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Surfaces with $K^2 = 2$, $p_g = 1$, $q = 0$

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Introduction

In the classification of algebraic surfaces of general type, one of the most challenging problems is the complete description of the surfaces with small K^2.

Also, surfaces with $K^2 = 1, 2$; $p_g = 1$, $q = 0$ have provided counter-examples to both the local and global Torelli problems ([Ca 1], [Ca 2], [Ch], [To 1], [Us 2]). But, whereas the geometry of surfaces with $K^2 = 1$, $p_g = 1$, $q = 0$ is completely understood ([Ca 1], [To 2]), there is no systematic treatment of the case $K^2 = 2$.

This paper is devoted to the study of the geometry of these surfaces. The results published here have already partly been used in the study of Torelli problems ([Ca 3], [Ol], [S-S-U], [Us 1]).

These surfaces were investigated first by Enriques ([En], p. 316—321) and then Bombieri established some general properties ([Bo], Theorems 6, 8, 15). Special constructions appear also in [Ca 4], [Ch] and [To 1]. However, Enriques for instance claims that the bicanonical map cannot be of degree 4 and therefore overlooks the entire family of surfaces with torsion $\mathbb{Z}/2\mathbb{Z}$. Todorov was the first to construct examples with torsion $\mathbb{Z}/2\mathbb{Z}$ in [To 1].

Our investigation is based on a detailed study of the bicanonical map, thereby following Enriques' original ideas. We first prove that the bicanonical map Φ is a morphism onto a surface Σ in $\mathbb{P}^3$ and we are then led to consider the following 5 cases (each of which does, in fact, occur):

0) Φ has degree 4 and Σ is a quadric cone.

1) Φ has degree 4 and Σ is a smooth quadric.

2) Φ has degree 2 and Σ is a rational quartic.

3) Φ has degree 2 and Σ is a $K3$ quartic.

4) Φ is birational and Σ has degree 8.

Recall also that by Theorem 15 of [Bo], the torsion of $\mathrm{Pic}(S)$ is either 0 or $\mathbb{Z}/2\mathbb{Z}$.

We show that the torsion is $\mathbb{Z}/2\mathbb{Z}$ if and only if the surface is of type 0), and accordingly we examine this case first.

Using methods similar to those employed by Reid in [Re], we prove that these surfaces are quotients of weighted complete intersections of type (4, 4) in the weighted projective space $\mathbb{P}(1, 1, 1, 2, 2)$, by a fixed point-free involution. Proceeding as in [Ca 1], [Ca 2], we prove that their moduli space is an irreducible rational variety of dimension 16, and that their topological fundamental group is $\mathbb{Z}/2\mathbb{Z}$. The surfaces for which the bicanonical map Φ is a Galois covering correspond to an irreducible 12-dimensional subvariety. Todorov's examples quoted above belong to this subfamily.

In case 1), we prove that Φ is a $(\mathbb{Z}/2\mathbb{Z})^2$ Galois covering with branch locus consisting of three curves of respective types (1, 1), (3, 1) and (1, 3). These surfaces form an irreducible unirational family of dimension 11. We prove that they are all topologically simply connected. Their bicanonical map factors through two different double covers of rational surfaces and also through a double cover of a $K3$ surface. Using this property, we prove that case 1) is a specialization of cases 2) and 3).

In case 2), the rational quartic Σ has a double line and, in general, four nodes. It is the image of $\mathbb{P}^2$ under the system of plane quartics having a fixed double point and passing through 4 more given points with preassigned tangents. The surface S is expressed as a double plane branched along a curve of degree 10 with 4 points of type (3, 3) and a quadruple point. These surfaces form an irreducible unirational family of dimension 13.

In case 3), the $K3$ quartic Σ has in general 10 nodes. Its equation is expressible as a symmetric determinant of linear forms (classically called a symmetroid). The surface S is a double cover of Σ branched over the 10 nodes and a hyperplane section (cf. Theorem 3. 15 for a precise statement). These surfaces form an irreducible unirational family of dimension 12.

In cases 0), 1) and 2), the canonical curve of S is hyperelliptic (cf. 1. 3 for a precise definition) but not in case 3). When Φ is a birational, either of the two situations can occur.

Notice that by Kuranishi's theorem, the dimension of the moduli space is at least 16 at each point. It follows that in the zero torsion case, there is an everywhere dense Zariski open subset of the moduli space which corresponds to surfaces for which Φ is birational.

For surfaces of type 4), explicit determinantal equations for the bicanonical image Σ are given in § 4.

This fourth paragraph is devoted to a general study of the canonical ring $\mathscr{R}$ of a surface S. We describe generators and relations for this ring, using a method inspired by [Ca 4] and [Ci 1], and developed in [Ca 5], where it is applied to any regular surface of general type. Since [Ca 5] has already appeared, we will freely refer to it. Here is a rough outline of this method.

Let $\mathscr{A}$ be the ring $\mathbb{C}[W, Y_1, Y_2, Y_3]$ graded by $\deg W=1$, $\deg Y_i=2$ for $i=1, 2, 3$. Then $\mathscr{R}$ is a Cohen-Macaulay $\mathscr{A}$-module and therefore admits a minimal resolution by two free $\mathscr{A}$-modules of rank $h+1=5$ if the canonical curve is not hyperelliptic, 6 otherwise. The determinant of the square matrix (α) giving this resolution is an equation of Σ raised to the power $(\deg \Phi)$.

Using the fact that $\mathscr{R}$ is a commutative $\mathscr{A}$-algebra, we show that the matrix (α) can be chosen to be symmetric and to satisfy the following Rank Condition:

(R.C.) Let (α') be the $h\times(h+1)$ matrix obtained by deleting the first row of (α). Then the ideal I generated by the $(h\times h)$-minors of (α') contains the $(h\times h)$-minors of (α).

Conversely, in cases 3) and 4), the datum of such a matrix (α) determines an $\mathscr{A}$-algebra structure on $\mathscr{R}$. If $X=\operatorname{Proj}(\mathscr{R})$ has at most rational double points as singularities (open condition), it is the canonical model of a surface S. For surfaces of type 4), I is the conductor ideal of Φ (classically called the adjoint ideal), X is the blow-up of Σ along I, and S is the minimal desingularization of X.

We also work out the special forms that the matrix (α) takes in cases 1) to 4).

Now it follows from the smoothness of the moduli space at a point corresponding to a surface of type 1) that there is a unique 16-dimensional irreducible component $\mathscr{M}_{\mathrm{E}}$ of the moduli space which contains surfaces of type 1), 2) and 3). The corresponding surfaces are all topologically simply connected.

In § 5, we give a geometric construction of the surfaces corresponding to the general point of $\mathscr{M}_{\mathrm{E}}$, following [En]. In this case, the surface Σ has ordinary singularities. The ideal I is the ideal of its double curve Γ, of degree 14. There is a surface Ξ of degree 7, again with ordinary singularities, which is a projection of a Del Pezzo surface of degree 7 in $\mathbb{P}^7$ and has Γ as double curve. There is a quartic symmetroid F (adjoint of smallest degree) such that $F\cdot\Xi=2\Gamma$. The surface Ξ is the biadjoint of smallest degree.

There is a 15-dimensional subvariety of $\mathscr{M}_{\mathrm{E}}$ for which the corresponding surfaces have a hyperelliptic canonical curve.

Finally, we consider in § 6 étale double covers S of numerical Godeaux surfaces ($K^2=1$, $p_g=q=0$) with torsion $\mathbb{Z}/2\mathbb{Z}$, the existence of which was shown in [Ba] and [O-P]. The surface S has invariants $K^2=2$, $p_g=1$, $q=0$ and torsion 0. We show that the bicanonical map is birational and that the canonical curve is hyperelliptic. We expect the corresponding family to be irreducible, 8-dimensional and contained in $\mathscr{M}_{\mathrm{E}}$ but could not prove it.

Our initial goal was to prove that the moduli space of our surfaces with torsion zero is *irreducible*, hence equal to $\mathscr{M}_{\mathrm{E}}$. Some of the consequences of the connectedness of $\mathscr{M}_{\mathrm{E}}$ would then be:

— All surfaces with $K^2=2$, $p_g=1$, $q=0$ and torsion zero are topologically simply connected.

— All numerical Godeaux surfaces ($K^2=1$, $p_g=q=0$) with torsion $\mathbb{Z}/2\mathbb{Z}$ have topological fundamental group isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

However, after some unsuccessful efforts (which explains the delay in the publication of this article), we will state this as a conjecture, hoping to return to this matter in a subsequent paper.

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§ 0. Notation

We work over the field of complex numbers $\mathbb{C}$. For any scheme X proper over $\mathbb{C}$, its canonical sheaf will be denoted by ω_X. If X is Gorenstein, ω_X is invertible and an element of $|\omega_X|$ (if any), is called a canonical divisor, usually written as K_X.

For any projective smooth surface X, we write $p_g=h^0(X,\omega_X)$ and $q=h^1(X,\mathcal{O}_X)$.

Throughout this article, S will be a projective smooth minimal surface of general type with $K_S^2=2$, $p_g=1$, $q=0$. It will be shown in § 1 that the bicanonical system $|2K_S|$ is base-point-free. Therefore, the bicanonical map:

$$\Phi : S \longrightarrow \mathbb{P}^3$$

is a morphism; the image of Φ is a surface Σ.

The canonical ring $\mathcal{R}$ of S is the graded ring $\bigoplus_{m\in\mathbb{N}} H^0(S,mK_S)$. It is finitely generated and $X=\operatorname{Proj}(\mathcal{R})$ is the canonical model of S. There is a canonical morphism $\pi : S\longrightarrow X$ through which every pluricanonical morphism factorizes.

The vector space $\mathcal{R}_1\cong H^0(S,K_S)$ is generated by w and we will write:

$$\operatorname{div}(w)=C=\Gamma+Z$$

where Z and Γ are effective and $Z\leqq C$ is maximal such that $K_S\cdot Z=0$.

The vector space $\mathcal{R}_2\cong H^0(S,2K_S)$ is generated by $y_0=w^2$, y_1, y_2 and y_3.

The graded ring $\mathcal{A}$ is the ring $\mathbb{C}[W,Y_1,Y_2,Y_3]$ graded by

$$\deg W=1,\ \deg Y_1=\deg Y_2=\deg Y_3=2.$$

The scheme $\mathbb{P}=\operatorname{Proj}(\mathcal{A})$ is a weighted projective space $\mathbb{P}(1,2,2,2)$ and is isomorphic to $\mathbb{P}^3$.

The natural homomorphism of graded rings $\mathcal{A}\longrightarrow\mathcal{R}$ induces a morphism $\Psi : X=\operatorname{Proj}(\mathcal{R})\longrightarrow\mathbb{P}=\operatorname{Proj}(\mathcal{A})$ and Φ is the composition:

$$S\xrightarrow{\pi}X\xrightarrow{\Psi}\mathbb{P}\cong\mathbb{P}^3.$$

Finally, to any graded $\mathcal{A}$-module $\mathcal{M}$, one can associate ([Gr], 2. 5) a sheaf $\tilde{\mathcal{M}}$ on $\mathbb{P}$.

Segre-Hirzebruch surfaces. The surface $\mathbb{F}_d$ ($d \in \mathbb{N}$) is the rational ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$. It is minimal for $d \neq 1$. To the two quotients $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(d)$ of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)$ correspond two sections of $\mathbb{F}_d \to \mathbb{P}^1$, the images of which are denoted respectively by B_∞ and B_0. If F is a fibre, we have:

$$F \cdot B_0 = F \cdot B_\infty = 1,$$

$$B_\infty \equiv B_0 - dF,$$

$$B_0^2 = d, \quad B_\infty^2 = -d,$$

$$\operatorname{Pic} \mathbb{F}_d \cong \mathbb{Z}[B_0] \oplus \mathbb{Z}[F].$$

Double coverings. If $\pi: X \to Y$ is a finite morphism of degree two between two connected smooth varieties, we will usually write Δ for the branch locus of π (on Y). It is a smooth reduced divisor and we have:

$$\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-\delta) \quad \text{and} \quad \Delta \in |2\delta|.$$

Conversely, given a connected smooth variety Y and a divisor Δ on Y with $\Delta \in |2\delta|$, $\delta \in \operatorname{Div}(Y)$, one can construct a double cover $\pi: X \to Y$ with ramification Δ, where $X = \operatorname{Spec}(\sigma_Y \oplus \sigma_Y(-\delta))$, the algebra structure being given by Δ. We have:

$$X \text{ normal} \Leftrightarrow \Delta \text{ reduced},$$

$$X \text{ smooth} \Leftrightarrow \Delta \text{ smooth}.$$

§ 1. The bicanonical map

Let S be a minimal surface of general type with $K_S^2 = 2$, $p_g = 1$, $q = 0$.

We will write the canonical curve C of S (i.e. the unique element of the linear system $|K_S|$) as $C = \Gamma + Z$ where $KZ = 0$ and either:

a) Γ is irreducible and $K \cdot \Gamma = 2$, or

b) $\Gamma = \Gamma_1 + \Gamma_2$ and $K \cdot \Gamma_1 = K \cdot \Gamma_2 = 1$.

We first show the following basic result.

Proposition 1. 1. *The linear system $|2K_S|$ has no base-point.*

Proof. By Lemma 12 of [Bo], $|2K_S|$ has no fixed part. Therefore, if x is a base-point, we have $x \in C$ (because $2C \in |2K_S|$) and $x \notin Z$ (any component of Z through x would be fixed), hence $x \in \Gamma \backslash Z$.

There is an exact sequence:

(1. 2) $$0 \longrightarrow H^0(S, K_S) \longrightarrow H^0(S, 2K_S) \longrightarrow H^0(C, \omega_C) \longrightarrow 0,$$

so that x is a base-point for $|\omega_C|$.

Assume first that x is smooth on C. Then, by Riemann-Roch, we have

$$h^0(C, \mathcal{O}_C(x)) = 2.$$

Let C_0 be any irreducible component of C which meets the irreducible component of Γ on which x lies. Then there is a non-zero section s of $\mathcal{O}_C(x)$ which vanishes identically on C_0. By Proposition A of [B-C], there exists a decomposition $C=C_1+C_2$, $C_i>0$, with $C_1\cdot C_2\leqq \deg_{C_2}(\mathcal{O}_C(x)\otimes\mathcal{O}_{C_2})\leqq 1$, which contradicts the 2-connectedness of C ([Bo], Lemma 1).

Assume x is singular on C but Γ is reduced. Again, there is a surjection

$$H^0(S, 2K_S-Z)\longrightarrow H^0(\Gamma, \omega_\Gamma)$$

so that x is a base-point for $|\omega_\Gamma|$. It follows from Theorem D of [Ca6] that Γ is reducible, $\Gamma_1\cdot\Gamma_2=1$ and $\Gamma_1\cap\Gamma_2=\{x\}$. But then, for any decomposition $C=C_1+C_2$, one has $C_1C_2>1\geqq(C_1\cdot C_2)_x$ (by 2-connectedness of C), and, by Proposition B ii) of [B-C], x cannot be a base-point for $|\omega_C|$.

If Γ is non-reduced, we write $\Gamma=2\Gamma'$. Consider the blow-up $\pi:\tilde{S}\longrightarrow S$ of x. Let E be the exceptional divisor, $\tilde{\Gamma}'=\pi^*\Gamma'-E$. By Lemma 6 of [Bo], $D=2\tilde{\Gamma}'+\pi^*Z$ is numerically connected; therefore, by Lemma 3 of [Ra], $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}})\cong H^0(D, \mathcal{O}_D)$ and, since $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}})=0$, we obtain $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(-D))=0$. By Serre duality, we get:

$$0=H^1(\tilde{S}, \pi^*(2K_S)-E)=H^1(S, \mathfrak{m}_x\otimes\mathcal{O}(2K_S))$$

and x cannot be a base-point for $|2K_S|$. Q.E.D.

Now, since $h^0(S, 2K_S)=K^2+\chi(\mathcal{O}_S)=4$ ([Bo], page 185), the bicanonical system defines a morphism $\Phi: S\longrightarrow \mathbb{P}^3$ onto a surface Σ such that $\deg\Sigma\cdot\deg\Phi=8$. Moreover, the surface Σ has irregularity 0 and geometric genus 0 or 1. Therefore, we are in one of the following cases, each of which will be shown later to occur:

0) The morphism Φ is 4-to-1 onto a quadric cone.

1) The morphism Φ is 4-to-1 onto a smooth quadric.

2) The morphism Φ is 2-to-1 onto a rational quartic.

3) The morphism Φ is 2-to-1 onto a $K3$ quartic.

4) The morphism Φ is one-to-one onto an octic surface.

The morphism Φ induces a morphism $\varphi: C\longrightarrow\mathbb{P}^2$ with image C', which contracts each connected component of Z to a point. The induced map $\Gamma\longrightarrow C'$ is finite. Since φ is the morphism associated to the canonical sheaf ω_C, we set the following definition.

Definition 1. 3. We shall say that C is hyperelliptic if for a general point y of $C'=\Phi(C)$, the scheme theoretic fibre $\Phi^{-1}(y)$ has length two.

Proposition 1. 4. *The canonical curve C is hyperelliptic if and only if there exists an irreducible quadric Q in $\mathbb{P}^3$ such that $\Phi^*Q\geqq C$.*

Proof. We start with two remarks:

i) Let X be the canonical model of S. There is a factorization (cf. Notation and [Bo], page 176):

$$\Phi: S\xrightarrow{\pi} X\xrightarrow{\Psi}\Sigma\subset\mathbb{P}^3,$$

and $\pi_* \Gamma \in |\omega_X|$, $\pi^* \omega_X \cong \omega_S$ hence $\pi^*(\pi_* \Gamma) = C$. Therefore, if H is any divisor in $\mathbb{P}^3$ such that $\Phi^*(H) \geqq \Gamma$, we have:

$$\Phi^* H \geqq \Gamma \Rightarrow \Psi^* H \geqq \pi_* \Gamma$$
$$\Rightarrow \Phi^* H \geqq \pi^*(\pi_* \Gamma) = C.$$

ii) If Γ is non-reduced, we write $C = 2\Gamma' + Z$. By the Index Theorem, $(\Gamma')^2 \leqq 0$, and we are in one of the following two cases:

a) $p(\Gamma') = 0$, and $\varphi|_{\Gamma'}$ is an isomorphism onto a conic,

b) $p(\Gamma') = 1$, and $\varphi|_{\Gamma'}$ is a double cover of a line.

Now, assume there is an irreducible quadric Q such that $\Phi^* Q \geqq C$. Let Π be the plane corresponding to the element y_0 of $H^0(S, 2K_S)$ with divisor $2C$, and let C'' be the scheme theoretic intersection $(Q \cdot \Pi)$. Since Q is irreducible, C'' is a curve of degree 2 in Π. For a general hyperplane H in $\mathbb{P}^3$, $H \cdot C''$ is a scheme of length 2, while

$$\Phi^{-1}(H \cdot C'')$$

has as subscheme $\Phi^*(H) \cdot C$, which is of length four. Since $C' \subset C''$, we have proved that C is hyperelliptic in the sense of Definition 1.

Conversely, if C is hyperelliptic, $\deg C'$ is less than 2. Let Q be an irreducible quadric distinct from Σ such that $C'' = Q \cdot \Pi$ is either C' or $2C'$ if C' is a line. In case Γ is reduced, it is clear that $\Phi^*(Q) \geqq \Gamma$ hence $\Phi^*(Q) \geqq C$ by Remark i) above. If $\Gamma = 2\Gamma'$, we are in case ii) above. In case ii) a), C is hyperelliptic if and only if C' is contained in the ramification locus of Φ, hence $\Phi^*(Q) \geqq 2\Gamma'$. In case ii) b), C' is a line hence $Q \cdot \Pi = 2C'$ and $\Phi^*(Q) \geqq 2\Gamma'$. Q.E.D.

Remark 1.5. i) If C is not hyperelliptic, $\deg \Phi \leqq 2$ and the only quadrics Q such that $\Phi^* Q \geqq C$ are the ones which are divisible by y_0.

ii) If C is hyperelliptic, the affine space of quadrics Q such that $\Phi^* Q \geqq C$ has dimension exactly 4.

Proof. Complete y_0 to a basis $\{y_0, y_1, y_2, y_3\}$ of $H^0(S, 2K_S)$. Then one can always add to Q a product $y_0 \cdot y$ with $y \in H^0(S, 2K_S)$. But if the dimension were bigger than 4, one would get two independent quadratic forms $q_1(y_1, y_2, y_3)$, $q_2(y_1, y_2, y_3)$ such that $q_i \geqq C'$. Therefore q_1 and q_2 would be reducible, say $q_1 = y_1 y_2$, and $C' = \{y_0 = y_1 = 0\}$. Then $\dim(\Gamma \cap \Phi^*(y_2)) = 0$ hence $\Phi^*(q_1) \geqq C \Rightarrow \Phi^*(y_1) \geqq \Gamma \Rightarrow \Phi^*(y_1) \geqq C$. We would obtain $y_0, y_1 \in H^0(2K - C)$, which is absurd. Q.E.D.

Finally, we recall here that the torsion group of the Picard group of S (which we shall often call the torsion group of S) is, by Theorem 15 of [Bo], either $\{0\}$ or $\mathbb{Z}/2\mathbb{Z}$.

We begin with the following characterization of case 0) of the above list.

Proposition 1.5. *The surface Σ is a quadric cone if and only if the torsion group of S is $\mathbb{Z}/2\mathbb{Z}$.*

Proof. If $\eta \in \mathrm{Pic}(S)$ is such that $\eta \not\equiv 0$, $2\eta \equiv 0$, let $\tilde{S} \xrightarrow{p} S$ be the associated unramified double covering. We have $K_{\tilde{S}}^2 = 4$, $\chi(\mathcal{O}_{\tilde{S}}) = 4$, therefore $q(\tilde{S}) = 0$ by Lemma 14 of [Bo].

Then $H^1(K_S+\eta)=H^1(\eta)=0$, and, by R.R., $h^0(K+\eta)=2$. Let $\{x_1, x_2\}$ be a basis for $H^0(K_S+\eta)$. One can complete $\{x_1^2, x_1x_2, x_2^2\}$ to a basis for $H^0(2K_S)$ and clearly Σ is a quadric cone.

Conversely, let Σ be a quadric cone, and v its vertex: composing Φ with the projection with center v we obtain a rational map from S to $\mathbb{P}^1$, hence a pencil M without fixed part such that $2K_S \equiv 2M+G$.

