2$ must be odd, thus $\Lambda N' = 1$ and $A \cdot \Lambda = 0$: hence $A \cdot (2N' + \Lambda) = 0$, violating the connectedness of divisors in $|K + \eta|$.

Q.E.D.

COROLLARY 1.5. The general curve Γ in $|K + \eta|$ is smooth irreducible.

Proof. It is a direct consequence of Bertini's theorems, since $|K + \eta|$ is not composed of a pencil by 1.3 and has no fixed part by 1.4: $|K + \eta|$ cannot have a singular base point, otherwise we would have $2 \geqslant \deg(\psi) \deg(\Sigma)$, absurd.

Q.E.D.

LEMMA 1.6. If $|K|$ has a fixed part F, $K \cdot F = 0$.

Proof. Write $|K| = F + |M|$, and assume $K \cdot F \geqslant 1$. Consider the exact sequence

$$0 \to \mathcal{O}_S(\eta) \to \mathcal{O}_S(K) \to \mathcal{O}_\Gamma(K) \to 0 \,. \tag{1.7}$$

By lemma 1.1 the sequence is also exact on global sections, and the canonical map ϕ induces a morphism $\phi' : \Gamma \to \mathbb{P}^2$ such that (by our assumption) $\deg(\phi') \cdot \deg(\phi'(\Gamma)) \leqslant 5$. ϕ' cannot be birational since the arithmetic genus of a plane quintic is 6, while Γ is smooth of genus 7.

Since $|K - \Gamma| = \emptyset$, $\phi'(\Gamma)$ is not a line, therefore $\phi'(\Gamma)$ is a conic, $F \cdot K = 2$, $M \cdot K = 4$: by 2-connectedness again $M \cdot F \geqslant 2$, hence $M^2 = MF = 2$, $|M|$ is not composed of a pencil (if $|M|$ were composed of a pencil, since $\phi(S)$ is a conic, one would have M^2 divisible by 4, a contradiction), and $|M|$ has no base points.

Let b be the degree of the branch locus of ϕ: since $\phi(M)$ is a line, it must be, by Hurwitz's formula, $b - 4 = M(K + M) = 6$ on the one hand, on the other, since $\phi(\Gamma)$ is a conic, and ϕ' has degree 2, it must be $2b - 4 = \Gamma(\Gamma + K) = 12$, absurd.

Q.E.D.

PROPOSITION 1.7. $|K|$ is not composed of a pencil.

Proof. Assume the contrary: then $|K| \equiv F + dN$, where N is irreducible, and clearly $F \neq 0$.

By 1.6 $d = \deg(\phi'(\Gamma))$, hence $d = 2,3$, or 6, and $6/d = \Gamma \cdot N = K \cdot N = N \cdot F + dN^2$, and by connectedness $N \cdot F \geqslant 1$.

For $d = 6$ we get $N \cdot F = 1$, $N^2 = 0$, but then $(K+N) \cdot N = 1$, ansurd; for $d = 2$, $N^2 = 0$, $N \cdot F = 3$ is impossible since then $(K+N)N = 3$, while if $N^2 = N \cdot F = 1$, we have $6 = K^2 = F^2 + 4 + 1$, while $F^2 < 0$ since by 1.6 $K \cdot F = 0$ (cf. [Bo], prop. 1); for $d = 3$ we get $N \cdot F = 2$, $N^2 = 0$ and $|N|$ is a base point free pencil of genus 2 curves: but then $\mathcal{O}_N(K + \eta) \cong \mathcal{O}_N(K)$ gives rise to the same contradiction as in 1.3.

Q.E.D.

In the following proposition and in the sequel also we shall often consider (cf. [Bo], p. 176), instead of the smooth minimal model S, the canonical model X, obtained by contracting the (finitely many) curves E with $K \cdot E = 0$: X is normal, and with rational double points as singularities, and the dualizing sheaf ω_X is such that, $\pi : S \to X$ being the contraction map, $\mathcal{O}_S(K_S) \cong \pi^*(\omega_X)$. Also η is the pull-back of a Cartier divisor on X, that we shall denote by the same symbol, and the important feature is that the rational maps ϕ, ψ factor through π.

PROPOSITION 1.8. The twisted canonical map cannot be birational.

Proof. Let K_X be the Cartier divisor on X such that $\omega_X = \mathcal{O}_X(K_X)$. Then, by 1.6, 1.7 and Bertini's theorem, the general curve C in $|K_X|$ is irreducible of arithmetic genus 7. By the exact sequence (cf. 1.1)

$$\begin{array}{ccccccc} 0 = H^0(\mathcal{O}_X(\eta)) & \to & H^0(\mathcal{O}_X(K_X + \eta)) & \to & H^0(\mathcal{O}_C(K_X + \eta)) & \to & 0 \\ & & \| & & & & \\ & & H^0(\mathcal{O}_S(K + \eta)) & & & & \end{array}$$

we have on C a linear system of degree 6 and dimension 3, and by (an obvious extension of) the lemma of Clifford, C is hyperelliptic and $\psi : C \to \psi(C)$ is a double cover of a twisted cubic in $\mathbb{P}^3$.

