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Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations

Fabrizio Catanese

Dipartimento di Matematica della Università di Pisa, via Buonarroti 2, I-56121 Pisa, Italia

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0. Introduction

One of the main sources of inspiration for the present research is the notion of surfaces with irrational pencils, which was amply present in the work of several classical italian algebraic geometers, such as De Franchis, Castelnuovo, Enriques and Severi.

An irrational pencil is a morphism of the given surface X to a curve C of genus $\geqq 1$, and it was clear to the classical geometers how the existence of such pencils would imply the existence of exorbitant linear systems on X (cf. [Sev], and [Ca 1], [Ca 2], [Be 2] for a modern account). Castelnuovo [Cas 2] also claimed that for surfaces without irrational pencils the number of moduli would be rather low (cf. [Ca 1], [Ca 2], [Rei] for related discussion and results), and I have been curious for some time (cf. [Ca 3], p. 72) about the possible bounds which can be given for the number of moduli of surfaces possessing an irrational pencil: indeed this question was the motivation for the present research.

What really turns out, is that the classical geometers never realized about the fact that, if the genus of the target curve C is at least 2, the existence of such an irrational pencil is indeed a topological property of X. As far as I know, Siu was the first [Siu] to give a topological characterization in terms of fundamental groups, and his result was independently rediscovered with an entirely different proof by Beauville (cf. appendix to the present paper). After I had some partial results in this direction, Beauville's communication about his result stimulated my search for a cohomological characterization of irrational pencils of genus $\geqq 2$ (note that the existence of an irrational pencil of genus 1 is not a topological property: possibly this fact has been hiding the above result for quite a long time), and when I obtained it I pretty soon realized that it could have a higher dimensional analogue.

The present paper rallies thus around a very clean cut leit-motiv: essentially via Hodge theory, the existence of certain fibrations between irregular Kaehler manifolds and varieties is merely a multilinear algebra property of the real cohomology algebra of the source manifold.

To be more precise, given a compact Kaehler manifold X of $\mathbb{C}$-dimension equal to n, which is irregular, i.e., such that there are some non zero holomorphic

1-forms in $H^0(\Omega_X^1)$, we consider the Albanese morphism α: $X \to \mathrm{Alb}(X)$, given by integration of the above 1-forms.

We define X to be of **Albanese general type** if α is not surjective but it has a n-dimensional image. Although the above property seems to be depending upon the complex structure of X, we immediately notice (Prop. 1.4) that in fact it is only a property of the exterior graded subalgebra $\Lambda(X) \subset H^*(X, \mathbb{C})$ generated by $H^1(X, \mathbb{C})$. The class of Albanese general type manifolds is, in the realm of irregular manifolds, a good generalization of the class of curves of genus at least 2.

The fibrations whose existence is "dictated by topology" are the morphisms f: $X \to Y$, where Y is normal, of dimension $k<n$, and has a smooth model which is of Albanese general type.

The most general result we have concerning the previous rather vague statements is Theorem 2.25, stating that, for fixed X, there is a bijection between the set of morphisms as above and the set of saturated real $2k$-wedge subspaces F of $H^1(X, \mathbb{C})$.

Since this result is technically complicated, we first prove a very special but archetypical result, Theorem 1.10, stating that for given X there exists a non constant holomorphic map f: $X \to C$, where C is a curve of genus $g \geqq 2$, if and only if there is a g-dimensional maximal 1-wedge subspace V of $H^1(X, \mathbb{C})$.

Here, a k-wedge subspace is a subspace V of dimension at least $(k+1)$ such that all exterior products of $(k+1)$ elements of V are zero, but there is a k-fold non zero exterior product of elements of V.

The link of the above notion with geometry stems from the well known theorem proven by Castelnuovo and De Franchis at the turn of the century ([Cas 1],[D-F]), and relating the existence of a (non constant) holomorphic map f: $X \to C$ to the existence of two linearly independent 1-forms such that $\omega_1 \wedge \omega_2 = 0$.

The main line of ideas is rather straight and runs as follows: first of all, by the Kaehler condition, the type decomposition of forms passes to cohomology, and therefore the Hodge decomposition, together with complex conjugation, relates properties of the exterior algebra $H^*(X, \mathbb{C})$ to properties of the ("holomorphic") subalgebra $\Lambda^{\mathrm{hol}}(X)$ generated by $H^0(\Omega_X^1)$.

Thus the existence of wedge subspaces in $H^1(X, \mathbb{C})$ implies the existence of wedge subspaces in $H^0(\Omega_X^1)$. Once again the Kaehler condition comes into play: since holomorphic forms are closed, any subspace of $H^0(\Omega_X^1)$ determines an integrable foliation, and essentially the existence of meromorphic non constant ratios between the 1-forms of a wedge subspace can be used to imply the closedness of the leaves of the foliation.

Speaking less informally, a key result we establish in § 1 is Theorem 1.14, a generalization of the theorem of Castelnuovo and de Franchis, and stating that there is an Albanese general type fibration f: $X \to Y$, with $\dim Y = k$, if and only if there is a $(k+1)$ dimensional k-wedge subspace $U \subset H^0(\Omega_X^1)$ such that $\Lambda^k(U)$ embeds into $H^0(\Omega_X^k)$. This theorem has been independently obtained by other authors (see § 1) with different proofs, as far as we understand, and firstly by Ran in [Ran 1]: here our proof is based on Dini's theorem on implicit functions and allows a "twisted" generalization (Theorem 1.17).

We end § 1 by defining a Kaehler manifold X to be **Albanese primitive** if it does not admit any non trivial Albanese general type fibration, and we illustrate the relevance of this notion to the classification problem of irregular manifolds: in particular we pose some problem related with Green and Lazarsfeld's important and inspiring work on generic vanishing theorems [G-L 1], [G-L 2], by which we hope that, via fibrations and unramified covers, the classification of irregular manifolds may be reduced to the classification of a particular class of primitive manifolds, the **simple** ones, for which all topologically but not analytically trivial line bundles $\mathscr{L}$ have cohomology groups $H^i(\mathscr{L})=0$ for $i<n$. It should be noted that Green and Lazarsfeld [G-L 2] have arrived also to a notion of primitive manifold, but following essentially the dual way, of considering the Picard instead of the Albanese variety of X.

