Proceedings of Symposia in Pure Mathematics
Volume 46 (1987)

Canonical Rings and "Special" Surfaces of General Type

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0. Introduction. These lecture notes are meant to survey some very recent results on surfaces of general type, focusing on what in Persson's talk has been called the botanical problem of surface geography: given a pair of integers K^2, χ in the allowed region, classify all the minimal surfaces of general type S with $K_S^2 = K^2$, $\chi(\mathcal{O}_S) = \chi$.

As pointed out in the title, this purpose can be achieved only for special values of the integers K^2, χ or by virtue of other special features of the surfaces under consideration; we are going to describe some general methods which can be used to attack the "botanical" problem, but even when the method is theoretically effective, the computations become rapidly in practice intractable for general values of (K^2, χ) (embarrassingly enough, even the classification of surfaces with $K^2 = \chi = 1$ has not yet been completely achieved!). On the other hand, as we shall see, special geometric properties of the surface under consideration (such as the existence of a fibration $f\colon S \to B$ onto a curve B of given genus, or with fibres of a given genus) are sometimes forced by the fact that the invariants (K^2, χ) satisfy some equalities or inequalities.

By one of the possible definitions, a surface S is of general type if there exists a positive integer m such that the rational map ϕ_m associated to the linear system $|mK_s|$ (the mth-canonical map) is a birational map onto its image $\Sigma_m \subset \mathbf{P}^{N(m)}$ (where $N(m) = P_m(S) - 1 = \chi(\mathcal{O}_S) + \frac{1}{2}m(m-1)K_S^2 - 1$); it should be rather clear, and one can see many beautiful examples in Enriques's book [**En**], that one should seek a small value of m (say $m = 1$, or 2) for which ϕ_m has a good behavior: if for instance ϕ_m is a birational morphism, then S is birational to a surface (Σ_m) of small codimension, and one has more hope to find an explicit description of S.

In fact, pre-Arcata work by Moishezon, Kodaira, and Bombieri was devoted to the fulfillment of Enriques's program, i.e., to the study of pluricanonical maps [**Sh**, **Ko**, **Bo 1**, **Bo 2**] and one of the achieved goals was to prove that there

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14J10.

Author partly supported by AMS and MPI for his participation at the AMS Summer Institute on Algebraic Geometry.

exists an integer, 5, "good" for all surfaces of general type, thereby showing in particular (cf. [**B-H**, **Gi**, **B-P-V**]) that the surfaces with given invariants K^2, χ belong to a finite number of algebraic families.

Especially Bombieri's results in [**Bo 2**] were rather sharp and indicated all possible exceptions to the statements, save for the case of surfaces with $K^2 = 2, 1$ and $p_g = 0$, which seemed to deserve a special treatment (at the time only two examples of surfaces with those invariants were known).

Bombieri's paper stimulated research by several authors devoted to the study of the structure of surfaces with low values of the invariants: the interested reader may find a very good account in Chapter VII of the recent book *Complex surfaces* by Barth–Peters–Van de Ven [**B-P-V**].

We refer also to [**B-P-V**] and to the lecture notes by Barth and Persson (this volume) for the treatment of other very important progress in the theory of surfaces of general type, in particular (referring to our "botanical" theme) Horikawa's work [**Ho 2**] on surfaces with small K^2 (i.e., K^2 small with respect to χ, in particular $K^2 = 2\chi - 6$, $K^2 = 2\chi - 5$), and surfaces of positive index. An important new idea, first introduced by Mumford [**Mu 1**], and concretely exploited then by Reid (cf., e.g., [**Re 1**]), made its way in the meantime: to look at the finitely generated graded ring

$$\mathcal{R}(S) = \bigoplus_{m=0}^{\infty} H^0(S, \mathcal{O}_S(mK_S)),$$

the so-called canonical ring, as the main object of study.

In fact $X = \operatorname{Proj}(\mathcal{R}(S))$, the so-called canonical model of S, is a normal surface of which S is a minimal resolution of singularities: moreover the dualizing sheaf ω_X is invertible and the resolution morphism $\pi \colon S \to X$ is such that $\omega_S = \mathcal{O}_S(K_S) = \pi^* \omega_X$ (in particular, X has only rational double points as singularities). Hence all the pluricanonical mappings ϕ_m factor through π. X is a subvariety of a weighted projective space (cf. [**Do 2**]) and, for extremely special surfaces, it can turn out to be a (weighted) complete intersection. We shall give several simple and explicit examples of how one can describe the canonical ring, explaining also in §3 a couple of general methods (Godeaux-Reid's method of using torsion and unramified covers, our method of quasigeneric canonical projection) which can be applied with success to determine the canonical rings (hence also the minimal models) of surfaces of general type.

On the other hand, going back to the classical point of view, general results about the pluricanonical and canonical maps are not simply technical tools used to determine the mode of generation of the canonical ring.

In fact, there is an underlying philosophy that I'll try now to explain, starting to compare with the case of curves. It is well known that the bicanonical map of a curve C of genus $g \geq 3$ is an isomorphism onto its image, and the canonical map is also an isomorphism except if the curve is hyperelliptic, i.e., if C is a double cover of $\mathbf{P}^1$ branched on $2g+2$ points. Instead, for curves of genus 2, neither the canonical nor the bicanonical map are birational. The outcome is the following: the worse the behavior of the canonical and bicanonical maps, the easier to describe the curve in very simple terms. Does the same occur for surfaces?

Already from Bombieri's paper [**Bo 2**] one sees that the exceptions to the birationality of the tricanonical map are only two: these surfaces are to be thought of as the closest analogues of genus 2 curves and, as we shall see in §1, they admit a very simple description.

What is then the analogue of hyperelliptic curves? The basic observation is that if $C \subset S$ is a curve of genus two and $C^2 = 0$, then $K \cdot C = 2$ and the restriction to C of the canonical map of S either sends C to a point, or sends C to a projective line, and in this case $\mathcal{O}_C(C) \cong \mathcal{O}_C$; furthermore, if $\mathcal{O}_C(C) \cong \mathcal{O}_C$, then also the bicanonical map of S restricts to a projection of the bicanonical map of C, which has degree at least 2.

In particular, if C belongs to a pencil of curves of genus 2, $\mathcal{O}_C(C) \cong \mathcal{O}_C$ and then the bicanonical map of S must have degree at least 2 (as we shall see in §1, the image Σ_2 of the bicanonical map ϕ_2 of S is almost always a surface). Likewise, if C belongs to a pencil of hyperelliptic curves, then the canonical map cannot be birational onto its image.

The point is (cf. §§1, 2) that, while the exceptions to the birationality of ϕ_2, apart from a finite number of families (cf. [**Bo 2**, **Fr**]), are due to the existence of a pencil of genus 2 curves (and then ϕ_2 yields a double cover of a ruled or rational surface), the situation for the canonical map is more complicated, sometimes nasty, and it is still dubious whether a totally clear picture will eventually emerge. In fact, when ϕ_1 is not birational, several things can happen, as has been shown by Beauville [**Be 1**, **Be 3**] and Xiao [**X 2**, **X 3**]: Σ_1 can be any surface with $p_g = 0$ (this is in some sense still an analogue of the case of curves, where the canonical image is $\mathbf{P}^1$, i.e., a curve of genus 0, if C is hyperelliptic), but moreover there are infinite examples where Σ_1 is also a canonically embedded surface.

In case Σ_1 is a curve, Xiao has proved that the genus of Σ_1 is at most 1, while Beauville had proved that the genus of the fibres of ϕ_1 is bounded (effective bounds can be given, e.g., 5 holds for $p_g \geq 20$).

But, up to now, it is totally unknown whether there can be any region in the surface geography where the canonical map is birational, and it is only known [**X 3**] that, for large values of K^2, χ, the degree of ϕ_1 is at most 6; from the point of view of surface classification, however, the case when ϕ_1 has degree at least 3 does not seem for the time being feasible to get hold and elucidate the structure of special surfaces.