Q.E.D.

PROPOSITION 1.9. $\Sigma = \psi(S)$ is a cubic surface.

Proof. By 1.3, 1.8, we have first to show that deg $\psi = 3$ is impossible.

For a general $C \in |K_X|$, $C \xrightarrow{\psi} \psi(C)$ is of degree 2; if $C \neq \psi^{-1}(\psi(C))$, then $\psi^{-1}(\psi(C)) = C + A$, where $\psi : A \to \psi(C)$ is birational: this is a contradiction since A is rational, and S would be then ruled.

But then $C = \psi^{-1}(\psi(C))$: if Σ were a quadric surface, $\psi(C)$ would belong to a 5-dimensional linear system, while $\dim |K_X| = 2$, a contradiction.

Q.E.D.

COROLLARY 1.10. $K + \eta$ has no base points, and $\psi : X \to \Sigma$ is a finite morphism (of degree 2).

REMARK. Let $y_0, \dots, y_3$ be a basis of $H^0(\mathcal{O}_S(K+\eta))$; since Σ is not a quadric, the 10 monomials $y_i y_j$ are linearly independent sections of $H^0(\mathcal{O}_S(2K))$, which has, though, dimension equal to $\chi + K^2 = 10$ ([Bo], corollary p. 185).

PROPOSITION 1.11. Σ is a normal cubic surface.

Proof. If Σ were not normal, $\psi : \Gamma \to \psi(\Gamma)$ would be a double cover of an irreducible singular plane cubic curve, and Γ would be hyperelliptic. Then there is a morphism $u : \Gamma \to \mathbb{P}^1$ of degree 2, and $\mathcal{O}_\Gamma(K+\eta) = u^*(\mathcal{O}_{\mathbb{P}^1}(3))$. But, since $\omega_\Gamma = u^*(\mathcal{O}_{\mathbb{P}^1}(6))$, and $\omega_\Gamma \cong \mathcal{O}_\Gamma((K+\eta)+K)$, we would have $\mathcal{O}_\Gamma(K) \cong \mathcal{O}_\Gamma(K+\eta)$, i.e. $\mathcal{O}_\Gamma \cong \mathcal{O}_\Gamma(\eta)$.

Since $H^1(\mathcal{O}_S(-K)) = 0$, the exact sequence

$$0 \to \mathcal{O}_S(-K) \to \mathcal{O}_S(\eta) \to \mathcal{O}_\Gamma(\eta) \to 0$$

gives rise to a contradiction.

Q.E.D.

PROPOSITION 1.12. The canonical map ϕ has degree 2 and determines the same birational involution as ψ (i.e., there is a birational map $\xi : \Sigma \to \mathbb{P}^2$ such that $\phi = \xi \circ \psi$).

Proof. By 1.11, if Γ is a general curve in $|K+\eta|$, $\psi : \Gamma \to \psi(\Gamma) = E$ is a double cover of a smooth cubic curve. Therefore $\mathcal{O}_\Gamma(K+\eta) = \psi^*(\mathcal{O}_E(L))$, where L is a divisor of degree 3 on E. Also, $\omega_\Gamma = \psi^*(\mathcal{O}_E(M))$, where M is a divisor of degree 6 on E, and then we have that $\mathcal{O}_\Gamma(K) \cong \psi^*(\mathcal{O}_E(M-L))$. By standard facts on double covers, $\psi_*(\mathcal{O}_\Gamma) = \mathcal{O}_E \oplus \mathcal{O}_E(-M)$, hence $H^0(\mathcal{O}_\Gamma(K)) = H^0(\psi_* \mathcal{O}_\Gamma(K)) = H^0(\mathcal{O}_E(M-L) \oplus \mathcal{O}_E(-L)) = H^0(\mathcal{O}_E(M-L))$. Hence, in view of 1.7, $\phi|_\Gamma$ factors as $j \circ \psi$, where $j : E \to \mathbb{P}^2$ is the new embedding of E as a plane cubic, corresponding to the divisor $(M-L)$ (of degree 3).

We have thus proven, up to now, that there is a factorization $\phi = \xi \circ \psi$ and we have to show that ξ is birational, or, equivalently, that $\deg \phi = 2$.

Let $C \in |K_X|$ be a general curve: then $C \xrightarrow{\psi} \psi(C) \cong \mathbb{P}^1 \xrightarrow{\xi} \mathbb{P}^1$ and if $\xi|_{\psi(C)}$ were not an isomorphism, one would have $\dim_{\mathbb{C}} H^0(\mathcal{O}_C(K_X)) \geqslant 3$, contradicting the exact sequence

$$0 \to H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X(K_X)) \to H^0(\mathcal{O}_C(K_X)) \to 0 .$$

Q.E.D.