§ 2 is entitled "Kaehler-Hodge exterior algebra" and is the most technical part of the paper: it is devoted to the definitions and theorems which prove the topological nature of the existence of Albanese general type fibrations. We only remark here that the origin of the technical difficulties encountered stems from the fact that (as one can already see in the case where X is a curve) the natural homomorphism of $\Lambda^{\text{hol}}(X) \otimes \Lambda^{\text{hol}}(X)^*$ (* denoting complex conjugation), into $H^*(X, \mathbb{C})$ has a very huge kernel, hence recovering relations in $\Lambda^{\text{hol}}(X)$ from relations in $H^*(X, \mathbb{C})$ is not immediate.

§ 3 is instead devoted to the applications to moduli of algebraic surfaces which were my first motivation for dealing with this type of questions: the basic idea is very simple also in this case.

In fact, we notice that if a surface S is fibred over a curve C of genus $\geqq 2$, then, by the previous results, all its deformations possess such a fibration, and, if the fibre curves are not generically isomorphic, then we can apply Arakelov's theorem, implying that the moduli space of S has a quasi finite map to the moduli space for the pairs given by the curve C and the finite subset Σ of critical values of the fibration. In this way, via the classical Zeuthen-Segre computation of the number of critical values in terms of the second Chern class of X, we obtain some rather general upper bounds for the number of moduli of algebraic surfaces with an irrational pencil of genus at least 2.

Finally, after dealing with topological and transcendental methods, we turn to the problem whether one can give purely algebraic proofs of many of the previous statements.

Thus, in § 4, we consider the problem of stability under deformations of X of the existence of irrational pencils or higher dimensional analogues. We frame the problem in a rather general context, using the theory of Deformation of maps, as developped by Horikawa [Ho 1–3], Flenner [Fl] and Ran [Ran 2], obtaining sufficient conditions valid also in positive characteristic. Unfortunately these sufficient conditions, always verified in characteristic zero, by virtue of results on Variation of Hodge structure by Fujita and Kollar [Fu], [Ko], are not verified in positive characteristic, and we plan to return in the future on the problem of deciding about possible counterexamples. We also treat, but not in complete generality, the problem of bounds for moduli in higher dimension or char $=p>0$.

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Notation

$H^i(X, \mathscr{F}) = H^i(\mathscr{F})$	For a coherent sheaf $\mathscr{F}$ on an algebraic variety, or a compact Kaehler manifold
Ω^i_X	The sheaf of holomorphic i-forms on X
$A = \mathrm{Alb}(X)$	The Albanese variety of X
$\alpha\colon X \to A$	The Albanese morphism of X
$F = F_a$	The fibre of the Albanese map at $a \in A$
$K = K_X$	A canonical divisor of X
$[D]$	The invertible sheaf associated to a Cartier divisor D
$\lvert D\rvert = \mathbb{P}(H^0([D]))$	The complete linear system of effective divisors linearly equivalent to a divisor D
$\mathscr{P}$	A Poincaré sheaf on $X \times A^*$, where
$A^* = \mathrm{Pic}^0(X)$	is the dual Abelian variety of $A = \mathrm{Alb}(X)$, and since $\mathrm{Pic}^0(X) = H^1(\mathcal{O}^*_X)^0 \cong H^1(X, U(1))$, we shall usually denote by
$\mathbb{L}$	a locally constant sheaf determined by a cocycle in $H^1(X, U(1))$, by
$\mathscr{L}$	the holomorphic invertible sheaf $\mathbb{L} \otimes \mathcal{O}_X$, whose isomorphism class is an element of $\mathrm{Pic}^0(X)$, and by
$t \in A^*$	(the linear equivalence class of) a Cartier divisor D such that $[D] \cong \mathscr{L}$.
$j_t\colon X \to X \times \{t\}$	The natural isomorphism, for any $t \in A^*$
$\mathscr{P}_t$	The sheaf $j_t^*(\mathscr{P})$
$\bigoplus_{t\in A^*} (\bigoplus_{m\in\mathbb{N}} H^0(X, [mK] \otimes \mathscr{P}_t))$	$=$ The paracanonical algebra of X
$\lvert mK + t\rvert$	The complete linear system $\mathbb{P}(H^0([mK] \oplus \mathscr{P}_t))$, called a twisted m-th canonical system
$\pi_X\colon X \times A^* \to X$	The projection onto the first factor
$\mathscr{K} = \pi_X^*([K]) \otimes \mathscr{P}$	The paracanonical sheaf on $X \times A^*$
$\mathbb{P}(W)$	$\mathrm{Proj}(W^\vee)$, for a locally free sheaf W
V^*	The complex conjugate subspace of V

1. Primitive irregular Kaehler manifolds of Albanese general type, and the classification of irregular Kaehler manifolds

Let X be a compact Kaehler manifold of complex dimension n, which we assume throughout to be irregular, i.e., such that

(1.1) $$q = h^0(\Omega^1_X) = h^1(\mathcal{O}_X) > 0.$$

A fundamental geometric object for the study of X is given by the Albanese morphism of X

(1.2) $\alpha\colon X \to A$, where $A = \mathrm{Alb}(X)$ is the Albanese variety of X, the q-dimensional complex torus $(H^0(\Omega^1_X))^\vee / j(H_1(X, \mathbb{Z}))$, $j\colon H_1(X, \mathbb{Z}) \to (H^0(\Omega^1_X))^\vee$ being given by integration.

A leit-motiv in this paper shall be the one of pointing out how much of the holomorphic geometry of X is dictated by topology or by the differentiable structure.