As a final remark, we have mostly been trying in this exposition to present statements of theorems and concrete examples (without these last, general results about surfaces can hardly be understood). Also, in this survey, we have not tried by any means to strive for completeness; in particular, the references appear mainly when cited in the text: we apologize for the many omissions we have made.

Moreover, it is our belief that the major progress in the field done in the last ten years owes much to a lot of humble but skillful work on very special classes of surfaces: all this experimental material has been (and is) a rich humus and has allowed a better understanding of general properties of surfaces of general type.

For more broad recent surveys on surfaces of general type, we refer also to [**Ca 6**] and [**Ci 1**].

NOTATION.

S: a minimal model of a surface of general type (over $\mathbf{C}$).

For D, C Cartier divisors, $D \equiv C$ denotes linear equivalence, $D \sim C$ denotes numerical equivalence, and $|D|$ is the linear system of effective divisors $C \equiv D$.

Furthermore, an effective divisor D is said to be m-connected if, for every decomposition $D = A + B$, where A, B are effective divisors, one has $A \cdot B \geq m$.

$K = K_S$: a canonical divisor.

$\mathcal{R} = \mathcal{R}(S) = \bigoplus_{m=0}^{\infty} H^0(\mathcal{O}_S(mK)) = \bigoplus_{m=0}^{\infty} \mathcal{R}_m$: the canonical ring of S.

$X = \operatorname{Proj}(\mathcal{R}(S))$: the canonical model of X, with $\pi\colon S \to X$ the canonical morphism.

$\phi_m =$ the mth canonical map (i.e., the rational map associated to the linear system $|mK_S|) \cdot \Sigma_m = \phi_m(S)$. For x, y points of S, $|D - x - y|$ is the linear system of divisors of sections of $\mathcal{O}_S(D - x - y) = M_x M_y \mathcal{O}_S(D)$.

$q = \dim_{\mathbf{C}} H^1(\mathcal{O}_S)$, the irregularity of S.

$p_g = \dim_{\mathbf{C}} H^2(\mathcal{O}_S)$, the geometric genus of S.

$p_a = p_g - q = \chi(\mathcal{O}_S) - 1$, the arithmetic genus of S.

P_m: the mth plurigenus of S, i.e., $\dim_{\mathbf{C}} H^0(S, \mathcal{O}_S(mK))$, equals, by [**Ko**], $\chi(\mathcal{O}_S) + (m(m-1)/2)K^2$.

For a normal C.M. variety Z of dimension n, if Z^0 is the smooth part of Z, the dualizing sheaf ω_Z is $j_*(\Omega^n_{Z^0})$, where $j\colon Z^0 \to Z$ is the inclusion morphism.

Since $\pi^*(\omega_X) = \omega_S = \mathcal{O}_S(K)$, the pluricanonical maps $\phi_m\colon S \to \Sigma_m$ factor as $\phi_m = \psi_m \circ \pi$, where ψ_m is the rational map $\psi_m\colon X \to \Sigma_m$ given by the sections of $(\omega_X^{\otimes m})$.

π contracts the so-called (-2) curves: irreducible curves E with $E^2 = -2$, $E \cdot K = 0$ (hence $E \cong \mathbf{P}^1$).

$\operatorname{Tors}(S)$: the torsion group of S, i.e., the torsion subgroup of $H_1(S, \mathbf{Z})$ (= torsion subgroup of $H^2(S, \mathbf{Z})$).

For C a divisor in S the dualizing sheaf ω_C is $\mathcal{O}_C(K + C)$ (adjunction formula).

$\mathbf{P}(e_0, \ldots, e_n)$: the weighted projective space of degrees $e_0, \ldots, e_n$ ($e_i \in \mathbf{N}$) is the quotient $\mathbf{C}^{n+1} - \{0\}/\mathbf{C}^*$, where $\lambda \in \mathbf{C}^*$ acts on $x = (x_0, \ldots, x_n)$ by the formula $\lambda x = (\lambda^{e_0} x_0, \ldots, \lambda^{e_n} x_n)$

$\pi_1^{\mathrm{alg}}(S)$: the algebraic fundamental group, the inverse limit of the quotients of the topological fundamental group $\pi_1(S)$ by normal subgroups of finite order.

The Iitaka dimension $K(S, D)$ of a divisor D is, as usual, equal to $-\infty$ if $|mD| = \varnothing\ \forall m \geq 1$, and otherwise equal to the maximum of the dimensions of the images of the rational maps associated to the linear systems $|mD|$.

1. Pluricanonical maps. At the Arcata conference the following theorem was the best available result on pluricanonical maps (cf. [**Bo 2**]).

1.1. BOMBIERI'S THEOREM. *Let X be the canonical model of a surface of general type, and let $\psi_m\colon X \to \Sigma_m$ be the mth canonical map. Then*

(i) *ψ_m is an isomorphism for $m \geq 5$, for $m = 4$ if $K^2 \geq 2$, for $m = 3$ if $K^2 \geq 6$ or $K^2 \geq 3$ and $p_g \geq 4$.*

(ii) *ψ_m is birational for $m \geq 3$ except if $K^2 = 2$, $p_g = 3$ $(m = 3)$ or if $K^2 = 1$, $p_g = 2$ $(m = 3, 4)$ and, possibly, if $K^2 = 1$, $p_g = 0$ $(m = 3, 4)$ or $K^2 = 2$, $p_g = 0$ $(m = 3)$.*

(iii) *ψ_2 is birational for $K^2 \geq 10$, $p_g \geq 6$, except if S has a pencil of curves of genus 2.*

(iv) *ψ_m is a morphism for $m \geq 4$, for $m = 3$ and $K^2 \geq 3$ (or $K^2 \geq 2$, $p_g \geq 1$), for $m = 2$ and $K^2 \geq 5$, $p_g \geq 3$ (or $p_g \geq 3$, $q = 0$).*

At the end of the section we shall summarize, for the reader's convenience, the best results available nowadays for the pluricanonical maps ψ_m $(m \geq 2)$.

1.2. REMARK. The possible exceptions mentioned in the statement (ii) of Bombieri's theorem have in fact later been shown not to occur, through work of several authors (see e.g. [**Mi**, **B-C**, **Ca 1**]).

1.3. REMARK. As we stressed in the introduction, the presence of a pencil of curves of genus 2, $f\colon S \to B$ (where f can be just a rational map if $B \cong \mathbf{P}^1$), forces ψ_2 not to be birational, since, if F is a fibre of f, then $\mathcal{O}_F(K) \cong \omega_F$. Nevertheless these surfaces are very special and amenable to a very detailed description (cf. [**X 1**]).

In fact the hyperelliptic involution which is defined for each fibre is induced by a biregular involution $i\colon S \to S$, and S/i is a ruled surface Y (this is the point of view adopted by Horikawa in [**Ho 1**]): more precisely, if S' is a blow-up of S on which f becomes a morphism (cf. [**X 1**]), $Y = \mathbf{P}(f_*\omega_{S'/B})$ is a $\mathbf{P}^1$-bundle and S' has a double cover $g'\colon S' \to Y$ (g' is not necessarily finite, due to the presence of (-2) curves), yielding a finite double cover $g\colon X \to Y$.

Since a genus 2 curve is a double cover of $\mathbf{P}^1$ branched in 6 points, it follows that the branch curve Δ of g is a reduced curve which has 6 points of intersection with the general fibre of Y. Conversely, every such double cover $g\colon X \to Y$ gives a surface with a pencil of curves of genus 2, and one of the next problems is whether the pencil of curves of genus 2 is unique for a surface of general type.