§ 2. The canonical ring of the double cover Y.

Let $f: Y \to S$ be the unramified double cover of S such that $f_*(\mathcal{O}_Y) = \mathcal{O}_S \oplus \mathcal{O}_S(\eta)$.

Since $\mathcal{O}_Y(K_Y) = f^*(\mathcal{O}_S(K_S))$, we have that the canonical ring of Y

$$R(Y) = \bigoplus_{m=0}^{\infty} H^0(\mathcal{O}_Y(mK_Y))$$

splits as $\left[\bigoplus_{m=0}^{\infty} H^0(\mathcal{O}_S(mK))\right] \oplus \left[\bigoplus_{m=0}^{\infty} H^0(\mathcal{O}_S(mK+\eta))\right]$ where the first summand is the subring $R(S)$, the canonical ring of S, (and the invariant part of $R(Y)$ for the automorphism determined by the covering involution $i: Y \to Y$). We shall describe S by describing $R(Y)$ and the action of the involution i. Since ψ is a morphism, whereas ϕ is not, it is more convenient to split $R(Y)$ in a different way

(2.1) $$R(Y) = \left[\bigoplus_{m=0}^{\infty} H^0(\mathcal{O}_S(m(K+\eta)))\right] \oplus \left[\bigoplus_{m=0}^{\infty} H^0(\mathcal{O}_S(K+(m-1)(K+\eta)))\right]$$

and we denote the first summand by B, the second by G.

From now on we shall refer almost verbatim to [Ca 4] (see also [Ca 5]). Since $\mathcal{O}_S(K+\eta) = \psi^*(\mathcal{O}_\Sigma(1))$, we have that, A being the polynomial ring $\mathbb{C}[y_0, y_1, y_2, y_3]$, where $y_0, \dots, y_3$ are a basis of $H^0(\mathcal{O}_S(K+\eta))$,

(2.2) B and G are graded A-modules, and they are the respective full modules of sections associated, according to the Serre correspondence ([Se], p. 251), to the coherent sheaves $\psi_*\mathcal{O}_S$, $\psi_*\omega_S$.

REMARK 2.3. Let $j: S \to S$ be the birational involution determined by $\psi: S \to \Sigma$; since S is a minimal model j is a biregular automorphism, and clearly we have an automorphism $j^*: \Omega_S^2 \to \Omega_S^2$.

Furthermore, since $\mathcal{O}_S(K+\eta) = \psi^*(\mathcal{O}(1))$ there are two natural actions of $\mathbb{Z}/2$ also on the sheaf $\mathcal{O}_S(K+\eta)$, and hence on the modules B and G.

Furthermore we may choose the one of the two actions which is trivial on $H^0(\mathcal{O}_S(K+\eta))$, and then $\mathbb{Z}/2$ acts on B, G as an automorphism of graded A modules.

We have thus a splitting $B = B^+ \oplus B^-$, $G = G^+ \oplus G^-$, according to the $(+1)$, respectively (-1) eigenspace of j^*.

If we consider the corresponding sheaves on $\mathbb{P}^3$ (supported on Σ), we have splittings (since Σ is normal every j^*-invariant function on S is a pull--back from Σ, and the same holds for 2-forms regular in codimension 1)

(2.4)
$$\begin{cases} \psi_* \mathcal{O}_S = \mathcal{O} \oplus B \\ \psi_* \omega_S = \omega_\Sigma \oplus G. \end{cases}$$

It is clear now from (2.4) that B^+, G^+ are (aside from the grading) principal A-modules of the form $A/(F)$, where F is the equation of Σ.

For the modules B^-, G^- we use the fact that, (proof as in [Ca 4], prop. 2.8)

(2.5) B^-, G^- are Cohen-Macaulay modules.

The above statement (2.5) implies, for instance, that if $x_1, \ldots, x_h$ is a minimal system of homogeneous generators of G^- as an A-module, and

(2.6) $$\sum_{j=1}^{h} \alpha_{ij}(y) x_j = 0 \qquad (i = 1, \ldots, r)$$

is a minimal system of relations among the x_j's, then $r = h$ and $\det(\alpha_{ij}(y))$ is an equation for Σ (cf. [Ca 4], pp. 77-78, [Ca 5], 16-17).

(2.7) Now $H^0(\mathcal{O}_S(K))$ has dimension 3, and clearly it consists entirely of antiinvariant sections, therefore we have 3 generators x_1, x_2, x_3 of G^- in degree 1. On the other hand, $x_1, \ldots, x_h$ being a minimal system of generators, all the non zero $\alpha_{ij}(y)$'s have degree at least 1: the fact that $\det(\alpha_{ij}(y))$ has degree 3 implies immediately that $h = 3$ and that the $\alpha_{ij}(y)$'s are linear forms.

