# Surfaces with pg = q = 1



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# SURFACES WITH $p_g = q = 1$

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## INTRODUCTION

Surfaces of general type with $p_g = q = 1$ are the irregular surfaces with the lowest geometric genus. Therefore they are natural candidates for starting the investigation of irregular surfaces with $q = 1$ or, more generally, with an irrational pencil. We hope that the methods presented here could be useful to shed some light on the more general case.

For these surfaces one has $2 \leq K^2 \leq 9$ , and the analysis of the case $K^2 = 2$ , suggested by Bombieri, was worked out completely by the first author in [Ca]. For these the fibres of the Albanese map have genus $g = 2$ and the canonical maps of the fibres fit together to give a $2:1$ cover of the symmetric square $A^{(2)}$ of the Albanese curve $A$ .

Later on the second author proposed to obtain more costructions of these surfaces by looking at complete intersections in the $K^2$-symmetric power of $A$ , giving fibrewise canonical curves. We realized later that the possible cases are many if $K^2 \geq 3$ , and the research has been pending for a few years.

A further stimulus for its development came from the result proved by P. Francia in 1984, but published in this volume, to the effect that the bicanonical map of a surface of general type is a morphism if $p_g \geq 1$ , with the possible exception of the case $p_g = q = 1$ .

Soon after, in 1985, I. Reider proved that $|2K|$ is base points free as soon as $K^2 \geq 5$ . Therefore the only cases left out by the union of Francia's and Reider's results were $p_g = 0$ and $p_g = q = 1$ , $K^2 = 3,4$ (the case $K^2 = 2$ had been previously settled by the results in [Ca], which imply that $|2K|$ has no base points if $K^2 = 2$ ).

As for $p_g = 0$ , one knows that in some case (for instance if $K^2 = 1$ , see [Ca2], Remark 1.5) the bicanonical system may very well have base points. In any event a complete and clear picture of the behaviour of $|2K|$ is here still missing.

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As for $p_g = 0$, one knows that in some case (for instance if $K^2 = 1$, see [Ca2], Remark 1.5) the bicanonical system may very well have base points. In any event a complete and clear picture of the behaviour of $|2K|$ is here still missing.

We exclude in this paper the remaining two cases $p_g = q = 1$, $K^2 = 3, 4$ showing that for these too $|2K|$ is base point free. Therefore, as a consequence of the quoted results by Francia and Reider and ours, we can state the:

THEOREM. *If $S$ is a minimal surface of general type with $p_g > 0$ then the bicanonical map is a morphism for $S$.*

But, more importantly, we establish a number of general results concerning these surfaces which make their classification more accessible and we produce more examples.

More precisely, we consider, together with $K^2$, other basic numerical invariants such as the genus $g$ of the Albanese fibres, the index $\iota$ which is the number of paracanonical curves (curves algebraically equivalent to $K$) passing through a general point of $S$, $\lambda$ the «number» of paracanonical curves which move in a linear system and we give relations holding among them.

An interesting result which conjugates algebraic with geometric ideas states that a «projection» of the relative canonical mapping yields the paracanonical map $\omega^\iota : S \to A^{(\iota)}$ namely the rational map which, roughly speaking, associates to a general point the $\iota$-tuple of paracanonical curves passing through it. The methods we employ include the Grothendieck-Riemann-Roch theorem, Atiyah's classification of vector bundles on elliptic curves, part of the algebraic structure of the relative canonical algebra, and some complicated geometric argument based on the iterated use of linear series of degree 2 on an elliptic curve.

After these general results, we briefly recall the classification for $K^2 = 2$ and soon after we treat in detail the case $K^2 = 3$, in which we also get a complete classification. The main classification results can be summarized in the following:

THEOREM. *Let $S$ be a minimal surface of general type with $p_g = q = 1$. Then:*

(i) *(Catanese, [Ca]) if $K^2 = 2$ then one also has $\iota = g = 2$ and the paracanonical map, coinciding with the relative canonical map, is a $2:1$ morphism $\omega : S \to A^{(2)}$ whose branch divisor is described in § 5;*

(ii) *if $K^2 = 3$ then either $\iota = g = 2$ or $\iota = g = 3$, but in both cases the paracanonical map coincides with the relative canonical map.*

*More precisely, in the first case $\iota = g = 2$, $\omega : S \to A^{(2)}$ is not a morphism but its indeterminacy can be resolved by blowing up a single point of $S$. Moreover $\omega$ is generically $2:1$ and the branch curve is described in § 5.*

*Finally if $\iota = g = 3$ then $\omega : S \to A^{(3)}$ is a morphism which is birational onto its image and it is an isomorphism of the canonical model of $S$ onto $\omega(S)$. The divisor class of $\omega(S)$ in $A^{(3)}$ is described in § 5.*

The three types of surfaces listed above do in fact exist: this was proved in [Ca] for the case $K^2 = 2$, where it is also shown that the moduli space for these surfaces is irreducible, generically smooth of dimension 7. The existence in the case $K^2 = 3$, $\iota = g = 2$ is proved here in Theorem 5.4 below, but we postpone to the forthcoming paper [CaCi] the proof of the existence of surfaces with $K^2 = 3$, $\iota = g = 3$, as well as the description of the moduli spaces and other related matters.

## DEFINITIONS AND NOTATION

| | |
|---|---|
| $H^i(X,\mathcal{F}) = H^i(\mathcal{F})$ | For a coherent sheaf $\mathcal{F}$ on a variety $X$ over $\mathbb{C}$ |
| $h^i(\mathcal{F})$ | The $\mathbb{C}$- dimension of $H^i(\mathcal{F})$ |
| $S$ | A minimal algebraic surface of general type with $p_g = q = 1$ |
| $\Omega^i_S$ | The sheaf of holomorphic $i$- forms on $S$ |
| $A = A(S)$ | The Albanese variety of $S$, a curve of genus 1 |
| $\epsilon_t : A \to A$ | The translation morphism $\epsilon_t(s) = s + t$ |
| $\alpha : S \to A$ | The Albanese morphism of $S$ |
| $\Gamma_\alpha$ | The graph of $\alpha$ in $S \times S$ |
| $F = F_t$ | The fibre of the Albanese map at $t \in A$ |
| $g = g(S)$ | The arithmetic genus of a fibre of $\alpha$ |
| $K = K_S$ | A canonical divisor of $S$ |
| $K^2$ | The self-intersection of $K$ $(2 \le K^2 \le 9)$ |
| $[D]$ | The invertible sheaf associated to a Cartier divisor $D$ |
| $\equiv$ | The relation of linear equivalence between divisors |
| $\sim$ | The relation of algebraic equivalence between divisors |
| $\mathcal{P}'$ | A Poincare' sheaf on $A \times A$ giving an isomorphism $A \to \mathrm{Pic}^0(A)$, which makes $A$ an elliptic curve. Explicitly, fixing a point $0 \in A$, $\mathcal{P}' = [\Delta_A - (\{0\} \times A)]$, where $\Delta_A$ is the diagonal in $A \times A$ |
| $\mathcal{P} = (\alpha \times id)^*(\mathcal{P}')$ | A Poincare' sheaf on $S \times A$. Explicitly $\mathcal{P} = [\Gamma_\alpha - F_o \times A]$ |
| $j_t : S \to S \times \{t\}$ | The natural isomorphism, for any $t \in A$ |
| $\mathcal{P}_t$ | The sheaf $j_t^*(\mathcal{P})$; notice that $\mathcal{P}_t = [F_t - F_o]$ |

| | |
|---|---|
| $\lvert D\rvert = \mathbf{P}(H^\circ([D]))$ | The complete linear system of effective divisors linearly equivalent to a divisor $D$ |
| $\lvert iK + t\rvert$ | The complete linear system $\mathbf{P}(H^\circ([iK]\otimes\mathcal{P}_t))$, called a twisted $i$- th canonical system |
| $\varphi_t$ | The rational map $S \to \lvert 2K + t\rvert^\vee$, here called a twisted bicanonical map |
| $\{K\} = \{C_t\}_{t\in A}$ | The paracanonical system of $S$ (see § 1) |
| $Y \subset S \times A$ | The paracanonical incidence correspondence, i.e. for every $t \in A$, $Y \cap (S \times \{t\}) = C_t \times \{t\}$ |
| $\pi_S : S \times A \to S$ | The projection onto the first factor |
| $\pi_A : S \times A \to A$ | The projection onto the second factor |
| $\iota = \iota(S)$ | The degree of $\pi_{S\mid Y} : Y \to S$, called the index of the paracanonical system. Roughly speaking this is the number of distinct curves of $\{K\}$ passing through the general point of $S$ |
| $\mathcal{K} = \pi_S{}^*([K]) \otimes \mathcal{P}$ | The paracanonical sheaf on $S \times A$ |
| $V = \alpha_*[K]$ | The (locally free) direct image of the relative canonical sheaf for $\alpha$ on $A$ |
| $V = \oplus_{i=1,\dots,k} W_i$ | A decomposition of $V$ into indecomposable summands of degree $d_i$ and ranks $r_i$, for $i = 1, \dots, k$ |
| $V_i = \alpha_*[iK]$ | The direct image of $[iK]$, $i \geq 2$ |
| $\mathbf{P}(W)$ | Proj $(W^\vee)$, for a locally free sheaf $W$ |
| $p : \mathbf{P}(V^\vee) \to A$ | The projection on $A$ |
| $\omega : S \to \mathbf{P}(V^\vee)$ | The (rational) relative canonical map |
| $\Lambda^i$ | $R^i(\pi_A)_*\mathcal{K}$ |
| $\lambda = \lambda(S)$ | The length of $\Lambda^1$ |
| $\mathbf{P}_t$ | $\lvert 2K + t\rvert$, a projective space of dimension $K^2$ |
| $\psi_t : A \to \mathbf{P}_t$ | The morphism associating to $u \in A$ the divisor $C_u + C_{t-u}$ |
| $\Delta_t \subset \mathbf{P}_t$ | The image of $\psi_t$, an irreducible rational curve |
| $A^{(r)}$ | The $r$- th symmetric product of $A$ |
| $\beta : A^{(r)} \to A$ | The Abel-Jacobi map, realising $A^{(r)}$ as a $\mathbf{P}^{r-1}$ bundle over $A$ |
| $E = E_t$ | The fibre of $\beta$ at $t \in A$ |

| | |
|---|---|
| $D = D_t$ | $A^{(\iota-1)}$ embedded in $A^{(\iota)}$ as the set of all effective divisors of degree $\iota$ of $A$ containing $t \in A$ |
| $\omega' : S \to A^{(\iota)}$ | The rational map which, on the maximal open subset $U$ of $S$ over which $Y \to S$ is finite, is the classifying morphism for divisors of degree $\iota$ on $A$; $\omega'$ will be called the paracanonical map |

## 1. A RELATION BETWEEN INVARIANTS

Let us consider the torsion free sheaf $\Lambda^0$ on $A$. Observe that

$$h^0(|K + t|) = 1 + h^1(|K + t|)$$

for any $t \in A - \{0\}$. Moreover $p_g = 1$, hence by semicontinuity there is a Zariski open subset $A'$, containing $0$, such that for any $t \in A'$, one has $h^0(|K + t|) = 1$. For any $t \in A'$ we denote by $C_t$ the unique curve in $|K + t|$.