Since $\Phi(G)=v$, G is a sum of irreducible curves Z_i with $KZ_i=0$, $Z_i^2=-2$. We can write G uniquely as $2G'+Z_1+\cdots+Z_r$, where $Z_i \neq Z_j$ for $i \neq j$. Let $\Delta=Z_1+\cdots+Z_r$. Then $\Delta \equiv 2\delta$, where $\delta \equiv K_S-M-G'$.

Let $\Delta'=Z_1+\cdots+Z_s$ be a connected component of Δ. We have

$$\Delta' \cdot Z_i = \Delta \cdot Z_i \equiv 0 \pmod 2.$$

If $s \geqq 2$, since Δ' is connected and reduced, we also have $(\Delta'-Z_i)\cdot Z_i>0$ for $i=1,\ldots,s$, hence $\Delta'\cdot Z_i \geqq 0$ and $(\Delta')^2=\Delta'\cdot\left(\sum_{i=1}^{s} Z_i\right)\geqq 0$. Since $\Delta'\cdot K_S=0$, this contradicts the Index Theorem. Therefore $s=1$ and we have proven that Δ is a disjoint union of smooth curves $Z_1,\ldots,Z_r$.

We can therefore consider the smooth double cover $\hat{S}'$ of S in δ branched over the smooth curve Δ and the surface $\hat{S}$ obtained from $\hat{S}'$ by contracting the r exceptional curves of the first kind lying over the Z_i's. We have:

$$\begin{cases} K_{\hat{S}}^2 = r+2(K_S+\delta)^2=4, \\ \chi(\mathcal{O}_{\hat{S}})=4+\dfrac{1}{2}\,\delta(K_S+\delta)=4-\dfrac{r}{4}, \\ p_g(\hat{S})=h^0(K_S+\delta)+h^0(K_S)=h^0(M+G'+\Delta)+1\geqq 3. \end{cases}$$

Since $K^2<2p_g$ for $\hat{S}$, it follows from [De], Theorem 6.1, that $q(\hat{S})=0$, therefore $\chi(\mathcal{O}_{\hat{S}})=1+p_g(\hat{S})=4-\dfrac{r}{4}\geqq 4$. It follows that $r=0$, $p_g(\hat{S})=3$, hence $\hat{S}$ is a connected unramified double covering of S, and δ is a generator of the 2-torsion of S. Q.E.D.

§ 2. Surfaces with torsion $\mathbb{Z}/2\mathbb{Z}$ such their moduli

In this paragraph, we study the surfaces S for which the Picard group has torsion $\mathbb{Z}/2\mathbb{Z}$. Todorov was the first to construct examples of such surfaces ([To 1], p. 303, Remark 2) but some of his assertions are inaccurate (Lemma 2.5 of loc. cit.). We have already proven in 1.5 that this is exactly the case when the bicanonical image Σ is a quadric cone. We shall prove that the canonical model of such a surface can be written as the quotient of a complete intersection V in a weighted projective space $\mathbb{P}(1,1,1,2,2)$ by a fixed point-free involution. Explicit „canonical“ equations for V will allow us to prove that the moduli space for these surfaces S is an irreducible 16-dimensional rational variety. Their topological fundamental group is $\mathbb{Z}/2\mathbb{Z}$. We shall also study the surfaces for which the covering $S \to \Sigma$ is Galois.

Let $p: \tilde{S} \to S$ be the double étale covering associated to the unique element η of order 2 in $\operatorname{Pic}(S)$ and let τ be the involution induced on $\tilde{S}$ by p.

The canonical ring $\tilde{\mathscr{R}}$ of $\tilde{S}$ splits into eigenspaces for τ^* as $\tilde{\mathscr{R}} = \tilde{\mathscr{R}}^+ \oplus \tilde{\mathscr{R}}^-$, where:

$$\tilde{\mathscr{R}}^+ = \bigoplus_{m=0}^{\infty} \tilde{\mathscr{R}}_m^+ \cong \bigoplus_{m=0}^{\infty} H^0(mK_S) \cong \mathscr{R},$$

$$\tilde{\mathscr{R}}^- = \bigoplus_{m=0}^{\infty} \tilde{\mathscr{R}}_m^- \cong \bigoplus_{m=0}^{\infty} H^0(mK_S + \eta).$$

Let w be a generator of $\tilde{\mathscr{R}}_1^+$, $\{x_1, x_2\}$ a basis of $\tilde{\mathscr{R}}_1^-$. Complete $\{wx_1, wx_2\}$ to a basis of $\tilde{\mathscr{R}}_2^-$ by adding $\{z_3, z_4\}$. Note that, by R.R., $\dim \tilde{\mathscr{R}}_m^+ = \dim \tilde{\mathscr{R}}_m^- = 2 + m(m-1)$ for $m \geqq 2$.

Lemma 2. 1. *$\{w^2, x_1^2, x_1x_2, x_2^2\}$ is a basis of $\tilde{\mathscr{R}}_2^+$.*

Proof. By virtue of the exact sequence (1. 2) it suffices to prove that $x_1 \cdot x_2|_C$ cannot be identically zero (since any quadratic form in two variables is a product of two linear forms). Choose $C_i \leqq C$ maximal such that $x_i|_{C_i} \equiv 0$. By Proposition A of [B-C], $C_i(C-C_i) \leqq (K+\eta)\cdot(C-C_i)$. But C is 2-connected, therefore $2 \leqq C_i(C-C_i)$, hence

$$4 \leqq K(2C - C_1 - C_2) = 4 - K(C_1 + C_2).$$

We conclude that $KC_i = 0$, and in particular $x_1x_2|_C \not\equiv 0$. Q.E.D.

Remark 2. 2. Lemma 2. 1 also ensures that $C' = \Phi(C)$ is a smooth conic.

Lemma 2. 3. *$\{w^3, wx_1, x_2, x_1^2w, wx_2^2, x_1z_3, x_1z_4, x_2z_3, x_2z_4\}$ is a basis of $\tilde{\mathscr{R}}_3^+$.*

Proof. By the exact sequence:

$$0 \longrightarrow H^0(2K_S) \longrightarrow H^0(3K_S) \longrightarrow H^0(\mathcal{O}_C(3K_S)) \longrightarrow 0,$$

it suffices to prove that if

$$\left(\sum_{\substack{i=1,2\\ j=3,4}} \lambda_{ij} x_i z_j\right)\Bigg|_C \equiv 0,$$

then the λ_{ij}'s are all zero.

Notice that $\{x_1, x_2\}$ restricts to a basis of $H^0(\mathcal{O}_C(K+\eta))$, $\{z_3, z_4\}$ to a basis of $H^0(\mathcal{O}_C(2K+\eta))$. By changing bases, if necessary, we can assume that our relation is either

a) $x_1z_3|_C \equiv 0$

or

b) $(x_1z_3 + x_2z_4)|_C \equiv 0$.

In case a), z_3 would vanish identically on Γ, since we saw in Lemma 2. 1 that x_1 vanishes only on components C_1 with $K \cdot C_1 = 0$.

Taking $C_1 \leqq C$ maximal where z_3 vanishes, we apply again Proposition A of [B-C]. Since $C_1 \geqq \Gamma$, we have $KC_1=2$ and $C_1 \cdot (C-C_1) \leqq (2K+\eta)(C-C_1)=0$. Therefore $C_1=C$ and w divides z_3, absurd.

In case b), observe that $H^0(C, \mathcal{O}_C(K_S))=0$.

But, if $x_1 z_3 + x_2 z_4|_C \equiv 0$, since x_1 and x_2 have no common zeros on C, $z_3/x_2 = -z_4/x_1$ would define a regular section of $\mathcal{O}_C(K_S)$, a contradiction. Q.E.D.

Lemma 2. 4. *$\{x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, w^2 x_2, wz_3, wz_4\}$ form a basis of $\tilde{\mathcal{R}}_3^-$.*

Proof. By virtue of the following exact sequence

$$0 \longrightarrow H^0(2K_S+\eta) \longrightarrow H^0(3K_S+\eta) \longrightarrow H^0(\mathcal{O}_C(3K_S+\eta)) \longrightarrow 0. \qquad \text{Q.E.D.}$$

Let $\mathcal{R}'$ be the ring $\mathbb{C}[w', x_1', x_2', z_3', z_4']$ graded in such a way that

$$\deg w' = \deg x_i' = 1, \quad \deg z_j' = 2.$$

Let $\mathbb{P} = \mathbb{P}(1, 1, 1, 2, 2)$ (see [Dl], [Do]) be $\operatorname{Proj}(\mathcal{R}')$. The homomorphism $f^*: \mathcal{R}' \longrightarrow \tilde{\mathcal{R}}$ such that $f^*(w')=w$, $f^*(x_i')=x_i$, $f^*(z_j')=z_j$, defines a rational map $f: \tilde{S} \longrightarrow \mathbb{P}$; and the involution $(\tau')^*: \mathcal{R}' \longrightarrow \mathcal{R}'$ defined by $(\tau')^*(w')=w'$ and $(\tau')^*(x_i')=-x_i'$, $(\tau')^*(z_j')=-z_j'$ induces an involution τ' of $\mathbb{P}$ such that $f \circ \tau = \tau' \circ f$.

Proposition 2. 5. *The map $f: \tilde{S} \longrightarrow \mathbb{P}$ is a morphism, birational onto its image.*

Proof. We already know that f is a morphism, by (1. 1). Actually f is birational since, by [Bo], Theorem 4, the tricanonical map of $\tilde{S}$ is birational, and $f^*: \mathcal{R}_m' \longrightarrow \tilde{\mathcal{R}}_m$ is an isomorphism for $m \leqq 3$. Q.E.D.

Observe that $\dim(\mathcal{R}')_4^+ = 16$, $\dim \tilde{\mathcal{R}}_4^+ = 14$; therefore f^* has at least a 2-dimensional kernel in $(\mathcal{R}')_4^+$. Let F, G be two independent polynomials in $(\mathcal{R}')_4^+ \cap \ker f^*$, and V be the complete intersection in $\mathbb{P}$ defined scheme theoretically by the two hypersurfaces F, G. It is clear that $f(\tilde{S}) \subset V$, but in fact we have:

Theorem 2. 6. *The scheme V is isomorphic to the canonical model of $\tilde{S}$ via $f: \tilde{S} \longrightarrow V$, hence $\mathcal{R} \cong \mathcal{R}'/(F, G)$.*

Proof. V is of pure dimension 2, since, if F, G had a common factor, then $f^*: \mathcal{R}_m' \longrightarrow \mathcal{R}_m$ would not be an isomorphism for $m \leqq 3$. Since f is birational, $f(\tilde{S})$ and V have the same degree, hence $f(\tilde{S}) \subset V$ implies $f(\tilde{S}) = V$. Finally, since $\omega_V = \mathcal{O}_V(1)$ by Proposition 3. 3 of [Mo] and $f^*(\omega_V) = K_{\tilde{S}}$, V has only rational double points as singularities and is the canonical model of $\tilde{S}$. Q.E.D.

Remark 2. 7. Conversely, if $V \subset \mathbb{P}$ is a (weighted) complete intersection of type (4, 4) with at most rational double points, then its minimal desingularization $\tilde{S}$ has $K_{\tilde{S}}^2 = 4$. If moreover $V = \{F = G = 0\}$ and $F, G \in (\mathcal{R}_4')^+$, then the involution τ' preserves V and extends to $\tilde{S}$. If τ is free from fixed point, then $\tilde{S}/\tau = S$ is one of our surfaces. Notice that τ on $\tilde{S}$ is free if and only if V does not intersect the fixed locus of τ', i.e., the point (1, 0, 0, 0, 0) and the lines $w' = z_3' = z_4' = 0$ and $w' = x_1' = x_2' = 0$. In fact it is easy to show that any involution preserving a fundamental cycle has a fixed point there (look at the associated Dynkin diagram first, then remember that an involution on $\mathbb{P}^1$ always has a fixed point).

Assume moreover that there exists an isomorphism $\varphi: S_1 \xrightarrow{\sim} S_2$. Then φ determines $\tilde{\varphi}: \tilde{S}_1 \xrightarrow{\sim} \tilde{S}_2$ such that $\tilde{\varphi} \circ \tau_1 = \tau_2 \circ \tilde{\varphi}$, hence there exists $\varphi \in \operatorname{Aut}(\mathbb{P})^{\tau'}$ such that Ψ gives an isomorphism of V_1 with V_2, and the converse also holds.

In what follows, we are going to use almost the same arguments used in [Ca 2]: we want to put the pairs of equations F, G in a canonical form in order to have precise information about the moduli of our surfaces. We are also going to drop the prime ′ for the coordinates in $\mathbb{P}$.

Theorem 2. 8. *Any canonical model X of a surface S with $K_S^2=2$, $q=0$, $p_g=1$, torsion $\mathbb{Z}/2\mathbb{Z}$ occurs as the quotient V/τ' of a weighted complete intersection $V \subset \mathbb{P}$ with only rational double points as singularities, given by a pair of canonical equations*:

$$\begin{cases} F = z_3^2 + w z_4 l(x_1, x_2) + b w^4 + w^2 q(x_1, x_2) + B(x_1, x_2) = 0, \\ G = z_4^2 + w z_3 l'(x_1, x_2) + b' w^4 + w^2 q'(x_1, x_2) + B'(x_1, x_2) = 0, \end{cases}$$

where l, l' are linear forms, q, q' quadratic forms, B, B' quartic forms without common factors, and b, b' are constants not both zero. Moreover two such pairs of canonical equations (F_1, G_1), (F_2, G_2) give rise to isomorphic surfaces if and only if there exists a projectivity $h: \mathbb{P} \to \mathbb{P}$ such that

i) $h(w, x_1, x_2, z_3, z_4) = (dw, d_{11}x_1 + d_{12}x_2, d_{21}x_1 + d_{22}x_2, c_{33}z_3, c_{44}z_4)$,

ii) *either* $c_{33}^2 F_2 = F_1 \circ h$, $c_{44}^2 G_2 = G_1 \circ h$, *or* $c_{33}^2 F_2 = G_1 \circ h \circ i$, $c_{44}^2 G_2 = F_1 \circ h \circ i$, *where $i: \mathbb{P} \to \mathbb{P}$ is the involution permuting z_3 and z_4.*

Proof. Since V does not intersect the line $\{w = x_1 = x_2 = 0\}$, and F, G are in $(\mathscr{R}_4')^+$, which is spanned by the monomials $z_j z_k$, $z_j w x_i$, w^4, $w^2 x_i x_j$, $x_1^r x_2^{4-r}$, we see that, if we write $F = Q_1'(z_3, z_4) +$ (terms of degree ≤ 1 in z_3, z_4), and $G = Q_2'(z_3, z_4) + \cdots$, then Q_1', Q_2' do not have a common factor.

Therefore the pencil $\lambda Q_1' + \mu Q_2'$ gives a double cover $\mathbb{P}^1 \to \mathbb{P}^1$ with 2 distinct ramification points, i.e. there are two distinct members Q_1, Q_2 of the pencil and new coordinates (z_3, z_4), such that $Q_1 = z_3^2$, $Q_2 = z_4^2$.

Then, by a transformation of the type:

$$\begin{cases} z_3 \to z_3 + a_{31} w x_1 + a_{32} w x_2, \\ z_4 \to z_4 + a_{41} w x_1 + a_{42} w x_2 \end{cases}$$

one can put F, G in the desired form.

By Remark 2. 7, assume that there exists $\Psi \in \operatorname{Aut}(\mathbb{P})^{\tau'}$ such that the ideal (F_2, G_2) is the same as the ideal $(F_1 \circ \Psi, G_1 \circ \Psi)$.

Then Ψ is a projectivity of the form:

$$\begin{cases} w \longrightarrow dw, \\ x_i \longrightarrow \sum_{j=1}^{2} d_{ij} x_j, \\ z_h \longrightarrow \sum_{k=3}^{4} c_{hk} z_k + \sum_{j=1}^{2} a_{hj} w x_j. \end{cases}$$

If (F_1, G_1), (F_2, G_2) are in canonical form, then $\begin{pmatrix} c_{33} & c_{34} \\ c_{43} & c_{44} \end{pmatrix}$ gives an automorphism of $\mathbb{P}^1$ which preserves the pencil $\lambda z_3^2 + \mu z_4^2$. Therefore either $0, \infty$ are fixed or they are permuted.

Hence, we can assume, considering if necessary $(F_2 \circ i, G_2 \circ i)$ instead of (F_2, G_2), that $c_{34} = c_{43} = 0$, and that $F_1 \circ \Psi = c_{33}^2 F_2$, $G_1 \circ \Psi = c_{44}^2 G_2$.

We want to show that $a_{hj} = 0$ $(h = 3, 4,\ j = 1, 2)$, but this is an immediate consequence of the fact that F_2 does not contain any monomial divisible by wz_3 and that G_2 contains no monomial divisible by wz_4. The above conditions on b, b', B, B' ensure that V does not meet the fixed points of τ'. Q.E.D.

Theorem 2. 9. *In the notations of Theorem* 2. 8, $\Phi: S \to \Sigma$ *is a Galois covering if and only if one can choose* $l \equiv l' \equiv 0$. *The Galois group is* $(\mathbb{Z}/2\mathbb{Z})^2$.

Proof. If Φ is a Galois covering, since its restriction to C is a double cover of a conic (2. 2), the Galois group has an image of order 2 in $\mathrm{Aut}(\Gamma)$. Hence there exists an automorphism σ of order 2 leaving Γ pointwise fixed. The action of the Galois group lifts to $\tilde{S}$, hence to $\mathbb{P}$, and clearly commutes with τ. We have therefore an action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{P}$ (or on $\mathscr{R}'$) such that:

i) w is an eigenvector,

ii) $\mathscr{R}_2'^{+}$, $\mathscr{R}_2'^{-}$, $\mathscr{R}_1'^{-}$ are invariant subspaces,

iii) the action on $\mathscr{R}_2'^{+}$ is trivial.

We can therefore choose x_1, x_2, z_3, z_4 to be eigenvectors and, by iii), x_1, x_2 correspond to the same character of $(\mathbb{Z}/2\mathbb{Z})^2$. Since the projectivity $w \mapsto -w$, $x_i \mapsto -x_i$, $z_j \mapsto z_j$ is trivial, and we can assume σ to leave the double cover of Γ pointwise fixed, our group must consist of the projectivities $x_i \mapsto x_i$, $w \mapsto \varepsilon w$, $z_j \mapsto \varepsilon' z_j$ with $\varepsilon, \varepsilon' = \pm 1$. Since the ideal of V is invariant by the group action, we can assume that F, G are eigenvectors; this can happen, since z_3^2, z_4^2 are invariant monomials, if and only if $l \equiv l' \equiv 0$. But in this last case, $(\mathbb{Z}/2\mathbb{Z})^3$ acts by $x_i \mapsto x_i$, $w \mapsto \varepsilon w$, $z_j \mapsto \varepsilon_j z_j$ with a trivial action on $\mathscr{R}_2'^{+}$, hence the Galois group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Q.E.D.

Remark 2. 10. Let us explain the geometrical meaning of Th. 2. 9. Incidentally we will also prove that a general V is smooth so that our family is not empty. Assume that $l \equiv l' \equiv 0$, and consider that the Galois group action on V is generated by the 2 involutions $z_3 \mapsto -z_3$, $z_4 \mapsto -z_4$. Then $\Phi: S \to \Sigma$ is given by $y_0 = w^2$, $y_1 = x_1^2$, $y_2 = x_2^2$, $y_3 = x_1 x_2$.

It is then easy to see that the branch locus of Φ consists of the 3 images of the fixed loci of the 3 non-trivial involutions of the Galois group:

$$\{z_3 = 0\},\ \{z_4 = 0\},\ \{w = 0\} \cup \{x_1 = x_2 = 0\}.$$

Let β be a homogeneous polynomial of degree two such that:

$$\beta(w^2, x_1^2, x_2^2, x_1 x_2) = b w^4 + w^2 q(x_1, x_2) + B(x_1, x_2),$$

and analogously define β'.

Then the branch locus of Φ on Σ consists of:

$$\{\beta=0\},\ \{\beta'=0\},\ \{y_0=0\}\cup\{y_1=y_2=y_3=0\}.$$

The surface S can be obtained in 3 ways, one of which expresses a general S as a double cover of a $K3$ surface with 10 "even" nodes. Assume that the 3 curves $\{y_0=0\}=H$, β, β' have only transversal intersections, and do not pass through the vertex v of Σ.

Take in $\mathcal{O}_\Sigma(2)$ the double covering of Σ branched over $\beta+\beta'$; you get two nodes lying over v, and 8 nodes coming from the 8 intersections of β and β'; we get thus a $K3$ surface M with a morphism $M \xrightarrow{p} \Sigma$ such that M admits a double cover S ramified on $p^*(H)$ and on the 10 nodes. S is smooth and K_S is ample, therefore the corresponding V is clearly smooth (cf. the proof of Theorem 3. 12, which still applies in case Σ is a quadratic cone). Q.E.D.

Theorem 2. 11. *Surfaces S with $K^2=2$, $q=0$, $p_g=1$, and torsion $\mathbb{Z}/2\mathbb{Z}$ have fundamental group isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and are all diffeomorphic. Their moduli space is an irreducible rational variety $\mathcal{M}_T$ of dimension* 16.

Proof. The assertion on the fundamental group results from the fact that if V is smooth, by Lefschetz's Theorem ([Do]), V is simply connected. By Theorem 2. 8, all the canonical models of our surfaces appear as fibres of a flat family with base a Zariski open set in an affine space, hence they are diffeomorphic by G. Tjurina's result ([Tj]), and their coarse moduli space $\mathcal{M}_T$, which is a quasi-projective variety by ([Gi]), is irreducible. Consider the 16-dimensional family:

$$\begin{cases} F=z_3^2+wz_4x_1+w^4+w^2q(x_1,x_2)+B(x_1,x_2)=0,\\ G=z_4^2+wz_3x_2+w^4+w^2q'(x_1,x_2)+B'(x_1,x_2)=0\end{cases}$$

such that (q, q', B, B') lie in a Zariski open subset $\mathcal{U}$ of the 16-dimensional affine space defined by the conditions on F, G mentioned in Theorem 2. 8. Assume that

$$(F_1, G_1), (F_2, G_2)\in\mathcal{U}$$

are equivalent under a projectivity h as in Theorem 2. 8.