(1.3) **Definition.** *Let $Y = \alpha(X)$: then* $\dim(Y)$ *is called the Albanese dimension of X, and shall be denoted by $a = \mathrm{alb}(X)$.*

(1.4) **Proposition.** *The Albanese dimension a of X is a topological invariant of X. More precisely, let $\Lambda(X) \subset H^*(X, \mathbb{C})$ be the exterior graded subalgebra generated by $H^1(X, \mathbb{C})$, respectively $\Lambda^{\mathrm{hol}}(X)$ be the subalgebra generated by $H^0(\Omega^1_X) = H^{1,0}(X)$. Then $\Lambda(X)_{2a} \neq 0$, while $\Lambda(X)_{2a+1} = 0$ (resp. $\Lambda^{\mathrm{hol}}(X)_a \neq 0$, while $\Lambda^{\mathrm{hol}}(X)_{a+1} = 0$).*

Proof. By the Hodge decomposition $H^1(X, \mathbb{C}) = H^0(\Omega^1_X) \oplus (H^0(\Omega^1_X))^*$, where $*$ denotes complex conjugation. Hence if $\Lambda^{\mathrm{hol}}(X)_{a+1} = 0$, certainly $\Lambda(X)_{2a+1} = 0$.

Since $H^0(\Omega^1_X) = \alpha^*(H^0(\Omega^1_A))$, clearly $a = \dim(\alpha(X))$ is the maximal integer m with $\Lambda^{\mathrm{hol}}(X)_m \neq 0$.

To show that $\Lambda(X)_{2a} \neq 0$, we apply the following

(1.5) **Lemma.** *Let $\omega_1, \omega_2, \ldots, \omega_r$ be holomorphic forms in $H^0(\Omega^1_X)$ such that $\omega = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_r$ is $\neq 0$ in $H^0(\Omega^r_X)$. Then $\omega \wedge \omega^* \neq 0$ in $H^{2r}(X, \mathbb{C})$, and $\eta = \eta_1 \wedge \ldots \wedge \eta_r \neq 0$ in $H^r(X, \mathbb{C})$, for every choice of $\eta_i = \omega_i$ or ω_i^*.*

Proof. Let ξ be the Kaehler closed (1, 1) form on X.

If $\omega \wedge \omega^* = 0$ in $H^{2r}(X, \mathbb{C})$, then we would have

$$\int_X \omega \wedge \omega^* \wedge \xi^{n-2r} = 0.$$

But a constant times the integrand is positive, and strictly positive at each point where $\omega \neq 0$, a contradiction.

The second assertion follows immediately from the first, since $\eta \wedge \eta^* = \pm \omega \wedge \omega^* \neq 0$ in $H^{2r}(X, \mathbb{C})$. Q.E.D. for 1.4 and 1.5.

Consider now the Albanese map $\alpha\colon X \to A = \mathrm{Alb}(X) = (H^0(\Omega^1_X))^\vee / j(H_1(X, \mathbb{Z}))$, $j\colon H_1(X, \mathbb{Z}) \to (H^0(\Omega^1_X))^\vee$ being the homomorphism given by integration.

Clearly α depends upon the complex structure, but only to a certain extent. Once fixed a base point x_0, $\alpha(x)$ is given, up to translation, by integration on any path from x_0 to x, so this choice does not depend upon the complex structure.

The complex space $H^0(\Omega^1_X)$, via taking real and imaginary parts of forms belonging to it, determines a subspace V of the space of d-closed 1-forms which maps isomorphically to the De Rham cohomology group $H^1_{\mathrm{DR}}(X, \mathbb{R})$; viewing A as a differentiable torus, a change of choice for V alters α by adding $[g]$, the projection onto A of a differentiable vector valued function $g\colon X \to \mathbb{R}^{2q}$.

The map $\beta = (\alpha + [tg], id_{[0,1]})\colon X x [0, 1] \to A x [0, 1]$ is proper and of maximal rank, equal to $2a+1$, over a locally closed submanifold M of dimension $2a+1$, and with $M \cap (A x \{t\}) \neq \emptyset$ for each t in view of 1.4: hence $\beta^{-1}(M)$ is a differentiable fibre bundle over M, and if M were connected it would be true that the differentiable structure of the general fibre of the Albanese map of X is completely determined by the differentiable structure of X. One has, though, the following counterexample, due to Bogomolor and Kollàr.

(1.6) **Proposition.** *There do exist diffeomorphic Kaehler 3-folds, such that the respective Albanese fibres are not diffeomorphic.*

Proof. By a theorem of Donaldson ([Do]) there do exist simply connected homeomorphic but not diffeomorphic algebraic surfaces S_1, S_2. Let C be a curve of genus $g \geqq 2$. Then $S_1 \times C$ and $S_2 \times C$ are diffeomorphic by Smale's h-cobordism theorem, but their Albanese fibres S_1, S_2 are not diffeomorphic. Q.E.D.

Remark. It remains an interesting question to see what happens in the case of surfaces.

(1.7) **Definition.** *An irregular Kaehler manifold X of dimension n is said to be of Albanese general type if $q > n$ and its Albanese dimension a equals n (that is, its Albanese image Y has dimension n and is a proper subvariety).*

An entirely similar notion can be given for a normal algebraic variety or normal complex space bimeromorphic to a Kaehler manifold.

(1.8) *Remark.* By a theorem of Ueno ([Ueno] Thm. 10.9), given a proper subvariety Y of a complex torus A, letting A_1 be the complex subtorus $A_1 = \{x \in A \mid x + Y = Y\}$, and $u\colon A \to A_2 = A/A_1$ be the quotient map, then A_2 is an Abelian variety, $Z = u(Y)$ is an algebraic variety of general type, and u makes Y an analytic bundle over Z with fibre A_1.

Kawamata [Kaw] extended Ueno's result as to imply an analogous structure theorem for X of Albanese general type.

(1.9) *Remark.* In the case of curves ($\dim X = 1$), we are just considering curves of genus at least 2.

The reason why curves of genus 1, although being irregular, are not interesting from our present point of view, is that the condition of admitting a (non constant) holomorphic map to a curve of genus 1 is not stable by deformation, as it is shown by the example of Abelian surfaces with a fixed polarization type. In fact these manifolds form an irreducible family, but the Abelian surfaces admitting a (non constant) holomorphic map to a curve of genus 1 are exactly the ones isogenous to a product of elliptic curves, hence they form a countable union of proper algebraic subsets of the parameter space of the family.