By the base change theorem, $\Lambda^0$ is invertible. $\Lambda^2$ is a skyscraper sheaf of length 1 supported at the origin. $\Lambda^1$ is zero at the origin and supported at the set of points $t \in A$ such that $h^0(|K + t|) > 1$ (see [Ha], Theorem 12.11).

We define the *paracanonical incidence correspondence* to be the surface $Y$ in $S \times A$ which is the schematic closure of the set $\{(x, t) \in S \times A' | x \in C_t\}$. Henceforth we define $C_t$ for any $t$ as the fibre of $Y \to A$ over $t$, thus $Y$ provides a flat family of curves on $S$, which we denote by $\{K\}$ and call, according to the classical terminology, the *paracanonical system* of $S$.

The *index* $\iota$ of $\{K\}$ is the intersection number of $Y$ with the curves $\{x\} \times A$, with $x \in S$.

REMARK 1.1. (i) $Y$ is the divisor $\mathrm{div}(\sigma)$ of the unique (up to contants) section of $\mathscr{K} \otimes \pi_A^*(\Lambda^0)^\vee$. In fact $(\pi_A)_*(\mathscr{K} \otimes \pi_A^*(\Lambda^0)^\vee)$ is trivial on $A$, therefore $H^0(\mathscr{K} \otimes \pi_A^*(\Lambda^0)) \cong \mathbb{C}$, and with $\sigma \in H^0(\mathscr{K} \otimes \pi_A^*(\Lambda^0))$ non zero, $\mathrm{div}(\sigma)$ has no components of the form $S \times \{t\}$, hence it is irreducible and coincides with $Y$.

(ii) As in [Be] the paracanonical system is the irreducible component of the Hilbert scheme of curves on $S$ algebraically equivalent to $K$ which dominates $A$.

The result we prove in this section is the following:

THEOREM 1.2. *The index $\iota$ of $\{K\}$ is related to the genus $g$ of the Albanese fibres by the formula*

$$\iota = g - \lambda$$

*where $\lambda$ is the length of $\Lambda^1$.*

PROOF. By Grothendieck-Riemann-Roch for the morphism

$$\pi = \pi_A : S \times A \to A$$

we have

$$\begin{aligned} ch(\pi_! \mathcal{K}) &= 1 + [c_1(\Lambda^0) - c_1(\Lambda^1) + c_1(\Lambda^2)] = \\ &= \pi_*[(1 + \mathcal{K} + (1/2)\mathcal{K}^2 + (1/6)\mathcal{K}^3) \cdot (1 - (1/2)K + \\ &\quad (1/12)(K^2 + c_2(S))] = \\ &= \pi_*[(1/2)\mathcal{K}^2 + (1/12)(K^2 + c_2(S)) - (1/2)K \cdot \mathcal{K} + \\ &\quad + (1/6)\mathcal{K}^3 - (1/4)K \cdot \mathcal{K}^2 + (1/12)(K^2 + c_2(S)) \cdot \mathcal{K}] \end{aligned}$$

where we still denote by $K = K_S$ and $c_2(S)$ their pull-back to $S \times A$ via $\pi_S$. By Noether's formula and taking into account that $\mathcal{K} = \pi_S^*([K]) \otimes \mathcal{P}$ and $\mathcal{P} = (\alpha \times \mathrm{id})^*(\mathcal{P}')$ where $\mathcal{P}' = [\Delta_A - (\{0\} \times A)]$, we have

$$\pi_*[(1/12)(K^2 + c_2(S))] = 1$$

$$\pi_*[(1/12)\mathcal{K}^2 - (1/2)K \cdot \mathcal{K}] = (1/2)\pi_*(\mathcal{K} \cdot \mathcal{P}) = 0$$

since

$$(K \cdot \mathcal{P}) \cdot (S \times \{t\}) = K \cdot (\Gamma_\alpha - F_0 \times A) \cdot (S \times \{t\}) = K \cdot (F_t - F_0) = 0$$

Moreover

$$\mathcal{P}^2 = -2(F \times \{t\}), \quad K^3 = \mathcal{P}^3 = 0, \quad K^2 \cdot \mathcal{P} = K^2, \quad K \cdot \mathcal{P}^2 = -4(g-1)$$

whence

$$\pi_*[(1/6)\mathcal{K}^3 - (1/4)K \cdot \mathcal{K}^2] = \pi_*[(1/4)K \cdot \mathcal{P}^2] = 1 - g$$

We also have

$$\pi_*[(1/12)(K^2 + c_2(S)) \cdot \mathcal{K}] = \pi_*[(\{x\} \times A) \cdot \mathcal{P}] = 1$$

Summing up

$$\deg \Lambda^0 - \lambda = 1 - g$$

Concluding, by Remark 1.1, (i), we have

$$\iota = (\{z\} \times A) \cdot Y = (\{z\} \times A) \cdot (\mathscr{K} \otimes \pi_A^*(\Lambda^0)^\vee) =$$

$$= 1 - \deg \Lambda^0 = g - \lambda.$$

Q.E.D.

REMARK 1.3. We explicitly notice that no new information is gotten by using the projection onto $S$.

An interesting observation is given by the following:

THEOREM 1.4. *The invariant $\lambda$ is a topological invariant.*

PROOF. Notice that

$$\lambda = \text{length } (\Lambda^1) = \sum_{t \neq 0} h^1([K + t]) = \sum_{t \neq 0} h^1([-t])$$

In fact for each $t \neq 0$ there exists a unique unitary flat line bundle $\mathscr{L}_t$ such that $[t] = \mathscr{L}_t \otimes \mathscr{O}_S$. By Hodge theory with flat twisted coefficients, we have

$$H^1(\mathscr{L}_t) \cong H^1([t]) \oplus H^0(\Omega_S^1 \otimes \mathscr{L}_t) \cong$$

$$\cong H^1([t]) \oplus H^0(\Omega_S^1 \otimes t) \cong H^1([t]) \oplus H^1([-t])$$

The proof is over if we observe that the correspondence $t \to \mathscr{L}_t$ gives a group isomorphism $\mathrm{Pic}^0(S) \to H^1(S, U(1))$, and that the last group is a topological invariant. Q.E.D.

## 2. THE RELATIVE CANONICAL MAP

The main purpose of this section is to relate the *relative canonical map* $\omega : S \to \mathbb{P}(V^\vee)$ with the *paracanonical map* $\omega' : S \to A^{(\iota)}$. In particular we will show that there exists a commutative diagram of rational maps

$$\begin{array}{ccccc} & & S & & \\ & \omega \swarrow & & \searrow \omega' & \\ \mathbb{P}(V^\vee) & & -\ell \to & & A^{(\iota)} \\ & p \searrow & & \swarrow \beta & \\ & & A & & \end{array}$$

where $\ell : \mathbb{P}(V^\vee) \to A^{(r)}$ is a suitable linear (rational) map of projective bundles.

We start with the:

LEMMA 2.1. *$V$ is a locally free sheaf of rank $g$ on $A$ such that:*

(i) *for every locally free quotient $Q$ of $V$, one has* $\deg(Q) \geq 0$

(ii) $h^0(V) = p_g = 1$, $h^1(V) = 0$, $\deg(V) = p_g = 1$.

PROOF. The rank of $V$ is $h^0(K_{|F}) = g$. (i) is Fujita's theorem [Fu]. (ii) follows immediately since the Leray spectral sequence degenerates and by relative duality one has

$$R^1\alpha_*(K) = \alpha_*(\mathcal{O}_S)$$

The assertion about $\deg(V)$ follows by Riemann-Roch. Q.E.D.

Consider now a decomposition

$$V = \oplus_{i=1,\dots,k} W_i$$

of $V$ into indecomposable summands of degrees $d_i, i = 1,\dots,k$. By Lemma (2.1), (i) we may assume that $d_1 = 1$, and $d_i = 0$, $i = 2,\dots,k$. By [At], Lemma 15, one has $h^0(W_1) = 1$, hence $h^0(W_i) = 0$, for $i = 2,\dots,k$. Moreover $h^1(W_i) = 0$, $i = 1,\dots,k$.

By ibidem, theorem 6, $W_1$ is uniquely determined by its rank $r = r_1$ and its determinant bundle. More precisely there is a unique point $u \in A$ such that, defining inductively $E_1(u) = \mathcal{O}_A(u)$ and $E_i(u)$ to be the isomorphism class of the non trivial extension

$$0 \to \mathcal{O}_A \to E_i(u) \to E_{i-1}(u) \to 0$$

then $W_1 = E_r(u)$.

As it is known (cf. [At], p. 451), $\mathbb{P}(E_r(u)^\vee)$ is the symmetric product $A^{(r)}$, and the meaning of the above extension is that on $A^{(r)}$ the sheaf $\mathcal{O}(1)$ is $[D_u]$, where $D_t$ is the set of divisors of degree $r$ containing $t$.

Again by ibidem, theorem 5, if $i \geq 2$ then $W_i$ can be written in a unique way as $W_i = L_i \otimes F_{r_i}$, where $F_1 = \mathcal{O}_A$, and $F_i$ is the isomorphism class of the non trivial extension

$$0 \to \mathcal{O}_A \to F_i \to F_{i-1} \to 0$$

and $L_i$ is of degree $0$. Furthermore if $L$ is a line bundle of degree $0$ on $A$, then $h^0(L \otimes F_i) = 1$ if $L$ is trivial, and is $0$ otherwise (ibidem, lemma 15 and theorem 5): therefore $L_i$ is non trivial and $h^0(L \otimes W_i) = 1$ if $L = L_i^\vee$ and is $0$ otherwise.