Since the projectivity

$$\begin{pmatrix} d_{11} & d_{12}\\ d_{21} & d_{22}\end{pmatrix}: \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

preserves the 2 points $\{x_1=0\}$, $\{x_2=0\}$ by 2. 8. ii), h must be given by a diagonal matrix. Conditions ii) in 2. 8 read out as follows:

$$\begin{cases} c_{33}^2=dc_{44}d_{11},\ c_{33}^2=d^4,\\ c_{44}^2=dc_{33}d_{22},\ c_{44}^2=d^4,\end{cases} \Leftrightarrow \begin{cases} d_{11}^2=d_{22}^2=d^2,\\ d_{11}c_{44}=d^3,\\ d_{22}c_{33}=d^3,\end{cases}$$

$$q_2=c_{33}^{-2}d^2q_1(d_{11}x_1, d_{22}x_2)$$

i.e.

$$q_2(x_1,x_2)=q_1(\pm x_1, \pm x_2),$$

and analogously for q', B, B'.

On the other hand, if h is the projectivity which exchanges x_1 and x_2, and i is the involution which exchanges z_3 and z_4, then, according to 2. 8. ii), $(F \circ h \circ i, G \circ h \circ i)$ gives an isomorphic surface, given by

$$\begin{cases} z_3^2 + w z_4 x_1 + w^4 + w^2 q(x_2, x_1) + B(x_2, x_1) = 0, \\ z_4^2 + w z_3 x_2 + w^4 + w^2 q'(x_2, x_1) + B'(x_2, x_1) = 0. \end{cases}$$

Therefore the image of $\mathscr{U}$ in $\mathscr{M}_T$ (which is a Zariski open set in $\mathscr{M}_T$) is birational to the quotient of $\mathbb{C}^{16}$ by a linear action of $(\mathbb{Z}/2\mathbb{Z})^3$, and therefore $\mathscr{M}_T$ is rational. Q.E.D.

§ 3. Surfaces with torsion 0 for which the bicanonical map is not birational

In this paragraph, we will give geometric descriptions of the surfaces S with no torsion and for which the bicanonical map Φ is not birational. According to the results of § 1, we are in one of the three following cases:

1) Φ is of degree 4 onto a smooth quadric,

2) Φ is of degree 2 onto a rational quartic,

3) Φ is of degree 2 onto a $K3$ quartic.

Using these descriptions, we will also prove that the corresponding subvarieties $\mathscr{M}_1$, $\mathscr{M}_2$ and $\mathscr{M}_3$ of the moduli space $\mathscr{M}$ are irreducible and unirational of respective dimension 11, 13, and 12, that $\mathscr{M}$ is smooth at a general point of $\mathscr{M}_1$, and that $\mathscr{M}_1 \subset \bar{\mathscr{M}}_2 \cap \bar{\mathscr{M}}_3$.

The end of the paragraph will be devoted to the construction of a common flat degeneration of surfaces of type 2) and of certain surfaces of type 0) (studied in § 2) to non-normal rational surfaces.

We begin with the analysis of case 1).

Proposition 3. 1. *Suppose that the image Σ of the bicanonical map of S is a smooth quadric. Then S is a Galois covering of Σ with group $(\mathbb{Z}/2\mathbb{Z})^2$, branched along $\Delta = \Delta_1 + \Delta_2 + \Delta_3$, where Δ is reduced with at most double points as singularities, Δ_1 is of type* (3, 1), *Δ_2 of type* (1, 3), *Δ_3 of type* (1, 1).

Conversely, if Δ is a curve with the above decomposition and properties, then, taking the double cover S_k of the quadric ramified along $\Delta_i + \Delta_j$, S is the minimal resolution of the double cover of S_k branched along the pullback of Δ_k on S_k, and the nodes of S_k lying over $\Delta_i \cap \Delta_j$ (where $\{1, 2, 3\} = \{i, j, k\}$).

Proof. Let F_1 and F_2 be the inverse images on S of the two pencils of lines of Σ. We have

$$2K \equiv F_1 + F_2, \quad F_1 \cdot F_2 = 4, \quad F_1^2 = F_2^2 = 0.$$

Therefore $KF_1 = KF_2 = 2$ and $|F_1|$ and $|F_2|$ are two pencils of genus 2 curves without base points. The hyperelliptic involutions on the fibers induce two involutions σ_1 and σ_2 on S, with S/σ_1 and S/σ_2 ruled.

We first show that σ_1 and σ_2 are distinct. If $\sigma_1=\sigma_2=\sigma$, the surface $T=S/\sigma$ is ruled with two distinct rulings $G_1=F_1/\sigma$, $G_2=F_2/\sigma$, which satisfy:

$$\begin{cases} h^0(T, \mathcal{O}_T(G_i))=2 \text{ because } h^0(S, \mathcal{O}_S(F_i))=2, \\ K_T \cdot G_i=-2, \text{ by the genus formula.} \end{cases}$$

The R.R. theorem on T gives $H^1(T, \mathcal{O}_T(G_i))=0$ and there is an exact sequence:

$$0 \longrightarrow H^0(T, \mathcal{O}_T(G_1)) \longrightarrow H^0(T, \mathcal{O}_T(G_1+G_2)) \longrightarrow H^0(G_2, \mathcal{O}_{G_2}(G_1+G_2|_{G_2})) \longrightarrow 0.$$

But this is impossible, since:

$$\begin{aligned} h^0(T, \mathcal{O}_T(G_1+G_2)) &\leqq h^0(S, \mathcal{O}_S(F_1+F_2))=4, \\ h^0(G_2, \mathcal{O}_{G_2}(G_1+G_2|_{G_2})) &= h^0(G_2, \mathcal{O}_{G_2}(2))=3, \end{aligned}$$

because G_2 is rational smooth and $G_1 \cdot G_2=\frac{1}{2} F_1 \cdot F_2=2$. Hence σ_1 and σ_2 are distinct.

Now, a general fiber $\Phi^{-1}(\Phi(x))$ has four points, including x, $\sigma_1(x)$, $\sigma_2(x)$. Therefore, the last one is $\sigma_1\sigma_2(x)=\sigma_2\sigma_1(x)$ and Φ is a Galois covering with group

$$\{1, \sigma_1, \sigma_2, \sigma_1\sigma_2\} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Notice that, if w is a generator for $H^0(S, K_S)$, w is anti-invariant by σ_1^* and σ_2^*, because $p_g(S/\sigma_1)=p_g(S/\sigma_2)=0$, hence it is invariant by σ_3^*.

Let $\operatorname{Fix}\sigma_i$ be the pointwise fixed (smooth reducible) curve of σ_i, Δ_i its image by Φ. Let R be a component of $\operatorname{Fix}\sigma_3$. We can assume that, in local coordinates (x, y), we have:

$$\sigma_3(x, y)=(x, -y), \quad R=\{y=0\}, \quad w=f(x, y)\,dx \wedge dy.$$

Since $w=\sigma_3^* w=-f(x, -y)\,dx \wedge dy$, $w|_R=0$, i.e., $R \leqq C$. But the same argument shows that no component of C can be in $\operatorname{Fix}\sigma_1$ or $\operatorname{Fix}\sigma_2$. But, by Lemma 1.7, C is hyperelliptic, which proves that C' is in the branch locus of Φ. Therefore, we have proved that $\Delta_3=C'$, and, by the way, that $\operatorname{Fix}\sigma_i$ and $\operatorname{Fix}\sigma_j$ have no common components for $i \neq j$, and neither have Δ_i and Δ_j. Let us denote by Δ the curve $\Delta_1+\Delta_2+\Delta_3$. It is clear that the branch locus of $S/\sigma_i \overset{\pi_i}{\dashrightarrow} \Sigma$ is $(\Delta-\Delta_i)$.

Now, by construction, the inverse images on S/σ_1 of the fibres of type $(0, 1)$ are a ruling of S/σ_1. Therefore:

$$-2=\pi_1^*(0, 1)\, \pi_1^*\left(K_\Sigma+\frac{1}{2}(\Delta_2+\Delta_3)\right)$$

and Δ_2 is of type $(1, a_2)$. In the same way, Δ_1 is of type $(a_1, 1)$.

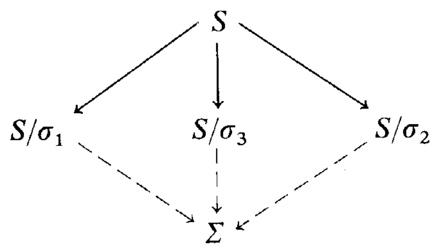
The inverse images on S of each of the rulings of Σ form a pencil of genus 2 curves. Therefore

$$2=\Phi^*(1,0)\cdot\Phi^*\left(K_\Sigma+\frac{1}{2}\Delta\right)=\Phi^*(0,1)\cdot\Phi^*\left(K_\Sigma+\frac{1}{2}\Delta\right)$$

and Δ is of type (5, 5).

Putting all these results together gives the types of the Δ_i's. For the singularities of Δ, we need to use the following two theorems. Notice first that because of their type, the Δ_i's are of arithmetic genus 0, therefore either they are irreducible smooth or they are reducible, in which case it is easy to verify that they have only nodes. Now, there are 10 curves of square (-2) on M, the resolution of the double cover of Σ branched along $\Delta_1+\Delta_2$, which don't meet the pullback of Δ_3 on M (as it will be shown in Theorem 3. 14) and which project to the points of intersection of Δ_1 and Δ_2, as it is easy to see. Therefore Δ_3 does not pass through the points of intersection of Δ_1 and Δ_2. The analogue holds for Δ_1 and Δ_2, using Theorem 3. 3. Q.E.D.

Remark 3. 2. We have shown that, if Σ is a smooth quadric, there is a commutative diagram:



where $p_g(S/\sigma_1)=p_g(S/\sigma_2)=0$ and $p_g(S/\sigma_3)=1$.

Therefore, the following theorem applies in cases 1) (Σ smooth quadric) and 2) (Σ rational quartic).

Theorem 3. 3. *All surfaces S without torsion and such that their bicanonical map factors through a rational map of degree 2, $S \dashrightarrow V$ with $p_g(V)=0$ are obtained in the following way:*

Let $\mathscr{P}=\{p_1,\ldots,p_5,q_1,\ldots,q_4\}$ be a set of nine points in $\mathbb{P}^2$, with q_i infinitely near to p_i for $i=1,\ldots,4$, such that:

i) *No line passes through five points of $\mathscr{P}$ and at most one line passes through four points, which are then of the form $\{p_i,q_i,p_j,q_j\}$ for two distinct elements i and j of $\{1,\ldots,4\}$.*

ii) *No conic passes through eight points of $\mathscr{P}$ and at most one conic passes through seven points, which are then of the form $\mathscr{P}\backslash\{p_i,q_i\}$, where $i\in\{1,\ldots,4\}$.*

iii) *There exists a curve Δ' of degree 10 whose singularities are, besides negligible ones (explained below), (3, 3) points at each (p_i,q_i) $i=1,\ldots,4$ and a point of multiplicity 4 at p_5.*

Then S is obtained by taking the minimal resolution of the double cover of $\mathbb{P}^2$ branched along Δ'.

Remarks 3. 4. 1) A point p_i, $1 \leqq i \leqq 4$, can also be infinitely near to another p_j, $1 \leqq j \leqq 5$.

2) Negligible singularities are:

— double points,

— triple points which resolve to at most a double point after one blow-up.

A (3, 3) point at (p_i, q_i) is a point of multiplicity 3 at p_i which resolves to a point of multiplicity 3 at q_i after blowing-up p_i, and which resolves to at most a negligible singularity after blowing-up q_i.

3) The surface Σ is the image of $\mathbb{P}^2$ under the complete linear system of quartics passing through the points of $\mathscr{P}$ and doubly through p_5. It follows (cf. proof of Theorem 3. 3) that:

a) If the points of $\mathscr{P}$ lie on a unique cubic, Σ is a rational quartic in $\mathbb{P}^3$ with a double line corresponding to this cubic and, in general, four distinct nodes. The canonical curve of S is hyperelliptic and its image is a conic passing through the four nodes of Σ, corresponding to the conic passing through $\{p_1, \ldots, p_5\}$ (unique by i)).

b) If the points of $\mathscr{P}$ lie on a pencil of cubics, Σ is a smooth quadric in $\mathbb{P}^3$. Its two rulings are given by this pencil of cubics, respectively by the pencil of lines through p_5. Notice that these pencils are distinct by ii).

4) For a generic set $\mathscr{P}$, iii) is always satisfied. More precisely, if there is a unique smooth cubic through the points of $\mathscr{P}$, there are curves Δ' satisfying property iii), irreducible and smooth outside $\mathscr{P}$.

Proof. We use the notations of the proof of the theorem. Let C_0 be the strict transform of this cubic on P. Then $C_0 \in |-K|$ and $\Delta_0 \equiv 3C_0 + (l-E)$. By considering, for $n > 1$, the exact sequences of cohomology of the exact sequences:

$$0 \longrightarrow \mathcal{O}_P(l-E+(n-1)C_0) \longrightarrow \mathcal{O}_P(l-E+nC_0) \longrightarrow \mathcal{O}_{C_0}(D_n) \longrightarrow 0,$$

where $\deg D_n = 2$, we get $H^1(P, l-E+nC_0)=0$. Hence $H^1(\Delta_0)=0$ and, by Riemann-Roch, $h^0(\Delta_0)=8$. Moreover, the possible base-points of $|\Delta_0|$ are on C_0. But there is a surjection:

$$H^0(P, \Delta_0) \longrightarrow H^0(C_0, D_3),$$

and $|D_3|$ is base-point-free on C_0.

Hence $|\Delta_0|$ is also base-point-free. The result now follows from Bertini's theorem.

Q.E.D.

Proof of Theorem 3. 3. Let σ be the involution of S associated to $S \dashrightarrow V$ and let $\varepsilon: \hat{S} \to S$ be the blow-up of the isolated fixed points of σ. Let $E'_1, \ldots, E'_e$ be the exceptional curves of the blow-up and let $\hat{\sigma}$ be the involution of $\hat{S}$ induced by σ. The surface $\hat{P} = \hat{S}/\hat{\sigma}$ is smooth and the branch locus Δ of the double cover $\pi: \hat{S} \to \hat{P}$ is smooth and reduced. One has $\Delta \in |2\delta|$, where $\pi_* \mathcal{O}_{\hat{S}} \cong \mathcal{O}_{\hat{P}} \oplus \mathcal{O}_{\hat{P}}(-\delta)$. The following equalities hold:

$$2 = \chi(\mathcal{O}_S) = 2[p_g(\hat{P}) - q(\hat{P}) + 1] + \frac{1}{2}\delta(\delta + K_{\hat{P}}),$$

$$2 - e = K_{\hat{S}}^2 = 2(K_P + \delta)^2.$$

They yield:

$$\delta(\delta+K_{\hat P})=0, \quad K_{\hat P}(\delta+K_{\hat P})=1-\frac{e}{2}. \tag{3.5}$$

Moreover, $\Phi_{|2K_{\hat S}|}$ factors through π, therefore either $H^0(2K_{\hat P}+2\delta)$ or $H^0(2K_{\hat P}+\delta)$ is zero. Since $p_g(\hat P)=0$, we have $1=p_g(S)=h^0(K_{\hat P}+\delta)$ hence $H^0(2K_{\hat P}+\delta)$ is zero and we can apply Riemann-Roch to $|2K_{\hat P}+\delta|$ on $\hat P$, which gives ($H^2(2K_{\hat P}+\delta)\cong H^0(-K_{\hat P}-\delta)^*$ is zero since $K_{\hat P}+\delta\equiv 0$ would imply $K_{\hat S}\equiv 0$):

$$h^1(2K_{\hat P}+\delta)=-\chi(2K_{\hat P}+\delta)=-2+\frac{e}{2}$$

hence

$$e\geqq 4. \tag{3.6}$$

Let us denote by A_i the image of E_i' under π. The curves A_i are smooth, rational, of self-intersection -2 and contained in Δ. Since $\left|2K_{\hat S}-2\sum_{i=1}^{e}E_i'\right|$ is base-point-free, the linear system $|H|=\left|2K_{\hat P}+\Delta-\sum_{i=1}^{e}A_i\right|$ is also base-point-free and:

$$\Phi\circ\varepsilon=\Phi_{|H|}\circ\pi. \tag{3.7}$$

We deduce from (3. 7) that:

$$h^0(H)=4, \quad H^2=4. \tag{3.8}$$

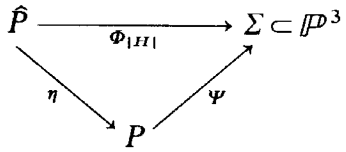
We apply Riemann-Roch to $|H|$ on $\hat P$, which gives, since $h^2(H)=h^0(K_{\hat P}-H)\leqq h^0(K_{\hat P})=0$:

$$\begin{aligned} h^1(H)&=-\chi(H)+4\\ &=4-\left[1+\frac{1}{2}(4-K_{\hat P}(2K_{\hat P}+2\delta))\right]=2-\frac{e}{2}. \end{aligned}$$

Therefore, from (3. 6), we get $e=4$, $H^1(\hat P, H)=0$ and:

$$K_{\hat P}\cdot H=-2. \tag{3.9}$$

Now, there exists a commutative diagram:



where η is a birational morphism onto a smooth surface P, and Ψ does not contract any exceptional curve of the first kind of P (i.e. a smooth rational curve of square -1). We want to show that no curve contracted by η meets $A=\sum_{i=1}^{4}A_i$.

Since η is a composition of blow-ups, it is enough to show that no exceptional curve of the first kind E contracted by η meets A. Suppose that this is not true and that E meets A_1. Since the intersection form on curves contracted by $\Phi_{|H|}$ is negative definite, we have $(E+A_1)^2<0$, hence $E\cdot A_1\leqq 1$. Since E is irreducible and meets A_1, we have $E\cdot A_1=1$ and A_1 is contracted by η. But this is impossible because P is smooth and the quotient of S by the involution σ is not smooth at the point corresponding to the isolated fixed point $\varepsilon(E_1')\in S$ of σ.

We still denote by the same letters the image of H, δ, Δ, A_i under η_*. We get:

$$\Psi=\Phi_{|H|},$$
$$\Delta=\Delta_0+A_1+\cdots+A_4,$$

where $\Delta_0, A_1, \ldots, A_4$ are disjoint, A_i is rational smooth of square (-2) and Δ_0 has negligible singularities, due to the fact that, for every exceptional divisor E of $\hat{P}$ contracted by η, we have:

$$\Delta\cdot E=(H+A_1+\cdots+A_4-2K_{\hat{P}})\cdot E=2.$$

Moreover, the equalities (3. 5), (3. 7), (3. 8), (3. 9) remain true on P. We will now show that $|K_P+H|$ is a pencil of rational curves without base-point.

First, the cohomology sequence associated to

$$0\longrightarrow \mathcal{O}_P(K)\longrightarrow \mathcal{O}_P(K+H)\longrightarrow \omega_H\longrightarrow 0$$

gives an isomorphism

(3. 10) $$H^0(P, K_P+H)\cong H^0(H, \omega_H).$$

But, by (3. 8) and (3. 9), a generic element H of $|H|$ is smooth of genus 2, therefore, if G_0 is irreducible and contained in the fixed part of $|K_P+H|$, we get, by (3. 10) and since $|\omega_H|$ has no base-point, $G_0H=0$. Therefore G_0 is contracted by $\Phi_{|H|}$, and $G_0^2<0$. By construction, G_0 cannot be exceptional. Therefore, $K_PG_0\geqq 0$.

If G is the fixed part of $|K_P+H|$, we have shown:

$$K_PG\geqq 0,\quad HG=0,\ \text{and therefore}\ G^2<0.$$

On the other hand, we have:

(3. 11) $$0\leqq (K_P+H-G)^2=K_P^2-2K_PG+G^2,$$

and by the Index Theorem:

$$H(H+2K_P)=0\Rightarrow (H+2K_P)^2=-4+4K_P^2\leqq 0,$$

equality holding if and only if $H \equiv -2K_P$. It is easy to check that there are only two possibilities:

(3. 12) $$K_P^2=0, \quad G=0, \quad (K_P+H)^2=0$$

and $|K_P+H|$ has no fixed part and no base-point.

(3. 13) $K_P^2=1$, $G=0$, $H\equiv -2K_P$ and by (3. 10), $h^0(P, -K_P)=2$.

In this case, clearly, $\Sigma=\Phi_{|H|}(P)$ is a quadric cone.

Since we have supposed S without torsion, we are in case (3. 12). We denote by F a fibre of the ruling of P given by $|K_P+H|$. We have:

$$g(F)=1+\frac{1}{2}(K_P+H)(2K_P+H)=0,$$

by (3. 8) and (3. 12). Hence P is a rational ruled surface. Recall that for any fibre F, F_{red} is a divisor with normal crossings.

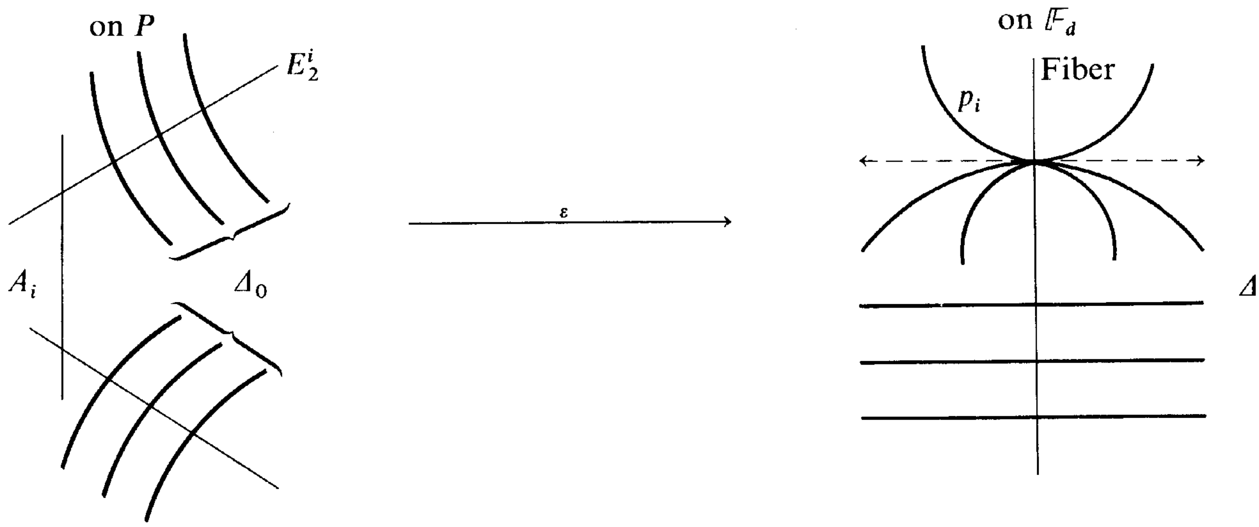
Now, since $H\cdot A_i=K_P\cdot A_i=0$, each A_i is contained in a reducible fibre. If E is an irreducible exceptional curve contained in a fibre, we have:

$$FE=0, \quad K_PE=-1$$

$$1+EA=(F-K_P+A)E=(H+A)E=(2K_P+\Delta)E \text{ is even.}$$

But E meets at most two other components of a fibre, therefore E meets exactly one A_i, say A_1. We can contract E and then A_1 and perform the same operation on the new surface. Thus, we can perform eight contractions $\varepsilon: P\longrightarrow P'$, with corresponding exceptional divisors $\{E_1^i, E_2^i\}_{1\le i\le 4}$ such that $A_i=E_1^i-E_2^i$, until the fibration $P'\longrightarrow \mathbb{P}^1$ is relatively minimal (in fact, by (3. 12), $K_{P'}^2=8$). Therefore, P' is a Segre-Hirzebruch surface $\mathbb{F}_d$ (cf. Notation).