Now, $\psi_2\colon X \to \Sigma_2$ clearly factors through $g\colon X \to Y$; hence the pencil is unique if Σ_2 is a surface and ψ_2 has degree 2. Xiao [**X 1**, 6.4, 6.5] proves that either S is a product of 2 curves of genus 2, or the pencil is unique if $K^2 \geq 5$ (he also classifies the possible exceptions).

It is curious to observe that, in spite of the simple nature of these surfaces, the only restriction to which their invariants (χ, K^2) are subject is the inequality $K^2 \leq 8\chi$ (sharper than the Bogomolov-Miyaoka-Yau inequality $K^2 \leq 9\chi$); in fact Persson [**Pe**] was able to solve the geographical question posed by Van de Ven in [**Ve**] by showing that all the invariants (χ, K^2) with $\chi, K^2 \geq 1$, $K^2 \geq 2\chi - 6$,

$K^2 \leq 8\chi - 20$, occur for some surface S carrying a pencil of curves of genus 2. This fact partly justifies the analogy set up in the introduction with hyperelliptic curves.

1.3. EXAMPLE. The surfaces which give exceptions to statement (ii) of Bombieri's theorem admit a very simple description, as we are going to show.

Surfaces with $p_g = 3$, $K^2 = 2$ belong to the I Horikawa line $K^2 = 2p_g - 4$ and as we saw in Persson's lecture (cf. [**Ho 2**, I]) ψ_1 has no base points; thus $\psi_1 : X \to \mathbf{P}^2$ is a double cover branched on a curve Δ of degree 8 (and with negligible singularities). If $f_8(x_0, x_1, x_2) = 0$ is an equation for Δ, the canonical ring $\mathcal{R} = \mathcal{R}(S)$ is $\mathbf{C}[x_0, x_1, x_2, z]/(z^2 - f_8(x_0, x_1, x_2))$; therefore X is a hypersurface of degree 8 in the 3-dimensional weighted projective space $\mathbf{P}(1, 1, 1, 4)$.

Similarly, in the case $p_g = 2$, $K^2 = 1$, one takes a basis $\{y_0, y_1\}$ of $\mathcal{R}_1 = H^0(\mathcal{O}_S(K))$, and completes the independent set $\{x_0 = y_0^2,\ x_1 = y_0 y_1,\ x_2 = y_1^2\}$ to a basis $\{x_0, x_1, x_2, x_3 = x\}$ of $\mathcal{R}_2$. One has the obvious relation $x_0 x_2 = x_1^2$ and, since one can show that $|2K|$ has no base points, the image of the bicanonical map is the quadric surface Σ_2 of equation $x_0 x_2 - x_1^2 = 0$.

Since $P_m = \dim_{\mathbf{C}} \mathcal{R}_m = 3 + m(m-1)/2$, a quick computation shows that the monomials in y_0, y_1, and x span $\mathcal{R}_3$ and $\mathcal{R}_4$, but one more generator, call it z, is needed for $\mathcal{R}_5$. We have seen that $\mathcal{R}$ contains the subring $\mathbf{C}[y_0, y_1, x]$; moreover $\mathbf{C}[y_0, y_1, x]$ and $z\mathbf{C}[y_0, y_1, x]$ are direct summands, since they belong to the $(+1)$, respectively (-1), eigenspace for the involution on $\mathcal{R}$ determined by the covering involution $i : S \to S$ associated to the degree 2 map $\phi_2 : S \to \Sigma_2$. Thus $\mathcal{R} \supset \mathbf{C}[y_0, y_1, x] \oplus z\mathbf{C}[y_0, y_1, x]$, but the graded pieces of degree m have the same dimension $\forall m \geq 1$; hence equality holds. Since z^2 belongs to the $(+1)$ eigenspace, there exists a (weighted homogeneous) polynomial $F_{10}(y_0, y_1, x)$ of degree 10 s.t.

$$\mathcal{R} = \mathbf{C}[y_0, y_1, x, z]/(z^2 - F_{10}(y_0, y_1, x)).$$

Again X is a hypersurface, of degree 10 in $\mathbf{P}(1, 1, 2, 5)$, and all such hypersurfaces with R.D.P.'s (= Rational Double Points) as singularities give rise to the minimal model of a surface with $K^2 = 1$, $p_g = 2$.

The 2 surfaces we have considered in this example belong to the I and II Horikawa lines; surfaces with $K^2 = 1$, $p_g = 2$ will be considered later in this paper also for the slow generation of their canonical ring, and are treated here in a different way than in [**Ho 2**]: this approach to Horikawa surfaces has been adopted by Iliev and Griffin [**Il 3**, **Gri**].

Improvements upon Bombieri's result have been recently obtained by Francia [**Fr**] and Reider [**Rei**].

1.4. FRANCIA'S THEOREM. (i) *ϕ_m is a morphism for $m = 2$ provided $p_g \geq 1$, except possibly when $p_g = q = 1$ (in this case, though, $|2K|$ has no fixed part).*

(ii) *If $K^2 \geq 10$, ϕ_2 is birational unless S has a pencil of curves of genus 2.*

(iii) *$\psi_2 : X \to \Sigma_2$ is an isomorphism either if*

(iiia) *$p_g \geq 6$ and divisors $D \in |K_X|$ are 3-connected, or if*

(iiib) *$K^2 \geq 10$, $p_g \geq 6$, unless there exists a curve C with either $C^2 = 0$, $KC = 2$ (a genus 2 curve!) or with $C^2 = -1$, $KC = 1$ (elliptic).*

Before giving an idea of the new ingredients employed in the proof, let's comment again on the result.

1.5. REMARK. Surfaces with $q = p_g = 1$ must have $2 \leq K^2 \leq 9$. The case $K^2 = 2$ has been completely described in [**Ca 7**], and in fact $|2K|$ is base point free; Ciliberto, Francia, and the author can prove existence of such surfaces for $K^2 = 3$, but the same ideas don't work properly for $K^2 \geq 5$. Furthermore on surfaces with $K^2 = 1$, $p_g = 0$, clearly $P_2 = 2$ and thus $|2K|$ has base points, but also it is known (cf. [**Mi 1**, **Bo 2**]) that $|3K|$ has base points, when the torsion group T has elements of order different from 2.

1.6. REMARK. It is clear that the existence of curves of in (iiib), as we also noticed in the introduction, forces ψ_2 not to be an isomorphism. Ciliberto (in [**Ci 2**], to show existence of surfaces for which $\mathcal{R}$ is not generated by elements of degree ≤ 2) has constructed an infinite number of families of surfaces where curves of this sort do in fact appear (clearly then K is not 3-connected).

The hypothesis $K^2 \geq 10$ is essential in statement (ii) of the theorem, as was shown by Bombieri who exhibited in [**Bo 2**, pp. 193–194] an example of a minimal surface with $K^2 = 9$, $p_g = 6$, $q = 0$, with ϕ_2 not birational and without any curve of genus 2, but with a pencil (with one base point) of hyperelliptic curves of genus 3.

Francia proved (unpublished) some results inspired by this example: for instance, if $p_g \geq 4$, $q = 0$, $K^2 \neq 8$, and ϕ_2 is not birational, S has either a genus 2 pencil, or a pencil of hyperelliptic curves of genus 3 (having a base point), and then $6 \leq K^2 \leq 9$.

Sketch of proof of (ii), 1.4. Since $H^1(\mathcal{O}_S(2K)) = 0$ by the Mumford-Ramanujam vanishing theorem [**Mu 2**, **Ra**], and since we have the exact sequence of sheaves

$$0 \to \mathcal{O}_S(2K - x - y) \to \mathcal{O}_S(2K) \to \mathbf{C}^2 \to 0,$$

ϕ_2 separates the two points x and y if and only if

$$H^1(\mathcal{O}_S(2K - x - y)) = 0.$$

Let $\sigma : \tilde{S} \to S$ be the blow-up of S at the two points x, y, and let L, M be the 2 exceptional divisors. We have

$$H^1(\mathcal{O}_S(2K - x - y)) = H^1(\mathcal{O}_{\tilde{S}}(\sigma^*(2K) - L - M)),$$

which is the Serre-dual of $H^1(\mathcal{O}_{\tilde{S}}(-(\sigma^* K) + 2L + 2M))$.