Concerning curves of genus $\geqq 2$, we shall show that the existence of a (non constant) holomorphic map to a curve of genus $\geqq 2$ is a topological property of X, which admits several characterizations. Later we shall extend this result to higher dimensional image manifolds.

In this respect the prototype of the results we are going to use is the well known (cf. [D-F], [Cas 1], [Bo], [B-P-V])

CdF theorem (Theorem of Castelnuovo-de Franchis). *Let X be a compact Kaehler manifold, and let $\omega_1, \omega_2 \in H^0(\Omega^1_X)$ be two linearly independent forms such that $\omega_1 \wedge \omega_2 = 0$: then there exists a holomorphic map $f\colon X \to C$ to a curve such that $\omega_1, \omega_2 \in f^*(H^0(\Omega^1_C)$ (hence C has genus $g \geqq 2$). More generally, an entirely analogous assertion holds for $\omega_1, \ldots, \omega_r$, linearly independent and such that $\omega_i \wedge \omega_j = 0$ for each i, j.*

We give now, as a direct application of the above CdF theorem, a first proof of a result we shall show later in a greater generality

(1.10) **Theorem.** *Let X be a compact Kaehler manifold: then there exists a non constant holomorphic map $f\colon X \to C$, where C is a curve of genus $g \geqq 2$, if and only if there is a g-dimensional maximal isotropic subspace V of $H^1(X, \mathbb{C})$. Here, saying that V is isotropic, means that $\Lambda^2 V$, the natural image of $\Lambda^2(V)$ into*

$H^2(X, \mathbb{C})$, is zero (we shall later call an isotropic subspace a 1-wedge subspace). Moreover, any maximal isotropic subspace V as above occurs as a pull back $f^*(V')$ of a maximal isotropic subspace V' of $H^1(C, \mathbb{C})$ for some f: $X \to C$ as above.

Proof. Assume that $\varphi_1, \varphi_2, \ldots, \varphi_g$ are a basis of V.

Since $H^1(X, \mathbb{C}) = H^0(\Omega_X^1) \oplus (H^0(\Omega_X^1))^*$, we can write $\varphi_1 = \omega_1 + \eta_1^*$, $\varphi_2 = \omega_2 + \eta_2^*, \ldots, \varphi_g = \omega_g + \eta_g^*$, and we let U be the span of $\omega_1, \omega_2, \ldots, \omega_g$, and W be the span of $\eta_1, \eta_2, \ldots, \eta_g$.

We know that, for $i, j = 1, \ldots, g$, the following form gives a zero cohomology class:

$$(1.11) \qquad (\omega_i + \eta_i^*) \wedge (\omega_j + \eta_j^*) = 0 \quad \text{in } H^2(X, \mathbb{C}).$$

But, on a Kaehler manifold, the type decomposition passes to cohomology, hence (1.11) is equivalent to

$$(1.12) \qquad \omega_i \wedge \omega_j = 0, \quad \eta_i \wedge \eta_j = 0, \quad (\omega_i \wedge \eta_j^*) + (\eta_i^* \wedge \omega_j) = 0.$$

Clearly (1.12) imply $\Lambda^2 U = \Lambda^2 W = 0$, and we can apply CdF to U and W, provided their dimension is at least 2.

Assume $\dim U = 1$: then we may assume $\omega_2 = \ldots = \omega_g = 0$, and again (1.12) plus Lemma 1.5 imply $\omega_1 \wedge \eta_j = 0$ for each j (for $j = 1$, $\omega_1 \wedge \eta_1 = 0$ follows from $\eta_1 \wedge \eta_j = 0$ for each j). Thus CdF can be applied to the subspace $U + W$, whose dimension is $\geqq 2$, hence V pulls back from a curve of genus $\geqq g$, but indeed $= g$ by the maximality of V. The same argument applies if $\dim W = 1$. If $\dim U$, $\dim W$ are both $\geqq 2$, we get f: $X \to C$, f': $X \to C'$, by applying CdF to U, respectively to W.

Consider the product map $\varphi = (f \times f')$: $X \to C \times C'$.

If the image of φ is a curve C'', we are done as before, since V pulls back from $H^1(C'', \mathbb{C})$.

If φ is surjective, we derive a contradiction against (1.12), since $H^*(C \times C', \mathbb{C}) = H^*(C, \mathbb{C}) \otimes H^*(C', \mathbb{C})$ by the Kuenneth formula, and φ^* is injective. Q.E.D.

As mentioned in the introduction, the previous result was inspired to us by the following result proven by Siu [Siu] and Beauville (appendix to the present paper) by entirely different methods.

(1.13) **Theorem** (*Siu-Beauville*). *Let X be a compact Kaehler manifold and C be a curve of genus $g \geqq 2$: then there is a non constant holomorphic map f: $X \to C'$, where C' is a curve of genus $g' \geqq g$, if and only if there is a surjective homomorphism $\pi_1(X) \to \pi_1(C)$.*

To extend these results, we need a generalization of the classical Castelnuovo de Franchis theorem, which has indeed been obtained independently by several authors, first of all by Ran [Ran 1], then later, around the same time, by Green-Lazarsfeld [G-L 2], by Peters, and by ourselves, but with at least 3 different proofs, as far as we understand. We present here our proof, which allows, as we shall soon see, a further generalization.

(1.14) **Theorem (GCdF).** *Let X be a compact Kaehler manifold, and let $U \subset H^0(\Omega_X^1)$ be a $(k+1)$-dimensional strict k-wedge subspace,* i.e., *a subspace with* *a basis given by forms* $\omega_1, \omega_2, \ldots, \omega_{k+1}$, *and such that* $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_{k+1} = 0$, *while* $\Lambda^k(U)$ *embeds into* $H^0(\Omega^k_X)$ *under the natural homomorphism.*

Then there is a holomorphic map $f\colon X \to Y$ *to a k-dimensional normal variety such that* $U \subset f^*(H^0(\Omega^1_Y))$ *(hence Y is of Albanese general type).*

Proof. U determines a rank k saturated subsheaf $\mathscr{H}$ of Ω^1_X, inducing a subbundle of the cotangent bundle in an open set whose complement Σ has codimension at least 2.