We denote by p_i the image of A_i on $\mathbb{F}_d$. Notice that the p_i's can be infinitely near but of odd order only. The picture is:



Since the Picard group of $\mathbb{F}_d$ is spanned by the class $[F]$ a fibre and the class $[B_0]$ of a section with self-intersection 0 (cf. Notation), we can set:

$$\delta'=\varepsilon_*\delta\equiv 3B_0+\alpha F, \quad \Delta'=\varepsilon_*\Delta.$$

Then, recalling that, on P, $F\equiv K_P+H\equiv 3K_P+\Delta-A$, we get:

$$\delta\equiv\varepsilon^*\delta'-\sum_{i=1}^{4}(E_1^i+2E_2^i).$$

We also have:

$$K_P\equiv\varepsilon^*K_{\mathbb{F}_d}+\sum_{i=1}^{4}(E_1^i+E_2^i)$$

and

$$K_{\mathbb{F}_d}\equiv -2B_0+(d-2)F.$$

The equality (3. 5) gives $2\alpha=7-3d$. Therefore d is odd and if $d\neq 1$, α is negative and, since $\Delta'\cdot B_\infty=2\alpha$, B_∞ is a component of Δ'. Since Δ' is reduced, we then get:

(3. 13) $$0\leqq(\Delta'-B_\infty)B_\infty=2\alpha+d=7-2d \quad \text{i.e.,} \quad d=3,$$

If $d=3$, we want to show that B_∞ cannot pass through a p_i which is actually on $\mathbb{F}_3$ (i.e., which is not an infinitely near point). Otherwise, if we perform two blow-ups as shown on the above picture, then $(F-E_2^i)$ is an exceptional divisor, which we can contract, a contraction which leads us on $\mathbb{F}_5$ (in fact, we have done two elementary transformations), contradicting (3. 13). Therefore, we can suppose that p_1 is not on B_∞. We perform again two blow-ups as above, but now the centers are not on B_∞, therefore, the contraction of $(F-E_2^1)$ leads us on $\mathbb{F}_1$. We can therefore assume that we are on $\mathbb{F}_1$. Then, contracting B_∞ leads us on $\mathbb{P}^2$ and we denote by p_5 the image of B_∞ on $\mathbb{P}^2$, by l (resp. E) the inverse image on P of a line (resp. of p_5). We get:

$$A_i=E_1^i-E_2^i,$$

$$\Delta\equiv 10l-4E-3\sum_{i=1}^{4}(E_1^i+E_2^i)+\sum_{i=1}^{4}A_i,$$

$$K_P+\delta\equiv 2l-E-\sum_{i=1}^{4}E_1^i+\sum_{i=1}^{4}A_i,$$

$$H\equiv 4l-\sum_{i=1}^{4}(E_1^i+E_2^i)-2E.$$

It remains to decide whether $\Sigma=\Phi_{|H|}(P)$ is a quadric or a quartic.

Notice that the linear system $|F|$ is the system of lines through p_5 and that the anticanonical system $|-K_P|$ is the linear system of cubics passing through

$$\mathscr{P}=\{p_1,\ldots,p_5,q_1,\ldots,q_4\}.$$

Since the fibres F are not contracted by $\Phi_{|H|}$, we have $h^0(-K_P)=h^0(H-F)\leqq 2$.

If $|-K_P|$ is a pencil, the equivalence $H \equiv -K_P + F$ shows that Σ is a quadric. Conversely, if Σ is a smooth quadric, we know from the proof of 3. 1 that the inverse image of one of its pencils of lines is the ruling $|F|$ on P and therefore, the inverse image of the other one is $|-K_P|$. This proves Remark 3. 4. 3).

Let us prove condition i). If L is a line through four points of $\mathscr{P}$, we have $\Delta_0 \cdot L < 0$ and therefore L is a component of Δ_0. Since $\Delta_0 \cap A_i = \emptyset$, we deduce that if L passes through p_i, then it passes also through q_i. Finally, L cannot contain 5 points since H is numerically non-negative. In the same way, we prove ii) except that the conic Q could contain $\mathscr{P} \setminus \{p_5\}$. But then we would have $H \equiv 2F + Q$ and Σ would be a cone (cf. Remark 3. 4. 3) b)).

Conversely, if $\mathscr{P}$ and Δ' are given, we define P as the blow-up of $\mathscr{P}$ on $\mathbb{P}^2$, Δ_0 as the strict transform of Δ' and A_i, A, Δ, δ, H by the above formulae. Suppose that the reduced curve Δ_0 has negligible singularities and does not meet A; then the double cover of P branched along Δ has at most ordinary double points as singularities. A minimal desingularization $\hat{S}$ is a surface carrying 4 exceptional curves corresponding to the inverse images of the A_i's. By contracting them, we get a smooth surface S satisfying:

$$K_S^2 = 2, \quad \chi(\mathcal{O}_S) = 2, \quad p_g(S) = h^0(P, K_P + \delta).$$

Furthermore, if condition i) of Theorem 3. 3 is satisfied, there is a unique conic through $\{p_1, \ldots, p_5\}$ hence $p_g(S) = 1$, $q(S) = 0$.

Let us show that the surface S is minimal. If this were not true, there would be a smooth irreducible curve G on $\hat{P}$ satisfying one of the following properties:

1) $G \subset \Delta$, $\quad G^2 = \Delta \cdot G = -2$, $\quad K_{\hat{P}} \cdot G = 0$,

2) $G \cap \Delta = \emptyset$, $\quad G^2 = K_{\hat{P}} \cdot G = -1$,

3) $G \cdot A_1 = 1$, $\quad G^2 = K_{\hat{P}} \cdot G = -1$, $\quad (K_{\hat{P}} + \delta) \cdot G = 0$,

4) $G \cdot A_1 = 2$, $\quad (K_{\hat{P}} + \delta) \cdot G = 0$, $\quad K_{\hat{P}} \cdot G < 0$.

All these cases are ruled out if G is not a component of A by the inequality

$$\begin{aligned} \Delta \cdot G &= [-3K_{\hat{P}} + (l - E) + A + D] \cdot G \\ &\geqq -3K_{\hat{P}} \cdot G + D \cdot G, \end{aligned}$$

where D is a sum of divisors contracted by $\eta : \hat{P} \to P$, by noticing that no component of D meets A, that any common component of D and Δ is of square (-4) and that any component of D not meeting Δ is of square (-2), hence $D \cdot G \geqq 0$ in each case 1) to 4).

It remains to show that S has no torsion. If this were not true, by (1. 8), Σ would be a cone. By Remark 3. 4. 4), this would mean that the pencil of cubics passing through $\mathscr{P}$ would be the pencil of lines passing through p_5. This could only happen if the eight points of $\mathscr{P} \setminus \{p_5\}$ were on a conic, which is excluded by hypothesis ii).

Q. E. D.

The following theorem applies to both cases 1) (Σ quadric) and 3) (Σ $K3$), according to Remark 3. 2. Its proof is completely analogous to the one of the preceding theorem and we shall only indicate the main steps.

Theorem 3. 15. *Suppose that the bicanonical map Φ of S factors through a rational map of degree 2, $S \dashrightarrow V$, with $p_g(V)=1$. Then S is obtained in the following way.*

Let M be a smooth minimal $K3$ surface with a set of 10 disjoint smooth rational curves $\{A_1, \dots, A_{10}\}$ and a smooth curve H of genus 3 such that:

$$\begin{cases} H \cdot A_i = 0, \\ H + A_1 + \cdots + A_{10} \equiv 0 \ (\mathrm{mod}\ 2). \end{cases} \tag{3. 16}$$

Let Δ_0 be an element of $|H|$ with at most negligible singularities (defined in 3. 4. 2)), which does not meet the A_i's. Then S is obtained by contracting the exceptional curves of the minimal resolution of the double cover of M branched along $\Delta_0 + \sum_{i=1}^{10} A_i$.

Remark 3. 17. The image Σ of Φ is $\Phi_{|H|}(M)$. Therefore we have two cases:

1) The generic member of $|H|$ is hyperelliptic. Then Σ is either a quadratic cone (cf. § 2) or a smooth quadric (cf. 3. 1).

2) The generic member of $|H|$ is non-hyperelliptic. Then Σ is a quartic surface in $\mathbb{P}^3$ with only rational double points as singularities. We will prove in 4. 9 that Σ is the locus of zeros of a symmetric 4×4 determinant of linear forms, the curves A_i lying over the locus N of common zeros of the 3×3 minors; and that, conversely, given a quartic Σ as above, a minimal desingularization M of Σ satisfies (3. 16) for some disjoint smooth rational curves A_i lying over N. In general, the set N is a set of 10 distinct nodes, the inverse image of which gives on M the 10 curves A_i. The fact that (3. 16) holds follows from [Ca 4].

We can construct a surface S as explained in 3. 15 and according to 1. 8, S has torsion 0. The canonical curve of S is non-hyperelliptic.

Proof of Theorem 3. 15. We keep the same notations as in the preceding proof except that we denote by $\hat{M}$ the surface $\hat{S}/\hat{\sigma}$. The beginning is the same. We get:

$$\delta(\delta + K_{\hat{M}}) = -4, \tag{3. 18}$$

$$(K_{\hat{M}} + \delta)^2 = 1 - \frac{e}{2}, \tag{3. 19}$$

$$h^1(2K_{\hat{M}} + \delta) = -\chi(2K_{\hat{M}} + \delta) = -5 + \frac{e}{2} \quad \text{i.e.} \quad e \geqq 10, \tag{3. 20}$$

$$\Phi_{|2K_{\hat{S}}|} = \Phi_{|H|} \circ \pi, \quad |H| \text{ base-point-free}, \tag{3. 21}$$
$$h^0(H) = 4, \quad H^2 = 4.$$

$$K_{\hat{P}} \cdot H = 0 \tag{3. 22}$$

$$h^1(H) = 5 - \frac{e}{2}. \tag{3. 23}$$

It follows from (3. 20) and (3. 23) that $e=10$. From (3. 22), it follows on one hand that $h^0(nK_{\hat{M}})=1$ for $n\geqq 0$, which proves that $\hat{M}$ is a $K3$ surface, and on the other hand that $\Phi_{|H|}$ contracts every exceptional divisor on $\hat{M}$. We denote as in the proof of 3. 3 by $\eta:\hat{M}\longrightarrow M$ the contraction of all exceptional divisors of $\hat{M}$ contracted by $\Phi_{|H|}$. We have proved that M is a minimal $K3$ surface, with a set of ten rational curves of square (-2), $\{A_1,\dots,A_{10}\}$, such that (3. 16) holds (because $H\equiv\Delta_0$ on M, by definition of H, since $K_M\equiv 0$).

Conversely, if M, H, $A_1,\dots,A_{10}$ are given as in the theorem, it is easy to compute the numerical invariants of the surface S obtained as explained in the statement. The fact that S is minimal follows again from the fact that $\Phi_{|H|}$ has no base-point (cf. [Be], page 129).
Q.E.D.

Our next aim is to prove that all the surfaces considered in this paragraph, i.e. the ones with torsion 0 for which Φ is not birational, form a connected proper subvariety of an irreducible component of the moduli space $\mathcal{M}$ of surfaces with $K^2=2$, $p_g=1$, $q=0$. Moreover, we shall show that they are all simply connected.

Recall that the surfaces with torsion 0 for which Φ is not birational fall in one of the following families:

1) The family of surfaces for which the image of Φ is a smooth quadric. It defines a locally closed subvariety $\mathcal{M}_1$ of $\mathcal{M}$.

2) The family of surfaces for which the image of Φ is a rational quartic. It defines a locally closed subvariety $\mathcal{M}_2$ of $\mathcal{M}$.

3) The family of surfaces for which the image of Φ is a $K3$ quartic. It defines a locally closed subvariety $\mathcal{M}_3$ of $\mathcal{M}$.

Theorem 3. 24. *The varieties $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are irreducible unirational of dimension respectively* 11, 13 *and* 12. *Moreover, $\mathcal{M}_1$ is in the closure of both $\mathcal{M}_2$ and $\mathcal{M}_3$ and the moduli space $\mathcal{M}$ is smooth of dimension* 16 *at the general point of $\mathcal{M}_1$.*

Proof. We first consider the space $\mathcal{M}_1$.

Referring to Proposition 3. 1, if S and S' are Galois covers of $\Sigma=\mathbb{P}^1\times\mathbb{P}^1$ with branch locus Δ, Δ', resp., and they are isomorphic, there exists $g\in\mathrm{Aut}(\Sigma)$ such that $g(\Delta_i)=\Delta_i'$, and conversely, if such a g exists, one can lift g to an isomorphism of S to S' in four possible ways (corresponding to the choice of a base-point on S' in $\Phi'^{-1}(p_0)$, where p_0 is a not critical point for Φ').

Hence we have a 4-to-1 cover of $\mathcal{M}_1$ by the quotient of a Zariski open set of $\mathbb{P}(H^0(\mathcal{O}_\Sigma(3,1)))\times\mathbb{P}(H^0(\mathcal{O}_\Sigma(1,3)))\times\mathbb{P}(H^0(\mathcal{O}_\Sigma(1,1)))$ by $\mathrm{Aut}(\Sigma)$. This proves, by easy arguments, that $\mathcal{M}_1$ is irreducible, unirational, of dimension $11=7+7+3-6$.

When the Δ_i's are smooth and intersect transversally, K_S is ample and $H^2(S,\Theta_S)=0$ (Example 3. 5 of [Ca 7]), hence $\mathcal{M}$ is smooth of dimension $16=10\chi-2K^2$, at the corresponding point.

Let S now be a surface such that there exists an involution σ for which the bicanonical morphism $\Phi: S \longrightarrow \Sigma$ factors through $S \longrightarrow S/\sigma$. If σ acts as $-\mathrm{id}$ (resp. $+\mathrm{id}$) on $H^0(S, K_S)$, we will construct an effective deformation of S together with the involution σ, with base space of dimension at least 13 (resp. 12). This will prove:

— $\mathcal{M}_1$ is contained in the closure of $\mathcal{M}_2$ and $\mathcal{M}_3$ (by Remark 3. 2).

— The dimension of $\mathcal{M}_2$ (resp. $\mathcal{M}_3$) is $\geqq 13$ (resp. $\geqq 12$) at every point.

Take any smooth surface S. We recall a way of defining its Kuranishi family. Fix an Hermitian metric on S, denote by $A^{0,q}(\Theta_S)$ the space of Θ_S-valued $\mathscr{C}^\infty$ differential forms of type $(0, q)$ and identify the space of harmonic elements of $A^{0,q}(\Theta_S)$ with $H^q(S, \Theta_S)$; then there exists an holomorphic map ϕ defined in a neighborhood U of 0 in $H^1(S, \Theta_S)$ with values in $A^{0,1}(\Theta_S)$, such that $\phi(0)=0$, $\phi(t)-\frac{1}{2}G\bar\partial^*[\phi(t), \phi(t)]=t$, where G is the Green operator, $\bar\partial^*$ the adjoint of $\bar\partial$.

Then B is defined as $\{t \in U \mid p_H[\phi(t), \phi(t)]=0\}$, where p_H is the orthogonal projection $A^{0,2}(\Theta_S) \longrightarrow H^2(S, \Theta_S)$.

Any element t of B defines a complex structure of the $\mathscr{C}^\infty$ manifold $S_{\mathbb{R}}$ underlying S: the anti-holomorphic tangent space $T_s^{0,1}$ at $s \in S_{\mathbb{R}}$ is defined as

$$\{L-\phi(t)_s L \mid L \in T_s^{0,1} S\}$$

where $\phi(t)_s$ is considered as an homomorphism $T_s^{0,1} S \longrightarrow \overline{T_s^{0,1} S}$.

The $\mathscr{C}^\infty$ manifold $\mathscr{S} = S_{\mathbb{R}} \times B$ can be endowed with a complex structure such that the mapping $p: \mathscr{S} \longrightarrow B$ is holomorphic and that the fibre $p^{-1}(t) = S_t$ has the complex structure defined by $\phi(t)$. The family $p: \mathscr{S} \longrightarrow B$ is a universal deformation for $S = S_0$.

Let now σ be an automorphism of S. Then σ induces a $\mathscr{C}^\infty$ automorphism of $S_{\mathbb{R}}$, which in turn induces an analytic automorphism of S_t if and only if $\sigma_*(T_s^{0,1} S_t) = T_{\sigma(s)}^{0,1} S_t$ for all $s \in S_{\mathbb{R}}$, i.e. if and only if $\sigma_* \phi(t) = \phi(t)$. Therefore, if we set:

$$B^\sigma = \{t \in B \mid \sigma_* \phi(t) = \phi(t)\},$$
$$\mathscr{S}^\sigma = \mathscr{S} \cap p^{-1}(B^\sigma),\ p^\sigma = p|_{\mathscr{S}^\sigma},$$

the family $p^\sigma: \mathscr{S}^\sigma \longrightarrow B^\sigma$ is such that σ induces an automorphism on each fibre. If we denote by $H^q(S, \Theta_S)^\sigma$ the subspace of elements of $H^q(S, \Theta_S)$ invariant by the action of σ_*, we have

$$B^\sigma = \{t \in U \mid \phi(t) \in H^1(S, \Theta_s)^\sigma,\ p_{H^\sigma}[\phi(t), \phi(t)] = 0\},$$

where p_{H^σ} is the orthogonal projection $A^{0,2}(\Theta_s)^\sigma \longrightarrow H^2(S, \Theta_s)^\sigma$, because the bracket of two invariant forms is invariant.

Therefore, B^σ is a subset of $H^1(S, \Theta_S)$ defined by

$$[\dim H^1(S, \Theta_s) - \dim H^1(S, \Theta_s)^\sigma] + \dim H^2(S, \Theta_s)^\sigma$$

equations, hence:

$$\dim B^\sigma \geqq \dim H^1(S, \Theta_s)^\sigma - \dim H^2(S, \Theta_s)^\sigma. \tag{3. 25}$$

Now, it remains to compute the above right-hand side when σ is an involution. This is provided by the following lemma:

Lemma 3. 26. *Let $\hat{\sigma}$ be an involution of a surface $\hat{S}$, without isolated fixed points. Let Δ be the branch locus of $\pi : \hat{S} \longrightarrow \hat{S}/\hat{\sigma} = T$. Then*:

$$\sum_{i=0}^{2} (-1)^i \dim H^i(\hat{S}, \Theta_{\hat{S}})^{\hat{\sigma}} = -10\chi(\mathcal{O}_T) + 2K_T^2 - \frac{1}{2}(\Delta^2 - K_T\Delta). \tag{3. 27}$$

Proof. A local computation (cf. [Ca 3], Proposition 3. 1) shows that:

$$\pi_*(\Omega^1_{\hat{S}}(K_{\hat{S}})) \cong [\Omega^1_T(\log \Delta) \otimes \mathcal{O}_T(K_T)] \oplus \Omega^1_T(K_T + \delta),$$

where $\Delta \in |2\delta|$ and the first (resp. second) term corresponds to the invariant (resp. anti-invariant) part.

Therefore, the left-hand side of (3. 27) is equal to:

$$\chi^{\hat{\sigma}} = \chi[\Omega^1_T(\log \Delta) \otimes \mathcal{O}_T(K_T)].$$

The following exact sequences:

$$0 \longrightarrow \Omega^1_T \otimes \mathcal{O}(K_T) \longrightarrow \Omega^1_T(\log \Delta) \otimes \mathcal{O}(K_T) \longrightarrow \mathcal{O}_T(K_T) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}_T(K_T - \Delta) \longrightarrow \mathcal{O}_T(K_T) \longrightarrow \mathcal{O}_\Delta(K_T) \longrightarrow 0$$

yield:

$$\begin{aligned}\chi^{\hat{\sigma}} &= \chi(T, \Omega^1_T \otimes \mathcal{O}(K_T)) + \chi(T, \mathcal{O}_T(K_T)) - \chi(T, \mathcal{O}_T(K_T - \Delta))\\ &= -10\chi(\mathcal{O}_T) + 2K_T^2 - \frac{1}{2}(\Delta^2 - K_T\Delta).\end{aligned}$$

Q.E.D.

Now, we go back to a surface S of our family, such that the Galois group of $S \longrightarrow \Sigma$ contains an involution σ with $\sigma^* w = -w$ (resp. $\sigma^* w = w$), where w is a non-zero element of $H^0(S, K_S)$.

We want to prove that the subvariety $\mathcal{M}_1 \cup \mathcal{M}_2$ (resp. $\mathcal{M}_1 \cup \mathcal{M}_3$) of the moduli space corresponding to the family of these surfaces is everywhere of dimension at least 13 (resp. 12).

Let $\hat{S} \to S$ be the blow-up of the 4 (resp. 10) isolated fixed points of σ [cf. (3. 3) (resp. (3. 15))], $\hat{\sigma}$ the fixed point-free involution induced on $\hat{S}$.

It follows from the above arguments that there exists a family $\hat{p}: \mathscr{S}^{\hat{\sigma}} \to B^{\hat{\sigma}}$ of deformations of $\hat{S} = \hat{p}^{-1}(0)$ such that there exists on each fibre an involution $\hat{\sigma}_t$ with $\hat{\sigma}_t^* w_t = -w_t$ (resp. $\hat{\sigma}_t^* w_t = w_t$).

In the notations of (3. 3) (resp. (3. 15)), $\hat{S}/\hat{\sigma}$ is the surface $\hat{P}$ (resp. $\hat{M}$). We can use the formula (3. 27), after noticing that its right-hand side can be computed on P (resp. M) as well. We get $\dim B^{\hat{\sigma}} \geqq 13$ (resp. $\dim B^{\hat{\sigma}} \geqq 12$). Now, by a well-known theorem of Kodaira ([Ko]), the 4 (resp. 10) exceptional curves of $\hat{S}$ are deformed into exceptional curves that we can blow down. This yields a deformation $p: \mathscr{S}^{\hat{\sigma}} \to B^{\hat{\sigma}}$ of S, which proves what we wanted.