If we set $D \equiv \sigma^* K - 2L - 2M$, what is wanted is the vanishing of $H^1(\mathcal{O}_{\tilde{S}}(-D))$, which would be implied, by the Bombieri-Ramanujam vanishing theorem [**Bo 2**, **Ra**], by the existence of an effective 1-connected divisor in $|D|$, provided $D^2 > 0$.

To have the existence of such an effective divisor, Bombieri has to assume $p_g \geq 6$, whereas Francia can relax this assumption by using the following restatement of Miyaoka's [**Mi 2**] generalization of the Mumford-Ramanujam vanishing theorem.

VANISHING THEOREM II. *Let D be a (not necessarily effective) divisor on a surface $\tilde{S}$ such that its Iitaka dimension $K(\tilde{S}, D)$ equals 2, and such that there exists a positive integer n with $|nD|$ containing a 1-connected divisor D'. Then $H^1(\mathcal{O}_{\tilde{S}}(-D)) = 0$.*

The hypothesis $K^2 \geq 10$ applies first to ensure that $P_2 = \chi + K^2$ is at least 11: then, if x and y are mapped to the same smooth point of Σ_2, then at most 10 linear equations have to be satisfied in order that a section of $H^0(\mathcal{O}_S(2K))$ vanish of order 4 at x and y. Therefore there exists an effective divisor $D' \in |\sigma^*2K - 4L - 4M|$, and one can apply the vanishing theorem unless D' is not connected (in fact $D'^2 > 0$).

In this last case, analyzing the disconnecting decomposition of D', and using $K^2 \geq 10$, Francia concludes the proof.

Another very useful vanishing theorem, used for the proof of statement (i), is the following:

VANISHING THEOREM I. *Let $\sigma : \tilde{S} \to S$ be a birational morphism, and D an effective 1-connected divisor on $\tilde{S}$ with $D \cdot \sigma^*(\sigma_* D) > 0$. Then $H^1(\mathcal{O}_{\tilde{S}}(-D)) = 0$.*

1.7. REIDER'S THEOREM. *ϕ_m is a morphism for $m = 2$ and $K^2 \geq 5$, $m = 3$ and $K^2 \geq 2$, $m \geq 4$. ϕ_m is an embedding for $m = 4$ and $K^2 \geq 2$, $m = 3$ and $K^2 \geq 3$.**

We reproduce Reider's beautiful and simple proof of the first assertion, noting that the last assertions follow with similar proof. Still he makes use of the connectedness property of pluricanonical divisors (used from [**Ko**] on), but there is also a new idea.

Sketch of proof of 1.7. If p is a base point of $|2K|$, by the Residue Theorem of [**G-H**] there exists a rank 2 locally free sheaf $\mathcal{E}$ on S, such that $c_1(\mathcal{E}) = K$, together with a section ξ whose scheme-theoretical zero locus is the reduced point p. Hence one has the Koszul exact sequence for $\mathcal{E}$

$$0 \to \mathcal{O}_S \xrightarrow{\xi} \mathcal{E} \xrightarrow{\wedge\xi} \mathcal{O}_S(K-p) \to 0. \tag{1.8}$$

Since $c_1^2(\mathcal{E}) = K^2 \geq 5 > 4c_2(\mathcal{E}) = 4$, $\mathcal{E}$ is Bogomolov unstable and there exists (cf. [**Re 4**]) an extension

$$0 \to \mathcal{O}_S(L) \to \mathcal{E} \to \mathcal{O}_S(K-L-Z) \to 0, \tag{1.9}$$

where Z is an effective 0-cycle, and L is a divisor such that $\Delta = 2L - K$ is in the positive cone of $NS(S)$. (Hence L is also a fortiori in the positive cone.) Hence it follows $(K-L) \cdot L > 0$: in fact, tensoring (1.8) and (1.9) with $\mathcal{O}_S(-L)$, we get $H^0(\mathcal{E}(-L)) \neq 0$, so that, since $H^0(\mathcal{O}_S(-L)) = 0$, $H^0(\mathcal{O}_S(K-L-p)) \neq 0$.

Thus there exists $D \in |K-L|$, and, L being positive, for $m \gg 0$, there exists also $D' \in |mL|$; now $|mK| \ni mD + D'$, and $(mD) \cdot D' = m^2 L(K-L)$, and $(K-L) \cdot L > 0$ since pluricanonical divisors are connected.

Finally, by Riemann-Roch, $(K-L) \cdot L$ is an even number, and we have a contradiction since, by (1.9), $1 = c_2(\mathcal{E}) = \deg(Z) + L(K-L)$. Q.E.D.

1.10. REMARK. By 1.5 we see that for $m = 3$ this is the best possible result; moreover, if $K^2 = 1$ and $p_g > 0$, then $|2K|$ has no base points. (In fact, $p_g \leq 2$ by Noether's inequality, and the case $p_g = 2$ follows from [**Ho 2**], $p_g = 1$ from [**Ca 8**].)

We recapitulate the above results, also keeping track of the case $p_g = K^2 = 3$ appearing in [**Ho 2**, **Il 1**, **Il 2**].

1.11. THEOREM. *Let X be the canonical model of a surface of general type, and let $\psi_m : X \to \Sigma_m$ be the mth canonical map. Then*

(i) ψ_m is a morphism for $m \geq 4$, $m = 3$ and $K^2 \geq 2$, for $m = 2$ if $K \geq 5$, or if $p_g \geq 1$ (except possibly if $p_g = q = 1$ and $K^2 = 3, 4$).

(ii) ψ_m is birational for $m \geq 3$ except if $K^2 = 2$, $p_g = 3$ $(m = 3)$ or if $K^2 = 1$, $p_g = 2$ $(m = 3, 4)$.

(iii) ψ_2 is birational for $K^2 \geq 10$ unless S has a pencil of curves of genus 2.

(iv) ψ_m is an isomorphism for $m \geq 5$, for $m = 4$ if $K^2 \geq 2$, for $m = 3$ if $K^2 \geq 3$, for $m = 2$ if $p_g \geq 6$, $K^2 \geq 10$ except if there are curves C with either $C^2 = 0$, $CK = 2$, or with $C^2 = -1$, $KC = 1$.

2. The canonical map and its pathologies. We have tried in the previous section, at the risk of boring the reader, to give an idea of now necessarily complicated have to be statements concerning all surfaces of general type (and of how, though, the surfaces giving exceptions have a structure full of interesting geometry). It can be expected then that the exceptions should rapidly go out of control when one wants a good behavior of the canonical map. Therefore, after briefly surveying the general pattern which has emerged through work of several authors, we shall describe a couple of significant examples.

Let's assume $p_g \geq 3$ (this way only a finite number of families are left out), and let's consider $\psi_1 : X \to \Sigma_1$. Two cases can a priori occur:

(A) $|K|$ is composed with a pencil (i.e., Σ_1 is a curve, whose geometric genus we shall denote by b);

(B) Σ_1 is a surface.

Let's start with case (A), which at first sight looks the more difficult to occur, and let's remark right away that there are an infinite number of families for which $|K|$ is composed of a pencil, according to classical examples of Pompilij and others (cf. [**Be 1**]). Nevertheless, some finiteness statements still hold true in this situation; namely, if B is the normalization of Σ_1, $f : S \to B$ is the pencil

*Reider also re-proves (ii) of Francia's theorem.

of which $|K|$ is composed, g denotes the genus of a general fibre of f, and b denotes the genus of B, one can give bounds for g, b.