Let $\mathscr{F}$ be the foliation defined by $\mathscr{H}$ in the holomorphic tangent bundle of $X - \Sigma$: since X is Kaehler, the holomorphic forms are closed, hence the foliation is integrable. We are going to show in particular that the leaves of the foliation are closed.

To this purpose, let $\mathbb{C}(\mathscr{F})$ be the field of meromorphic functions on X which are constant on the leaves of the foliation.

Furthermore, let Y' be a smooth birational model for $\mathbb{C}(\mathscr{F})$ and let $\pi\colon X \to Y'$ be the rational map associated to the inclusion of fields $\mathbb{C}(\mathscr{F}) \subset \mathbb{C}(X)$.

Let $\mathscr{V}$ be the "good" open set where π is a holomorphic submersion, and where not all the k-forms of the subspace $\Lambda^k U$ of $H^0(\Omega^k_X)$ vanish.

We let now $\omega_1, \omega_2, \ldots, \omega_{k+1}$, be any basis of U, and let x be a point of $\mathscr{V}$ where $\omega_1 \wedge \ldots \wedge \omega_k \neq 0$: then, since the ω_i's are d-closed, there are local holomorphic coordinates $z_1, z_2, \ldots, z_n$ around x, and a function ψ such that

$$\omega_1 = dz_1, \quad \omega_2 = dz_2, \ldots, \omega_k = dz_k, \quad \omega_{k+1} = d\psi. \tag{1.15}$$

Moreover, $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_{k+1} = 0$ implies that $\psi = \psi(z_1, z_2, \ldots, z_k)$, i.e., ψ depends only upon the first k coordinates.

Another important remark is that, even if ψ is a local holomorphic function, its partial derivatives $\partial\psi/\partial z_j$ are global meromorphic functions, and thus in $\mathbb{C}(\mathscr{F})$.

In fact, $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_{j-1} \wedge \omega_{j+1} \wedge \ldots, \omega_{k+1} = (-1)^{k-j}(\partial\psi/\partial z_j) \ \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_k$.

Similarly, for every meromorphic function $w \in \mathbb{C}(\mathscr{F})$, $(\partial w/\partial z_j) \in \mathbb{C}(\mathscr{F})$ (remember that $w = w(z_1, z_2, \ldots z_k)$).

Let now $r = \dim(Y')$, and $w_1, w_2, \ldots, w_r$ be meromorphic functions $\in \mathbb{C}(\mathscr{F})$ which give local holomorphic coordinates at $\pi(x) \in Y'$. In particular, $w_1, w_2, \ldots, w_r$ are holomorphic at x and we may assume (up to replacing $\omega_1, \omega_2, \ldots, \omega_k$ by suitable linear combinations) that $w_1, \ldots, w_r, z_{r+1}, \ldots, z_n$ are local holomorphic coordinates at x. Since, though, we are dealing with local holomorphic functions constant on the leaves, it suffices to work in k variables, where we shall apply the following

(1.16) **Lemma.** *Let* $\psi = \psi(z_1, z_2, \ldots, z_k)$, $w_i = w_i(z_1, z_2, \ldots, z_k)$, *for* $i = 1, \ldots, r$, *be germs of holomorphic functions around the origin in* $\mathbb{C}^k$, *such that*

(i) $w_1, \ldots, w_r, z_{r+1}, \ldots, z_k$ *are local holomorphic coordinates at the origin in* $\mathbb{C}^k$

(ii) for each i, j, $(\partial\psi/\partial z_j)$ *and also* $\partial w_i/\partial z_j$ *are functions of* $(w_1, w_2, \ldots, w_r)$.

Then, ψ *can be written uniquely as* $\varphi(w_1, w_2, \ldots, w_r) + \gamma$, *where* γ *is a linear function of* $z_{r+1}, \ldots, z_k$.

Proof. Denote by $y = (y_1, \ldots, y_r, y_{r+1}, \ldots, y_k)$ the new coordinate vector given by the functions $w_1, w_2, \ldots, w_r, z_{r+1}, \ldots, z_k$.

Then $(\partial\psi/\partial y_i)=\sum_{j=1,\ldots,k}(\partial\psi/\partial z_j)(\partial z_j/\partial y_i)$, but the matrix $(\partial z_j/\partial y_i)$ is the inverse of the matrix

$$(\partial y_i/\partial z_j)=\begin{pmatrix}(\partial w_i/\partial z_j) & \\ 0 & I_{k-r}\end{pmatrix}$$

where I_{k-r} is the identity matrix of order $k-r$: therefore, $(\partial z_j/\partial y_i)$ and also, by (ii), $(\partial\psi/\partial y_i)$ are functions of $w=(w_1, w_2, \ldots, w_r)$.

Set now $u=(z_{r+1}, \ldots, z_k)$, and write now ψ as a power series in the $y=(w, u)$ coordinate

$$\psi(w, u)=\sum_{\alpha,\beta}\psi_{\alpha,\beta}\, w^\alpha u^\beta;$$

since $(\partial\psi/\partial w_i)=(\partial\psi/\partial y_i)$ is just a function of w, we see that if the multiindex β is $\neq 0$, and if $\psi_{\alpha,\beta}\neq 0$, then necessarily $\alpha=0$.

Hence we can write ψ uniquely was $\psi=\varphi(w)+\gamma(u)$; finally, since also $(\partial\psi/\partial u_i)=(\partial\psi/\partial y_i)$ is again a function of w alone, we see that $(\partial\psi/\partial u_i)=(\partial\gamma/\partial u_i)$ is a constant, whence γ is a linear function. Q.E.D.

We can immediately apply the previous lemma, since it implies that, given any basis $\omega_1, \omega_2, \ldots, \omega_{k+1}$ of U, there are constants $c_1, c_2, \ldots, c_k$ such that $\omega_{k+1}-\sum_{j=1,\ldots,k} c_j\omega_j=\omega'_{k+1}$ is (at least locally) the pull back of a holomorphic differential on Y'. Repeating the same argument inductively for a new basis $\omega_1, \omega_2, \ldots, \omega_h, \omega'_{h+1}, \omega'_{k+1}$ of U (letting ω_h play the role of ω_{k+1}), we find a basis of U, $\omega'_1, \omega'_2, \ldots, \omega'_{k+1}$ such that all the ω'_i's, hence all of U, pull back from Y'.