The fact that $\mathscr{M}_3$ is irreducible unirational 12-dimensional will be proven in § 4, using the description of the canonical ring of the corresponding surfaces.

We now prove that $\mathscr{M}_2$ is irreducible unirational and 13-dimensional.

Following [Ran], § 1, one can define a scheme structure on the set:

$$\mathscr{U} = \{\mathscr{P} = \{p_1, \ldots, p_5, q_1, \ldots, q_4\},\ q_i \text{ infinitely near to } p_i \text{ for } 1 \leqq i \leqq 4 \text{ and:}$$
$$(p_i \text{ infinitely near to } p_j) \Rightarrow i<j\}.$$

To define an element of $\mathscr{U}$, one has to choose successively $p_5, p_4, q_4, \ldots, p_1, q_1$, each time in an irreducible subset of some blow-up of $\mathbb{P}^2$. Therefore $\mathscr{U}$ is irreducible, 14-dimensional.

There is a variety $\bar{\mathscr{B}}$ together with a morphism $\bar{p}: \bar{\mathscr{B}} \to \mathscr{U}$ such that for $\mathscr{P} \in \mathscr{U}$, the fibre $(\bar{p})^{-1}(\mathscr{P})$ is the projective space $|\Delta_0|$ (defined in the proof of Theorem 3. 3). Consider the following open subset of $\bar{\mathscr{B}}$ (which, by Remark 3. 4. 4), dominates $\mathscr{U}$):

$$\mathscr{B} = \{\Delta_0 \mid \Delta_0 \text{ does not meet } A = \Sigma A_i \text{ and has negligible singularities}$$
$$\text{(on the blow-up of } \mathscr{P} \text{ on } \mathbb{P}^2)\}.$$

Then, to each point of $\mathscr{B}$, we can associate a surface S. We get a map:

$$q: \mathscr{B} \longrightarrow \mathscr{M},$$

the image of which contains $\mathscr{M}_2$ by Theorem 3. 3. Since the projectivities of $\mathbb{P}^2$ act on $\mathscr{U}$, the dimension of the fibres of q is at least $\dim \mathrm{PGL}(3) = 8$.

But we have just proven above that the dimension of $\mathscr{M}_2$ is everywhere $\geqq 13$. It follows that:

(3. 28) The dimension of $q^{-1}(\mathscr{M}_2)$ is everywhere $\geqq 21$.

We estimate now the dimension of the fibres of $p: q^{-1}(\mathscr{M}_2) \to \mathscr{U}$, in a lemma which will be proven later on.

Lemma 3. 29. *Let S be a surface corresponding to a point of $\mathcal{M}_2$. We adopt the notations of the proof of Theorem 3. 3. Then $h^0(P, \Delta_0)=8$ except when:*

— *there is a line through $\{p_i, q_i, p_j, q_j\}$*

or,

— *there is a conic through $\mathcal{P}\backslash\{p_i, q_i\}$,*

in which cases $h^0(P, \Delta_0)=9$.

Let $\mathcal{V}$ be the image of $p: q^{-1}(\mathcal{M}_2)\longrightarrow \mathcal{U}$ and $\mathcal{V}_s$ the subset of $\mathcal{V}$ corresponding to the special cases of the lemma. We have:

$$\dim \mathcal{V}_s \leqq 12,$$
$$\forall \mathcal{P}\in\mathcal{V}_s, \quad \dim p^{-1}(\mathcal{P})=8$$

hence $\dim p^{-1}(\mathcal{V}_s)\leqq 20$.

It follows from (3. 28) that $p^{-1}(\mathcal{V}_s)$ is a proper subset of a component of $q^{-1}(\mathcal{M}_2)$ and that $p^{-1}(\mathcal{V}-\mathcal{V}_s)$ is dense in $q^{-1}(\mathcal{M}_2)$. But we know that the fibres of p over $\mathcal{V}-\mathcal{V}_s$ are open subsets of 7-dimensional projective spaces. It follows that $p^{-1}(\mathcal{V}-\mathcal{V}_s)$ is irreducible, unirational and 21-dimensional.

Therefore $\mathcal{M}_2$ is irreducible, unirational and 13-dimensional. Q.E.D.

Proof of the Lemma. Recall from the proof of 3. 3 that the notations are

$$\Delta=\Delta_0+A_1+\cdots+A_4$$

on P, where $A_i=E_1^i-E_2^i$ are smooth rational curves with self intersection -2 and Δ_0 is disjoint from $A=\sum_{i=1}^{4} A_i$, with negligible singularities (cf. Remark 3. 4 2)). Since P is regular, there is an exact sequence:

$$0\longrightarrow H^0(P, \mathcal{O}_P)\longrightarrow H^0(P, \Delta_0)\longrightarrow H^0(\Delta_0, \mathcal{O}_{\Delta_0}(\Delta_0))\longrightarrow 0.$$

We also have the following numerical relations:

$$\Delta_0^2=12, \quad \Delta_0 K_P=-2.$$

Using Riemann-Roch, we find:

$$h^0(P, \Delta_0)=8+h^0(\Delta_0, K_{P|\Delta_0}).$$

The line bundle $\mathcal{O}_{\Delta_0}(K_P)$ has total degree -2. We first look for irreducible curves D contained in Δ_0 such that $\deg(\mathcal{O}_D(K_P))=D\cdot K_P>0$.

We observe that since Δ_0 is the strict transform of $\Delta'\subset\mathbb{P}^2$ on P, every component of Δ_0, and in particular D, projects birationnally to a plane curve in $\mathbb{P}^2$.

Let $C_0 \in |K_P|$ be the unique cubic through the points of $\mathscr{P}$. Since $C_0^2=0$, $D \cdot C_0 < 0$ implies that C_0 is reducible, D is a component of C_0 and $D^2<0$. In particular, D is a curve of degree $\leqq 2$ and is rational. Therefore $D^2=-2-DK_P \leqq -3$ and either D is a line through 4 points of $\mathscr{P}$ or a conic through 7 points (at least). We conclude with properties i) and ii) of Theorem 3. 3, that we are in one of the exceptional cases of the lemma. Notice that in these cases, D is unique and satisfies:

$$D^2=-3, \quad K_P \cdot D = 1, \quad \Delta_0 \cdot D = -2.$$

Hence D is the fixed part of $|\Delta_0|$. We will write $\Delta_0 = D + \Delta_0'$. We get:

$$h^0(P, \Delta_0) = h^0(P, \Delta_0') = 9 + h^0(\Delta_0', K_{P|\Delta_0'}).$$

Now, the degree of K_P on any component of Δ_0 (resp. Δ_0' in the exceptional cases) is non-positive. Therefore, if the sheaf $\mathscr{O}_{\Delta_0}(K_P)$ (resp. $\mathscr{O}_{\Delta_0'}(K_P)$) had a section, there would be a decomposition $\Delta_0 = \Gamma + \Delta_1$ with $K_P \cdot \Gamma = 0$ and $\Gamma \cap \Delta_1 = \emptyset$ (resp. with Δ_0 replaced by Δ_0'), with Γ irreducible. But now, by the Index Theorem, we have:

$$K_P^2 = K_P \cdot \Gamma = 0 \Rightarrow \Gamma^2 < 0 \quad \text{or} \quad \Gamma \equiv K_P.$$

In the non-exceptional case, notice that $\Delta_0 \equiv -3K_P + F$, hence:

$$\Gamma^2 = \Gamma \cdot (\Delta_0 - \Delta_1) = \Gamma \cdot \Delta_0 = \Gamma \cdot F \geqq 0 \text{ since } F \text{ is moving.}$$

Therefore $\Gamma \equiv K_P$ and $\Gamma^2 = 0$. But this contradicts the equalities

$$\Gamma^2 = \Gamma \cdot \Delta_0 = K_P \cdot \Delta_0 = -2.$$

We have thus proven that $H^0(\Delta_0, \mathscr{O}_{\Delta_0}(K_P))$ is zero hence the lemma in this case.

In the exceptional cases, one has $\Gamma^2 = \Gamma \cdot F - \Gamma \cdot D$. Notice that

$$\Gamma D + \Delta_1' D = \Delta_0' D = \Delta_0 D - D^2 = 1.$$

If $\Delta_1' D < 0$, D is a component of Δ_1' and $\Gamma \cdot D = 0$, $\Gamma^2 \geqq 0$. If $\Delta_1' D \geqq 0$, one has $\Gamma D = 0$ or 1. If $\Gamma D = 0$, we have again $\Gamma^2 \geqq 0$. If $\Gamma D = 1$, one has

$$\Gamma \cdot (C_0 - D) = -\Gamma \cdot K_P - 1 = -1,$$

hence Γ is a component of C_0 and $\Gamma \cdot F \geqq 1$. Therefore, we always have $\Gamma^2 \geqq 0$, hence $\Gamma \equiv K_P$. As above, we are led to the contradiction $0 = \Gamma^2 = \Gamma \cdot \Delta_0' = K_P \cdot \Delta_0' = -3$.

We have thus proven that $H^0(\Delta_0', \mathscr{O}_{\Delta_0'}(K_P))$ is zero hence the lemma in the exceptional cases. Q.E.D.

Corollary 3. 30. *The surfaces S for which* $\Phi = \Phi_{|2K|}$ *is not birational, and* $\Sigma = \Phi(S)$ *is not a quadric cone are simply connected.*

Proof. It follows from the preceding theorem that those surfaces form a connected family. Therefore it is sufficient to construct a simply connected member of this family; this construction can be found in [Ch]. We can also remark that it is enough to prove that the fundamental group is abelian, since there is no torsion; but if S is a Galois cover of $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ and the Δ_i's are smooth and intersect transversally, then $\pi_1(\Sigma - \Delta_1 - \Delta_2 - \Delta_3)$ is abelian (cf. [Ca 7]). Q.E.D.

We end up this section with the following proposition, which links the surfaces studied in this section to the ones for which the torsion is $\mathbb{Z}/2\mathbb{Z}$, studied in § 2.

Proposition 3. 31. *There is a flat family of surfaces $\mathcal{S} \to \bar{\mathcal{B}} = \bar{\mathcal{B}}_0 \cup \bar{\mathcal{B}}_2$ where $\bar{\mathcal{B}}_0$ and $\bar{\mathcal{B}}_2$ (resp. $\bar{\mathcal{B}}_0 \cap \bar{\mathcal{B}}_2$) are smooth irreducible of dimension* 13 *(resp.* 12*), such that:*

- *For a general point b of $\bar{\mathcal{B}}$, the fibre S_b is a smooth minimal surface of general type with $K^2 = \chi(\mathcal{O}) = 2$. The induced rational map from $\bar{\mathcal{B}}$ to the moduli space $\mathcal{M}$ is generically finite on its image.*

If b is general in $\bar{\mathcal{B}}_0$, S_b is of type 0) *(Σ quadric cone).*

If b is general in $\bar{\mathcal{B}}_2$, S_b is of type 2) *(Σ rational quartic).*

- *If b is general in $\bar{\mathcal{B}}_0 \cap \bar{\mathcal{B}}_2$, S_b is not normal. Its normalization is a smooth rational ruled surface.*

Proof. Consider the situation of Theorem 3. 3 but suppose that the 8 points of $\mathcal{P} \setminus \{p_5\}$ lie on an irreducible conic C_1'. We want to construct a surface S according to the procedure described in Theorem 3. 3, and we keep the same notations. Let C_1 be the strict transform of C_1' on the blow-up P of $\mathbb{P}^2$ along $\mathcal{P}$. Then $\Delta_0 \cdot C_1 = -4$ and C_1 is in the fixed part of $|\Delta_0|$. We write $\Delta_0 \equiv C_1 + \Delta_1$ and $\Delta_1 \equiv 2H$, $\Delta_1 \cdot C_1 = 0$. Notice that $H \equiv C_1 + 2(l - E)$ and that there is an exact sequence:

$$0 \longrightarrow H^0(P, 2(l-E)) \longrightarrow H^0(P, H) \longrightarrow H^0(C_1, \mathcal{O}_{C_1}) \longrightarrow 0.$$

Therefore $|H|$ (hence $|\Delta_1|$) has no base-point. A general element of $|\Delta_0|$ is smooth (but not connected) and we can construct a surface S as described in Theorem 3. 3. It is easy to see that S is a minimal surface with $K^2 = 2$, $p_g = 1$, $q = 0$. Its bicanonical map factors through $S \dashrightarrow P$ and its bicanonical image is $\Phi_{|H|}(P)$, which is a quadric cone. Hence S has torsion $\mathbb{Z}/2\mathbb{Z}$.

We construct now a family of such surfaces. Fix the points p_1, p_2, p_3, p_4 projectively independent. Take $\mathcal{U}$ to be an open 6-dimensional family of sets $\mathcal{P}$ containing p_1, p_2, p_3, p_4; and let $\mathcal{W}$ be the 3-dimensional subvariety corresponding to the sets $\mathcal{P}$ such that $\mathcal{P} - \{p_5\}$ lie on an irreducible conic. We shall suppose moreover that for $\mathcal{P} \in \mathcal{U} - \mathcal{W}$, there is a unique smooth cubic passing through the points of $\mathcal{P}$. Let $\bar{p}: \bar{\mathcal{B}} \to \mathcal{U}$ be a morphism such that the fibre of $\mathcal{P}$ is the projective space $|\Delta_0(\mathcal{P})|$, and let $p: \mathcal{B} \to \mathcal{U}$ be its restriction to the open set of *smooth* Δ_0's. It follows from Remark 3. 4 4) that over $\mathcal{U} - \mathcal{W}$, the fibres of p are 7-dimensional, irreducible. This yields an irreducible 13-dimensional component $\mathcal{B}_2$ of $\mathcal{B}$, which dominates $\mathcal{U}$. It follows from the discussion above that if $\mathcal{P} \in \mathcal{W}$, the fibre $p^{-1}(\mathcal{P})$ is an open subset of $|\Delta_1| \cong \mathbb{P}^{10}$. This yields another 13-dimensional component $\mathcal{B}_0$ of $\mathcal{B}$, which dominates $\mathcal{W}$, and

$$\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_2.$$

There is a generically finite map from $\mathscr{B}$ to the moduli space $\mathscr{M}$ of our surfaces, which sends $\mathscr{B}_0$ onto a 13-dimensional locally closed subset of $\mathscr{M}_T$ (surfaces with torsion $\mathbb{Z}/2\mathbb{Z}$) and $\mathscr{B}_2$ onto a dense open subset of $\mathscr{M}_2$ (surfaces of type 2)).

What is the intersection of $\bar{\mathscr{B}}_0$ and $\bar{\mathscr{B}}_2$ (closure in $\mathscr{B}$)?

First, $\bar{\mathscr{B}}_0=(\bar{p})^{-1}(\mathscr{W})$. On the other hand, since a limit of connected sets is connected, any $(\Delta_0, \mathscr{P})$ in $\bar{\mathscr{B}}_2$ is such that Δ_0 is connected. In case $\mathscr{P}\in\mathscr{W}$, it can be written as:

$$\Delta_0=2C_1+\Delta_2,$$

$$\Delta_2\equiv H+2(l-E).$$

Therefore, $(\bar{p})^{-1}(\mathscr{P})\cap\bar{\mathscr{B}}_2$, for $\mathscr{P}\in\mathscr{W}$, is contained into the 9-dimensional projective space $2C_1+|\Delta_2|$. It follows that the intersection $\bar{\mathscr{B}}_0\cap\bar{\mathscr{B}}_2$ is contained in the smooth irreducible divisor $\bar{\mathscr{B}}_{02}$ of $\bar{\mathscr{B}}_0$ defined by:

$$\bar{\mathscr{B}}_{02}=\bigcup_{\mathscr{P}\in\mathscr{W}}\{2C_1+|\Delta_2|\subset|\Delta_0|\}.$$

We will now show that there is in fact equality. We will adopt the notations of [We]. If Σ_{Δ_0} is the sheaf of germs of differential operators of degree ≤ 1 of $\mathcal{O}_P(\Delta_0)$, the isomorphism classes of linear infinitesimal deformations of the couple $(P, \mathcal{O}_P(\Delta_0))$ are parametrized by $H^1(P, \Sigma_{\Delta_0})$. If $s\in H^0(P, \mathcal{O}_P(\Delta_0))$, the isomorphism classes of linear infinitesimal deformations of the triple $(P, \mathcal{O}_P(\Delta_0), s)$ are parametrized by the first hypercohomology group $\mathbb{H}^1(d^1s)$ of the complex $(0\longrightarrow\Sigma_L\xrightarrow{d^1s}L\longrightarrow 0)$ (Proposition 1. 2 of [We]). There is an exact sequence (1. 6 of loc. cit.):

$$\text{(3. 32)}\qquad 0\longrightarrow H^0(P, \Delta_0)/\mathbb{C}s\longrightarrow\mathbb{H}^1(d^1s)\xrightarrow{\beta}H^1(P, \Sigma_{\Delta_0})\xrightarrow{d^1s}H^1(P, \Delta_0).$$

Notice that in our case, the exact sequence:

$$0\longrightarrow\mathcal{O}_P\longrightarrow\Sigma_{\Delta_0}\longrightarrow\Theta_P\longrightarrow 0$$

yields an isomorphism $H^1(P, \Sigma_{\Delta_0})\xrightarrow{\sim}H^1(P, \Theta_P)$.

Let $\mathscr{P}$ be an element of $\mathscr{U}$ and let $s\in H^0(P, \mathcal{O}_P(\Delta_0))$ correspond to a point x of $p^{-1}(\mathscr{P})$. There are natural inclusions $T_x\mathscr{B}\hookrightarrow\mathbb{H}^1(d^1s)$ and $T_{\mathscr{P}}\mathscr{U}\hookrightarrow H^1(P, \Sigma_{\Delta_0})$ which, together with the exact sequence (3. 32), induce the exact sequence:

$$0\longrightarrow T_x|\Delta_0|\longrightarrow T_x\mathscr{B}\xrightarrow{T_xp}T_{\mathscr{P}}\mathscr{U}\xrightarrow{d^1s}H^1(P, \Delta_0).$$

Notice that for any $x\in\bar{\mathscr{B}}_0$, $T_xp(T_x\mathscr{B})\supset T_{\mathscr{P}}\mathscr{W}$ hence d^1s induces a morphism $\varphi_x: T_{\mathscr{P}}\mathscr{U}/T_{\mathscr{P}}\mathscr{W}\longrightarrow H^1(P, \Delta_0)$. Both of these vector spaces are 3-dimensional. Now, if $x\in\bar{\mathscr{B}}_0\backslash\bar{\mathscr{B}}_2$, since $\bar{\mathscr{B}}_0$ is a projective bundle over the smooth base $\mathscr{W}$, we have $T_xp(T_x\mathscr{B})=T_x\mathscr{W}$ and φ_x is an isomorphism. Therefore, the closed subset $\bar{\mathscr{B}}_0\cap\bar{\mathscr{B}}_2$ of $\bar{\mathscr{B}}_0$ contains the divisor $\{x\mid\det\varphi_x=0\}$.

It follows that $\bar{\mathscr{B}}_0\cap\bar{\mathscr{B}}_2=\bar{\mathscr{B}}_{02}$ is smooth, irreducible, and 12-dimensional.

Let $\mathscr{S}$ be the family of double coverings of the P's, branched along

$$\Delta=\Delta_0+A\equiv 2\delta.$$

We have:

$$\forall n \in \mathbb{N}, \quad \chi(\mathcal{O}_{\hat{S}}(n\pi^* l)) = \chi(\mathcal{O}_P(nl)) + \chi(\mathcal{O}_P(nl-\delta)),$$

independent of Δ and P. Therefore the family is flat.

On each fibre of $\hat{\mathcal{S}} \longrightarrow \bar{\mathcal{B}}$, we can contract the 4 exceptional curves composing the inverse image of A: we obtain the desired flat family $\mathcal{S} \longrightarrow \bar{\mathcal{B}}$.

For a general point b of $\bar{\mathcal{B}}_0 \cap \bar{\mathcal{B}}_2$, S_b is not normal, since $\Delta = 2C_1 + \Delta_2 + A$ is not reduced. Its normalization N is obtained by contracting 4 exceptional curves on the double cover of P branched along $\Delta_2 + A$.

The inverse image of the pencil $|l-E|$ is still a pencil of rational curves (because $(l-E)(\Delta_2+A)=2$). Therefore N is rational ruled. The surface S_b is obtained by identifying the points of an elliptic bisection which are on the same fibre of the ruling.

Q.E.D.

§ 4. The canonical ring

Let S be a minimal surface with $K^2=2$, $p_g=1$ and $q=0$. Its canonical ring is defined as:

$$\mathcal{R} = \bigoplus_{m=0}^{\infty} H^0(S, mK_S).$$

This paragraph will be devoted to the study of this ring, in the lines of [Ca 5]. In particular, we will get determinantal equations for the bicanonical images.

It was shown by Mumford ([Mu]) that $\mathcal{R}$ is a finitely generated graded ring. We will denote by $\mathcal{R}_m$ the homogeneous part of degree m i.e. $\mathcal{R}_m = H^0(S, mK_S)$.

The canonical model X of S is by definition $X = \operatorname{Proj}(\mathcal{R})$. It is a normal surface with at most rational double points as singularities ([Mu]). There is a morphism $\pi: S \longrightarrow X$ through which every pluricanonical morphism factorizes.

We also introduce the graded $\mathbb{C}$-algebra $\mathcal{A} = \mathbb{C}[W, Y_1, Y_2, Y_3]$ with grading defined by:

$$\deg W = 1, \quad \deg Y_j = 2 \quad \text{for} \quad i = 1, 2, 3.$$

There is a morphism of graded rings, $\mathcal{A} \longrightarrow \mathcal{R}$ obtained by sending W to a generator w of $H^0(S, K_S)$ and $\{W^2, Y_1, Y_2, Y_3\}$ to a basis $\{y_0 = w^2, y_1, y_2, y_3\}$ of $H^0(S, 2K_S)$, which makes $\mathcal{R}$ into a graded $\mathcal{A}$-module.