Proposition 2.2. *One has* $\lambda = k - 1$ *and* $h^0(K \otimes \alpha^* L) > 1$ *iff* $L = L_i^\vee$ *for some* $i = 2, \dots, k$.

Proof. We have

$$\lambda = \text{length } (\Lambda^1) = \sum_{t \neq 0} h^1([K + t]) =$$

$$= \sum_{t \neq 0} h^0([K + t]) - 1 = \sum_{t \neq 0,\, i = 2, \dots, k} h^0(W_i \otimes t) .$$

The assertion follows now easily. Q.E.D.

Theorem 2.3. *One has* $r = r_1 = \iota$ *and* $r_i = 1$, $i = 2 \dots, k$.

Proof. The proof shall be given in two steps.

*Step 1.* We prove that if $r = \iota$, then $r_i = 1$, $i = 2 \dots, k$. One has, by Theorem 1.2 and Proposition 2.2, that

$$g = r + \sum_{i = 2 \dots k} r_i \geq \iota + k - 1 = g - \lambda + k - 1 = g$$

whence the assertion follows.

*Step 2.* We prove that $r = \iota$. Consider the commutative diagram of rational maps

$$\begin{array}{ccc} S & -\omega \rightarrow & \mathbf{P}(V^\vee) \\ \omega'' \searrow & & \swarrow \varphi \\ & \mathbf{P}(W_1^\vee) = A^{(r)} & \end{array}$$

where $\varphi$ is induced by the inclusion $W_1 \to V$. This diagram shows that the divisors of the system $\mathscr{D} = \{D_t\}_{t \in A}$ on $A^{(r)}$ pull back on $S$ to the movable part of $\{K\}$ (see § 3 and more precisely Lemma 3.1 below). In particular for a general point $x \in S$, if $\omega''(x) = t_1 + \dots + t_r$, then the paracanonical curves passing through $x$ are exactly $C_{t_1}, \dots, C_{t_r}$, and thus $\iota = r$. Q.E.D.

Remark 2.4. (i) For any $t \in A$, we denote by $W_t$ the direct sum of summands $W_i$, $i \geq 2$, such that $W_i^\vee = t$ (remember that we identified $A$ with $\mathrm{Pic}^0(A)$, hence $t$ stands for $\mathscr{O}_A(t - 0)$ ).

Notice that

$$W_1 \otimes t = E_r(u) \otimes t = E_r(u')$$

where $u' = u + rt$ (the addition is on the elliptic curve). Hence $V \otimes t$ has a trivial subbundle of rank equal to $h^0([K + t])$. Correspondingly we have a rational map $\ell_t : \mathbf{P}(V^\vee) \to \mathbf{P}(H^0([K+t])^\vee)$, such that $\ell_t \circ \omega$ is the rational map $\varphi_t$ associated to the linear system $|K+t|$.

(ii) For $i \geq 2$, the sheaf $V_i = \alpha_*[iK]$ is locally free of rank $(2i-1)(g-1)$, whereas $R^1\alpha_*[iK] = 0$ and

$$\deg V_i = \chi(V_i) = \chi([iK]) = 1 + [i(i-1)/2]K^2$$

It should be interesting to study the relative canonical algebra $\oplus_i V_i$. For example if $g = 2, 3$, and one has $V_2 = \mathrm{Sym}^2(V)$ (as happens if the fibres are irreducible, and non hyperelliptic in genus 3 ), then by extracting degrees we infer that $K^2 = g$. We shall come back to this in § 5.

(iii) It is amusing that one can provide equations for the paracanonical curves $C_t$, such that $\dim |C_t| \geq 1$. For such a curve consider in fact the cohomology sequence of

$$0 \to \mathcal{O}_S \to [C_t] \to N_{C_t} \to 0$$

The map $H^0(N_{C_t}) \to H^1(\mathcal{O}_S)$ is surjective: in fact the paracanonical system provides a smooth section of $\mathrm{Hilb}_S$ on $A = \mathrm{Pic}^0(S)$ (later on, in proving Proposition 3.4, we essentially give another proof of this fact). Hence, if $s_t$ is such that $\mathrm{div}(s_t) = C_t$, the map $H^1(\mathcal{O}_S) \to H^1([C_t])$ given by multiplication by $s_t$ is the zero map. Since $H^1([C_t]) \cong H^1(t^\vee)^\vee$, we obtain that for any $\eta' \in H^1(t^\vee)$ and for any $\eta \in H^1(\mathcal{O}_S)$, $(\eta \cup \eta') \cdot s_t \in H^2(\Omega^2_S)$ is zero. We can interpret these as $h^0([C_t]) - 1$ independent linear equations, provided we show that $\eta \cup \eta' \neq 0$ if $\eta$ and $\eta'$ are both non zero. Interpret now, by Hodge theory with twisted flat coefficients, $\eta$, resp. $\eta'$, as conjugates of sections of $\Omega^2_S$, resp. $\Omega^2_S \otimes t$. By the exact sequence

$$0 \to \alpha^*(\Omega^1_A) \cong \mathcal{O}_S \to \Omega^1_S \to \Omega^1_{S/A} \to 0$$

we have

$$0 \to H^0(t) \to H^0(\Omega^1_S \otimes t) \to H^0(\Omega^1_{S/A} \otimes t)$$

Therefore since $h^0(t) = 0$, a non zero $\eta' \in H^0(\Omega^1_S \otimes t)$ has non zero image in $H^0(\Omega^1_{S/A} \otimes t)$, thus $\eta'$ does not vanish on vertical tangent vectors, and this yields that one cannot have $\eta \wedge \eta' = 0$.

## 3. A REMARKABLE CURVE INSIDE THE TWISTED BICANONICAL SYSTEMS

We denote by $\mathbf{P}_t$ the linear system $|2K + t|$. Consider then the morphism $\psi_t : A \to \mathbf{P}_t$ associating to $u \in A$ the divisor $C_u + C_{t-u}$, and let $\Delta_t \subset \mathbf{P}_t$ be the image of $\psi_t$. $\Delta_t$ is an irreducible rational curve, since $\psi_t$ factors through the rational involution $\varepsilon_t$ which exchanges $u$ and $t-u$ and defines a map $\overline{\psi}_t : \mathbf{P}^1 \to \mathbf{P}_t$.

We shall now study $\Delta_t$. Before this we prove a lemma which will be also useful later. Notice that $Y$ uniquely decomposes as $Y' + (\pi_S)^*(X)$, where every component of $Y'$ dominates $S$. $X$ is a divisor on $S$, called the *fixed part* of $\{K\}$. We shall write $C_t = X + M_t$, and call $\{M\} = \{M_t\}_{t\in A}$ the *movable part* of $\{K\}$.

LEMMA 3.1. *The general member of $\{M\}$ is an irreducible curve, hence $Y'$ is irreducible.*

PROOF. Assume the contrary. Then there exist $M_1$, $M_2$ irreducible divisors and an effective divisor $M'$ such that:

(i) $M_1 + M_2 + M'$ is algebraically equivalent to $M$;

(ii) $M_1$ and $M_2$ move in positive dimensional algebraic systems $\mathcal{M}_1$ and $\mathcal{M}_2$ which dominate $\mathrm{Pic}^0(S)$.

But then for every $t \in A$, we find that $|K + t|$ is positive dimensional, a contradiction. In fact for each curve $M_1' \in \mathcal{M}_1$ we could find a curve $M_2' \in \mathcal{M}_2$ with $M_1' + M_2' + M' + X$ linearly equivalent to $K + t$. Q.E.D.

COROLLARY 3.2. *For any $t \in A$, $\overline{\psi}_t$ is birational; moreover for general $t \in A$, $\overline{\psi}_t$ is injective.*

PROOF. Let $\sum = \{u \in A | M_u \text{ is reducible}\}$ and let us choose $v \in A$ such that $v$ is not in $\sum$. Assume we have a point $v' \in A$ such that $\psi_t(v) = \psi_t(v')$. Then

$$2X + M_v + M_{t-v} = 2X + M_{v'} + M_{t-v'} \Rightarrow M_v + M_{t-v} = M_{v'} + M_{t-v'}$$

By the irreducibility of $M_v$, $M_v$ is a component of $M_{v'} + M_{t-v'}$. Hence either $v = v'$ or $v = t - v'$, which proves the first assertion. The same argument proves that injectivity holds unless both $v$ and $t - v$ are in $\sum$. Hence $t$ belongs to a finite set. This proves the second assertion too. Q.E.D.

We are now able to prove the:

PROPOSITION 3.3. *The degree of $\Delta_t$ equals the index $\iota$ of $\{K\}$.*

PROOF. Let us set $Y_t = (\mathrm{id} \times \varepsilon_t)^*(Y)$ where $\varepsilon_t : A \to A$ is the translation map $\varepsilon_t(s) = s + t$. As one can easily verify computing classes in $A \times A$, we have

$$[Y + Y_t] = (\pi_S)^*[2K + t] \otimes (\pi_A)^*([0] \otimes [t])^{\otimes \iota}$$

Let $\sigma = \sigma(x, u)$ be a section such that $Y = \mathrm{div}(\sigma)$. Thus $\sigma(x, u) \cdot \sigma(x, t-u)$ is a section which represents $\psi_t(u)$. We choose $x$ such that $Y \cup Y_t \to S$ is étale on $x$. Then $\sigma(x, u) \cdot \sigma(x, t-u)$ has $2\iota$ simple zeroes $u$ for which $u \neq t - u$. This shows that

$$\iota = (\deg \Delta_t) \cdot (\deg \overline{\psi}_t) = \deg \Delta_t$$

by Lemma 3.2. Q.E.D.