Hence $\dim Y'=k$, and the leaves of $\mathscr{F}$ are closed (and smooth) in the good open set $\mathscr{V}$, and thus also in $X-\Sigma$.

A standard argument (on the universal cover X' of X, the leaves are the inverse images of a holomorphic function $\psi\colon X'\to\mathbb{C}^{k+1}$, and we just showed that $\pi_1(X)$ operates properly discontinuously on the image $\psi(X')$), shows that there is a holomorphic quotient $\pi\colon X\to Y$, with connected fibres equal to the closures of the leaves of $\mathscr{F}$, with Y a normal variety of dimension $=k$, and π a submersion on $X-\Sigma$. Moreover, since $\mathbb{C}(\mathscr{F})$ is algebraically closed inside $\mathbb{C}(X)$, $\mathbb{C}(\mathscr{F})=\mathbb{C}(Y)$, and finally the holomorphic forms in U are pull-backs from Y. Q.E.D.

(1.17) *Problem.* When can one improve the statement in 1.14, and have a holomorphic map to a smooth manifold Y?

At the Bergen Conference, in July '89, we gave the previous detailed proof of the Generalized Castelnuovo de Franchis theorem, and Ziv Ran kindly pointed out that he had obtained a similar result some years ago in his paper [Ran 1], dealing with subvarieties of Abelian varieties. The idea in that paper, of defining a notion of non degeneracy for subvarieties of Abelian varieties, is rather close to our idea of defining a notion (see later) of primitive nondegenerate Kaehler manifolds. Ran's proof of what we call GCdF is based also on the fact that Abelian subvarieties of Abelian varieties are rigid, whereas our proof is more differential geometric, and in fact allows us to obtain a further "twisted" generalization, which we hope may turn out to be useful for some conjectures and problems we are going to raise in the sequel.

(1.18) **Theorem (Twisted GCdF).** *Let X be a compact Kaehler manifold, and let $\omega_1, \omega_2, \ldots, \omega_{k+1}$ be respective sections of $H^0(\Omega^1_X\otimes\mathscr{L}_i)$, where $\mathscr{L}_i$, for i*

$=1, \ldots, k+1$, *is an invertible sheaf in* $\mathrm{Pic}^0(X)$. *Assume that* $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_{k+1} = 0$, *while* $\Lambda^k(\bigoplus_{i=1,\ldots,k+1} \mathbb{C}\omega_i)$ *embeds into* $H^0(\Omega^k_{X^*})$, *where* X^* *is the universal cover of* X.

Then there is a holomorphic map $f\colon X \to Y$ *to a k-dimensional normal variety.* $\mathscr{L}_i$ *is trivial on the generic fibre of* f, *and* ω_i *is a section of* $(f^*\Omega^1_Y)\otimes\mathscr{L}_i$ *outside an analytic set of codimension* $\geqq 2$.

Proof. As in 1.14, we obtain a rank k saturated subsheaf $\mathscr{H}$ of Ω^1_X, and a foliation $\mathscr{F}$ defined by $\mathscr{H}$. We argue as in 1.14, replacing the field $\mathbb{C}(\mathscr{F})$ by the field of local meromorphic functions at x obtained by meromorphic sections of the $\mathscr{L}_i$'s, constant on the leaves of the foliation.

We proceed with the same local argument mutatis mutandis, that is, we fix local flat trivializations of the $\mathscr{L}_i$'s, so that the ω_i's are represented locally by d-closed holomorphic 1-forms, and we can take linear combinations of them (with $\mathbb{C}$-coefficients). We choose $w_1, w_2, \ldots, w_r$ to be meromorphic functions $\in\mathbb{C}(\mathscr{F})$ such that they give a holomorphic map of highest possible rank at some point p in a neighbourhood U of x.

The only difference now is that the partial derivatives $\partial\psi/\partial z_j$, $(\partial w/\partial z_j)$ (in the chosen system of coordinates) are represented by sections σ of some flat bundle $\mathscr{L}$, and are constant on the leaves of $\mathscr{F}$.

But then $d\log(\sigma)$ is a meromorphic 1-form belonging to the differentials of the field $\mathbb{C}(\mathscr{F})$, and thus σ can locally be written as a holomorphic function of $w_1, \ldots, w_r$. Then the argument is the same as in 1.14, yielding $r=k$.

Therefore the leaves are closed and the same argument applies: on the universal cover X' of X, the leaves are the inverse images of a holomorphic function $\psi\colon X' \to \mathbb{C}^{k+1}$, and since $\pi_1(X)$ operates properly discontinuously on the image $\psi(X')$, there is a holomorphic quotient $\pi\colon X \to Y$, with Y a normal variety of dimension $=k$, and with π having connected fibres equal to the closures of the leaves of $\mathscr{F}$ and being a submersion on $X-\Sigma$. Q.E.D.

(1.19) **Corollary.** *Let* X *be a compact Kaehler manifold,* X' *an Abelian unramified covering of* X, $\pi\colon X' \to X$ *with group* G, $U' \subset H^0(\Omega^1_{X'})$ *a strict G-invariant k-wedge subspace, with associated holomorphic map* $f'\colon X' \to Y'$. *Then there are holomorphic maps* $f\colon X \to Y$ (*with* Y *a k-dimensional normal variety*), $g\colon Y' \to Y$, *such that* $f\circ\pi = g\circ f'$.

Proof. Since G is Abelian, we have a natural splitting $\pi_*(\mathcal{O}_{X'}) = \mathcal{O}_X \oplus (\bigoplus_{i=1,\ldots,r} \mathscr{L}_i)$, where $\mathscr{L}_i$, for $i=1, \ldots, r$, is a torsion element of $\mathrm{Pic}^0(X)$.

Then $H^0(\Omega^1_{X'}) = H^0(\Omega^1_X) \oplus (\bigoplus_{i=1,\ldots,r} H^0(\Omega^1_X \otimes \mathscr{L}_i))$ and Theorem 1.18 ensures then the existence of f. It is moreover clear that π carries leaves of the foliation $\mathscr{F}'$ on X' to leaves of $\mathscr{F}$ on X. Then π induces a holomorphic map g between the respective quotients of the holomorphic relations induced by the leaves of the respective foliations. Q.E.D.