We set $\mathbb{P} = \operatorname{Proj}(\mathcal{A})$. In the notations of [D 1] and [Do], $\mathbb{P}$ is the weighted projective space $\mathbb{P}(1, 2, 2, 2)$. It is isomorphic to $\mathbb{P}^3$ and via this isomorphism, one has ([B-R], Proposition 3 C.7):

$$\forall n \in \mathbb{Z}, \quad \mathcal{O}_{\mathbb{P}}(2n+1) \cong \mathcal{O}_{\mathbb{P}}(2n) \cong \mathcal{O}_{\mathbb{P}^3}(n).$$

There is a morphism:

$$\Psi : X = \mathrm{Proj}(\mathscr{R}) \longrightarrow \mathbb{P} = \mathrm{Proj}(\mathscr{A}),$$

which is finite over its image $\Sigma \subset \mathbb{P}$, such that the composition:

$$S \xrightarrow{\pi} X \xrightarrow{\Psi} \mathbb{P} \underset{\sim}{\xrightarrow{u}} \mathbb{P}^3$$

is the bicanonical map Φ that we have studied in the first parts of this article.

To the graded $\mathscr{A}$-module $\mathscr{R}$ is associated a sheaf $\tilde{\mathscr{R}}$ on $\mathbb{P}$ ([Gr], 2. 5) and we have ([Gr], Proposition 2. 8. 7):

$$\Psi_* \mathcal{O}_X \cong \tilde{\mathscr{R}}. \tag{4. 1}$$

This situation has been studied in general by the first author in [Ca 5]. Notice that (1, 2, 2, 2) is not normalized in the sense of Definition 2. 4 of loc. cit. However, this hypothesis is used there only to write 2. 5 and to prove 2. 7, which is just the isomorphism (4. 1) above, and again in Theorem 4. 24, which we won't use. Therefore, we will use freely the results of [Ca 5]. For example, it follows from Proposition 2. 8 of loc. cit. that:

Proposition 4. 2. *$\mathscr{R}$ is a Cohen-Macaulay $\mathscr{A}$-module.*

Therefore, it admits a length one minimal resolution by free $\mathscr{A}$-modules which, according to 2. 9 and 2. 18 of loc. cit., can be written as:

$$0 \longrightarrow \bigoplus_{i=0}^{h} \mathscr{A}(-8+l_i) \xrightarrow{\alpha} \bigoplus_{i=0}^{h} \mathscr{A}(-l_i) \longrightarrow \mathscr{R} \longrightarrow 0$$

with $0 = l_0 < l_1 \leqq \cdots \leqq l_h$.

The morphism α is given by a square matrix (α) of homogeneous elements α_{ij} of $\mathscr{A}$, such that, for $0 \leqq i, j \leqq h$:

$$\deg \alpha_{ij} = \begin{cases} 8 - l_i - l_j & \text{if } \ 8 - l_i - l_j > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The determinant of (α) is an homogeneous element of $\mathscr{A}$ of degree 16, equal to $(\Sigma)^{\deg \Psi}$, where $\Sigma \in \mathbb{C}[Y_0, Y_1, Y_2, Y_3] \subset \mathscr{A}$ is an equation of the bicanonical image $\Phi(S)$.

In our particular case, it is now easy to get the two possible forms of the minimal resolution for $\mathscr{R}$:

Theorem 4. 3. *The $\mathscr{A}$-module $\mathscr{R}$ has a minimal free resolution of the form:*

i) $0 \longrightarrow \mathscr{A}(-8) \oplus \mathscr{A}(-5)^4 \xrightarrow{\alpha} \mathscr{A} \oplus \mathscr{A}(-3)^4 \longrightarrow \mathscr{R} \longrightarrow 0$

if the canonical curve is non-hyperelliptic,

ii) $0 \longrightarrow \mathscr{A}(-8) \oplus \mathscr{A}(-5)^4 \oplus \mathscr{A}(-4) \xrightarrow{\alpha} \mathscr{A} \oplus \mathscr{A}(-3)^4 \oplus \mathscr{A}(-4) \longrightarrow \mathscr{R} \longrightarrow 0$

otherwise.

Proof. It follows from Proposition 4. 2 that if we select a minimal set of homogeneous generators for $\mathscr{R}$ as an $\mathscr{A}$-module, we get a minimal free resolution for $\mathscr{R}$. Notice that:

$$\dim_C \mathscr{R}_0 = \dim_C \mathscr{R}_1 = 1,$$
$$\dim_C \mathscr{R}_m = 2 + m(m-1) \quad \text{for} \quad m \geqq 2.$$

When choosing a minimal set of generators for $\mathscr{R}$, we need to take $1 \in \mathscr{R}_0$ and $v_1, \ldots, v_4$ which complete $\{wy_0, \ldots, wy_3\}$ to a basis for $\mathscr{R}_3$ (recall that $\{y_0 = w^2, y_1, y_2, y_3\}$ is a basis for $\mathscr{R}_2$). In degree four, we have $\dim \mathscr{R}_4 = 14$, hence either the 14 elements $y_i y_j$, wv_h form a basis for $\mathscr{R}_4$ or there is a relation:

$$Q(y) + wv = 0 \tag{4. 4}$$

where Q is a quadratic form in y_0, y_1, y_2, y_3 and v a linear combination of $v_1, \ldots, v_4$.

Lemma 4. 5. *There exists such a relation if and only if the canonical curve is hyperelliptic. In this case there is, up to constant, only one such relation. The element v of $\mathscr{R}_3$ is zero if and only if the bicanonical image Σ is a quadric.*

Proof. Suppose that the canonical curve C is not hyperelliptic. If there exists a relation $Q(y) + wv = 0$ then we have $\Phi^* Q \geqq C$. By Remark 1. 6, we can conclude that Q is divisible by y_0. Hence v can be written as $v = wy$ where y is a linear combination of $y_0, \ldots, y_3$. But since $\{v_1, \ldots, v_4, wy_0, \ldots, wy_3\}$ is a basis for $\mathscr{R}_2$ this is possible only if $v = 0$. But then we have $Q(y) = 0$ and Σ is a quadric. This contradicts the fact that C is non-hyperelliptic.

If C is hyperelliptic then there is, up to constant, only one quadric $Q(y_1, y_2, y_3)$ such that $Q(y_1, y_2, y_3) = w \cdot s$, where $s \in H^0(S, 3K_S)$ (Remark 1. 6). If we write s as a linear combination of the v_i's and the wy_j's we get a relation of type (4. 4). Q.E.D.

At this point, in the non-hyperelliptic case, we have obtained a basis for $\mathscr{R}_4$. We know that:

$$\begin{cases} l_0 = 0, \ l_1 = l_2 = l_3 = l_4 = 3, \ l_5 \geqq 5 \quad \text{if} \quad h \geqq 5, \\ 8 - l_h \geqq 5 \ \text{(no relation in degrees } \leqq 4). \end{cases}$$

Therefore $h = 4$ and the theorem is proved in this case.

In the hyperelliptic case, we need one new generator v_5 in degree 4. Hence:

$$\begin{cases} l_0 = 0, \ l_1 = l_2 = l_3 = l_4 = 3, \ l_5 = 4, \ l_6 \geqq 5 \quad \text{if} \quad h \geqq 6, \\ 8 - l_h \geqq 4 \ \text{(no relation in degrees } \leqq 3). \end{cases}$$

Therefore $h = 5$ and the theorem is proved in this case. Q.E.D.

Following the arguments of [Ca 5] we now want to describe more precisely the structure of the canonical ring.

Let us recall the following definition from [Ca 5] (rank condition):

(R.C.) *Let $h+1$ be the size of the matrix (α) (here $h=4$ or 5), and (α') be the matrix obtained from (α) by erasing its first row. Then the $(h\times h)$-minors of (α) belong to the ideal generated in $\mathscr{A}$ by the $(h\times h)$-minors of (α').*

We will now state our results, postponing the proofs until the end of this section.

Theorem 4. 6. *The matrix (α) of the preceding theorem can be chosen symmetric and satisfying* (R.C.).

Our results are in fact much more precise and we give in particular the special forms the matrix (α) takes, depending on the bicanonical image Σ.

We also show that conversely, in most cases, one can recover the ring structure of $\mathscr{R}$ from the matrix (α), and therefore the surface S itself as a minimal desingularization of $\mathrm{Proj}(\mathscr{R})$. In these cases, we also show that, to any matrix (α) belonging to a Zariski open subset of the set of matrices of the right form satisfying (R.C.), one can associate a surface S.

The existence problem will not be considered here. Existence follows from the results of § 3 when Φ is not birational. The existence of surfaces for which Φ is birational also follows from Theorem 3. 24. They will be explicitly constructed in § 5 (with a non-hyperelliptic canonical curve) and in § 6 (with a hyperelliptic canonical curve).

We give below the results case by case.

1) *Σ is a smooth quadric.* Then the matrix (α) can be chosen of the following form:

$$(\alpha)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & l_1 l_4 - l_2 l_3 \\ 0 & 0 & 0 & l_1 & l_2 & 0 \\ 0 & 0 & 0 & l_3 & l_4 & 0 \\ 0 & l_1 & l_3 & 0 & 0 & 0 \\ 0 & l_2 & l_4 & 0 & 0 & 0 \\ l_1 l_4 - l_2 l_3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the l_i's are independent linear forms in $y_0, \ldots, y_3$.

The equation of Σ is $\Sigma = l_1 l_4 - l_2 l_3$ and $\det(\alpha) = (\Sigma)^4$. Notice that given a matrix (α) of the above form, it is not possible to recover the corresponding surface S. This is because even if the $\mathscr{A}$-module $\mathscr{R}$ is well defined, we cannot recover the *ring structure* from (α) alone (for example Proposition 4. 7 of [Ca 5] does not apply). Geometrically, in fact, it is easy to see that the datum of (α) determines only Σ and the branch curve $\Delta_3 = \Sigma \cap \{y_0 = 0\}$, not Δ_1 and Δ_2 (cf. proof of Proposition 3. 1 for the notations).

2) *Σ is a rational quartic.* In this case, the canonical curve is hyperelliptic. The matrix (α) can be chosen of the form:

$$(\alpha)=\begin{pmatrix} 0 & 0 & 0 & wq_3 & wq_4 & Q \\ 0 & 0 & 0 & l_1 & l_2 & w \\ 0 & 0 & 0 & l_3 & l_4 & 0 \\ wq_3 & l_1 & l_3 & 0 & 0 & 0 \\ wq_4 & l_2 & l_4 & 0 & 0 & 0 \\ Q & w & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the l_i's are linear forms, q_3, q_4, Q quadratic forms such that $Q=l_1 l_4 - l_2 l_3$, $q_3, q_4 \in (l_3, l_4)$.

Notice that (α) can be written as $\begin{pmatrix} 0 & M \\ {}^tM & 0 \end{pmatrix}$ where M is a 3×3 matrix.

An equation of Σ is $\det(M)=0$. The double line of Σ has equations $l_3=l_4=0$.

The matrix (α) only depends on the surface Σ in $\mathbb{P}^3$ and *not on the choice of the ramification divisor* of the map $S \to \Sigma$. Therefore, it is not possible to recover the surface S from the matrix (α).

3) *Σ is a $K3$ surface.* The canonical curve is not hyperelliptic. The matrix (α) can be chosen of the form:

$$(\alpha)=\begin{pmatrix} \det(a) & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & (a) & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}$$

where (a) is a symmetric 4×4 matrix of linear forms. An equation for Σ is $\Sigma=\det(a)$. The rank condition is obviously always satisfied.

Conversely, to any symmetric 4×4 matrix of linear forms (a) such that the surface defined in $\mathbb{P}$ by the equation $\det(a)=0$ is irreducible with only rational double points as singularities, we can associate a minimal surface S with $K^2=2$, $p_g=1$ and $q=0$ such that the bicanonical image of S is Σ and such that the double cover $S\to\Sigma$ is branched over the plane section $\Sigma\cap\{y_0=0\}$ and some singular points of Σ (notice that the hyperplane section Δ_0 of Σ over which $\Psi: X\to\Sigma\subset\mathbb{P}^3$ is ramified (cf. 3. 15) is fixed by our choice of an isomorphism $\mathbb{P}\xrightarrow{\sim}\mathbb{P}^3$).

Therefore, the family of such surfaces is irreducible of dimension 12 in the moduli space.

4) Φ *is birational.* This case splits into two subcases. *If the canonical curve is hyperelliptic*, then the matrix (α) can be chosen of the form:

$$(\alpha)=\begin{pmatrix} y_0G+\det(a) & wq_1 & wq_2 & wq_3 & wq_4 & Q \\ wq_1 & \multicolumn{4}{c}{} & w \\ wq_2 & & & & & 0 \\ wq_3 & & (a) & & & 0 \\ wq_4 & & & & & 0 \\ Q & w & 0 & 0 & 0 & 0 \end{pmatrix}$$

where q_1, q_2, q_3, q_4, Q are quadratic forms in $(y_0, \ldots, y_3)$, G is a cubic form, (a) is a symmetric matrix of linear forms. All the 3×3-minors of (a) are in the ideal (y_0, Q) and $\det(a)\in(y_0, Q^2)$.

Furthermore, (α) satisfies (R.C.).

If the canonical curve is non-hyperelliptic, the matrix (α) can be chosen of the form:

$$(\alpha)=\begin{pmatrix} y_0G+\det(a) & wq_1 & wq_2 & wq_3 & wq_4 \\ wq_1 & & & & \\ wq_2 & & (a) & & \\ wq_3 & & & & \\ wq_4 & & & & \end{pmatrix}$$

with $G, q_1, \ldots, q_4$ as above, (a) a symmetric matrix of linear forms.

Again, (α) satisfies (R.C.).

Conversely, for any matrix (α) belonging to an open subset (which may a priori be empty!) of the set of matrices satisfying one of those two sets of conditions (which are all *closed* conditions), it is possible to define a *ring structure* on the $\mathscr{A}$-module $\mathscr{R}$ which (α) defines. The surface $X=\operatorname{Proj}(\mathscr{R})$ is the canonical model of a minimal surface S with $K_S^2=2$, $p_g=1$, $q=0$ for which the bicanonical map is birational onto an octic in $\mathbb{P}^3$ with equation $\det(\alpha)$.

Geometrically, the surface X is obtained as the blow-up of the surface Σ along the ideal generated by the $(h\times h)$-minors of (α). This ideal becomes the conductor of $\Psi: X\to\Sigma$.

Proof of the Theorem. Case 1). The situation is as follows:

$$\begin{array}{ccc} S & \xrightarrow{\ \Phi\ } & \mathbb{P}^3 \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle u}\,\wr \\ X & \xrightarrow{\ \Psi\ } & \mathbb{P}=\operatorname{Proj}(\mathscr{A}), \end{array}$$

and Φ induces a $(\mathbb{Z}/2\mathbb{Z})^2$-Galois covering of the smooth quadric Σ. Let $\mathcal{O}$ be the ring $\mathbb{C}[Y_0, Y_1, Y_2, Y_3]$ graded by $\deg Y_i = 1$ for $i=0,\dots,3$ (isomorphic to the graded ring $\mathcal{A}^{(2)}$ in the notations of [Gr], § 2), so that $\mathbb{P}^3 = \mathrm{Proj}(\mathcal{O})$. The $\mathcal{O}$-module $\mathcal{R}$ splits as the sum of two $\mathcal{O}$-modules $\mathcal{R}'$ and $\mathcal{R}''$ such that:

$$\mathcal{R}' = \bigoplus_{m=0}^{\infty} \mathcal{R}_{2m}, \qquad \tilde{\mathcal{R}}' = \Phi_* \mathcal{O}_S,$$

$$\mathcal{R}'' = \bigoplus_{m=0}^{\infty} \mathcal{R}_{2m+1}, \qquad \tilde{\mathcal{R}}'' = \Phi_* \omega_S.$$

More precisely, with the notations of the proof of 3. 1, set $R_i = \mathrm{Fix}\,\sigma_i$ for $1 \leqq i \leqq 3$. There exist $z_i \in H^0(S, \mathcal{O}(R_i))$, for $1 \leqq i \leqq 3$, such that:

$$\mathrm{div}(z_i) = R_i, \qquad z_3 = w \in H^0(S, K_S),$$
$$z_i^2 \in \Phi^* H^0(\Sigma, \mathcal{O}_\Sigma(3,1)); \quad z_2^2 \in \Phi^* H^0(\Sigma, \mathcal{O}_\Sigma(1,3)).$$

It follows from (2. 13) and (2. 15) of [Ca 7] that:

$$\mathcal{R}'_m = \mathcal{R}_{2m} \cong \Phi^* H^0(\mathcal{O}_\Sigma(m)) \oplus z_3 z_1 \Phi^* H^0(\mathcal{O}_\Sigma(m-2, m-1))$$
$$\oplus\, z_3 z_2 \Phi^* H^0(\mathcal{O}_\Sigma(m-1, m-2)) \oplus z_1 z_2 \Phi^* H^0(\mathcal{O}_\Sigma(m-2)),$$

$$\mathcal{R}''_m = \mathcal{R}_{2m+1} \cong z_3 \Phi^* H^0(\mathcal{O}_\Sigma(m)) \oplus z_1 \Phi^* H^0(\mathcal{O}_\Sigma(m-1, m))$$
$$\oplus\, z_2 \Phi^* H^0(\mathcal{O}_\Sigma(m, m-1)) \oplus z_1 z_2 z_3 \Phi^* H^0(\mathcal{O}_\Sigma(m-2)).$$

Let

$\{u_1, u_2\}$ be a basis for $H^0(\mathcal{O}_\Sigma(0,1))$,

$\{v_1, v_2\}$ be a basis for $H^0(\mathcal{O}_\Sigma(1,0))$.

Now $u_i v_j \in H^0(\mathcal{O}_\Sigma(1,1)) \cong H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ will be written as a linear form l_{ij} $(1 \leqq i, j \leqq 2)$. The $\mathcal{O}$-module $\mathcal{O}_\Sigma(-1,0)$ has the following minimal resolution:

$$0 \longrightarrow \mathcal{O}(-2)^2 \xrightarrow{\gamma} \mathcal{O}(-1)^2 \longrightarrow \mathcal{O}_\Sigma(-1,0) \longrightarrow 0,$$

corresponding to the generators u_1, u_2, with $(\gamma) = \begin{pmatrix} l_{21} & -l_{11} \\ l_{22} & -l_{12} \end{pmatrix}$. Notice that

$$Q = -l_{21} l_{12} + l_{22} l_{11}$$

is an equation for Σ. Accordingly, we have a minimal resolution for $\mathcal{O}_\Sigma(0,-1)$, corresponding to the generators $v_2, -v_1$:

$$0 \longrightarrow \mathcal{O}(-2)^2 \xrightarrow{{}^t\gamma} \mathcal{O}(-1)^2 \longrightarrow \mathcal{O}_\Sigma(0,-1) \longrightarrow 0.$$

It follows that $\{1, wz_1u_1, wz_1u_2, wz_2v_1, wz_2v_2, z_1z_2\}$ generate the $\mathcal{O}$-module $\mathcal{R}'$ and that $\{w, z_1u_1, z_1u_2, z_2v_1, z_2v_2, wz_1z_2\}$ generate $\mathcal{R}''$. Then

$$\{1, z_1u_1, z_1u_2, z_2v_2, -z_2v_1, z_1z_2\}$$

generate $\mathcal{R}$ as an $\mathcal{A}$-module and the corresponding matrix (α) is of the desired form (set $l_1=l_{21}$, $l_2=l_{22}$, $l_3=-l_{11}$, $l_4=-l_{12}$). Q.E.D.

Case 2). The morphism $\Phi: S \longrightarrow \Sigma$ is 2-to-1 onto a non-normal rational quartic. With the notations of the proof of 3. 3, the situation is as follows:

$$\begin{array}{ccccc} \hat{S} & \xrightarrow{\pi} & \hat{P} & \xrightarrow{\eta} & P \\ \Big\downarrow{\scriptstyle\varepsilon} & & & & \Big\downarrow{\scriptstyle\Phi_{|H|}} \\ S & & \xrightarrow{\quad\Phi\quad} & & \Sigma \subset \mathbb{P}^3 \end{array}$$

where ε, η and $\Phi_{|H|}$ are birational.

The canonical ring splits into a direct sum of two $\mathcal{A}$-modules:

$$\mathcal{R} = \mathcal{R}' \oplus \mathcal{R}'',$$

where

$$\mathcal{R}' = \bigoplus_{m=0}^{\infty} H^0(P, mK_P + m\delta),$$

$$\mathcal{R}'' = \bigoplus_{m=0}^{\infty} H^0(P, mK_P + (m-1)\delta).$$

Notice that since $w \in \mathcal{R}_1 \cong H^0(P, K_P+\delta)$ is σ-anti-invariant $(p_g(P)=0)$, $\mathcal{R}'_m$ is the eigenspace corresponding to the eigenvalue $(-1)^m$ for the action of σ on $\mathcal{R}_m$.

Finally, $\mathcal{R}'$ is a subring of $\mathcal{R}$ while $\mathcal{R}''$ is not a ring.

Lemma 4. 7. *$\mathcal{R}'$ is a Cohen-Macaulay $\mathcal{A}$-module.*

Proof. We follow the proof of Proposition 2. 8 of [Ca 5]. Choose

$$\{y_0, y_1, y_2\} \in H^0(S, 2K_S) \cong H^0(P, 2(K_P+\delta)) = \mathcal{R}'_2$$

such that $\bigcap_{i=0}^{2} \operatorname{div}(y_i) = \emptyset$. We show that $\{y_0, y_1, y_2\}$ is a $\mathcal{R}'$-regular sequence. Suppose $y_1r'_1 + y_0r'_0 = 0$ with $r'_0, r'_1 \in \mathcal{R}'$. We can suppose r'_0 and r'_1 to be homogeneous of the same degree m. Since y_1 is not a zero divisor in $\mathcal{R}/y_0\mathcal{R}$, there exists $r_1 \in \mathcal{R}_{m-2}$ such that $r'_1 = y_0r_1$. But then $\sigma r'_1 = (-1)^m r'_1 \Rightarrow \sigma r_1 = (-1)^m r_1 \Rightarrow r_1 \in \mathcal{R}'$.

Again, if $y_2r'_2 + y_1r'_1 + y_0r'_0 = 0$, with $r'_i \in \mathcal{R}'_m$, we get:

$$\exists r_2^0, r_2^1 \in \mathcal{R}_{m-2} \qquad r'_2 = y_0r_2^0 + y_1r_2^1$$

$$\Rightarrow \sigma r'_2 = y_0\sigma r_2^0 + y_1\sigma r_2^1$$

$$\Rightarrow 2r'_2 = y_0[r_2^0 + (-1)^m\sigma r_2^0] + y_1[r_2^1 + (-1)^m\sigma r_2^1] \in y_0\mathcal{R}'_{m-2} + y_1\mathcal{R}'_{m-2}.$$

Hence y_2 is not a zero divisor in $\mathscr{R}'/(y_0, y_1)\mathscr{R}'$ and $\{y_0, y_1, y_2\}$ is a $\mathscr{R}'$-regular sequence. Q.E.D.