Beauville [**Be 1**] uses the Bogomolov-Miyaoka-Yau inequality to prove the following

2.1. THEOREM. *If $|K|$ is composed of a pencil, the curves of the pencil have genus $2 \leq g \leq 5$ if $p_g \geq 20$ (moreover f is a morphism then); furthermore $g \leq 6$ if $p_g \geq 11$.*

The other bound is even nicer [**X 2**].

2.2. XIAO'S THEOREM. *If $|K|$ is composed of a pencil, then either $q(S) = b = 1$, or $b = 0$, $q(S) \leq 2$.*

As in [**X 1**] one of the basic tools employed is positivity, positivity of quotients of the locally free sheaf (of rank g) $f_*\omega_{S/B} = f_*\omega_S \otimes \omega_B^{-1}$, which follows from a theorem of Fujita [**Fu**], and positivity of $\omega_{S/B}$ (cf. [**Be 2**]).

The starting point of the argument is that the assumption that ϕ_1 factors through $f\colon S \to B$ implies the existence of a subline bundle $\mathcal{L}$ of $f_*\omega_S$ such that $H^0(B, \mathcal{L}) \cong H^0(B, f_*\omega_S)$: then positivity of the quotient bundle $f_*\omega_{S/B}/\mathcal{L} \otimes \omega_B^{-1}$ and Riemann-Roch imply $b \leq 1$. The bounds for q use the results of [**Be 2**] plus a careful analysis of two pencils existing on S, the given one $f\colon S \to B$ plus the pencil given by the Albanese map. Beyond the above bounds on b and g, there is also a "geographical" constraint in order that (A) may happen. The following results, after Castelnuovo's inequality [**C**] $K^2 \geq 3p_g - 7$, established under the assumption that ϕ_1 be an embedding, are due to the work of several authors: Horikawa, Reid, Beauville, Debarre [**Ho 2**, **Re 2**, **Be 2**, **De**].

2.3. THEOREM. *If ϕ_1 is birational, then $K^2 \geq 3p_g + q - 7$. If $K^2 < 3p_g - 7$, then ϕ_1 is of degree 2 and Σ_1 is a ruled surface. If Σ_1 is a curve, then $K^2 \geq 3p_g - 6$, and $K^2 \geq 4p_g + 4(b-1)$ if the curves of the pencil have genus g at least 3.*

As remarked in [**Be 1**] (this paper is, in our opinion, the best reference for 2.3), one sees then that surfaces with small K^2 are only in part easy to classify: in fact if $K^2 < 3p_g - 7$, S is only birational to the double cover of a geometrically ruled surface, and the study of the branch locus may become very fastidious. Beauville suggests calling "hyperelliptic" the surfaces which are birational to a double cover of a ruled surface; this is slightly confusing, since classically the hyperelliptic surfaces (bielliptic in Beauville's terminology) are the minimal elliptic surfaces with $12K \equiv 0$, $K \not\equiv 0$, and $b_1 > 0$ (in fact $b_1 = 2$), explicitly classified by Bagnera and De Franchis (see [**B-P-V**, p. 148]): nevertheless we shall temporarily adhere to this notation.

The "hyperelliptic" surfaces of general type have been studied by Xiao [**X 3**], who in particular states interesting results about unicity of the hyperelliptic pencil and about the structure of the fundamental group $\pi_1(S)$, thus confirming some conjecture of Reid. In particular Xiao proves the following.

2.4. THEOREM (XIAO). *Let S be a surface of general type with a hyperelliptic pencil $f\colon S \to B$. Then, if $K^2 < 4\chi - 28$, then $q(S) = b =$ genus of B.*†

2.5. REMARK. In particular, when $q > 0$, the fibres of the Albanese map give the hyperelliptic pencil. The above statement confirms a conjecture of Severi to the effect that if $K^2 < 4\chi$, then the image of the Albanese map is a curve. An important step in this direction had been done earlier by Reid [**Re 5**] and Horikawa [**Ho 2**, V].

2.6. THEOREM (HORIKAWA-REID). *Let S be a minimal surface of general type with $K^2 < 3\chi$. Then, if $q(S) \geq 1$, the image of the Albanese map is a curve and the curves of the pencil thus obtained have genus 2 or 3, and indeed equal to 2 if $K^2 < \frac{8}{3}\chi$. If $q = 0$, either $\pi_1(S)$ is finite, or there exists an unramified covering $u\colon \tilde{S} \to S$ and a morphism $f\colon \tilde{S} \to B$ with fibres hyperelliptic curves of genus ≤ 5, and with $\pi_1(B) \cong \pi_1(\tilde{S})$.*

The striking fact of Theorem 2.3 above is that the topological invariants (χ, K^2) force the canonical map to have a "bad" behavior, and in some sense surfaces with "small K^2," i.e., $K^2 < 3p_g - 7$, are another surface analogue of curves of genus 2: the connection with curve theory is here more transparent, since the main methods of proof are based on the analysis of the restriction of ϕ_1 to canonical curves, and on the application of classical lemmas (such as Clifford's or Commessatti's lemma) about special systems on curves. It is somehow expected, but to our knowledge not even made explicit by some conjecture, that ϕ_1 should be birational for "K^2 very large." In view of 2.1 and 2.2, it seems that most information is missing in the case (B) where Σ_1 is a surface. Beauville proved the following [**Be 1**].

2.7. THEOREM (BEAUVILLE). *If the image Σ_1 of the canonical map is a surface, then either*

(i) *$p_g(\Sigma_1) = 0$ or*

(ii) *Σ_1 is canonically embedded (i.e., there exists a surface S' whose canonical map is birational, and with image Σ_1).*

Moreover, in case (i) *$\deg \phi_1 \leq 9$ if $\chi \leq 31$, and in fact $\deg \phi_1 \leq 4$ if Σ_1 is not ruled; in case* (ii) *$\deg \phi_1 \leq 3$ if $\chi \geq 14$.*

2.8. REMARK. Class (i) should be thought of as the rule at least when ϕ_1 is not birational. As a matter of fact, all smooth surfaces Σ with $p_g = 0$ occur as canonical images under a morphism ϕ_1 of degree 2. To see this, let's consider divisors δ, Δ on Σ such that Δ is effective, reduced, smooth and $\Delta \equiv 2\delta$. If S is the double cover of Σ branched on Δ, S being a smooth surface in the line bundle associated to the invertible sheaf $\mathcal{O}_\Sigma(\delta)$, then, in general, by the Leray spectral sequence, $H^0(\mathcal{O}_S(K_S)) \cong H^0(\mathcal{O}_\Sigma(K_\Sigma)) \oplus H^0(\mathcal{O}_\Sigma(K_\Sigma + \delta))$. By the

†In a more recent preprint, entitled *Fibered algebraic surfaces with low slope*, Xiao extends the result to the case where $f\colon S \to B$ is any pencil with $K^2 < 4\chi + 4(b-1)(g-1)$.

assumption $p_g(\Sigma)=0$ the first summand is zero, which ensures that ϕ_1 factors through the double cover $S \xrightarrow{\pi} \Sigma$ and the map associated to the linear system $|K_\Sigma+\delta|$, which, for δ sufficiently ample, gives an embedding of Σ.

One should notice that even the classification of these simple "exceptions" is quite complicated, since a surface Σ with $p_g=0$ can be ruled, hyperelliptic, an Enriques surface, a properly elliptic surface, a surface of general type with $p_g=0$: thus, let alone the problem of studying the adjoint systems $|K_S+\delta|$, even the classification of the surfaces themselves with $p_g=0$ has not been yet achieved!