(2.8) Since $H^0(\mathcal{O}_S(2K+\eta))$ has dimension 10, and the antiinvariant part (for j^*) is generated by the 12 monomials $y_i x_j$, which however satisfy the 3 linear relations above (2.6), we see that the invariant part is generated by a section w. Since G^+ is principal, w is indeed a generator of G^+ as an A-module.

(2.9) Let's consider now the module B^-: since $H^0(\mathcal{O}_S(3K+\eta))$ has dimension 22, while $H^0(\mathcal{O}_\Sigma(3))$ has dimension 19, we see (in the same way as for G^-) that B^- is generated, as an A-module, by 3 independent antiinvariant sections of $H^0(\mathcal{O}_S(3K+\eta))$. But wx_1, wx_2, wx_3 are clearly such sections, and it is clear now that multiplication by w gives a (non graded) isomorphism of G^- with B^-.

(2.10) Since $H^0(\mathcal{O}_S(2K))$ has the 10 monomials $y_i y_j$ as a basis, $\forall h,k = 1,2,3$ the section $x_h x_k$ can be expressed in a unique way as a quadratic polynomial β_{hk} in the y_i's

(2.11) $x_h x_k = \beta_{hk}(y)$.

As in [Ca 1], [Ca 4], we notice that the relations $\sum_{j=1}^{3} \alpha_{ij}(y) x_j = 0$ imply $\sum_{j=1}^{3} \alpha_{ij}(y) \cdot \beta_{jk}(y) \equiv 0 \pmod{\det(\alpha_{ij}(y))}$ and we infer that one can replace the 3 relations by 3 independent linear combinations in order to obtain

(2.12) $$\alpha_{ij}(y) = \alpha_{ji}(y), \quad \text{and} \quad (\beta_{ij}) = \Lambda^2(\alpha_{ij}).$$

In particular, Σ is defined by $\det(\alpha_{ij}(y)) = 0$, and is then a symmetrical cubic according to our definition.

Similarly w^2 is an invariant section of $H^0(\mathcal{O}_S(4K))$, hence there is a homogeneous polynomial P of degree 4 in the variables $y_0, \dots, y_3$ such that

(2.13) $$w^2 = P(y).$$

It is now easy (cfr. [Ca 4], thm. 4.3) to see that the following holds.

PROPOSITION 2.14. The canonical ring $R(Y)$ of the double cover Y is a quotient of $\mathbb{C}[y_0, y_1, y_2, y_3, x_1, x_2, x_3, w]$ by the ideal I generated by the following relations

$$(2.15)\qquad \begin{cases} \sum_{j=1}^{3} \alpha_{ij}(y)x_j = 0 & (i = 1,2,3) \\ x_i x_j = \beta_{ij}(y) & (i,j = 1,2,3\ ,\ (\beta_{ij}) = \Lambda^2(\alpha_{ij})) \\ w^2 = P(y) \end{cases}$$

The involution $i: Y \to Y$ such that $S = Y/i$ acts on the canonical model of Y in the following way $(y,x,w) \mapsto (-y, +x, -w)$.

Furthermore the quartic surface $\{P(y) = 0\}$ does not intersect the singular locus of Σ, which equal the locus $\{y \mid \beta_{ij}(y) = 0\ \forall i,j = 1,2,3\}$.

Prof. By virtue of our previous considerations we are left only with proving the last assertions. The action of i^* on $\mathcal{R}(Y)$ has clearly the subspaces $H^0(\mathcal{O}_S(mK+\eta))$ as (-1) eigenspaces, therefore the involution sends $y_i \mapsto -y_i$, $x_j \mapsto x_j$, $w \mapsto -w$; for later use we notice that, since the variable w has degree two, while the other variables have degree one, the action above is projectively equivalent to the action $y \mapsto y$, $x \mapsto -x$, $w \mapsto -w$. The fixed points of i on the canonical model of Y are thus the points where $w = 0$, and either $y = 0$ or $x = 0$. But by cor. 1.10 ψ is a morphism and there is no point with $y = 0$ on the canonical model (or from the equations one sees that then also $w = x = 0$), whereas $w = 0$, $x = 0$, is equivalent to $\beta_{ij}(y) = 0\ \forall i,j$, and $P(y) = 0$.

That the locus given by $\{y \mid \beta_{ij}(y) = 0\}$ equals the singular locus of Σ follows e.g. by the list of symmetrical normal cubics given in [Ca 2] (pag. 32-34, cases ii), iii), v)), and if the quartic surface $\{P(y) = 0\}$ would pass through a singular point of Σ, i would have a fixed point on the canonical model of Y.

From the classification of rational double points it follows easily that i would have a fixed point on Y, a contradiction (in fact, let Z be the fundamental cycle of a rational double point: by looking at the corresponding Dynkin diagram one checks that either a singular point of Z_{red} is left fixed, or that a component of Z is left invariant, and then one uses the fact that every automorphism of $\mathbb{P}^1$ has a fixed point).

Q.E.D.

COROLLARY 2.15. The canonical ring of S, $\mathcal{R}(S)$, is generated in degrees 1, 2, 3.