Lemma 4.8. *If* $0 \longrightarrow \bigoplus_{j=0}^{h} \mathscr{A}(-r_j) \xrightarrow{\alpha'} \bigoplus_{j=0}^{h} \mathscr{A}(-l_j) \longrightarrow \mathscr{R}' \longrightarrow 0$ *is a free resolution for* $\mathscr{R}'$, *then*:

$$0 \longrightarrow \bigoplus_{j=0}^{h} \mathscr{A}(l_j-8) \xrightarrow{{}^t\alpha'} \bigoplus_{j=0}^{h} \mathscr{A}(r_j-8) \longrightarrow \mathscr{R}'' \longrightarrow 0$$

is a free resolution for $\mathscr{R}''$.

Proof. We follow the proof of (2.15) of [Ca 5]. If (y) is the maximal ideal $(y_0, \ldots, y_3)$ in $\mathscr{A}$, there is an exact sequence:

$$0 \longrightarrow H^3((y), \mathscr{R}') \longrightarrow H^4((y), \bigoplus \mathscr{A}(-r_j)) \xrightarrow{\alpha'} H^4((y), \bigoplus \mathscr{A}(-l_j)) \longrightarrow 0.$$

By duality

$$H^4((y), \mathscr{A}(-r)) \cong \mathscr{A}(r-7).$$

We also have:

$$H^3((y), \mathscr{R}')^\vee \cong H^2(P, \bigoplus_{n\in\mathbb{Z}} \mathscr{O}_P(-n(K_P+\delta)))$$

which is isomorphic, by Serre duality on P, to:

$$\bigoplus_{n\in\mathbb{Z}} H^0(P, (n+1)K_P+n\delta).$$

By transposing the above exact sequence, we get a free resolution for $\mathscr{R}''(1)$. Q.E.D.

Notice that we also get a resolution for $\mathscr{R}$, which is symmetric! The corresponding matrix (α) is:

$$(\alpha) = \begin{pmatrix} 0 & {}^t(\alpha') \\ (\alpha') & 0 \end{pmatrix}.$$

We construct now a minimal free resolution for $\mathscr{R}'$. We have:

$$\dim \mathscr{R}_3'' = h^0(P, 3K_P+2\delta) = h^0(P, F) = 2$$
$$\Rightarrow \dim \mathscr{R}_3' = 8-2 = 6.$$

Therefore we need to take as generators for $\mathscr{R}'$:

$$1 \in \mathscr{R}_0'; \quad v_1, v_2 \in \mathscr{R}_3'.$$

Since the canonical curve is hyperelliptic, we can furthermore choose v_1 such that there is a relation:

$$Q(y)+wv_1=0$$

(because $\sigma v_1=-v_1 \Rightarrow v_1 \in \mathscr{R}'_3$).

We also choose $v_3, v_4 \in \mathscr{R}''_3$ to get a basis for $\mathscr{R}_3$. Finally, we want to show that the last element $v_5 \in \mathscr{R}_4$ needed to generate $\mathscr{R}$ is in $\mathscr{R}''_4$. From the exact sequence (notations of the proof of Theorem 3. 3):

$$0 \longrightarrow \mathcal{O}_P(H) \longrightarrow \mathcal{O}_P(2H) \longrightarrow \mathcal{O}_H(2H) \longrightarrow 0$$

and

$$H^1(P, H)=0,$$
$$g(H)=2,$$

we get:

$$\dim \mathscr{R}'_4 = h^0(P, 2H)=4+8+1-2=11.$$

Therefore $y_i y_j$, wv_2 form a basis for $\mathscr{R}'_4$ and we can take $v_5 \in \mathscr{R}'_4$.

Now it is easy to see that since we have a minimal set of homogeneous generators for $\mathscr{R}$ (according to Theorem 4. 3) which are all invariant or anti-invariant under σ, the ones which are in $\mathscr{R}'$ generate this $\mathscr{A}$-module. Therefore, we have constructed a minimal resolution for $\mathscr{R}'$, of the form:

$$0 \longrightarrow \mathscr{A}(-r_0) \oplus \mathscr{A}(-r_1) \oplus \mathscr{A}(-r_2) \xrightarrow{\alpha'} \mathscr{A} \oplus \mathscr{A}(-3)^2 \longrightarrow \mathscr{R}' \longrightarrow 0.$$

By Lemma 4. 8, and the fact that to generate $\mathscr{R}''$, one needs at least two elements in degree 3 and one in degree 4, we get:

$$r_0=r_1=5, \quad r_2=4.$$

The matrix (α') can be written as:

$$(\alpha')=\begin{pmatrix} wq_3 & l_1 & l_3 \\ wq_4 & l_2 & l_4 \\ Q & w & 0 \end{pmatrix}.$$

Notice that:

$$\det(\alpha') \text{ is an equation for } \Sigma,$$
$$\Sigma \equiv Q^2 \pmod{y_0},$$

hence

$$Q \equiv l_1 l_4 - l_2 l_3 \pmod{y_0}.$$

By changing q_3, q_4 and Q, we can always suppose that:

$$Q = l_1 l_4 - l_2 l_3.$$

From the resolution of $\mathscr{R}'$, it is easy to get a resolution of $\mathscr{R}'_{\text{even}} = \bigoplus_{m=0}^{\infty} \mathscr{R}'_{2m}$ by two free $\mathscr{A}_{\text{even}}$-modules of rank 2, corresponding to the generators $\{1, wv_2\}$. The corresponding matrix is:

$$(\alpha'_{\text{even}}) = \begin{pmatrix} (y_0 q_3 - l_1 Q) & l_3 \\ (y_0 q_4 - l_2 Q) & l_4 \end{pmatrix}.$$

But it is easy to see that $\mathscr{A}_{\text{even}} = \mathcal{O}_{\mathbb{P}^3}$ and that $\mathscr{R}'_{\text{even}}$ is the $\mathcal{O}_{\mathbb{P}^3}$-module associated to the sheaf $\Phi_{|H|*}(\mathcal{O}_P)$ (cf. [Ca 5], Proposition 2. 7). Now, this sheaf has rank 2 on the double line of Σ, therefore (α'_{even}) must vanish there.

Hence the equation of the double line of Σ is $l_3 = l_4 = 0$ and $y_0 q_3, y_0 q_4 \in (l_3, l_4)$.

Now, any linear combination of l_3 and l_4 corresponds to the sum of a line through p_5 and the unique cubic through $\mathscr{P}$ (cf. 3. 4. 2) a)) while y_0 corresponds to twice the unique conic through $\{p_1, \ldots, p_5\}$ (as elements of $|H|$).

Therefore $\{y_0, l_3, l_4\}$ are independent and $q_3, q_4 \in (l_3, l_4)$.

Notice that any non-zero (5×5)-minor of the matrix:

$$(\alpha) = \begin{pmatrix} 0 & {}^t(\alpha') \\ (\alpha') & 0 \end{pmatrix}$$

is the product of a 2×2-minor of ${}^t(\alpha')$ by $\Sigma = \det(\alpha')$. Since ${}^t(\alpha')$ itself satisfies (R.C.), (α) also satisfies (R.C.).

However, it is not possible to recover the ring structure of the canonical ring $\mathscr{R}$ from the matrix (α). In fact (α) only depends on the surface Σ and not on the choice of the branch locus Δ of $\hat{S} \xrightarrow{\eta\pi} P$, since $\mathscr{R}'$ only depends on P, by its very definition. The divisor Δ comes in only in the definition of the multiplication in $\mathscr{R}$ of two elements of $\mathscr{R}''$:

$$H^0(P, mK_P + (m-1)\delta) \otimes H^0(P, m'K_P + (m'-1)\delta) \longrightarrow H^0(P, (m+m')(K_P+\delta),$$
$$s \otimes s' \qquad \longmapsto \qquad s \cdot s' \cdot s_\Delta.$$

Case 3). The form of the matrix (α) follows from [Ca 5], Theorem 3. 22, since Σ is normal. Notice that this reproves 3. 15 since Σ is a "symmetroid" which satisfies, according to [Ca 5], the conditions 3. 16.

The converse follows from Remark 3. 17. 2).

The matrix (a) must therefore belong to the open set $\mathscr{U}$ of the affine space of symmetric matrices of linear forms such that $\det(a) = 0$ has only rational double points as singularities. The matrix (a) is determined up to the action $g \cdot a = {}^t g a g$ of $\mathrm{GL}(4)$. The corresponding image in the moduli space is $\mathscr{U}/\mathrm{GL}(4) \times \mathrm{Aut}\,\mathbb{P}$, irreducible unirational of dimension $(10 \times 4) - 16 - 12 = 12$. Q.E.D.

Case 4). It follows from [Ca 5], Theorem 3. 8 that in this case, the matrix (α) can be chosen symmetric.

If the canonical curve is hyperelliptic, the degrees of the elements of (α) are, according to Theorem 4. 3:

$$\begin{pmatrix} 8 & 5 & 5 & 5 & 5 & 4 \\ 5 & & & & & 1 \\ 5 & & & & & 1 \\ 5 & & 2 & & & 1 \\ 5 & & & & & 1 \\ 4 & 1 & 1 & 1 & 1 & \alpha_{55}=0 \end{pmatrix}.$$

Since $\mathscr{A}_1=\mathbb{C}w$ and since $\alpha_{15},\dots,\alpha_{45}$ cannot be all zero ($\det(\alpha)=\Sigma$ would then be divisible by α_{05}, a quadratic form in the y_i's and Σ would be a quadric), we can suppose that $\alpha_{15}=w$, $\alpha_{i5}=0$ for $2\leqq i\leqq 5$.

Recall that the intersection of Σ with the hyperplane $\{y_0=0\}$ is the conic $\{y_0=Q=0\}$ counted four times. Therefore $Q^4\equiv\det(\alpha) \pmod{y_0}$ and $\det(a)\in(y_0, Q^2)$.

The matrix (α) satisfies the rank condition.

If we compute the 5×5-minors of (α) in $A/(w)$ we get:

$$I\subset(Q\det(a), w)=(Q^3, w)$$

hence

$$Q^2\cdot(\text{any } 3\times 3\text{-minor of } (a))\in(Q^3, y_0)$$

and

$$\alpha_{00}\cdot\det(a)\in(Q^3, y_0).$$

Therefore, α_{00} can be written as $y_0G'+QQ'$. By performing standard operations with the first and sixth rows and columns of (α) we can obtain

$$\alpha_{00}=y_0G'+\lambda Q^2=y_0G+\det(a).$$

In the non-hyperelliptic case, the intersection $\{\Sigma=y_0=0\}$ is twice the image of the canonical curve. Hence $\alpha_{00}\cdot\det(a)$ is a square modulo y_0. If $\det(a)$ does not contain a square modulo y_0, we can conclude that $\alpha_{00}\in(y_0, \det(a))$.

To conclude in general, remark that the conductor ideal I is contained in $(w, \det(a))$.

By the rank condition, we get:

$$\alpha_{00}\cdot(\text{any } 3\times 3\text{-minor of } (a))\in(y_0, \det(a)).$$

i) If $\det(a) \equiv Q^2 \pmod{y_0}$, Q irreducible quadric in y_1, y_2, y_3, then:

$$Q^2 \mid \alpha_{00} \cdot (\text{any } 3\times 3\text{-minor of } (a)) \ (\mathrm{mod}\,(y_0))$$
$$\Rightarrow Q \mid \alpha_{00} \ (\mathrm{mod}\,(y_0)).$$

Since $\alpha_{00} \cdot Q^2$ is a square modulo y_0, it follows that $\alpha_{00} \in (Q^2, y_0)$.

ii) If $\det(a) \equiv l_1 l_2 l^2 (y_0)$, $\alpha_{00} \equiv l_1 l_2 m^2 (y_0)$.

Since $\Phi(C)$ has at most two components, this case cannot happen.

Conversely, if we are given a square matrix (α) of homogeneous elements of $\mathscr{A}$ of one of the two forms above, such that $\det(\alpha)$ is an irreducible polynomial of degree 16 defining a surface Σ in $\mathbb{P} = \mathrm{Proj}(\mathscr{A})$, we can define an $\mathscr{A}$-module $\mathscr{R}$ by the exact sequence of Theorem 4. 3. If furthermore (α) satisfies (R.C.) and if $(\beta_{ij})_{0\leqq i,j\leqq h}$ are the maximal minors of (α), we can write:

$$\beta_{ij} = \sum_{k=0}^{h} l_{ij}^k \beta_{0k}, \qquad 1 \leqq i, j \leqq h,\ h=4 \text{ or } 5.$$

We can then make $\mathscr{R}$ into an $\mathscr{A}$-algebra by Proposition 4. 7 of [Ca 5]; the multiplication is defined on the generators $v_1, \ldots, v_h$ of $\mathscr{R}$ by the formulae:

$$v_i v_j = \sum_{k=0}^{h} l_{ij}^k v_k, \qquad 1 \leqq i, j \leqq h.$$

If moreover $X = \mathrm{Proj}(\mathscr{R})$ has at most rational double points as singularities, X is the canonical model of a minimal regular surface S with $K^2=2$, $p_g=1$, $q=0$, for which the bicanonical morphism is birational with image Σ (by Theorem 4. 22 of [Ca 5]).

§ 5. Construction of a component of the moduli space

It follows from Theorem 3. 24 that there is a unique irreducible component $\mathscr{M}_{\mathrm{E}}$ of the moduli space of our surfaces (E for Enriques) which contains $\mathscr{M}_1$ (surfaces for which Σ is a smooth quadric). Moreover, $\mathscr{M}_{\mathrm{E}}$ is 16-dimensional and contains $\mathscr{M}_2 \cup \mathscr{M}_3$ (surfaces for which Σ is a quartic). We conjecture that the moduli space of surfaces with $K^2=2$, $p_g=1$, $q=0$ and torsion zero is irreducible, hence equal to $\mathscr{M}_{\mathrm{E}}$.

In this paragraph, we are going to describe an irreducible family of these surfaces dominating a Zariski open set of $\mathscr{M}_{\mathrm{E}}$. These surfaces will be such that $\Phi: S \to \Sigma$ is birational and the canonical curve is non-hyperelliptic, and for a general surface, $\Sigma \subset \mathbb{P}^3$ will have ordinary singularities.

To explain the essence of this construction, due to Enriques ([En]) and based on the existence of a biadjoint surface Ξ to Σ, projection of a Del Pezzo surface Y in $\mathbb{P}^7$, we recall a few facts about the surfaces S.

Let S be a surface for which the canonical curve C is non-hyperelliptic and for which the bicanonical map Φ is birational. It follows from Theorem 4. 6 case 4) that the matrix (α) giving the resolution 4. 3. i) of the canonical ring $\mathscr{R}$ of S can be chosen of the form:

$$(5.1)\qquad (\alpha)=\begin{pmatrix} y_0G+\det(a) & wq_1 & wq_2 & wq_3 & wq_4 \\ wq_1 & & & & \\ wq_2 & & (a) & & \\ wq_3 & & & & \\ wq_4 & & & & \end{pmatrix}$$

and satisfies the rank condition (R.C.) (cf. § 4). Therefore, there exist polynomials $\lambda^i_{jk}(y)$, $0\leqq i\leqq 4$, $1\leqq j,\ k\leqq 4$ such that the cofactors β_{ij} (equal to $(-1)^{i+j}$ times the (i,j)-minor of (α)) satisfy:

$$(5.2)\qquad \beta_{jk}=\lambda^0_{jk}\beta_{00}+w\sum_{i=1}^4 \lambda^i_{jk}\beta_{0i}.$$

The canonical model X of S is then isomorphic to the subscheme of

$$\mathbb{P}(1,2,2,2,3,3,3,3)$$

defined by the following equations in the coordinates $w, y_1, y_2, y_3, v_1, \dots, v_4$ (with $y=(w^2, y_1, y_2, y_3)$):

$$(5.3)\qquad \begin{cases} \sum_{j=1}^4 a_{ij}(y)v_j+q_i(y)w=0, \\ \sum_{j=1}^4 q_j(y)wv_j+y_0G(y)+(\det a)(y)=0, \\ v_jv_k-\lambda^0_{jk}(y)-\sum_{i=1}^4 \lambda^i_{jk}(y)wv_i=0. \end{cases}$$

Leaving aside geometrical considerations, for which we refer to [En], we associate to the matrix (α) the following symmetric matrix $(\tilde{\alpha})$, with entries in $\mathbb{C}[y_0, \dots, y_3]$:

$$(5.4)\qquad (\tilde{\alpha})=\begin{pmatrix} G & q_1 & \cdots & q_4 \\ q_1 & a_{11} & \cdots & a_{14} \\ \vdots & \vdots & & \vdots \\ q_4 & a_{14} & \cdots & a_{44} \end{pmatrix}.$$

Lemma 5. 5. *If y_0 does not divide* $\det(a)$, *the matrix $(\tilde{\alpha})$ satisfies the rank condition if and only if (α) does.*

Proof. The cofactors $\tilde{\beta}_{ij}$ of $(\tilde{\alpha})$ satisfy $\tilde{\beta}_{00}=\det(a)=\beta_{00}$, $\beta_{0i}=w\tilde{\beta}_{0i}$ for $i\leqq 1$, $\beta_{jk}=b_{jk}\det(a)+y_0\tilde{\beta}_{jk}$ for $1\leqq j,\ k\leqq 4$, where b_{jk} is the (j,k)-cofactor of (a).

If (α) satisfies (R.C.), we have by (5. 2):

$$\beta_{00}\, b_{jk} + y_0\, \tilde{\beta}_{jk} = \beta_{jk} = \lambda^0_{jk}\, \beta_{00} + w \sum_{i=1}^{4} \lambda^i_{jk}\, w\, \tilde{\beta}_{0i}.$$

Since y_0 does not divide β_{00}, it does divide $(\tilde{\lambda}^0_{jk} - b_{jk})$ and we define $\tilde{\lambda}^0_{jk}$ to be the quotient. Then $\tilde{\beta}_{jk} = \tilde{\lambda}^0_{jk}\, \tilde{\beta}_{00} + \sum_{i=1}^{4} \lambda^i_{jk}\, \tilde{\beta}_{0i}$, so that $(\tilde{\alpha})$ satisfies (R.C.). The converse is immediate. Q.E.D.

To the matrix $(\tilde{\alpha})$, we associate a subscheme Y of $\mathbb{P}^7$, defined by the following equations in the coordinates $(y_0, \ldots, y_3, x_1, \ldots, x_4)$:

$$(5.6)\qquad \begin{cases} \sum_{j=1}^{4} a_{ij}(y)\, x_j + q_i(y) = 0, \\ \sum_{j=1}^{4} q_j(y)\, x_j + G(y) = 0, \\ x_j x_k - \tilde{\lambda}^0_{jk}(y) - \sum_{i=1}^{4} \lambda^i_{jk}(y)\, x_i = 0. \end{cases}$$

We let also $\Xi \subset \mathbb{P}^3$ be the surface with equation $\det(\tilde{\alpha}) = 0$. It is classically called a biadjoint surface ([En]).

We claim that the projection $\tilde{\psi}: Y \to \Xi$ is birational.

In fact, since $(\beta_{00}, \beta_{01}, \ldots, \beta_{04}) = (\beta_{00}, w\tilde{\beta}_{01}, \ldots, w\tilde{\beta}_{04}) \subset (w, \beta_{00}) \cap (\tilde{\beta}_{00}, \tilde{\beta}_{01}, \ldots, \tilde{\beta}_{04})$ (we are considering ideals in $\mathbb{C}[w, y_1, y_2, y_3]$) the zero set Γ of the ideal $(\tilde{\beta}_{00}, \ldots, \tilde{\beta}_{04})$ has codimension 1 in Σ. By the arguments of [Ca 5], 4. 7, 4. 15 and 4. 19, a rational inverse for $\tilde{\psi}$ is given outside Γ by $(y_0, \ldots, y_3) \to (y_0\, \tilde{\beta}_{00}, \ldots, y_3\, \tilde{\beta}_{00}, \tilde{\beta}_{01}, \ldots, \tilde{\beta}_{03})$ and Y is the blow-up of Ξ along Γ.

If Y is irreducible with only rational double points as singularities, a slight generalization of Theorem 4. 22 of [Ca 5] yields that Y is a Del Pezzo surface (i.e. $\omega_Y \cong \mathcal{O}_Y(-1)$).

We describe now the converse construction, starting from a smooth Del Pezzo surface Y of degree 7 in $\mathbb{P}^7$. A general projection $\tilde{\psi}: Y \to \mathbb{P}^3$ is birational and its image Ξ has ordinary singularities. This projection makes the projective coordinate ring $\mathscr{R}$ into a module over the polynomial ring $\mathscr{A} = \mathbb{C}[y_0, \ldots, y_3]$, thereby yielding ([Ca 5], Theorem 4. 3) a matrix $(\tilde{\alpha})$ as in (5. 4), satisfying (R.C.), and generators of the projective ideal of Y as in (5. 6). Setting $y_0 = w^2$ and $\lambda^0_{jk} = b_{jk} + y_0\, \tilde{\lambda}^0_{jk}$ (notations of the proof of 5. 5), we define a subscheme $\hat{X}$ of $\mathbb{P}(1, 2, 2, 2, 3, 3, 3, 3)$ by the equations (5. 3). The problem is now to show that for generic choice of $\tilde{\psi}$ and y_0, the surface $\hat{X}$ has at most rational double points as singularities (in fact, it will be smooth).

To simplify notations, we set $F = \tilde{\beta}_{00} = \det(a)$, and we identify a surface with its defining equation.

Remarks 5.7. 1) The curve Γ defined by the ideal $(\tilde{\beta}_{00}, \tilde{\beta}_{01}, \ldots, \tilde{\beta}_{04})$ is the singular locus of Ξ. Its only singularities are transversal triple points and, for a general projection $\tilde{\varphi}$, it is *irreducible* ([Fr] or [En], pages 8 to 13).

For generic choice of y_0, the surface Σ with equation $\det(\alpha)=y_0 \cdot \Xi + F^2$ is smooth outside $\Xi \cap F = \Gamma$ by Bertini's theorem; the intersection $\Gamma \cap \{y_0=0\}$ is a finite set of 14 points and $C' = F \cap \{y_0=0\}$ is a smooth quartic curve.

2) In the ring $\mathbb{C}[w, y_1, y_2, y_3]$, the equalities

$$(\beta_{00}, \beta_{01}, \ldots, \beta_{04}) = (\tilde{\beta}_{00}, w\tilde{\beta}_{01}, \ldots, w\tilde{\beta}_{04}) = (w, \tilde{\beta}_{00}) \cap (\tilde{\beta}_{00}, \ldots, \tilde{\beta}_{04})$$

hold. This follows easily from the finiteness of $\Gamma \cap \{y_0=0\}$.