PROPOSITION 3.4. *For a general $t \in A$, $\overline{\psi}_t$ is an embedding.*

PROOF. We already know that $\overline{\psi}_t$ is injective. So we only have to show that $\overline{\psi}_t$ has non zero differential at every point. At a point $w$ of $\mathbb{P}^1$ corresponding to a point $v \in A$ such that $v \neq t - v$, this amounts to prove that $\psi_t$ has non zero differential. Let us fix such a point $v \in A$. Let moreover $\mathbb{C} \to A$ be the universal covering of $A$ and let $u$ be a coordinate such that $\mathrm{d}/\mathrm{d}u$ is the translation invariant vector field. A section $\sigma$ such that $\mathrm{div}(\sigma) = Y$ lifts to a section on $S \times \mathbb{C}$ which by abuse of notation we still denote by $\sigma(x, u)$. There exists an open cover $\{U_i\}$ of $S$ such that $\sigma$ on $U_i \times \mathbb{C}$ is expressed by a holomorphic function $\sigma_i(x, u)$. Since, as is well known, any line bundle on $A \times A$ is linearizable on $\mathbb{C} \times \mathbb{C}$, one easily sees, by the definition of the Poincare' bundle, that one can choose the cover $\{U_i\}$ such that $\sigma$ satisfies the cocycle condition

$$(*) \qquad \sigma_i(x, u) = \sigma_j(x, u) \cdot g_{ij}(x) \cdot \exp 2\pi i(u \cdot f_{ij}(x))$$

where $\{g_{ij}(x)\}$ is a cocycle for $[K]$ and $\{f_{ij}(x)\}$ is an additive cocycle which, being the pull back via $\alpha$ of a generator of $H^1(\mathcal{O}_A)$, generates $H^1(\mathcal{O}_A)$.

The logarithmic derivative of $(*)$ with respect to $u$ gives the relation

$$(**) \qquad (\log \sigma_i(x, u))' = f_{ij}(x) + (\log \sigma_j(x, u))'$$

where, as usual, we denote $\mathrm{d}f/\mathrm{d}u$ by $f'$. Assume now that the derivative of $\psi_t$ is zero at $v$. Then there is a constant $\lambda$ such that

$$[\sigma_i(x, u) \cdot \sigma_i(x, t-u)]'_{|v} = \lambda \sigma_i(x, v) \cdot \sigma_i(x, t-v)$$

where with an abuse of notation we identified $t$ and $v$ with some lifts in $\mathbb{C}$. This amounts to

$$(\log \sigma_i(x,u))'_{|v} = \lambda + \log \sigma_j(x,u))'_{|t-v}$$

and therefore yields that the functions $(\log \sigma_i(x,u))'_{|v}$ are regular: in fact $v \not\equiv t-v$ in $A$, the fixed part of $\{K\}$ cancels out in the logarithmic derivative and the curves $M_v$ and $M_{t-v}$ have no common components. Whence $\{f_{ij}(x)\}$ turns out, by $(**)$, to be a coboundary in $H^1(\mathcal{O}_A)$, a contradiction.

Consider now a point $w \in \mathbf{P}^1$ corresponding to $v \in A$ such that $2v = t$ in $A$. One imediately sees that if $z$ is a local parameter around $w$ in $\mathbf{P}^1$, then $(\mathrm{d}\overline{\psi}_t/\mathrm{d}z)_{|w} = 0$ iff $\psi_t''(v) = 0$. Notice that $\psi_t'(v) = 0$ because $\overline{\psi}_t$ factors and there is ramification at $v$, accordingly with the following computation

$$[\sigma_i(x,u) \cdot \sigma_i(x,t-u)]'_{|v} =$$

$$= \sigma_i'(x,v) \cdot \sigma_i(x,t-v) - \sigma_i(x,v) \cdot \sigma_i'(x,t-v) = 0$$

Therefore $(\mathrm{d}\overline{\psi}_t/\mathrm{d}z)_{|w} = 0$ yields the existence of a constant $\mu$ such that

$$[\sigma_i(x,u) \cdot \sigma_i(x,t-u)]''_{|v} = \mu\sigma_i(x,v) \cdot \sigma_i(x,t-v) = \mu\sigma_i(x,v)^2$$

that is

$$2[\sigma''_i(x,v) \cdot \sigma_i(x,v) - \sigma_i'(x,v)^2] = \mu\sigma_i(x,v)^2$$

which we write as

$$\sigma''_i(x,v)/\sigma_i(x,v) = (\mu/2) + [\sigma_i'(x,v)/\sigma_i(x,v)]^2$$

Whence we conclude that $(\log \sigma_i(x,u))'_{|v}$ is regular: in fact $M_v$ is irreducible since $t$ is general, and if a general point of it were a pole for $(\log \sigma_i(x,u))'_{|v}$ then, by the above relation, we would have a meromorphic function with a pole on the one side of order at most one and on the other side of order exactly two, a contradiction. From $(**)$ we derive the same contradiction as before, namely that $\{f_{ij}(x)\}$ is a coboundary in $H^1(\mathcal{O}_A)$. Q.E.D.

We finish this paragraph by introducing a system of curves on $S$ which are related with the curves $\Delta_t$ and will be useful in the sequel. As in the proof of Proposition 3.3, we define $Y_t' = (\mathrm{id} \times \varepsilon_t)^*(Y')$, where $Y' \subset S \times A$ gives the movable part of $\{K\}$. We denote by $L_t^*$ the intersection scheme of $Y'$ and $Y_t'$ on $S \times A$: we notice that $L_t^*$ is the pull back of a scheme $L_t' \subset S \times \mathbf{P}^1$ where $\mathbf{P}^1 = A/\varepsilon_t$. We define $L_t$ to be the image of $L_t'$ on $S$ as a cycle.

We remark that $L_t^*$ is pure of dimension 1 for any $t \in A$.

Furthermore for general $t \in A$, $L_t^*$ contains no curve of the form $D \times \{u\}$ : in fact for $t$ general and for every $u$ in $A$, $M_u$ and $M_{t-u}$ have no common component. Thus for $t$ general $L_t$ is the union of the cycles of variable intersections of $M_u$ and $M_{t-u}$ as $u$ varies in $A$.

We denote by $\delta$ the degree of the projection of $L_t'$ onto $\mathbf{P}^1$. Then

$$M^2 = \delta + \mu$$

where $\mu$ is the sum of the intersection multiplicities of two general curves in $\{M\}$ at the base points of $\{M\}$.

## 4. THE BICANONICAL SYSTEM IS BASE POINT FREE

In the present paragraph we shall prove the following result:

THEOREM 4.1. *If $S$ is a minimal surface of general type with $p_g = q = 1$ and $K^2 = 3,4$ then $|2K|$ is base point free.*

This theorem together with results by Francia [Fr], Reider [Re] and Catanese [Ca], implies the:

THEOREM 4.2. *If $S$ is a minimal surface of general type with $p_g \geq 1$ then $|2K|$ is base point free.*

Before proceeding to the proof, we make the following:

REMARKS 4.3. (i) The fibres of the Albanese map are connected.

(ii) Let $f : S \to B$ be an irrational pencil of genus $b$ (i.e. $B$ has genus $b \geq 1$). Then $b = 1$, and $f$ is the composition of the Albanese morphism with an isogeny $A \to B$ whose degree equals the number of connected components of the general fibre of $f$. This follows by the fact that $H^0(\Omega_B^1)$ injects into $H^0(\Omega_A^1)$ and by the universal property of the Albanese morphism.

(iii) If $\iota = 1$ then $Y'$ is by definition birational to $S$ and it is easy to see that it is indeed isomorphic to $S$. By the previous remark (ii) and Lemma 3.1 it follows that the movable part $\{M\}$ of $\{K\}$ is given by the system of fibres of the Albanese morphism $\alpha$.

(iv) If there exists a paracanonical curve $C_t$ containing a fibre $F$ of $\alpha$, then we are in the previous case, since all paracanonical curves contain a full fibre.

(v) If all the fibres of the Albanese morphism are 2 - connected, then the relative canonical map is a morphism (cf. eg. [Ml], 1, 6.3).

Other easy but useful facts are given by the following lemmas:

LEMMA 4.4. *There is no system of curves on $S$, parametrized by an elliptic curve $E$, without fixed components and such that two general curves of it meet at one variable point.*

PROOF. The existence of such a system would yield the existence of a rational dominant map $E^{(2)} \to S$, contradicting the fact the $S$ is of general type. Q.E.D.

LEMMA 4.5. *If $x \in S$ is a base point for $|2K + t|$, it is also a base point for the paracanonical system $\{K\}$ (i.e. it belongs to $C_u$ for all $u \in A$).*

PROOF. If $x$ is a base point for $|2K + t|$, it belongs to all the curves corresponding to points of $\Delta_t$. If $x$ would belong only to a finite number of curves in $\{K\}$, then we would have only a finite number of curves corresponding to points of $\Delta_t$ containing $x$, a contradiction. Q.E.D.

REMARK 4.6. If $x$ is a base point for $|2K|$, then $x$ lies on the canonical curve $K$ of $S$ and it is known to be a simple point for $K$ (see [Fr], Lemma 3.2), therefore it is a simple point for the general paracanonical curve.

LEMMA 4.7. *If $\iota = 2$ and $\omega' : S \to A^{(2)}$ is a rational double cover, then any fibre of the Albanese pencil is hyperelliptic. Moreover if $\iota = 2$ and $\{K\}$ has no fixed part then, with the notation introduced at the end of § 3, we have $\delta = 2g - 2$ and therefore $K^2 = \mu + 2g - 2$.*

PROOF. The first assertion is trivial. Moreover if $\iota = 2$ and $\{K\}$ has no fixed part then the paracanonical map $\omega' : S \to A^{(2)}$ has degree $\delta$. The images of the fibres $F$ of the Albanese pencil are the fibres $E$ of $\beta : A^{(2)} \to A$, and $\omega'_{|F} : F \to E \cong \mathbf{P}^1$ is, for $F$ general, a morphism given by a subseries of the canonical series of $F$. Hence the assertion.

Q.E.D.

We start by proving the first part of Theorem 4.1, i.e. the following:

PROPOSITION 4.8. *If $S$ is a minimal surface with $p_g = q = 1$ and $K^2 = 3$, then $|2K|$ is base point free.*

PROOF. The proof will be given through a sequence of auxiliary results, some of them of independent interest. The first is the following:

LEMMA 4.9. *Let $K^2 = 3$ ; then the general paracanonical curve of $S$ is irreducible (of arithmetic genus 4 ).*

PROOF. By Lemma 3.1 we have to show that the fixed part $X$ is zero. Since $M^2 \geq 0$ , we have

$$3 = K^2 = K \cdot (X + M) = K \cdot X + K \cdot M = K \cdot X + (X + M) \cdot M =$$

$$= K \cdot X + X \cdot M + M^2 \geq K \cdot X + X \cdot M$$

Now $X \cdot M \geq 2$ by the 2 - connectedness of the paracanonical curves on $S$ (see [Bo], lemma 1), and $K \cdot X \geq 0$ : therefore either $K \cdot X = 0$ or $K \cdot X = 1$ .