Proof. The sections x_i generate $H^0(\mathcal{O}_S(K))$, the sections $y_i y_j$ generate $H^0(\mathcal{O}_S(2K))$, the sections wy_i complete the previous set to a set of generators, but they do not belong to the subring of $R(Y)$ generated by the x_i's, y_j's.

Q.E.D.

DEFINITION 2.16. We recall the notion of negligible singularities: if Δ is a curve lying on a smooth surface S and p is a singular point of Δ, the singularity is called negligible if the multiplicity is at most 3 and, in case the multiplicity equals 3, the proper transform of Δ in the blow-up of S in p has no points of multiplicity 3 lying above p.

It is well known (cf. [Hor 1] p. 49, [Per] 1.8) that Δ has negligible singularities if and only if a double cover of S branched over Δ has at most R.D.P.'s (rational double points) as singularities.

THEOREM 2.17. Let $(\alpha_{ij}(y_0, y_1, y_2, y_3))$ be a symmetric 3×3 matrix of linear forms such that the cubic surface $\Sigma = \{y \mid \det(\alpha_{ij}(y)) = 0\}$ has only isolated singularities.

Let moreover $P(y)$ be a homogeneous polynomial of degree 4 which does not vanish at any singular point of Σ, and moreover such that the curve $\Delta \subset \Sigma$ defined as the zero locus of $P(y)$ is reduced and has only negligible singularities.

Let Y' be the $(\mathbb{Z}/2)^2$ finite Galois cover of Σ defined, in the weighted projective space $\mathbb{P}(1,1,1,1,1,1,1,2)$ by the following equations

$$\begin{cases} \sum_{j=1}^{3} \alpha_{ij}(y) x_j = 0 & (i = 1,2,3) \\ x_i x_j = \beta_{ij}(y) & (i,j = 1,2,3) \ \ (\beta = \Lambda^2(\alpha)) \\ w^2 = P(y) \end{cases}$$

Then Y' is the canonical model of a simply-connected surface of general type Y with $K_Y^2 = 12$, $p_g(Y) = 7$.

The involution i on Y' such that $i(y,x,w) = (-y, x, -w)$ extends to a fixed point free involution i on Y s.t. $S = Y/i$ is a minimal surface of general type with $K_S^2 = 6$, $p_g(S) = 3$, and fundamental group $\pi_1(S) = \mathbb{Z}/2$.

Conversely, all the surfaces S with $K^2 = 6$, $\chi = 4$, and 2-torsion arise in this way.

Finally, given S and $\hat{S}$ associated to the data $(\alpha = (\alpha_{ij}), P)$ $(\hat{\alpha} = (\hat{\alpha}_{ij}), \hat{P})$, they are isomorphic if and only if there exists a projectivity $g : \mathbb{P}^3 \to \mathbb{P}^3$ such that $g(\Sigma) = \hat{\Sigma}$, and $g(\Delta) = \hat{\Delta}$.

Proof. Y' is the fibre product of two double covers.

The first one is $W \xrightarrow{\tau} \Sigma$ where $W \subset \mathbb{P}(1,1,1,1,2)$ is defined by $w^2 = P(y)$: W is branched on Δ hence, by our assumption on Δ, W has only R.D.P.'s as singumarities.

The other one is $\widetilde{\Sigma} \xrightarrow{\sigma} \Sigma$, given in $\mathbb{P}^6$ by the equations $\sum_{j=1}^{3} \alpha_{ij}(y) x_j = = 0$, $x_i x_j = \beta_{ij}(y)$.

For y a smooth point of Σ the matrix $(\alpha_{ij}(y))$ has rank 2, hence the vector (x) is a multiple of a generator of $\ker(\alpha(y))$ and the condition $x_i x_j = \beta_{ij}(y)$ determines the vector x up to multiplication by ± 1 (in the singular points of Σ, instead, $\beta_{ij}(y) = 0$, hence $x = 0$): also the projective coordinate ring of $\widetilde{\Sigma}$ is an integral extension of the one of Σ.

Therefore σ is a finite double cover branched over the singular set of Σ. Since $\omega_\Sigma = \mathcal{O}_\Sigma(-1)$, $\omega_W = \tau^* \mathcal{O}_\Sigma(1)$; again by [Ca 2], pp. 32-34, $\widetilde{\Sigma}$ has at most R.D.P.'s as singularities and $\omega_{\widetilde{\Sigma}} = \sigma^* \omega_\Sigma = \mathcal{O}_{\widetilde{\Sigma}}(-1)$, and $\widetilde{\Sigma}$ is a weak del Pezzo surface of degree 6.

We have clearly that $\omega_{Y'} = \mathcal{O}_{Y'}(1)$, since $\omega_{Y'}$ is the pull back of ω_W.

Therefore we have $K_{Y'}^2 = 12$, $p_g(Y') = 7$.