Another direction in which Theorem 1.18 can be used is to imply the existence of holomorphic maps to varieties possessing an unramified covering of Albanese general type.

(1.20) **Definition.** *A holomorphic map with connected fibres* $f\colon X \to Y$ *to a k-dimensional normal variety of Albanese general type, as in Theorem* 1.14 (*GCdF*), *shall be called an Albanese general k-fibration, or a higher irrational pencil.*

A compact Kaehler manifold X admitting no higher irrational pencil shall be said to be Albanese primitive.

The importance of the above concept of Albanese primitive manifold is due firstly to the fact that, as we shall show in the following paragraph, the property of X of being Albanese primitive is just a topological property, actually a property of the cohomology algebra $H^*(C, \mathbb{C})$. Our point of view is that, once given a more restricted notion of primitive irregular manifold, the classification of irregular manifolds should be attacked via the study of primitive manifolds and via the study of the fibrations over them.

Secondly, we believe to be of paramount importance, in the study of primitive irregular manifolds, the study (cf. notation) of the paracanonical algebra of X:

(1.21) $$\bigoplus_{t\in A^*}\Big(\bigoplus_{m\in\mathbb{N}} H^0(X,[mK]\otimes\mathscr{P}_t)\Big).$$

The last is not a graded algebra in the usual sense, but should be viewed as an algebra of finitely supported functions defined on A^* and with values in a system of coefficients, endowed with a convolution product induced by the natural bilinear pairings

$$H^0(X,[mK]\otimes\mathscr{P}_t)\times H^0(X,[m'K]\otimes\mathscr{P}_{t'}))\to H^0(X,[(m+m')K]\otimes\mathscr{P}_{t+t'}).$$

Of course

(i) knowledge of the structure of the above paracanonical algebra encodes, and is essentially equivalent, to knowledge of the family of twisted pluricanonical maps associated to the systems $|mK+t|$

(ii) the above system of coefficients is related, via the base change theorem, to the direct image sheaves

(1.22) $$(\pi_X)_*(\mathscr{K}^{\otimes m}\otimes\mathscr{P})\quad\text{on } A^*\qquad(\text{with } \pi_X\colon X\times A^*\to A^*).$$

Now, the best situation is the one when all the above sheaves are locally free, enjoying the base change property, and of rank equal to χ ($[mK]$), with the exception of the case $t=m=0$.

Fundamental work of Green-Lazarsfeld and Beauville [G-L1], [Be2], [G-L2] has almost settled this problem by considering (for $i<n$) subvarieties

(1.23) $$S^i(X)\subset \mathrm{Pic}^0(X),\qquad S^i(X)=\{\mathscr{L}\in\mathrm{Pic}^0(X)\,|\,H^i(X,\mathscr{L})\neq 0\},$$

and proving that the components of these subvarieties are translates of complex subtori, and indeed these subtori are pull backs $f^*(\mathrm{Pic}^0(Z))$ for a suitable holomorphic map with connected fibres $f\colon X\to Z$, where Z has maximal Albanese dimension $\leqq i$ (i.e., $a(Z)=\dim Z\leqq i$).

(1.24) **Definition.** *An irregular Kaehler manifold X is said to be simple if $H^i(X,\mathscr{L})=0$, for $i<n=\dim X$, and $\mathscr{L}\in\mathrm{Pic}^0(X)-\{0\}$*, and primitive if the $S^i(X)$'s are finite sets for $i<n$.

Thus if X is simple, then it is primitive, and we pose the following daring questions, which we would like to have a positive answer:

(1.25) *Problem.* Assume X is primitive, i.e., the subvarieties $S^i(X)$ are 0-dimensional: then, is $S^i(X)$ consisting of a finite set of torsion points?

(1.26) *Problem.* Assume X is primitive: then, does there exist an unramified Abelian covering X' of X which is simple?

As far as we understand, Problem 1.25 has also been formulated by Beauville: the evidence for conjecturing 1.25, 1.26 is just that (see Beauville's example in [Ca 2]) the only known cases where the $S^i(X)$ are non trivial and finite satisfy the conjectures, and some attempt of disproving 1.25 failed.

It should also be pointed out that both conditions, of being Albanese primitive, respectively primitive or simple, are of topological nature (the latter case follows by Hodge theory with flat twisted coefficients): hence, if 1.26 would hold, a basic object of research would be the study of simple irregular manifolds of general type, for which the sheaves 1.22 are vector bundles over Abelain varieties and enjoying special properties. As far as I understand, the only case where almost everything is known about vector bundles on an Abelian variety is the case of elliptic curves (see [At]): and in fact, this knowledge is one of the many tools which can be used for the classification of irregular manifolds with $q=1$ (see [C-C 1], [C-C 2]).

2. Kaehler-Hodge exterior algebra

In this section we shall establish a multilinear algebra glossary for translating the notions of Albanese general type fibrations, respectively fibrations to an Albanese primitive variety Y of Albanese general type, firstly to some notions concerning the structure of $\Lambda^{\mathrm{hol}}(X)$ (the exterior subalgebra generated by $H^0(\Omega^1_X)$), namely of maximal honest k-wedge and of maximal primitive k-wedge; secondly we shall translate these notions concerning the structure of $\Lambda^{\mathrm{hol}}(X)$ into notions concerning the structure of $H^*(X,\mathbb{R})$.

(2.1) **Definition.** *Let U be a vector subspace of $H^0(\Omega^1_X)=H^{1,0}(X)$. Then U is said to be a k-wedge subspace if $\Lambda^k U$, the natural image of $\Lambda^k(U)$ into $H^0(\Omega^k_X)=H^{k,0}(X)$, is non zero, whereas the dimension of U is at least $k+1$ and $\Lambda^{k+1}U=0$.*

(2.2) **Definition.** *Let U be a k-wedge subspace of $H^0(\Omega^1_X)=H^{1,0}(X)$. Then U is said to be a strict k-wedge subspace if moreover the natural map of $\Lambda^k(U)$ into $H^0(\Omega^k_X)=H^{k,0}(X)$ is an isomorphism onto its image $\Lambda^k U$.*

(2.3) **Definition.** *Let U be a k-wedge subspace of $H^0(\Omega^1_X)=H^{1,0}(X)$. Then U is said to be a honest k-wedge subspace if it contains a strict k-wedge subspace.*

(2.4) *Remark.* One can define, in a similar fashion, analogous notions for vector subspaces of $H^1(X,\mathbb{C})$.