We dispose immediately of the case where $K \cdot X = 0$ . In fact we would have $K \cdot M = 3$ , and therefore $M^2 > 0$ , $M^2$ being then an odd number. Since $3 = X \cdot M + M^2$ we find $X \cdot M = 2$ and $M^2 = 1$ , against Lemma 4.4.

We can therefore assume $K \cdot X = 1$ . We have

$$K \cdot X = 1, \quad X \cdot M = 2, \quad M^2 = 0, \quad K \cdot M = 2, \quad X^2 = -1$$

The first consequence of these relations is that $\{M\}$ coincides with the system of fibres of the Albanese pencil, which is therefore an elliptic pencil of curves of genus 2 (see Remarks 4.3).

Secondly we claim that $X$ must be irreducible. Assume the contrary. Let us then write $F = X_0 + X'$ , where $X_0$ is the union of all the irreducible components $D$ of $X$ such that $K \cdot D = 0$ . Since $S$ is minimal we have that:

(i) $X_0$ is a union of rational $(-2)$ - curves , which must be contained in fibres of the Albanese pencil, thus $X_0 \cdot M = 0$ ;

(ii) $X'$ is an irreducible curve such that

$$K \cdot X' = 1, \quad M \cdot X' = M \cdot X = 2 .$$

First we prove that $X_0 = 0$ . In fact

$$0 = K \cdot X_0 = (X + M) \cdot X_0 = X \cdot X_0 = (X' + X_0) \cdot X_0 = X' \cdot X_0 + X_0^2 \Rightarrow$$

$$\Rightarrow X' \cdot X_0 = X_0^2 \geq 2$$

(the last inequality holding by the criterion of Artin-Mumford, see [BPV], p. 75). But then

$$1 = K \cdot X' = (X + M) \cdot X' = X \cdot X' + M \cdot X' = (X_0 + X') \cdot X' + M \cdot X' =$$

$$= X_0 \cdot X' + X'^2 + M \cdot X' \geq X'^2 + 4 \Rightarrow X'^2 \leq -3$$

and so

$$1 = K \cdot X' = 2p_a(X') - 2 - X'^2 \geq 2p_a(X') + 1 \Rightarrow p_a(X') = 0$$

a contradiction, because $X'$ would then be a rational curve not contained in fibres of the Albanese pencil.

So $X_0$ must be zero, $X = X'$ is irreducible and $p_a(X) = 1$. Since $X$ dominates $A$, its geometric genus is $1$, hence $X$ is smooth and $\alpha_{|X}$ is an unramified double cover. By the adjunction formula and since $\{M_t\} = \{F_t\}$, $X$ intersects $F_t$ in a canonical divisor. This divisor, as we saw, consists of two distinct smooth points.

Let us consider the biregular involution $\sigma : S \to S$, which acts on the general curve $F_t$ (which has genus $2$) as the hyperelliptic involution. This involution clearly fixes $X$ and has no fixed points on $X$. Therefore $X$ should have even self intersection on $S$, and this contradicts the fact that $X^2 = -1$; thus also the case $K \cdot X = 1$ is excluded. Q.E.D. for 4.9.

LEMMA 4.10. *If $K^2 = 3$, then $\{K\}$ has at most one single smooth transversal base point $x$.*

PROOF. Let $x_1, \ldots, x_h$ be base points of $\{K\}$ and let $\mu$ be, as in § 3, the sum of the multiplicities of intersection at $x_1, \ldots, x_h$ of two general paracanonical curves, so that $1 \leq \mu \leq 3$.

If $\mu = 3$, then clearly $\iota = 1$ and $\{K\}$ would consist of the system of Albanese fibres, a contradiction (see Remark 4.3, (iii)). Moreover $\mu = 2$ is contradicted by Lemma 4.4.

So we are left with the case $\mu = 1$, which implies the assertion. **Q.E.D. for 4.10.**

LEMMA 4.11. *Let $K^2 = 3$ and assume $\{K\}$ has a base point $x$ as in **4.10**. Then if $y$ is a general point of $S$, there is one and only one point $z \neq y$ such that the set of the $\iota$ paracanonical curves passing through $y$ coincides with the set of paracanonical curves passing through $z$.*

PROOF. Let $C = C_t$ be a general paracanonical curve. Let us look at the general *twisted bicanonical system* $|2K + u|$, $u \in \mathrm{Pic}^0(S)$ of $S$. One has

$$\dim |2K + u| = 3$$

Furthermore, by the exact sequence

$$0 \to [K + u - t] \to [2K + u] \to \mathcal{O}_C(2K + u) \to 0$$

and from the genericity of $u$ and $t$, which yields

$$h^0(S, [K + u - t]) = 1, \quad h^0(S, [K + u - t]) = 0$$

we deduce that $|2K + u|$ cuts out on $C$ a complete, non special, linear series $g^2_6$. Inside this series we have the rational series of divisors cut out on $C$ by the curves of the system $\Delta_u$. This rational series, in turn, has $2x$ as its fixed divisor, therefore its variable part is a rational series $\gamma^1_4$ of dimension 1 of divisors of degree 4 of $C$.

Since this $\gamma^1_4$ is rational, it is certainly contained in a linear series $g^r_4$, where $r = 1$ or $r = 2$. If $r = 2$, $C$ is hyperelliptic and the $g^2_4$ is unique, which would imply that, as $u$ varies in $A = \mathrm{Pic}^0(S)$, $|2K + u|$ cuts out equivalent series on $C$. But if this happens then clearly $u_{|C} = \mathcal{O}_C$ for any $u \in \mathrm{Pic}^0(S)$. Looking at the sequence

$$0 \to [u - C] \to u \to u_{|C} \to 0$$

and taking into account that

$$h^0(S, [u - C]) = 0, \quad h^1(S, [u - C]) = h^1(S, [2K - u + t]) = 0$$

we see this is impossible.

Thus $r = 1$ and the $\gamma^1_4$ is actually a $g^1_4$. Take now $y \in C$ a general point on $S$ and let $C_2, \ldots, C_\iota$ be the paracanonical curves through $y$ other than $C$. We denote by $y_2, \ldots, y_\iota$ their residual intersection points with $C$ (subtracting $x + y$), and we want to prove that $y_2 = \ldots = y_\iota$, thereby proving the assertion. Let $C'$ be another general paracanonical curve which does not pass through any one of the points $y_2, \ldots, y_\iota$. Then we take the unique paracanonical curves $C'_3, \ldots, C'_\iota$, such that

$$C' + C_2 \equiv C_3 + C'_3 \equiv \ldots \equiv C_\iota + C'_\iota$$

Now all these curves cut out on $C$ the unique divisor of a $g^1_4$ passing through $y$, therefore $y_3, \ldots, y_\iota$ have to lie on $C' + C_2$. But since they do not lie on $C'$, they all lie on $C_2$, i.e. they all coincide with $y_2$. Q.E.D. for 4.11.

LEMMA 4.12. *Let $K^2 = 3$ and assume $\{K\}$ has a base point $x$ as in 4.10. Then $g = \iota = 2$.*

PROOF. In the notation introduced at the end of § 3, we take a general curve $L_t'$. By our assumption $\delta = 2$ and the contents of Lemma 4.11 can be rephrased as saying that the rational map $\omega' \circ \pi_S : S \times \mathbf{P}^1 \to A^{(\iota)}$, when restricted to $L_t'$, factors through the finite double cover $L_t' \to \mathbf{P}^1$. Hence $L_t$ is a fibre of the Albanese map. We proved by the way that $L_t$ is hyperelliptic, but we shall soon see that $\iota = 2$ and then $g = 2$ follows then by Lemma 4.7, since $\delta = 2$. Let indeed $x$ be a general point of $S$, and let $C_1, \dots, C_\iota$ be the $\iota$ paracanonical curves passing through $x$. The $\iota - 1$ curves $C_1 + C_2, \dots, C_1 + C_\iota$ provide $\iota - 1$ distinct curves $L_t$ containing $x$, therefore $\iota = 2$. Q.E.D. for 4.12.

We conclude the proof of Proposition 4.8 with the following:

LEMMA 4.13. *If $\iota = g = 2$ and $\{K\}$ has no fixed part, then there are no base points of $|2K|$.*

PROOF. The canonical series of an Albanese fibre $F$ is the $g_2^1$ cut out by the paracanonical system, since $F$ is not contained in any paracanonical curve.

We claim that a base point $x$ of $|2K|$ cannot be a smooth point of the Albanese fibre $F$ containing it. In fact, $x$ would be a base point of the canonical series of $F$, thus $x$ would belong to a component of $F$ isomorphic to $\mathbf{P}^1$, which would be in the base locus of the canonical series of $F$ (cf. [BC], proposition B), against our assumption.

But, if $x$ is a singular point of $F$ the unique holomorphic 1-form on $S$ vanishes at $x$, hence Lemma 3.5 of [Fr] applies, whence $x$ is not a base point of $|2K|$, a contradiction. Q.E.D. for 4.13 and 4.8.

REMARK 4.14. The argument given above shows that if $K^2 = 3$ then $\{K\}$ has a base point $x$ if and only if there is a base point for the relative canonical map $\omega$, in which case we also saw that $g = \iota = 2$. More precisely then the fibre $F$ of the Albanese morphism through $x$ is singular at $x$ and has a decomposition $F = F_1 + F_2$ such that $x \in F_1 \cap F_2$, $F_1 \cdot F_2 = 1$, and $F_1$, $F_2$ are therefore of arithmetic genus 1.

We turn now to surfaces with $p_g = q = 1$, $K^2 = 4$ and we want to prove the second part of Theorem 4.1, namely:

PROPOSITION 4.15. *If $S$ is a minimal surface with $p_g = q = 1$ and $K^2 = 4$, then $|2K|$ is base point free.*

PROOF. As in the proof of Proposition 4.8 we proceed by showing a series of auxiliary results.

LEMMA 4.16. *Let $S$ be a surface with $p_g = q = 1$ and $K^2 = 4$; if $|2K|$ has some base point, then the general paracanonical curve of $S$ is irreducible (of arithmetic genus 5).*

PROOF. It sufficies to prove that $\{K\}$ has no fixed part $X$. There are two cases to be examined.