Since $\widetilde{\Sigma}$ is a rational variety (in particular simply connected) and $\sigma^{-1}(\Delta)$ is a divisor which moves in a linear system without base points and with positive self-intersection, by [MM] (appendix), Y' (hence also Y) is simply connected.

That the involution i has no fixed points on Y' (hence also on Y) was already remarked in the proof of 2.14, hence $\chi(\mathcal{O}_S) = \frac{1}{2}\chi(\mathcal{O}_Y) = 4$, $K_S^2 = = \frac{1}{2} K_Y^2 = 6$.

By 2.14 every surface S with $K_S^2 = 6$, $\chi = 4$, and an unramified double cover Y, arises in the previous way. Furthermore, if there is an isomorphism $\rho : S \to \hat{S}$, $\rho^*(\mathcal{O}_{\hat{S}}(K + \eta)) \cong \mathcal{O}_{\hat{S}}(K_{\hat{S}} + \hat{\eta})$, hence ρ induces a projectivity g carrying Σ to $\hat{\Sigma}$, and Δ to $\hat{\Delta}$. Conversely, since every symmetrical cubic carries only one symmetrization ([Ca 2], thm. 1.3), such an isomorphism g lifts to give isomorphisms of W to $\hat{W}$, and of $\widetilde{\Sigma}$ to $\hat{\widetilde{\Sigma}}$, hence also to give an isomorphism of Y to $\hat{Y}$ commuting with the respective involutions $i, \hat{\imath}$.

Q.E.D.

§ 3. The equation of the general surface and the moduli space.

Let M be the moduli space of the surfaces of general type with $K^2 = 6$, $\chi = 4$ and 2-torsion, which is a quasi-projective variety by Gieseker's theorem ([Gi]).

By theorem 2.17 M is irreducible and unirational, and from now on we shall only consider the Zariski open set $V \subset M$ of the surfaces for which Σ is a 4-nodal cubic surface, hence projectively equivalent to the Cayley cubic of equation

$$\sum_{i=0}^{3} y_0 y_1 y_2 y_3 / y_i = 0 . \tag{3.1}$$

It is well-known that the group of automorphisms of Σ equals the symmetric group $\mathcal{S}_4$, permuting the 4-coordinates y_0, y_1, y_2, y_3.

The Cayley surface is the image of $\mathbb{P}^2$ under the birational map ξ^{-1} given by the system of cubics passing through the 6 vertices of a complete quadrilateral in $\mathbb{P}^2$.

If we choose coordinates (x_1, x_2, x_3) in $\mathbb{P}^2$, and set $x_0 = (x_1 + x_2 + x_3)$, ξ^{-1} is given in the following way

$$y_0 = x_0 x_2 x_3 \, , \; y_1 = x_0 x_1 x_3 \, , \; y_2 = x_0 x_1 x_2 \, , \; y_3 = -x_1 x_2 x_3 \, . \tag{3.2}$$

Since the following relations hold

$$\begin{cases} (y_0 + y_3) x_1 + y_3 x_2 + y_3 x_3 = 0 \\ y_3 x_1 + (y_1 + y_3) x_2 + y_3 x_3 = 0 \\ y_3 x_1 + y_3 x_2 + (y_2 + y_3) x_3 = 0 \end{cases} \tag{3.3}$$

we see that Σ is the symmetrical determinantal cubic corresponding to the matrix $\alpha = \begin{bmatrix} (y_0 + y_3) & y_3 & y_3 \\ y_3 & (y_1 + y_3) & y_3 \\ y_3 & y_3 & (y_2 + y_3) \end{bmatrix}$.

An easy computation gives that the adjoint matrix of α is the matrix

$$\beta = \begin{bmatrix} (y_1 y_2 + y_1 y_3 + y_2 y_3) & -y_2 y_3 & -y_1 y_3 \\ -y_2 y_3 & (y_0 y_2 + y_0 y_3 + y_2 y_3) & -y_0 y_3 \\ -y_1 y_3 & -y_0 y_3 & (y_0 y_1 + y_0 y_3 + y_1 y_3) \end{bmatrix} \tag{3.4}$$

Let moreover U be the vector space of polynomials P of the form

$$P(y) = \sum_{i=0}^{3} \left(\sum_{j=0}^{3} \lambda_{ij} y_i^3 y_j\right) + \sum_{0 \leq i < j \leq 3} \mu_{ij} y_i^2 y_j^2 + \tag{3.5}$$

$$+ \lambda y_0 y_1 y_2 y_3 + \sum_{i \neq j} c_{ij} (y_0 y_1 y_2 y_3) \cdot y_j / y_i \,,$$

$$\text{and with } \sum_{\substack{i=0 \\ i \neq j}}^{3} c_{ij} = 0 \qquad j = 0, \dots, 3 \,.$$

U is of dimension 31, is a $\mathfrak{S}_4$-invariant subspace of all the polynomial of degree 4, and restricts isomorphically to $H^0(\mathcal{O}_\Sigma(4))$.