On the other hand, case (ii) when $\deg\phi_1\geq 2$ was considered for a long time to be impossible (cf. [**Ca 2**]), and in fact basically the same example (with deg $=2$) was found independently by Van der Geer–Zagier, Beauville, and the author [**G-Z**, **Be 2**, **Ca 2**].

One more example was found by Ciliberto, but recently Beauville [**Be 3**] found an infinite series of examples where (ii) occurs, with $\deg\phi_1=2$. Whether (ii) can occur with $\deg\phi_1=3$ is still an open problem (moreover, in this case, $K^2>8\chi$, and the surface S must have positive index).

2.9. *Beauville's examples.* Let A be a principally polarized abelian surface, and $f\colon A\to K\subset \mathbf{P}^3$ the double finite cover of its Kummer surface given by the linear system $|2\theta|$. Let $u\colon K\to\mathbf{P}^1$ be the rational map obtained via a generic projection of $\mathbf{P}^3$ onto $\mathbf{P}^1$. (Thus $u\circ f$ is given by two theta-functions of the second order.)

Let $v_n\colon\mathbf{P}^1\to\mathbf{P}^1$ be the morphism such that $v_n(z)=z^n$, and consider the following diagram, obtained by taking Σ and X to be the respective fibre products of the pairs of morphisms (u,v_n), $(u\circ f, v_n)$.

$$\begin{array}{ccccc}
A\times\mathbf{P}^1 & & K\times\mathbf{P}^1 & & \\
\cup & & \cup & & \\
X & \xrightarrow{\pi} & \Sigma & \to & \mathbf{P}^1 \\
\downarrow & & \downarrow & & \downarrow v_n \\
A & \xrightarrow{f} & K & \xrightarrow{u} & \mathbf{P}^1 \\
 & & \cap & & \\
 & & \mathbf{P}^3 & &
\end{array}$$

Σ has $16n$ nodes, and $\pi\colon X\to\Sigma$ is the quotient morphism by an involution having exactly $16n$ fixed points: hence $\pi^*(\omega_\Sigma)=\omega_X$. To show that the canonical map of X factors through the canonical map of Σ, we simply compute and find $p_g(X)=p_g(\Sigma)$. In fact, since both X and Σ are hypersurfaces, by adjunction we obtain (with self-explanatory notation)

$$\omega_\Sigma=\mathcal{O}_K(1)\otimes\mathcal{O}_{\mathbf{P}^1}(n-2),\qquad \omega_X=\mathcal{O}_A(2\theta)\otimes\mathcal{O}_{\mathbf{P}^1}(n-2).$$

Therefore, for $n\geq 3$, $|\omega_\Sigma|$ clearly gives an embedding of Σ; furthermore a straightforward computation gives $p_g(X)=p_g(\Sigma)=1+4(n-1)=4n-3$.

Where are these surfaces geographically located? We have $K_X^2=2K_\Sigma^2=2(n+2(n-2))4=24n-32$; thus $K_X^2=6p_g(X)-14>6\chi(X)-8$, and K^2 is not really large.

The last problem we are going to touch in this section refers to the degree of ϕ_1 in the case when $p_g(\Sigma_1)=0$.

2.10. EXAMPLE. There is an example, due to Persson [**Per 2**, 5.8], where $\deg\phi_1=16$, but there is an infinite series of examples due to Beauville, where the degree is 8 (cf. Theorem 2.11, 8 is the best possible number). The following is taken from [**Be 1**]: let X be the surface $C\times\mathbf{P}^1$, C being a curve of genus 3, let $p\colon X\to C$ be the first projection, and F the inverse image of a point in $\mathbf{P}^1$ under the second projection. If η is a divisor on C with $\eta\not\equiv 0$, $2\eta\equiv 0$, consider the divisor δ on X such that $\delta\equiv p^*(\eta)+(a+2)F$, where a is a positive integer. Let Δ be the union of $2a+4$ disjoint fibres F_t, and let $\pi\colon S\to X$ be the double cover of X branched on Δ, where the surface S is a divisor in the line bundle associated to the invertible sheaf $\mathcal{O}_X(\delta)$: as in Remark 2.8 the canonical map of S factors as $f\circ\pi$, f being the map given by the linear system $|K_X+\delta|=|p^*(K_C+\eta)+aF|$. Since C has genus 3, and $K_C+\eta$ has degree 4, but is not $\equiv K_C$, $|K_C+\eta|$ defines a covering of degree 4 of $\mathbf{P}^1$; therefore ϕ_1 has degree 8 and Σ_1 is isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$.

The program of finding restrictions upon surfaces for which the degree of ϕ_1 is large has been carried on with success by Xiao [**X 4**], using a finer analysis of the linear systems on a ruled surface Σ which give a ruling of Σ (cf. also [**Re 3**, **Di**] for a more conceptual presentation of this result), and again the consideration of the sheaf $f_*\omega_{S/B}$ associated to a given pencil $f\colon S\to B$.

2.11. THEOREM (XIAO). (1) *If ϕ_1 has degree 3 and $K^2<4p_g-12$ (resp., 27 for $p_g=10$), then $q(S)\leq 1$, S has a pencil of curves of genus 3, $K^2\geq\frac{10}{3}p_g-8$.*

(2) *If $p_g>187$, ϕ_1 has degree ≤ 8, and if equality holds, $q(S)\leq 4$, and S has a pencil of curves of genus 5 or 6.*

(3) *If $\deg\phi_1\geq 5$, S has bounded irregularity and a pencil of curves of bounded genus.*

3. The canonical ring $\mathcal{R}$. From the fact that there exists m s.t. $|mK|$ is free from base points Mumford's theorem (cf. [**Mu 1**], also [**B-P-V**]) that the canonical ring $\mathcal{R}=\mathcal{R}(S)$ is finitely generated follows easily. $\mathcal{R}$ is "the" birational invariant of S, since S and S' are isomorphic iff the two rings $\mathcal{R}(S)$ and $\mathcal{R}(S')$ are isomorphic, and S can be recovered from $\mathcal{R}$ since S is the minimal resolution of singularities of $X=\mathrm{Proj}(\mathcal{R})$. (Notice that, $\mathcal{R}(S)$ being integrally closed in the field $\mathbf{C}(S)$ of rational functions of a surface S, by definition a surface of general type is just a surface for which $\mathbf{C}(S)$ equals the field of homogeneous fractions of $\mathcal{R}(S)$.) For the effective determination of $\mathcal{R}$, it is important to know an upper bound for the degree of a minimal set of homogeneous generators of $\mathcal{R}$: very relevant to this purpose are the results mentioned in §1, and also more general results on the pluricanonical images Σ_m, such as their projective normality.

After the work of Kodaira and Bombieri, dealing mostly with the problem of projective normality, there have been better results in the direction of determining this upper bound, and also upper bounds for the relations among the generators, by Gasbarrini, Green, Ciliberto, and the present author (cf. [**Ga**, **Gr 1**, **Gr 2**, **Ci 2**, **Ca 3**]). Some of Ciliberto's results are the sharpest, e.g., he proves (we shall give here only the most general result, referring the reader to [**Ci 2**] for more precise statements)

3.1. THEOREM (CILIBERTO). *$\mathcal{R}$ is generated by (homogeneous) elements of degree ≤ 5 if $p_g \geq 1$, and of degree ≤ 6 if $p_g = 0$.*

The bounds in Theorem 3.1 are sharp. For instance, we have seen in Example 1.3 that a generator in degree 5 is needed for surfaces with $K^2 = 1$, $p_g = 2$.

Even if one knows some bounds about the degrees of the generators in a minimal set, one has to give some general method in order to be also able to write down the relations among them.