CASE I. $M^2 > 0$. We cannot have $M^2 = 1$ by Lemma 4.4. Thus $M^2 \geq 2$ and therefore, since $X \cdot M \geq 2$, we have

$$4 = K^2 = K \cdot (X + M) = K \cdot X + (X + M) \cdot M = K \cdot X + X \cdot M + M^2 \geq 4$$

So

$$X \cdot M = 2, \quad M^2 = 2, \quad K \cdot X = 0, \quad X^2 = -2$$

and $X$ (supposed to be non zero) is a fundamental cycle (cf. [Bo], p. 176). Moreover the system $\{M\}$ cannot have base points because, in that case, $S$ would either possess an irrational pencil with base points, or we would have a contradiction in view of Lemma 4.4. Therefore if $x$ is a base point of $|2K|$ it has to lie on $X$, and since $(2K) \cdot X = 0$, some component of $X$ would be a fixed component for $|2K|$, another contradiction in view of Theorem 3.1, (i) of [Fr]. The conclusion is that $X$ is zero, namely $\{K\}$ has no fixed part.

CASE II. $M^2 = 0$. So $\{M\}$ coincides with the system of fibres of the Albanese pencil. Then $K \cdot M \leq 4$ and therefore either $K \cdot M = 2$ or $K \cdot M = 4$. In the latter case we have $K \cdot X = 0$, so $X$ is composed of rational curves which are contained in curves of the Albanese pencil. Therefore $X \cdot M = 0$, against the connectedness of the paracanonical curves. So we must have $K \cdot M = 2$ and therefore

$$K \cdot X = X \cdot M = 2, \quad X^2 = 0$$

Once again we write $X = X_0 + X'$, where $X_0$ is the union of the all irreducible components $D$ of $F$ such that $K \cdot D = 0$, and we have $K \cdot X' = 2$. Now we have to discuss three possibilities:

(i) $X'$ is irreducible;

(ii) $X' = X_1' + X_2'$, with $X_1'$, $X_2'$ distinct irreducible curves such that $K \cdot X_1' = K \cdot X_2' = 1$;

(iii) $X' = 2X_1'$, with $X_1'$ an irreducible curve such that $K \cdot X_1' = 1$.

In either case we start from the remark that a base point $x$ of $|2K|$ has to lie on $X$, and specifically on $X'$ and not on $X_0$, otherwise $|2K|$ would have, as we saw in case I, fixed components, a contradiction.

SUBCASE (i). Let $x \in X'$ be a base point of $|2K|$. By the results of Francia's we mentioned above (cf. Remark 4.6 and Lemma 3.5 of [Fr]) and by the argument in step 2 of the proof of Proposition 3.6 of [Fr], $x$ is a simple point for $X'$, and the curve $F = M_0$ of $\{M\}$ passing through $x$, has intersection multiplicity 1 with $X'$ at $x$. Since $M \cdot X' = 2$, this means that $F$ intersects $X'$ at $x$ and at a further point $y$ distinct from $x$.

Furthermore the curves of the Albanese pencil have genus 2, and therefore there is a biregular involution $\sigma : S \to S$ which fixes the curves of the Albanese pencil and acting on the general one as the hyperelliptic involution does. Of course $\sigma(X') = X'$ and in particular $\sigma(x) = y$. So $y$, as well as $x$, is a base point for $|2K|$. Moreover each curve of $|2K|$ is $\sigma$-invariant, because $|2K|$ cuts out on the general curve of the Albanese pencil a series contained in the bicanonical series, which is still composed with the hyperelliptic involution. This means that each pair of $\sigma$-conjugate points on $X'$ imposes at most one condition to the linear series cut out by $|2K|$ on $X'$. Let us now look at this series more closely. It has degree $(2K) \cdot X' = 4$ and dimension

$$r = h^0([2K]) - h^0([2K - X']) - 1 = 4 - h^0([K + X_0 + M'])$$

where we write $K = X' + X_0 + M'$ and $M' \in \{M\}$. Consider the exact sequence

$$0 \to [K + X_0] \to [K + M' + X_0] \to \omega_{M'}(X_0) \to 0$$

Since $h^0([K + X_0]) = h^0([K]) = 1$ (this easily follows from Artin's criterion, because $X_0$ is a sum of rational $(-2)$-curves), we deduce

$$\begin{aligned} h^0([K + X_0 + M']) &\leq h^0([K + X_0]) + h^0(\omega_{M'}(X_0)) \leq \\ &\leq h^0([K + X_0]) + p_a(M') = 3 \end{aligned}$$

(the last inequality follows by $M \cdot X_0 = 0$).

Then $r \geq 1$, in other words $|2K|$ cuts out on $X'$ a $g_4^r$, with $r \geq 1$, and with two distinct base points $x$ and $y$. Therefore $r = 1$ and the series is obtained from the $g_2^1$ on $X'$ (which has genus 2 ) by adding the fixed divisor $x+y$. But this $g_2^1$, as we saw, must be composed and hence coincide with the irrational (elliptic) series of divisors of order two cut out on $X'$ by the curves of $\{M\}$, a contradiction.

SUBCASE (ii). Suppose a base point $x$ of $|2K|$ lies on $X_1'$. We cannot have $M \cdot X_1' = 0$ because then $X_1'$ must be contained in some curve of the Albanese pencil, and therefore by using Lemma 3.5 of [Fr] once again we find a contradiction.

Neither can we have $M \cdot X_1' = 2$. In fact this would imply $M \cdot X_2' = 0$, namely $X_2'$ would be contained in some curve of the Albanese pencil. By the 2 - connectedness of the canonical curve and analysing the decomposition $(M + X_1') + (X_2' + X_0)$ of $\{K\}$, we would then have

$$X_1' \cdot (X_0 + X_2') \geq 2$$

Thus

$$1 = K \cdot X_1' = X_1'^2 + X_1' \cdot (X_2' + X_0) + X_1' \cdot M \geq X_1'^2 + 4 \Rightarrow X_1'^2 \leq -3$$

whence

$$1 = K \cdot X_1' = 2p_a(X_1') - 2 - X_1'^2 \geq 2p_a(X_1') - 2 + 3 = 2p_a(X_1') + 1 \Rightarrow$$

$$\Rightarrow 2p_a(X_1') \leq 0$$

a contradiction, since $X_1'$ dominates $A$.

So one has $M \cdot X_1' = 1$, therefore also $M \cdot X_2' = 1$, and a base point $x$ on $X_1'$ yields the existence of another base point $y$ on $X_2'$. Now we may repeat word by word the argument of subcase (i) which implies that on $X' = X_1' + X_2'$, the system $|x + y|$ is a $g_2^1$. Therefore $X_1'$ and $X_2'$ should be rational, a contradiction.

SUBCASE (iii). In this case no point of $X$ can be a base point of $|2K|$ since it is a singular point of $|K|$. Q.E.D. for 4.16.

Let us assume $|2K|$, hence $\{K\}$, has some isolated base points $x_1, \ldots, x_h$. In view of the previous Lemma 4.16 we may assume the general paracanonical curve is irreducible of arithmetic genus 5, smooth at the points $x_1, \ldots, x_h$, as we are going to see. Let in fact $\mu$ be, as usual, the sum of the multiplicities of intersection at $x_1, \ldots, x_h$ of two general curves in $\{K\}$; thus $1 \leq \mu \leq 4$.

If $\mu = 4$ then $\iota = 1$ and $\{K\}$ would consist of the system of Albanese fibres, a contradiction; the case $\mu = 3$ is ruled out by Lemma 4.4. So we are left with the two cases $\mu = 1$ and $\mu = 2$, which we discuss separately.

*The case* $\mu = 1$ $(h = 1, x := x_1)$.

Let $C = C_t \in |K + t|$ be a general paracanonical curve and, as in the proof of Proposition 4.11, we look at the linear series $g(u)$ cut out on $C$ by a general twisted bicanonical system $|2K + u|$, $u \in \mathrm{Pic}^0(S)$. This series is a complete, non special, $g^3_8$ and inside it we have the rational series $\mathscr{H}(u)$ of divisors cut out on $C$ by the curves of the system $\Delta_u$. The fixed divisor of $\mathscr{H}(u)$ is exactly $2x$, therefore the movable part is a rational series $\gamma^1_6$. Now we shall prove the:

LEMMA 4.17. (i) *The general twisted bicanonical system* $|2K + u|$ *does not have* $x$ *as a base point;*

(ii) *there are curves of a general twisted bicanonical system* $|2K + u|$ *passing through* $x$ *and having a simple point at* $x$.

PROOF. (i) We start noticing that a complete, non special, $g^3_8$ on an irreducible curve of arithmetic genus 5 can have, by Clifford's lemma, at most 2 base points, and if the bound is achieved then the curve is hyperelliptic.

Now assume the general twisted bicanonical system $|2K + u|$ has $x$ as a base point. Then the base locus of $g(u)$ is either $x$ or $2x$, in view of the above remark and since $x$ is the only base point of $\{K\}$. But if the base locus were $2x$ for general $u$, by Clifford's lemma all the series $g(u)$ would coincide, implying a contradiction as in the Proof of 4.11. We can therefore assume that for general $u \in A$, $x$ is a simple base point of $g(u)$; since the index of speciality of $g(u) - x$ is 1, there exists a unique smooth point $y(u)$ such that $g(u) - x = |K - y(u)|$ has no fixed points and in particular $C$ is not hyperelliptic. So we get a rational map $f: u \in \mathrm{Pic}^0(S) \to y(u) \in C$, and this map is not constant, because $g(u) \neq g(u')$ if $u \neq u'$.

On the other hand the fact that $f$ is not constant gives a contradiction, because $S$ would then have a 1 dimensional system of elliptic (or rational) curves, thus $S$ would not be of general type (see [Bo], p. 206).

(ii) Assume the general twisted bicanonical curve passing through $x$ is singular at $x$ (and in this case it must have a node at $x$, as it is shown by considering the twisted bicanonical curves which are sums of two curves of $\{K\}$.

Consider the subsystem of $|2K + u|$ formed by all curves passing through $x$.

This system cuts out on $C$ a $g^2_8$ contained in $g(u)$ and consisting of a complete series $g^2_6$, which we denote by $g'(u)$, plus the fixed divisor $2x$; notice the $g'(u)$ has no base points since it contains the movable part of $\mathscr{H}(u)$, in particular $C$ is not hyperelliptic.