We let U' be the locally closed set of U consisting of the P's such that $\lambda_{ii} \neq 0$ (hence P does not vanish at the nodes of Σ), $\sum_{i=0}^{3} \lambda_{ii} = 1$, and such that $\Gamma \subset \Sigma$, defined by $P = 0$, is reduced and with negligible singularities.

For $P(y) \in U'$, we get the following 30-dimensional family of surfaces (the canonical models of the double covers Y) in $\mathbb{P}(1,1,1,1,1,1,1,2)$

$$\left\{ \begin{array}{l} (y_0 + y_3) x_1 + y_3 x_2 + y_3 x_3 = 0 \\ y_3 x_1 + (y_1 + y_3) x_2 + y_3 x_3 = 0 \\ y_3 x_1 + y_3 x_2 + (y_2 + y_3) x_3 = 0 \\ \\ x_1^2 = (y_1 y_2 + y_1 y_3 + y_2 y_3) \\ x_2^2 = (y_0 y_2 + y_0 y_3 + y_0 y_3) \\ x_3^2 = (y_0 y_1 + y_0 y_3 + y_1 y_3) \\ \\ x_1 x_2 + y_2 y_3 = 0 \\ x_1 x_3 + y_1 y_3 = 0 \\ x_2 x_3 + y_0 y_3 = 0 \\ w^2 = P(y) \,. \end{array} \right. \tag{3.6}$$

An immediate consequence of theorem 2.17 is that there is a surjective morphism $\xi : U' \to V \subset \mathcal{M}$, such that, set theoretically, $V = U'/\mathfrak{S}_4$ (since $\mathfrak{S}_4 = \mathrm{Aut}(\Sigma)$).

Let U'' be the open set of U' consisting of the P's such that the curve $\Delta = \Sigma \cap \{P(y) = 0\}$ is smooth, and let V'' be the open set of V corresponding to the surfaces S s.t. the canonical divisor K_S is ample: indeed (in fact, if Σ has an A_3 or A_5 singularity, then $\tilde{\Sigma}'$ is not smooth) V'' is open in $\mathcal{M}$, and (again, set theoretically) $V'' \doteq U''/\mathfrak{S}_4$. We shall prove in the sequel the following

PROPOSITION 3.7. If $P(y) \in U''$, then, for the corresponding surface S, $H^1(S,\Theta_S)$ has dimension 30, and the Kuranishi family of S is smooth of dimension 30, and in fact given by the family of quotients of the family (3.6) by the involution i such that $i(y,x,w) = (-y, x, -w)$.

COROLLARY 3.8. The moduli space $\mathcal{M}$ is reduced, and V'' is isomorphic to the quotient $U''/\mathfrak{S}_4$.

We defer the reader to the thesis of the first author ([Ba], pp. 49-55), for a proof of the following theorem; we shall only remark that the proof uses elementary representation theory and the solvability of $\mathfrak{S}_4$ (cf. also [Ca 2]).

THEOREM 3.9. $U/\mathfrak{S}_4$, and hence $\mathcal{M}$, is a rational variety.

Proof of proposition 3.7.

By the Hirzebruch-Riemann-Roch formula $-\chi(\Theta_S) = 10\chi - 2K^2 = 28$, therefore it suffices to show that $H^0(\Omega_S^1 \otimes \Omega_S^2)$, which is the Serre dual of $H^2(\Theta_S)$, has dimension equal to 2. We notice that $H^0(\Omega_S^1 \otimes \Omega_S^2)$ is an invariant by birational transformations, therefore we shall perform the computation on the blow up $\tilde{S}$ of S at the 4 points mapping to the 4 nodes of Σ.

Let F be the minimal resolution of the cubic surface Σ, so that we have birational morphisms $a: F \to \Sigma$, $b: F \to \mathbb{P}^2$ such that $\xi = b \circ a^{-1}$, and such that b is the blow up of the 6 points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,-1,0)$, $(1,0,-1)$, $(0,1,-1)$.

We identily Δ with $a^{-1}(\Delta)$, and we denote by H the divisor such that $\mathcal{O}_F(H) = a^*(\mathcal{O}_\Sigma(1))$, by h the divisor such that $\mathcal{O}_F(h) = b^*(\mathcal{O}_{\mathbb{P}^2}(1))$.

We denote by E_{ij} the exceptional curve of the I kind obtained by blowing up the point $p_{ij} \in \mathbb{P}^2$ given by $x_i = x_j = 0$ $(0 \leqslant i < j \leqslant 3)$ (e.g. $p_{03} = (1,-1,0)$), and by A_i the proper transform of the line $\{x_i = 0\}$; hence $A_i^2 = -2$ and $a(A_i)$ is a node of Σ.