I *Method: the method of quasigeneric canonical projections* [**Ca 3**]. The main idea here is to understand in algebraic terms the classical picture of a generic projection, thus generalizing and systematizing the beautiful intuitions of Enriques (who often treated pluricanonical maps as being generic maps). To simplify things, a quasigeneric birational canonical projection $\Phi: S \to \mathbf{P}$ is defined to be (cf. [**Ca 3**]) a birational morphism of S into a 3-dimensional weighted projective space $\mathbf{P}$, given by 4 homogeneous elements y_0, y_1, y_2, y_3 of the canonical ring. The method we have introduced works for regular surfaces (i.e., with $H^1(\mathcal{O}_S) = 0$), and owes much to the work of several authors [**Ca 2**, **A-S**, **Ser**, **Ci 3**, **C-D**].

As noticed above, the results about pluricanonical maps ensure that, knowing K^2, χ, we can make an a priori choice of the degrees $e_0, \dots, e_3$ of such elements $y_0, \dots, y_3$. The trick is to complete $y_0, \dots, y_3$ to a huge set of generators of $\mathcal{R}$, but in such a way that the relations become very simple and closely related to the geometry of $\Sigma = \Phi(S)$: this is done (following an old idea of Petri about canonical curves) by considering $\mathcal{R}$ as a module over the polynomial ring $\mathbf{C}[y_0, y_1, y_2, y_3]$.

Then we pick a minimal set $v_1 = 1, v_2, \dots, v_h$ of (homogeneous) generators of $\mathcal{R}$ as a $\mathbf{C}[y_0, \dots, y_3]$-module: the hypothesis $H^1(\mathcal{O}_S) = 0$ guarantees that the module $\mathcal{R}$ is Cohen-Macaulay; hence the (linear) relations among the v_j's have no syzygies, and they are generated by h relations

$$\sum_{j=1}^{h} \alpha_{ij}(y) v_j = 0 \qquad (i = 1, \dots, h). \tag{3.1}$$

Moreover, since the v_j's generate $\mathcal{R}$ as a module, and $\mathcal{R}$ is a ring, $v_i v_j$ is an element of $\mathcal{R}$; hence there exist polynomials $\lambda_{ij}^k(y)$ such that, in $\mathcal{R}$, the following holds:

$$\sum_{k=1}^{h} \lambda_{ij}^k(y) v_k = v_i v_j. \tag{3.2}$$

$\mathcal{R}$ is generated by $y_0, \dots, y_3, v_2, \dots, v_h$ (where $\deg y_i = e_i$, $\deg v_j = l_j$), with the only relations (3.1) and (3.2), and the polynomials α_{ij}'s, λ_{ij}^k's can be chosen to satisfy certain properties, stated in the following theorem: but much more important is the converse result, which gives a systematic tool to classify all the regular surfaces with given numerical invariants χ, K^2.

3.3. THEOREM [**Ca 3**]. *Let $A = (\alpha_{ij}(y))$ be a symmetric $h \times h$ matrix of weighted homogeneous polynomials in $y_0, \dots, y_3$, such that, if $A_{\hat{i}}^{\hat{j}}$ is the minor obtained by deleting the ith row and the jth column of A, there do exist polynomials $\lambda_{ij}^k(y)$ with $\det(A_{\hat{i}}^{\hat{j}}) = \sum_{k=1}^{h} \lambda_{ij}^k(y) \det(A_{\hat{1}}^{\hat{k}})$.*

Assume further that $\det(A)$ is an irreducible polynomial, and that the surface X defined by equations (3.1) and (3.2) has only R.D.P.'s as singularities: then, if the degrees of the α_{ij}'s are suitable (there do exist integers $l_1 = 0 < l_2 \leq \dots \leq l_h$ with $\deg(\alpha_{ij}) = (\sum_{i=0}^{3} e_i) + 1 + l_i - l_j$), X is the canonical model of a regular surface of general type.

3.4. EXAMPLE. We illustrate the previous theorem by considering surfaces with $q = 0$, $p_g = 4$, $K^2 = 6$, assuming for simplicity that ϕ_1 is a morphism and that Σ_1 is not a quadric (cf. [**Ci 3**, **Ca 3**]). Let $y_0, \dots, y_3$ be a basis of $\mathcal{R}_1 = H^0(\mathcal{O}_S(K))$: then $\mathcal{R}$ is generated as a module by 1 and an element v of degree 2. The degrees of the matrix A are $\binom{5\ 3}{3\ 1}$, and the above condition on the minors of A simply means that there do exist homogeneous polynomials $G(y)$, $Q(y)$, of respective degrees 4 and 2, such that $\alpha_{11}(y) = G(y)\alpha_{22}(y) + Q(y)\alpha_{12}(y)$.

The canonical ring is generated by y_0, y_1, y_2, y_3, v, and the ideal of relations among them is generated by the simple relations

$$\begin{cases} \alpha_{12}(y) + \alpha_{22}(y)v = 0, \\ G(y)\alpha_{22}(y) + \alpha_{12}(y)(Q(y) + v) = 0, \\ v^2 = G(y) + Q(y)v. \end{cases}$$

The four polynomials α_{12}, α_{22}, G, Q can be chosen generically; hence the moduli space for these surfaces is unirational. (Notice, though, that the map ϕ_1 is birational only when $\alpha_{22}(y) \not\equiv 0$.)

The following method, although applicable to a much more restricted class of surfaces, is in practice very useful for effectively enlarging the class of surfaces which have a simple algebraic description: it was basically found by Godeaux, revived by Reid who also considered nonabelian groups, extended by Barlow to the case of a nonfree action.

II *Method: Torsion, unramified coverings, and group actions.* The method is applicable when the fundamental group has a subgroup (which we may assume normal) of finite index, e.g., when the torsion group $T \subset H_1(S, \mathbf{Z})$ is nontrivial.

Then there exists an unramified Galois cover $p\colon Y \to S$ with group G, so that $S = Y/G$.

The canonical ring $\mathcal{R}(Y)$ is a representation of G and $\mathcal{R}(S)$ is (implicitly) described if one can describe $\mathcal{R}(Y)$ and the action of G. Let's give the most classical example, due to Godeaux, where Y is a hypersurface.

3.5. EXAMPLE [**Mi 1**, **Re 1**]. Assume S has $K^2 = 1$, $p_g = 0$, $T \cong \mathbf{Z}/5$. Since $q(S) = 0$, one has a $\mathbf{Z}/5$-cover Y with $p_g = 4$, $K^2 = 5$: one proves that ϕ_1 is a morphism for Y. Hence Y is a quintic surface in $\mathbf{P}^3$, and $\mathcal{R}(Y) = \mathbf{C}[y_0, \ldots, y_3]/f_5(y_0, \ldots, y_3)$. It is easy to see that $\mathcal{R}_1(Y)$ is the direct sum of the 4 nontrivial characters of $\mathbf{Z}/5$, since $p_*\mathcal{O}_Y(K_Y) = \bigoplus_{\eta \in T} \mathcal{O}_S(K + \eta)$, and $H^0(\mathcal{O}_S(K + \eta))$ has dimension 1 for $\eta \not\equiv 0$, 0 for $\eta \equiv 0$. Conversely, if $f_5(y_0, \ldots, y_3)$ is invariant for the action of $\mathbf{Z}/5$, and this action has no fixed points on Y, then $Y/(\mathbf{Z}/5) = S$ is the desired surface.

The previous example shows what has to be the strategy of the method:

(i) Describing the canonical ring of Y;

(ii) Using the representation theory of G plus the geometry of S to determine what can be the possible actions of G on $\mathcal{R}(Y)$;

(iii) Conversely, given Y and an allowed action of G on $\mathcal{R}(Y)$, checking whether there exists Y with such a *fixed-point-free* action (this is often the hardest step, at least computationally, cf. [**Bar 2**]).