Now by the Riemann-Roch theorem the residual of the $g'(u)$ with respect to the canonical system of $C = C_t$, is a divisor $D(u)$ of degree two. Thus we have an elliptic 1 dimensional family $\mathscr{D}$ of divisors of degree 2 on $C$, given by $\{D(u)\}_{u \in \mathrm{Pic}^0(S)}$. $\mathscr{D}$ has no base points on $C$, as we can see by repeating the same argument we already gave before.

Notice that

$$|K_C - D(u)| + 2x = g'(u) + 2x = |K_C + (u - t)_{|C}|$$

therefore

$$D(u) \equiv 2x + (t - u)_{|C}$$

thus

$$D(u) + D(u') \equiv 4x + (2t - u - u')_{|C} \equiv D(v) + D(v')$$

as soon as $u + u' = v + v' = w$. Since $C$ is not hyperelliptic, the above divisors give a complete $g^1_4(w)$.

We imitate now the argument we gave in the proof of Lemma 4.11. Let us fix a general point $y$ in $C$, and let $D(u_1), \dots, D(u_h)$ be the divisors of the series $\mathscr{D}$ containing $y$; we put $D(u_i) = y + y_i$, $i = 1, \dots, h$. We choose a divisor $D(u'_1)$ which does not contain $y, y_1, \dots, y_h$, and fix $u'_2, \dots, u'_h$ so that

$$u_1 + u'_1 = u_2 + u'_2 = \ldots = u_h + u'_h$$

We have

$$D(u_1) + D(u'_1) \equiv D(u_2) + D(u'_2) \equiv \ldots \equiv D(u_h) + D(u'_h)$$

and in fact, since all the divisors $D(u_i) + D(u'_i)$ contain $y$ and are divisors of one and the same $g^1_4$, they must coincide. Therefore $y_2, \dots, y_h$ belong to $D(u_1) + D(u'_1)$, and since they do not belong to $D(u'_1)$, they do belong to $D(u_1)$, i.e. $y_2, \dots, y_h$ coincide with $y_1$ and $h = 1$. In other words $\mathscr{D}$ is an elliptic involution on $C$.

Now we have

$$g'(u) - D(u') \equiv K_C - D(u) - D(u')$$

and since $|D(u) + D(u')|$ is a $g^1_4$, also $|K_C - D(u) - D(u')|$ is a $g^1_4$. This shows that the divisors of $\mathscr{D}$ impose only one condition to the series $g'(u)$, for every choice of $u$; in other words $g'(u)$ is composed of $\mathscr{D}$ for every $u$.

The above yields that if $y$ is a general point of $C$ and $C_2, \ldots, C_\iota$ are the other paracanonical curves through $y$, then $C_2, \ldots, C_\iota$ also pass through the conjugate of $y$ in the involution defined by $\mathscr{D}$ on $C$.

This statement in turn implies the validity of the following:

ASSERTION 4.18. *Let $y$ be a general point of $S$ and let $C_1, \ldots, C_\iota$ be the paracanonical curves through $y$. Then there is one and only one point $z \neq y$ such that $z$ too lies on $C_1, \ldots, C_\iota$.*

This assertion yields the existence of a biregular involution $\sigma : S \to S$ which fixes $x$ and also fixes all paracanonical curves. In particular it leaves invariant the intersection of two general paracanonical curves. But these intersect at 4 points, one of which, $x$, is fixed by $\sigma$. This is clearly impossible. Q.E.D. for 4.17.

Lemma 4.17 implies that for general $u$ imposing vanishing of order 2 at $x$ to the divisors of $g(u)$ gives two conditions.

Look again at the series $\mathscr{H}(u)$ now. Since, as we know, $\mathscr{H}(u) = 2x + \gamma^1_6$, by the previous remark $\mathscr{H}(u)$ has to be linear, in other words the $\gamma^1_6$ is in fact a complete $g^1_6$. This observation enables us to prove the:

LEMMA 4.19. *Let $y$ be a general point of $S$, let $C, C'$ be the paracanonical curves through $y$. If $C'$ intersects $C$ in $x + y + z + z'$, then all the paracanonical curves passing through $y$ cut out on $C$ the same divisor.*

A moment of reflection shows that the proof of this lemma can be carried through with the same argument we use in the proof of Lemma 4.11; therefore we omit it.

LEMMA 4.20. *If $K^2 = 4$ and $\mu = 1$, then $\iota = 2$.*

PROOF. As in the Proof of 4.12 we consider the curves $L'_t$ and $L_t$. Since $L'_t \to \mathbf{P}^1$ is a triple cover and $S$ is of general type, then $L_t$ is irreducible for general $t$ and trigonal. Lemma 4.19 implies that the rational map $\omega' \circ \pi_S : S \times \mathbf{P}^1 \to A^{(\iota)}$, when restricted to $L'_t$, factors through the triple cover $L'_t \to \mathbf{P}^1$. Hence the curves $L_t$ are the fibres of the Albanese pencil. Therefore $\iota = 2$ (see the proof of Lemma 4.12). Q.E.D. for 4.20.

We can finally exclude the case $\mu = 1$, in view of Lemma 4.7, since we should have $3 = \delta = 2g - 2$, a contradiction.

*The case* $\mu = 2$ $(h \leq 2)$. By 4.7, 4.13 and 4.16 we only need to show that $\iota = 2$.

Let $C = C_t \in |K + t|$ be again a general paracanonical curve and look at the linear series $g(u)$ cut out on $C$ by a general twisted bicanonical system $|2K + u|$, $u \in \mathrm{Pic}^0(S)$ of $S$. The series $g(u)$ is a complete, non special, $g^3_8$.

Consider also the series $\mathcal{H}(u)$. This series has a fixed part formed by $2(x_1 + x_2)$ if $h = 2$, or by $4x$ if $h = 1$ (we put $x := x_1$ in this case), its variable part being a $\gamma^1_4$. Since this $\gamma^1_4$ is rational, it is certainy contained in a linear series $g^r_4$, where either $r = 1$ or $r = 2$. If $r = 2$, $C$ is hyperelliptic and the $g^2_4$ is unique, which implies the usual contradiction, since the linear series $g(u)$ are different. Thus $r = 1$ and the $\gamma^1_4$ is a $g^1_4$. Now the same arguments as in the case $\mu = 1$ carry over to prove assertion 4.18, whence the curves $L_t$ are the Albanese fibres and $\iota = 2$. Q.E.D. for 4.15.

## 5. SURFACES WITH LOW INVARIANTS

We start this paragraph by recalling the results of [Ca] giving a complete description of the case $K^2 = 2$.

The main remark was made by Bombieri ([Bo], p. 206) to the effect that if $K^2 = 2$, then:

(i) the general paracanonical curve is irreducible.

To prove this, notice that, with the usual notation, one has

$$2 = K^2 = K \cdot (X + M) = K \cdot X + X \cdot M + M^2$$

If $X$ is non zero, since $X \cdot M \geq 2$, $K \cdot X \geq 0$, $M^2 \geq 0$, one should have $X \cdot M = 2$, $K \cdot X = 0$, $M^2 = 0$. Hence $\{M\}$ should be the Albanese pencil and $K \cdot X = 0$ implies that all components of $X$ are contained in fibres of the Albanese pencil, contradicting $M \cdot X = 2$.

It is now easy to see that:

(ii) $\iota = g = 2$.

In fact (i) yields $\iota \geq 2$ and then $\iota = 2$ by Propositions 3.3 and 3.4. Moreover $\{K\}$ has no base points by Lemma 4.4, thus Lemma 4.7 implies $g = 2$.

Since $\{K\}$ has no base point (and consequently $|2K + t|$ is base point free, by 4.5), then $\omega : S \to A^{(2)}$ is a morphism of degree 2.

Let $B$ be the branch locus of $\omega$ on $A^{(2)}$. Since $F$ has genus 2, and $C_t$ has genus 3, $B$ intersects the fibres $E$ of the $\mathbf{P}^1$-bundle $A^{(2)} \to A$ with multiplicity 6 and the sections $D$ (cf. § 2) with multiplicity 4. Therefore $B \sim 6D - 2E$.

Observe now that the automorphism group of $A^{(2)}$ coincides with Aut$(A)$ acting naturally on the divisors of degree 2. We see that $u \in A$ acts on the set of classes of algebraically equivalent divisors by $E_t \to E_{t+2u}, D_t \to D_{t-u}$. Therefore Aut$(A)$ acts transitively on all algebraically equivalent classes of linear equivalence which are not algebraically equivalent to the canonical class (which is $\sim -2D + E$). Thus all these surfaces can be obtained by fixing a divisor $L = 3D - E$ and extracting the square root of an effective divisor $B \equiv 6D - 2E$ with only simple singularities.

Now by standard computations one sees that $h^0(|6D - 2E|) = 7$. Hence by varying the elliptic curve, one has a 7-dimensional irreducible family of surfaces with $p_g = q = 1$, $K^2 = 2$; we refer to [Ca], § V, for the proof of the following:

THEOREM 5.1. *The moduli space of surfaces with $p_g = q = 1$, $K^2 = 2$ is irreducible, generically smooth, of dimension 7. Moreover the Albanese morphism induces an isomorphism of fundamental groups.*

REMARK 5.2. The correction to a misprint at p. 282 of [Ca] never reached the printer. Line 7 from the bottom should read: « ... sections of $H^-$ correspond to sections of $\Omega^1_X(\log D) \otimes \Omega^2_X$. Secondly ... ». Accordingly the last line of the proof of proposition 22 should be: «... We exploit the residue sequence

$$0 \to H^0(\Omega^1_X \otimes \Omega^2_X) \to H^0(\Omega^1_X(\log D) \otimes \Omega^2_X) \to H^0([K_X]_{|D})$$

the vanishing of $H^0(\Omega^1_X \otimes \Omega^2_X)$ and $H^0([K_X]_{|D})$, since $K_X \cdot K = -2$, to show the vanishing of $H^-$ ». Notice: in [Ca], $A^{(2)}$ is denoted by $X$ and the branch locus $B$ by $D$.