We have

$$h \equiv A_i + \sum_{\substack{j=0 \\ j\neq i}}^{3} E_{ij}$$

$$H \equiv 3h - \sum_{i<j} E_{ij}$$

$\tilde{S}$ is the double cover of F branched on the divisor $\tilde{\Delta} = \Delta + \sum_{i=0}^{3} A_i$; $\tilde{\Delta} \equiv$ $\equiv 16h - 6\sum_{i<j} E_{ij} = 2L$, if we set $L = 8h - 3\sum_{i<j} E_{ij}$, and indeed, if

$\pi : \tilde{S} \to F$ is our double cover, $\pi_*(\mathcal{O}_{\tilde{S}}) \cong \mathcal{O}_F \oplus \mathcal{O}_F(-L)$.

Then (cf. e.g. [Ca 3], 3.1), $H^0(\Omega^1_{\tilde{S}} \otimes \Omega^2_{\tilde{S}}) \cong H^0(\Omega^1_F(\log \tilde{\Delta}) \otimes \Omega^2_F) \oplus$ $\oplus H^0(\Omega^1_F \otimes \Omega^2_F(L))$.

Since $\Omega^2_F \cong \mathcal{O}_F(-3h + \sum_{i<j} E_{ij}) = \mathcal{O}_F(-H)$, the first summand is zero a fortiori if we show that $H^0(\Omega^1_F(\log \tilde{\Delta})) = 0$.

This follows since, in the exact sequence (cf. ibidem, 3.7)

$$\begin{array}{ccccccc} 0 \to & H^0(\Omega^1_F) & \to H^0(\Omega^1_F(\log \tilde{\Delta})) & \to & H^0(\mathcal{O}_{\tilde{\Delta}}) & \xrightarrow{\partial} & H^1(\Omega^1_F) \\ & \| & & & & & \\ & 0 & & & & & \end{array}$$

the image of ∂ is spanned by the Chern classes of $\Delta, A_0, \ldots, A_3$, which are independent: hence ∂ is injective and our cohomology group vanishes. We are thus left with computing the dimension of

$$H^0(\Omega^1_F(5h - 2\sum_{i<j} E_{ij})) = H^0(b_*\Omega^1_F(5h - 2\sum_{i<j} E_{ij}))\,.$$

The sheaf $b_*\Omega^1_F(5h - 2\sum_{i<j} E_{ij})$ is a subsheaf of the sheaf $\Omega^1_{\mathbb{P}^2}(5)$, and is defined by suitable linear conditions at the points P_{ij}.

LEMMA 3.10. If b is the blow-up of the origin in $\mathbb{C}^2$, and E is the exceptional curve in $\tilde{\mathbb{C}}^2$, $b_*\Omega^1_{\tilde{\mathbb{C}}^2}(-2E)$ consists of the 1 forms whose Taylor series at 0 is of the form $\lambda(x\,dy - y\,dx)$ + terms of higher order.

Proof. If $u = y/x$, $dy = x\,du + u\,dx$, and $\omega = a_1\,dx + a_2\,dy + \mu_1 x\,dx +$ $+ \mu_2 y\,dy + \lambda_1 x\,dy + \lambda_2 y\,dx$ + terms of higher order, $b^*\omega = a_1\,dx +$ $+ a_2(x\,du + u\,dx) + \mu_1 x\,dx + \mu_2 ux(x\,du + u\,dx) + \lambda_1 x(x\,du + u\,dx) +$ $+ \lambda_2 xu\,dx + x^2(\ldots)$ is divisible by x^2 if and only if $a_2 = a_1 = \mu_1 = \mu_2 = 0$ and $\lambda_1 + \lambda_2 = 0$.

Q.E.D.

By the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^2}(5) \to \mathcal{O}_{\mathbb{P}^2}(4)^3 \to \mathcal{O}_{\mathbb{P}^2}(5) \to 0$$

one sees that the sections of $\Omega^1_{\mathbb{P}^2}(5)$ can be written as $\sum_{i=1}^{3} P_i(x)\,dx_i$ where the P_i's are homogeneous polynomials of degree 4 in (x_1, x_2, x_3), satisfying the relation $\sum_{i=1}^{3} x_i P_i(x) = 0$. $H^0(\Omega^1_{\mathbb{P}^2}(5))$ is thus easily seen to have dimension

24, and the subspace $H^0(b_* \Omega^1_F(5b - 2\Sigma E_{ij}))$ is defined by 30 linear conditions: an explicit computation (see [Ba] pp. 42-47) shows that indeed the given subspace has dimension 2.

By the semi-universality of the Kuranishi family, we have a holomorphic map f of U'' to the base B of the Kuranishi family, where B is an analytic subspace of $H^1(\Theta_S) = \mathbb{C}^{30}$.

Since the morphism of U'' to $\mathcal{M}$ factors through f, and f has finite fibres, B has dimension 30, and the Kuranishi family is smooth.

If f were not a local biholomorphism, f would ramify on a hypersurface, and there would be a permutation $\sigma \in \mathcal{S}_4$ leaving pointwise fixed a hypersurface in U'': it is easy to check that this does not happen.

Q.E.D.

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