3.6. EXAMPLE [**C-D**]. Let S be a surface with $K^2 = 2$, $p_g = 1$, $q = 0$, $T = \mathbf{Z}/2$, and let Y be the double cover. Step (ii) is very easy, since $\mathcal{R}(Y)$ just splits according to the two eigenspaces for $\mathbf{Z}/2$, and we write $\mathcal{R}(Y) = \mathcal{R}^+ \oplus \mathcal{R}^-$, where $\mathcal{R}^+ \cong \mathcal{R}(S)$. $\mathcal{R}(Y)$ is generated by $w \in \mathcal{R}_1^+$, $x_1, x_2 \in \mathcal{R}_1^-$, $z_3, z_4 \in \mathcal{R}_2^-$, and one proves that Y is a complete intersection of type $(4,4)$ in $\mathbf{P} = \mathbf{P}(1,1,1,2,2)$. $\mathbf{Z}/2$ acts on $\mathbf{P}$ by sending $(w, x, z) \to (w, -x, -z)$, and the 2 equations of Y must be $\mathbf{Z}/2$ invariant; with little work one can even show that the equations can be written in the very special form

$$\begin{cases} z_3^2 + wz_4 l(x_1, x_2) + G(w, x_1, x_2) = 0, \\ z_4^2 + wz_3 l'(x_1, x_2) + G'(w, x_1, x_2) = 0, \end{cases}$$

with l, l' linear forms, G, G' of degree 4 and containing only even powers of w. From this explicit form of the equations one can even show that the moduli space for our surfaces is a rational variety.

As the reader will have noticed, the method is based on the condition that one can already (step (i)) classify the surfaces Y which occur as unramified coverings, but in fact, from our geographical point of view of classifying the surfaces with given invariants K^2, χ, a preliminary question has to be answered:

(iv) what can be the torsion group T or the algebraic fundamental group $\pi_1^{\mathrm{alg}}(S)$ for a surface with given invariants, or at least is there a bound for the order of these groups?

For instance, Miyaoka and Reid [**Mi 1**, **Re 1**] proved that for a surface with $K^2 = 1$, $p_g = 0$, T has order at most five, and T cannot be isomorphic to $(\mathbf{Z}/2)^2$; then, since the representation theory of an abelian group is trivial, Reid [**Re 1**] was able to attack problems (ii) and (iii) determining all the surfaces with $K^2 = 1$, $p_g = 0$, order of $T = 3, 4, 5$.

Regarding the "preliminary" step (iv) we remind the reader of Theorem 2.6, and mention another result of the type one is looking for ([**Re 2**], cf. also [**Be 1**]).

3.7. THEOREM (REID). *If $K^2 \neq 0$, $p_g = 0$, then the order of π_1^{alg} is at most* 9.

REMARK. In fact Reid states the incorrect bound 8, but the following nice example gives a surface with $K^2 = 2$, $p_g = 0$, $\pi_1 = (\mathbf{Z}/3)^2$ (cf. [**X 1**], anyhow we shall give the more explicit description adopted by Beauville in [**Be 3**]).

3.8. EXAMPLE (XIAO). Consider $\mathbf{P}^2 \times \mathbf{P}^2$, with homogeneous coordinates (x_0, x_1, x_2), (y_0, y_1, y_2). Let Y_λ be the complete intersection of the two hypersurfaces

$$\left\{\sum_{i=0}^{2} x_i y_i = 0\right\}, \qquad \left\{\left(\sum_{i=0}^{2} x_i^3\right)\left(\sum_{j=0}^{2} x_j^3\right) + \lambda \prod_{i=0}^{2} x_i y_i = 0\right\}.$$

For a general value of λ, Y_λ is smooth and $\pi_1(Y_\lambda) = 0$ by Lefschetz's theorem. On the other hand, Y_λ is clearly invariant by the group $G \cong (\mathbf{Z}/3)^2$ generated by the following two transformations g_1, g_2 such that:

$$g_1(x, y) = ((x_0, \varepsilon x_1, \varepsilon^2 x_2), (y_0, \varepsilon^2 y_1, \varepsilon y_2)) \qquad (\varepsilon = \exp(2\pi i/3)),$$
$$g_2(x, y) = ((x_1, x_2, x_0), (y_1, y_2, y_0)).$$

$K_{Y_\lambda}^2 = 18$ and it is easy to see that the action of G is free; hence the surface S has the derived invariants.

But what happens if, after prescribing a surface Y and a possible action of a group G on Y, one sees that the action is not free? This situation has been considered by Barlow [**Bar 1**, **Bar 2**], especially the "good" case where the points with a nontrivial stabilizer G_y form a finite set, and where the differential at y of a $g \in G_y$ has determinant 1. In fact in this "good" case the quotient Y/G has only R.D.P.'s as singularities; hence it is still a candidate for being the canonical model of a surface of general type.

By using this method, Barlow [**Bar 1**] was able to produce a simply connected surface of general type with $p_g = 0$ (Dolgachev, see, e.g., [**Do 1**], had already given an example of an elliptic surface with $\pi_1 = p_g = 0$, against a conjecture of Severi that such a surface should be a rational surface).

3.9. REMARK. The only drawback of Barlow's method of construction is that in this way one usually obtains only proper subvarieties of the moduli space, since the covering does not exist by topological reasons, but by the existence of singular points forming in some sense a special configuration. The reason why we have mainly been focusing on the above two methods is that often the many existing methods (among which the classical method of double multiple planes, introduced by Campedelli and recently applied again with success by several people as Burniat, Oort, Peters (see, e.g., [**O-P**, **Pet 1**]) are based on special configurations which do not exist for the general surface in the moduli space; thus they are useful methods in order to show existence of certain surfaces, but they don't solve the classification problem. Still, the "local" moduli problem can be attacked via deformation theory (cf. [**Ca 4**]), showing that a certain family of surfaces given by an explicit construction does indeed yield a Zariski open set of an irreducible component of the moduli space; but it is harder (cf. [**Ca 5**]) to see what is the closure of this Zariski open set.

3.10. EXAMPLE [**Ba-Ca**]. This is an example where first a construction was made, and then the solution of the "local" moduli problem gave confidence to attack and solve the more difficult ("global") classification problem. Let Σ be a cubic surface with 4 nodes, R the rational surface which is the double cover of Σ ramified only at the 4 nodes, and Z the double cover of Σ branched on the intersection of Σ with a quartic surface G. The fibre product $Y = Z \times_\Sigma R$ is a $(\mathbf{Z}/2)^2$ Galois cover of Σ, and the diagonal action of $\mathbf{Z}/2$ is free on Y, thus giving as quotient a smooth surface $S = Y/(\mathbf{Z}/2)$. S has $K^2 = 6$, $p_g = 3$, $\pi_1 = \mathbf{Z}/2$, and the remarkable feature that ϕ_1 has 4 base points on S, so that ϕ_1 is a (rational) double cover of $\mathbf{P}^2$.

Varying G it was first seen that one had an irreducible reduced rational component of the moduli space: then, looking at the canonical system $|K|$ and the "twisted" canonical system $|K+\eta|$, where $2\eta \equiv 0$, $\eta \not\equiv 0$, it was shown that all surfaces with $K^2 = 6$, $\chi = 4$, and with 2-torsion would occur as above (just letting Σ degenerate to a normal cubic whose equation can be written as the determinant of a symmetric 3×3 matrix of linear forms).

Added in proof. We have just received a letter from G. Xiao sketching the construction of a series of simply connected "hyperelliptic" surfaces of general type with positive index (i.e., $K^2 > 8\chi$), and with the ratio K^2/χ asymptotic to 8.726.

A previous conjecture (by Bogomolov?) to the effect that surfaces with positive index should have infinite fundamental group had previously been disproven by B. Moishezon–M. Teicher ("Simply connected algebraic surfaces with positive index," to appear), but for this we refer to Persson's lecture. Xiao's construction shows in particular also that even when K^2 is rather large, still one can have ϕ_1 not birational.

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