Next we turn to the case where $g = 2$. By the basic formula $\iota = g - \lambda$, we have two cases:

(i) $g = 2$, $\iota = 1$, $\lambda = 1$;

(ii) $g = \iota = 2$;

in both of which $\omega$ yields a double cover over a $\mathbf{P}^1$-bundle $Y$ over $A$. Moreover an important information is provided by the following:

PROPOSITION 5.3. *If $g = 2$ then $K^2 = 2 + \nu$, where $\nu$ is the number of the Albanese fibres which are not 2-connected, counted with the appropriate multiplicity. If $g = 3$ and the general Albanese fibre is not hyperelliptic, then $K^2 \geq 3$.*

PROOF. Let $g = 2$. Let us go back to the notation of Remark 2.4, (ii). Both sheaves $V$ and $V_2$ are locally free of respective ranks 2 and 3 and enjoy the base change property. We have an injective sheaf map $\mathrm{Sym}^2(V) \to V_2$ with cokernel $J$ whose stalk $J_t$ at $t \in A$ is the cokernel of the map $\mathrm{Sym}^2(H^0([K_{F_t}]) \to H^0([2K_{F_t}])$. Therefore $\dim J_t \neq 0$ if and only if $F_t$ is not 2 - connected (cf. [Ml], Theorem 3.1). The assertion follows by extracting degrees. The case $g = 3$ can be treated similarly.

Q.E.D.

Now we discuss the two cases where $g = 2$.

CASE (i). We know form § 2 that $V = \mathcal{O}_A(u) \oplus t$ (cf. Remark 2.4, (i)), and $Y = \mathbf{P}(V^\vee)$ is obtained by adding a section at infinity to a line bundle $L = [u']$, where $u' = u - t$. If $\varepsilon : Y \to A \times \mathbf{P}^1$ is the elementary transformation at the 0 - point of the fibre of $L$ over $u'$, then $\varepsilon \circ \omega = \alpha \times f$, where $f$ is the rational map determined by the linear system $|K - t|$. As we know by the fact that if $K^2 \leq 3$ (see Lemma 4.9 and the above discussion for $K^2 = 2$) the general paracanonical curve is irreducible, this case could only occur if $K^2 \geq 4$. In fact (cf. also 5.10) there is an example by Xiao ([X2], Theorem 2.9, (i)) with $K^2 = 4$.

CASE (ii). Here $K^2 \geq 2$. Since we already discussed the case $K^2 = 2$, we may assume $K^2 \geq 3$. By Proposition 5.3 there are $K^2 - 2$ Albanese fibres which are not 2 - connected. Since the paracanonical system cuts out the complete canonical system of the genus 2 fibres, the base locus of $\{K\}$ consists, roughly speaking, of the $K^2 - 2$ base loci of the canonical systems of the fibres which are not 2 - connected.

Concerning this case we can prove the following:

THEOREM 5.4. *The case $K^2 = 3$ and $g = \iota = 2$ occurs if and only if $\{K\}$ has exactly one trasversal base point, lying in the only fibre of the Albanese pencil which is not 2 - connected. Such surfaces are double covers of $A^{(2)}$ and do in fact exist.*

PROOF. If $K^2 = 3$ we saw that $\{K\}$ has no fixed part. Thus if $g = \iota = 2$ then $\{K\}$ has exactly one trasversal base point $x$, and the Albanese fibre through $x$ decomposes into the sum $F = F_1 + F_2$ of two curves of arithmetic genus 1, such that $F_1 \cap F_2 = \{x\}$, $F_1 \cdot F_2 = 1$, and $F_i^2 = -1$, for $i = 1, 2$. Notice that this is the only Albanese fibre which is not 2 - connected in view of Proposition 5.3. Recall from § 4, that conversely we saw that if $\{K\}$ has a base point $x$, then $g = \iota = 2$ and we are in the above situation.

Consider the rational map $\omega : S \to A^{(2)}$, whose indeterminacy locus coincides with $x$. By blowing $x$ up we have a surface $S'$ and a morphism $S' \to A^{(2)}$ which contracts the proper transforms of $F_1$ and $F_2$ (two curves of arithmetic genus 1 and self intersection $-2$) to two elliptic singularities lying on one and the same fibre $E$ of $A^{(2)} \to A$. The branch locus contains this fibre because $\omega$ is injective on the exceptional divisor of $S'$. On the other hand since the general paracanonical curve is irreducible of arithmetic genus 4, we immediately see that the branch divisor $B$ is algebraically equivalent to $6D$.

In order to prove the existence, we seek for a curve $B = B' + E$, such that $B' \sim 6D - E$ has two ordinary triple points on $E$ and no other non simple singularity. Then $B$ has two ordinary quadruple points each of which produces a simple elliptic singularity corresponding to a smooth elliptic curve with self intersection $-2$.

We provide now an explicit construction of such an example. Notice that on the surface $A^{(2)}$ there are exactly 3 curves $T_\eta$, where $\eta$ is a point of order 2 on $A$, algebraically equivalent to $-K \sim 2D - E$. $T_\eta$ is the set of divisors $u + v$ such that $u - v = \eta$. Let $T = T_\eta$ be one of these curves and observe that $T$ intersects transversally each fibre $E$ at two distinct points which do not lie on the diagonal of $A^{(2)}$. Thus once $E$ has been fixed, we denote by $x$ and $y$ the points $T \cap E$. Since $x$ and $y$ are not on the diagonal, there are exactly two curves $D_t$, $D_{t+\eta}$ of the system $\mathscr{D} = \{D_t\}_{t \in A}$ through $x$ and analogously two curves $D_{t+\xi}$, $D_{t+\eta+\xi}$ through $y$ ($\xi$ is a 2-torsion point different from $\eta$). It is easy to check that the curve $B' = T + D_t + D_{t+\eta} + D_{t+\xi} + D_{t+\eta+\xi}$ has the required properties since its singularities are exactly: two ordinary triple points at $x$ and $y$ and 4 nodes at the points $t + (t+\xi)$, $t + (t+\xi+\eta)$, $(t+\eta) + (t+\xi)$, $(t+\eta) + (t+\xi+\eta)$. We omit the routine verification that, taking the minimal resolution $S'$ of the double covering of $A^{(2)}$ branched on $B = B' + E$, one finds the blow up at one point of a minimal surface with $p_g = q = 1$, $K^2 = 3$. Q.E.D.

PROBLEM 5.5. *Do the above surfaces with $p_g = q = 1$, $K^2 = 3$, $g = \iota = 2$ form an irreducible family?*

We believe the answer should be positive.

We continue the discussion of the case $K^2 = 3$, by obtaining further information:

PROPOSITION 5.6. (i) *If $\{K\}$ has no fixed part and $\delta$ is odd then the general fibre of the Albanese pencil is not hyperelliptic;*

(ii) *if $g = 4$, $\lambda = 0$, $\iota \geq 2$ and the general fibre of the Albanese fibration is not hyperelliptic then $K^2 \geq 4$;*

(iii) *if $g = K^2 = 3$ then no fibre of the Albanese pencil is hyperelliptic.*

PROOF. (i) The hyperelliptic involution on the fibres induces an involution $\sigma : S \to S$ such that $\sigma(C_t) = C_t$ for any $t \in A$. Since $\{K\}$ has no fixed part, then two general curves of $\{K\}$ intersect at $\delta$ variable points which are not fixed by $\sigma$ but are pairwise conjugate in $\sigma$. Then $\delta$ must be even, a contradiction.

(ii) We have an exact sequence of sheaves on $A$

$$0 \to \mathscr{B} \to \mathrm{Sym}^2 V \to V_2 \to \mathscr{T} \to 0$$

where $\mathscr{T}$ is a torsion sheaf and $\mathscr{B}$ is locally free of rank

$$\mathrm{rk}\ \mathscr{B} = \mathrm{rk}\ \mathrm{Sym}^2 V - \mathrm{rk}\ V_2 = [g(g+1)/2] - 3(g-1)$$

If $g = 4$ then $\mathscr{B}$ is a line bundle and we must have $\deg \mathscr{B} \leq 0$ since $|2 D_{E_t}|$ is empty (see [CaCi]). We conclude since

$$\deg \mathrm{Sym}^2 V = 1 + g, \quad \deg V_2 = 1 + K^2$$

(iii) The sheaf map $\mathrm{Sym}^2(V) \to V_2$ is now an isomorphism, hence the assertion follows. Q.E.D.

THEOREM 5.7. *If $K^2 = 3$, then $g \leq 3$ and if $g = 3$ then $\lambda = 0$.*

PROOF. By a Theorem by G. Xiao ([X1], Theorem 2) we have $g \leq 4$. If $g = 4$, Xiao proves moreover that $V$ is semistable, hence indecomposable, therefore $\lambda = 0$, contradicting Proposition 5.6. If $g = 3$ and $\lambda \neq 0$, then $\iota \leq 2$, but $\iota \geq 2$ by Lemma 4.9: whence $\iota = 2$ and by Lemma 4.7 we derive a contradiction. Q.E.D.

THEOREM 5.8. *If $g = K^2 = 3$, then $\omega : S \to A^{(3)}$ is a morphism which is birational onto its image. Moreover $\omega$ is an isomorphism of the canonical model of $S$ onto $\omega(S)$, which is a divisor with at most simple singuarities in a linear system homologous to $|4D - E|$. Such surfaces do in fact exist.*

PROOF. The first assertion is clear. For the rest of the theorem we refer to [CaCi]. Q.E.D.

We conclude with a few problems:

PROBLEM 5.9. *Describe the hyperelliptic case: i.e. the case in which the general Albanese fibre is hyperelliptic.*

PROBLEM 5.10. *Can one give contructions of surfaces with $g = 2$ and large $K^2$ ?*

Xiao ([X2], Thm. 2.2) proves that then $K^2 \leq 6$ , and gives (Thm. 2.9, (i)) an example with $K^2 = 4$ (there, in Xiao's notation, $e = 1$ and the divisor $D_2$ is empty).

PROBLEM 5.11. *Study the tricanonical map for surfaces with $p_g = q = 1$ (for $K^2 = 2$ , the tricanonical image has a double curve with 6 cuspidal points).*

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PROBLEM 5.10. *Can one give contructions of surfaces with $g = 2$ and large $K^2$ ?*

Xiao ([X2], Thm. 2.2) proves that then $K^2 \leq 6$ , and gives (Thm. 2.9, (i)) an example with $K^2 = 4$ (there, in Xiao's notation, $e = 1$ and the divisor $D_2$ is empty).

PROBLEM 5.11. *Study the tricanonical map for surfaces with $p_g = q = 1$ (for $K^2 = 2$ , the tricanonical image has a double curve with 6 cuspidal points).*

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