Proposition 5.8. *Let $Y \subset \mathbb{P}^7$ be a smooth Del Pezzo surface of degree 7 and let $\tilde{\varphi}: Y \longrightarrow \mathbb{P}^3$ be a linear projection inducing a birational morphism onto a surface Ξ with ordinary singularities. Then, for a general linear form y_0 on $\mathbb{P}^3$, the surface*

$$\Sigma = y_0 \cdot \Xi + F^2$$

has ordinary singularities.

(The surface F is defined above).

Proof. Recall from 5.7.1) that the singular locus of Σ is Γ. We must now study the nature of the singularities of Σ along Γ. We begin with the triple points of Γ.

Lemma 5.9. *If P is a triple point for Γ, then* $\operatorname{Rank}(a_{ij}(P))=2$ *and P is a conical double point for F. If y_0 does not vanish at P (which holds by 5.8), Σ has an ordinary triple point at P.*

Proof. Since $P \in \Gamma \subset F$, $\det(a_{ij}(P))$ vanishes. If the rank of $(a_{ij}(P))$ were 3 then ([Ca5], 5.4) the equation of Ξ would be locally of the form $\Delta y^2 - \zeta^2 = 0$, where $y, \zeta \in \mathfrak{m}_{\mathbb{P}^3,P}$. It is easy to check that the tangent cone of Ξ at P could not consist of 3 distinct planes.

If $\operatorname{Rank}(a_{ij}(P))=1$, $F=\det(a)$ would have multiplicity at least 3 at P and $2\Gamma = F \cdot \Xi$ would have multiplicity at least 9 at P, which is impossible.

Thus the rank of $(a_{ij}(P))$ is 2 and a local equation for F at P is of the form $xy - z^2 = 0$, for a suitable regular sequence (x, y, z) in $\mathcal{O}_{\mathbb{P}^3,P}$. By (5.10) of [Ca5], any local equation for Ξ at P belongs to $(x, y, z)^3$. Since Ξ has an ordinary triple point at P, it follows immediately that (x, y, z) are local holomorphic coordinates at P. Therefore, P is a conical double point for F.

Now, since F has multiplicity 2 at P, and if y_0 does not vanish at P, Ξ and $\Sigma = y_0 \cdot \Xi + F^2$ have the same tangent cones at P and Σ has therefore an ordinary triple point at P. Q.E.D.

Lemma 5.10. *The surface F is normal, smooth at the smooth points of Γ.*

Proof. By Kronecker's formula ([Ca4], 1.3)), 2Γ is the cycle intersection of F and Ξ. Since Ξ is singular along Γ (Remark 5.7.1)), it follows that F must be smooth at the smooth points of Γ.

Now, any one-dimensional component of Sing F would meet the ample divisor Ξ of $\mathbb{P}^3$, hence the curve Γ, at a point which would have to be singular on Γ. But this is impossible since, by Lemma 5.9, F has isolated singularities at the singular points of Γ.

Therefore, we have proved that F has only isolated singularities, hence is normal.

Q.E.D.

Let now Q be one of the 14 points of intersection of Γ with the plane y_0, which we can suppose not to be a pinch point of Ξ. We will prove that Q is a pinch point for Σ.

In fact, since Γ, F (by 5.10) and the two branches of Ξ at Q are smooth, we can choose local holomorphic coordinates x, y and z at Q such that:

i) $F=x$,

ii) $\Gamma=\{x=z=0\}$,

iii) $\Xi=z(ax+bz)$ with $b(Q)\neq 0$ (since $F\cdot\Xi=2\Gamma$),

iv) $y_0=y$.

Then we have $\Sigma=x^2+yz(ax+bz)$. By Weierstrass' preparation theorem, we can find new holomorphic coordinates such that $\Sigma=x^2+byz^2$. Since b does not vanish at Q, Q is a pinch point for Σ.

Let us now deal with the smooth points P of Γ where y_0 does not vanish, and which are not pinch points for Ξ. Then we can find local holomorphic coordinates (x, y, z) at P satisfying i) to iii) above, with $a(P)\,b(P)\neq 0$.

Then, again, $\Sigma=x^2+y_0z(ax+bz)$ and P is a transversal double point of Σ if

$$\Delta_P=y_0^2a^2-4by_0$$

is non-zero at P, and a pinch point if Δ_P has a simple root on Γ at P.

Since $H^0(\mathcal{O}_{\mathbb{P}^3}(1))\cong H^0(\mathcal{O}_\Gamma(1))$ generates the sheaf of 1-jets with values in $\mathcal{O}_\Gamma(1)$, at least at the smooth points of Γ, a simple dimension argument shows that the sections y_0 for which Δ_P has a double root at P for some point $P\in\Gamma$ form at most a hypersurface in $H^0(\mathcal{O}_\Gamma(1))$.

Therefore, we get only transversal double points or pinch points (for a generic choice of y_0).

Finally, let Q be a pinch point of Ξ. Then there are local holomorphic coordinates (x, y, z) at Q such that:

i) $\Xi=x^2-yz^2$ hence $\Gamma=\{x=z=0\}$,

ii) $F=ax+bz$.

Then $\Sigma=y_0(x^2-yz^2)+(ax+bz)^2$ and Q is a transversal double point for Σ if $(y_0+a^2)b^2-a^2b^2=y_0b^2$ does not vanish at Q. It is enough to show that $b(Q)^2\neq 0$. If $b(Q)$ were zero then $a(Q)$ would be non-zero (since F is smooth at Q by 5.10) and we would have $\Xi\equiv\left(\frac{b^2}{a^2}-y\right)z^2 \pmod F$, this would contradict $\Xi\cdot F=2\Gamma$ (cf. proof of 5.10).

Q.E.D.

Remark 5. 11. More precisely, we have shown that Γ is the double curve (of degree 14) of Σ. The 14 points of intersection of Γ with y_0 are pinch points for Σ (an easy computation shows that the total number of pinch points of Σ is 44). The triple points of Σ are the 10 points P where $\operatorname{Rank}(a_{ij}(P))=2$ (their number follows from the classical computation of Theorem 2. 2 of [Ca4]); they are conical double points on F.

Proposition 5. 12. *Let* Y, Ξ, F, Σ *and* (α) *be as defined above. Then the variety* $\hat{X}$ *defined by the equations* (5. 3) *is the normalization of* Σ *and is a smooth minimal surface with* $K^2=2$, $p_g=1$, $q=0$ *and torsion zero.*

Proof. By [Ca5], (4. 19), $\hat{X}$ is the blow-up of Σ in the ideal $(\beta_{00},\dots,\beta_{04})$, where we are identifying $\mathbb{P}(1,2,2,2)$ with $\mathbb{P}^3$. By 5. 7. 2), $\hat{X}$ is thus the blow-up of Σ along the double curve Γ and the smooth curve C'; therefore $\hat{X}$ is the normalization of Σ and is smooth since Σ has ordinary singularities. The rest follows as in Theorem 4. 24 of [Ca5]. Q.E.D.

We finally proceed to the description of a versal irreducible family for the surfaces we have constructed above.

First of all, fix a smooth Del Pezzo surface Y of degree 7 in $\mathbb{P}^7$: Y is isomorphic to the blow-up of $\mathbb{P}^3$ in two distinct points and $\operatorname{Aut}(Y)$ is a 4-dimensional connected complex Lie group acting linearly on $\mathbb{P}^7$.

Let G be the Grassmannian of 4-dimensional vector subspaces Λ of

$$H^0(\mathcal{O}_{\mathbb{P}^7}(1))\cong H^0(\mathcal{O}_Y(1))$$

and let U_0 be the complement of the zero section of the universal subbundle on G (U_0 is 20-dimensional).

Let $G'\subset G$ be the open set of linear subspaces Λ such that the linear system $|\Lambda|$ has no base-point on Y; any such Λ defines a birational morphism $\tilde{\varphi}: Y\to\Xi\subset\mathbb{P}^3$. Let U_1 be the open set $\{(\Lambda, y_0)\mid \Lambda\in G'\}$. We can define a family of surfaces $p:\mathcal{X}\to U_1$ by choosing, locally on U_1, a basis $\{y_0,\dots,y_3,x_1,\dots,x_4\}$ of $H^0(\mathcal{O}_{\mathbb{P}^7}(1))$ with first element y_0 and such that $y_0,\dots,y_3$ span Λ. With respect to this basis, we can find generators of the projective ideal of Y of the form (5. 6).

Setting $w^2=y_0$, we consider the family of subvarieties $\hat{X}$ of $\mathbb{P}(1,2,2,2,3,3,3,3)$ defined by the equations (5. 3). It is easy to see that two different choices of such basis give rise to isomorphic families, so that we can patch everything together to get a family $p:\mathcal{X}\to U_1$.

Let U_2 be the open set in U_1 corresponding to surfaces $\hat{X}$ with at most R.D.P.'s as singularities. By 5. 12 and deformation arguments, X is then the canonical model X of a surface with $K^2=2$, $p_g=1$, $q=0$.

There is a natural action of $\operatorname{Aut}(Y)$ on U_2. The stabilizers are finite, so that the orbits are smooth of dimension 4. The family $\mathcal{X}|_{U_2}$ yields a morphism $f: U_2\to\mathcal{M}$, where $\mathcal{M}$ is the coarse moduli space for surfaces with $K^2=2$, $p_g=1$ and $q=0$. The image $f(U_2)$ has dimension $\dim U_2-\dim\operatorname{Aut}(Y)=16$.

Let U_3 be the open set in U_2 consisting of the pairs (Λ, y_0) such that Ξ and Σ have ordinary singularities.

Proposition 5. 13. *The image $f(U_3)$ is open in the moduli space $\mathcal{M}$ hence its closure is an irreducible component $\mathcal{M}_E$ of $\mathcal{M}$.*

Proof. For each deformation $(X_t)_{t\in T}$ of a (smooth) surface X corresponding to a point of U_3, we get a family of bicanonical images Σ_t and of biadjoint surfaces Ξ_t. The set of t's for which both have ordinary singularities is open and so is the set of t's for which the surface Y_t in $\mathbb{P}^7$ is smooth. For those t's, there exists a pair $(\Lambda, y_0)\in U_3$ such that Ξ_t is isomorphic to the projection of $Y\cong Y_t$ from $|\Lambda|$ and $\Sigma_t = y_0\cdot\Xi_t + F_t^2$.

We have thus proven that $f(U_3)$ is open in $\mathcal{M}$. Q.E.D.

It remains to show that this component $\mathcal{M}_E$ contains the subvariety $\mathcal{M}_1$ of $\mathcal{M}$ (surfaces for which the bicanonical image is a smooth quadric).

Proposition 5. 14. *The component $\mathcal{M}_E=\overline{f(U_3)}$ contains $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$.*

Proof. Since $\mathcal{M}$ is smooth at a general point of $\mathcal{M}_1$ and since $\mathcal{M}_1$ is contained in the closure of $\mathcal{M}_2$ and of $\mathcal{M}_3$ (Theorem 3. 24), it is enough to show that $\mathcal{M}_3$ is contained in the closure of $f(U_3)$.

We first claim that a general quartic symmetroid occurs as the adjoint surface F of a surface S corresponding to a point of $f(U_3)$.

Start from any surface S corresponding to a point of $f(U_3)$. It follows from Lemma 5. 10 and Remark 5. 11 that the adjoint surface F is normal and has 10 nodes (at least) corresponding to the points P where $\mathrm{Rank}(a(P))=2$. Let $\tilde F\to F$ be a desingularization of F, let $A_1,\dots,A_{10}$ be the inverse images of the 10 nodes of F and let H be the pull-back on $\tilde F$ of a hyperplane section of F. Then the strict transform $\tilde\Gamma$ of Γ on $\tilde F$ is smooth and irreducible (Remark 5. 7. 1)). Moreover, since ω_F is trivial and since F is smooth at the smooth points of Γ (cf. proof of Lemma 5. 10) and has nodes at the singular points of Γ, we have $\omega_{\tilde F}\otimes\mathcal{O}_{\tilde\Gamma}\cong\mathcal{O}_{\tilde\Gamma}$ hence $\mathcal{O}_{\tilde\Gamma}(\tilde\Gamma)\cong\omega_{\tilde\Gamma}$. Since $\tilde\Gamma$ is irreducible, it follows that:

$$h^0(\tilde F, \mathcal{O}_{\tilde F}(\tilde\Gamma)) \leqq 1 + h^0(\tilde\Gamma, \mathcal{O}_{\tilde\Gamma}(\tilde\Gamma)) = 1 + g(\tilde\Gamma) = 2 + \frac{1}{2}\tilde\Gamma^2.$$

From $2\tilde\Gamma \equiv 7H - 3\sum_{i=1}^{10} A_i$, we get $\tilde\Gamma^2=4$ and $\dim|\tilde\Gamma|\leqq 3$.

We now conclude with a simple count of constants: since the Ξ's depend on "12 moduli", the F's that we get depend on at least "12-dim $|\tilde\Gamma|\geqq 9$ moduli", since the datum of the surface F and of the curve of constant Γ determines the surface Ξ ([Ca 5], page 104).

This proves that a general quartic symmetroid occurs as the adjoint surface F of a surface of the family we have constructed.

Therefore, for a general symmetric 4×4 matrix of linear forms (a), there exists a canonical model X corresponding to a point of $f(U_3)$ which is isomorphic to the subscheme of $\mathbb{P}(1,2,2,2,3,3,3,3)$ defined by the equations (5. 3). In particular, there exists a matrix (α) of type (5. 1) which satisfies the rank condition. For all $t\in\mathbb{C}$, we set:

$$(\alpha_t)=\begin{pmatrix} t^2y_0G+\det(a) & twq_1 \cdots twq_4 \\ twq_1 & \\ \cdot & (a(y)) \\ \cdot & \\ twq_4 & \end{pmatrix}.$$

Then (α_t) satisfies (R.C.) and the subscheme X_t of $\mathbb{P}(1,2,2,2,3,3,3,3)$ defined by the equations:

$$\begin{cases} \sum_{j=1}^{4} a_{ij}(y)v_j+tq_i(y)w=0, \\ \sum_{j=1}^{4} tq_j(y)wv_j+t^2y_0G(y)+(\det a)(y)=0, \\ v_jv_k-b_{jk}(y)-t^2y_0\tilde{\lambda}^0_{jk}-\sum_{i=1}^{4} t\lambda^i_{jk}(y)wv_i=0, \end{cases}$$

if it has only R.D.P.'s as singularities, is the canonical model of a surface S_t. Since $X_1=X$ corresponds to a point of the Zariski open subset $f(U_3)$ of the moduli space, X_t also corresponds to a point of $f(U_3)$ for $t\in\mathbb{C}-\Delta$ where Δ is a finite set.

But now, it is easy to see that X_0 is a double cover of the general symmetroid $F=\det(a)$ branched over a smooth hyperplane section and the 10 nodes of F. Therefore X_0 corresponds to a general point of $\mathcal{M}_3$ and since X_t corresponds to a point of $f(U_3)$ for $t\notin\Delta$ hence for t small non zero, we have proven that $\mathcal{M}_3\subset\overline{f(U_3)}$ hence the proposition. Q.E.D.

Our last result concerns the surfaces with $K^2=2$, $p_g=1$ and $q=0$ for which the bicanonical map is birational and the canonical curve is hyperelliptic. It shows, in particular, the existence of those surfaces (cf. also § 6 for a geometrical construction of a family of examples).

Proposition 5. 15. *There is a 15-dimensional (non-empty) subvariety of $\mathcal{M}_E$ corresponding to surfaces for which Φ is birational and C is hyperelliptic.*

Proof. Choose a general surface S_0 in $\mathcal{M}_1$, so that C is a genus 3 hyperelliptic smooth irreducible curve (§ 3). The rational map $\mathcal{M}_E\longrightarrow\bar{M}_3$ (moduli space of stable curves of genus 3) associating to a surface S the isomorphism class of its canonical curve is defined at the smooth point of $\mathcal{M}_E$ corresponding to S_0. The conclusion follows since hyperelliptic curves form a $\mathbb{Q}$-Cartier divisor in $\bar{M}_3$. Q.E.D.

Remark 5. 16. The paper [Ca 8], referring to a previous version of the present paper, contains two errors: in (1. 3) the term $B_{jk}(y)$ should be replaced by

$$B_{jk}(y)+y_0\, l^0_{jk}(y)$$

($b_{jk}(y)+y_0\,\bar{\lambda}^0_{jk}(y)$ in the present notations), whereas the equations (1. 4) do not generate the whole projective ideal of Y. These errors do not affect the validity of any statement of [Ca 8].

§ 6. Double étale covers of Godeaux surfaces

In this paragraph, we describe a certain class of our surfaces, obtained as double étale covers of numerical Godeaux surfaces (see below). They give examples of surfaces for which the canonical curve is hyperelliptic and the bicanonical map is birational.

A (numerical) Godeaux surface is a minimal surface of general type T such that $K_T^2=1$, $p_g(T)=q(T)=0$. We suppose moreover that the torsion of the Picard group of T is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Examples of such surfaces have been constructed in [Ba] and [O-P]. The double étale cover associated to the non-zero torsion element η of $\mathrm{Pic}(T)$ is a minimal surface S with $K_S^2=2$, $p_g(S)=1$, $q(S)=0$, which comes equipped with fixed point-free involution τ. We prove:

Theorem 6. 1. *Let S be the double étale cover of a numerical Godeaux surface with torsion $\mathbb{Z}/2\mathbb{Z}$. Then the bicanonical map of S is birational and its canonical curve is hyperelliptic.*

Proof. The involution τ acts on the canonical ring $\mathscr{R}$ of S, which splits into eigenspaces $\mathscr{R}=\mathscr{R}^+\oplus\mathscr{R}^-$. We have:

$$\mathscr{R}_m^+\cong H^0(T, mK_T), \quad \mathscr{R}_m^-\cong H^0(T, mK_T+\eta)$$

and

$$\dim\mathscr{R}_m^-=\dim\mathscr{R}_m^+=1+\binom{m}{2} \quad \text{for} \quad m\geqq 2.$$

We can choose a set of minimal generators for $\mathscr{R}$ as follows:

	$w\in\mathscr{R}_1^-$,
$y_0=w^2$, $y_1\in\mathscr{R}_2^+$,	$y_2, y_3\in\mathscr{R}_2^-$,
$wy_2, wy_3, v_3, v_4\in\mathscr{R}_3^+$	$wy_0, wy_1, v_1, v_2\in\mathscr{R}_3^-$.

Then $\{y_0^2, y_0y_1, y_1^2, y_2^2, y_2y_3, y_3^2, wv_1, wv_2\}$ are 8 elements of the 7-dimensional vector space $\mathscr{R}_4^+$ and there is a relation:

(6. 2) $$Q_1(y_0, y_1)+Q_2(y_2, y_3)+\lambda wv_1=0.$$

Therefore the canonical curve is hyperelliptic.

Since S has no torsion, there are a priori three cases:

1) Σ is a smooth quadric,

2) Σ is a rational quartic,

3) Φ is birational.

We want to exclude the first two cases.

In each of them, the map Φ is a Galois covering. The involution τ acts on $\mathbb{P}^3$ with signature (2, 2), hence on Σ and the map Φ is τ-equivariant. Therefore τ acts by conjugation on the Galois group G of Φ.

In case 1), $G \cong (\mathbb{Z}/2\mathbb{Z})^2$. It follows from the relation (6. 2) that Σ has equation $Q_1(y_0, y_1) + Q_2(y_2, y_3) = 0$ and therefore that τ does not exchange the pencils of Σ. In the notations of 3. 1, with $G = \{1, \sigma_1, \sigma_2, \sigma_3 = \sigma_1 \sigma_2\}$, if we had $\tau\sigma_i = \sigma_j \tau$ with $j \neq i$, we would get $\tau(\Delta_i) = \Delta_j$ which is impossible since τ conserves the types. Therefore τ acts trivially on G.

We deduce from this, that in each case 1) and 2), there exists an involution σ on S, which commutes with τ, such that Φ factors through $S \to S/\sigma$ and $p_g(S/\sigma) = 0$ (cf. Remark 3. 2 for case 1)).

We take the notations of the proof of 3. 3.:

$$\begin{array}{ccccc} \hat{S} & \xrightarrow{\eta\pi} & P & \xrightarrow{\Phi_{|H|}} & \Sigma \subset \mathbb{P}^3 \\ \big\downarrow{\scriptstyle\varepsilon} & & & \nearrow{\scriptstyle\Phi} & \\ S & & & & \end{array}.$$

Recall from the proof of Theorem 4. 6, case 2), that the σ-invariant part of $\mathcal{R}_3$ is $H^0(P, 3K_P + 2\delta) = H^0(P, F)$, where F is the pencil of lines through p_5. Since σ and τ commute, τ acts on $H^0(P, F)$ hence induces an τ-equivariant *morphism*:

$$p : S \dashrightarrow P \xrightarrow{\Phi_{|F|}} |F|^\vee \cong \mathbb{P}^1.$$

A generic fibre of p is a smooth curve of genus 2. If it were stable under τ, τ would have a fixed point on it, which is impossible. Therefore τ acts on $\mathbb{P}^1$ non-trivially:

$$\begin{array}{ccc} S & \xrightarrow{p} & \mathbb{P}^1 \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\bar{\alpha}} \\ T & \xrightarrow{q} & \mathbb{P}^1/\tau \cong \mathbb{P}^1. \end{array}$$

If F_p is a fibre of p over a point of ramification for $\bar{\alpha}$, we can write $\alpha_* F_p = 2F' =$ fibre F_q of q. But this is impossible since $F_q^2 = 0$, $K_T F_q = 2$ would imply $F'^2 + K_T F' = 1$. Q.E.D.

We can also prove that in this case, the 6×6 matrix giving a resolution for the canonical ring $\mathscr{R}$ of S can be chosen of the form (same notations as in § 4):

$$(\alpha)=\begin{pmatrix} y_0 G^- & wq_1^- & wq_2^- & wq_3^+ & wq_4^+ & Q^+ \\ wq_1^- & - & - & + & + & w \\ wq_2^- & - & - & + & + & 0 \\ wq_3^+ & + & + & - & - & 0 \\ wq_4^+ & + & + & - & - & 0 \\ Q^+ & w & 0 & 0 & 0 & 0 \end{pmatrix},$$

where + (resp. −) means that the polynomial is τ-invariant (resp. τ-anti-invariant).

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