More generally, these notions can be defined for subspaces of the degree 1 summand of an exterior graded algebra. In our case, we can consider the subalgebras $\Lambda(X)\subset H^*(X,\mathbb{C})$ generated by $H^1(X,\mathbb{C})$, respectively $\Lambda^{\mathrm{hol}}(X)$ generated by $H^0(\Omega^1_X)=H^{1,0}(X)$.

It is also clear how to define, via inclusions, the respective notions of minimal and maximal k-wedge subspaces.

(2.5) **Lemma.** *Let U be a k-wedge subspace: then there is an integer $k'\leqq k$, and $U'\subset U$ such that U' is a strict k'-wedge subspace.*

Proof. We may first of all assume that U is not strict. Secondly, if $\dim U \geqq k+2$, we may find a proper subspace $U'' \subset U$ which is still a k-wedge subspace. Proceeding in this way, either we find a k-wedge subspace, which is strict, or we find a $(k+1)$ dimensional one, U^0, such that the map of $\Lambda^k(U^0)$ into $\Lambda^k U^0$ has a non zero kernel. Since $\Lambda^k(U^0)$ is naturally isomorphic to the dual space $(U^0)^\vee$, there is a k-dimensional subspace W of U^0 with $\Lambda^k W = 0$, and then W must be a h-wedge subspace with $h \leqq (k-1)$. We finish the proof, inductively, by observing that every 1-wedge subspace is strict. We notice, parenthetically, that we shall see later (Lemma 2.20) that a $(k+1)$-dimensional k-wedge subspace U^0 contains a unique strict wedge subspace. Q.E.D.

(2.6) **Lemma.** *Assume U is a strict k-wedge subspace. Then there do not exist an integer $h<k$ and a subspace U'' of U which is a h-wedge.*

Proof. Otherwise, there is a subspace U' of U'', of dimension $h+1$ and with $\Lambda^{h+1} U' = 0$. It suffices to complete a basis of U' to an independent set of k vectors in U, and to take its wedge product, thus contradicting the hypothesis that U is strict. Q.E.D.

(2.7) **Lemma.** *Assume U_i is a strict k_i-wedge subspace, for $i=1,2$.*
Set $U = U_1 \cap U_2$, and assume $k_1 \leqq k_2$. If $\dim U = k \leqq k_1$, then $\Lambda^k U \neq 0$. Otherwise, $k_1 = k_2$, and U is also a strict k_2-wedge.

Proof. Since $U \subset U_1$, the wedge product of h independent vectors in U is non zero if $h \leqq k_1$, thus the first assertion is proved. For the other, assume that $k_1 < k$, $k_1 < k_2$: then one can choose k_1+1 independent vectors in U whose wedge product is zero, against the assumption that U_2 is strict. Q.E.D.

(2.8) **Definition.** *Let U be a honest k-wedge subspace: assume U_i is a strict k_i-wedge subspace of U, with $i=1,2$ and $k_1 = k$. Then, $k_2 \leqq k$, and if equality always holds, then U is said to be a primitive k-wedge.*

(2.9) *Remark.* G.C.d.F. can thus be rephrased as: there is a bijection between {maximal honest k-wedge subspaces U of $H^{1,0}(X)$} and {Albanese general type k-fibrations $f: X \to Y$}.

Moreover, in the above bijection Y is Albanese primitive (cf. 1.19), if and only if U is primitive according to 2.8.

(2.10) **Lemma.** *Either there is a strict k-wedge vector subspace of $H^0(\Omega^1_X) = H^{1,0}(X)$, for some $k < n = \dim X$, or, letting $h = \min\{q,n\}$, every h-dimensional subspace U of $H^0(\Omega^1_X) = H^{1,0}(X)$ has $\Lambda^h U \neq 0$. In particular, if $q>n$, and X is primitive, the subspace $(\Lambda^n H^0(\Omega^1_X))$ of $H^0(\Omega^n_X)$ has dimension at least $\geqq n(q-n)+1$.*

Proof. If U has dimension $h = \min\{q,n\}$, and $\Lambda^h U = 0$, then there is a $k<n$, such that U is a k-wedge. The other assertion follows since $n(q-n)$ is the projective dimension of the Grassmann variety, whose affine cone is the set of wedges of n vectors inside $\Lambda^n(H^0(\Omega^1_X))$. Q.E.D.

Our next purpose is to locate strict and maximal honest k-wedge subspaces of $H^0(\Omega^1_X)$ by inspecting wedge subspaces of $H^1(X,\mathbb{C})$, and using complex conjugation in $H^1(X,\mathbb{C}) = H^1(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

(2.11) **Definition.** *A k-wedge subspace of $H^1(X,\mathbb{C})$ is said to be a good k-wedge if $V \cap V^* = 0$ (V^* denoting the complex conjugate of V), and if moreover $\Lambda^{2k} W \oplus W^* \neq 0$ for each k-dimensional subspace W of V.*

(2.12) *Remark.* Notice firstly that a good $(k+1)$-dimensional k-wedge is strict. Secondly, let V be a strict k-wedge subspace of $H^0(\Omega_X^1)$: then V is also a good k-wedge subspace of $H^1(X,\mathbb{C})$, as it is easy to verify (cf. Lemma 1.5).

Assume instead that U is a strict k-wedge subspace of $H^0(\Omega_X^1)$. Then it is not true, even when $\dim U=k+1$, that every $(k+1)$-dimensional subspace V of $H^1(X,\mathbb{C})$, such that $V\oplus V^*=U\oplus U^*$, is a good k-wedge.

In fact, then, for every such V and for every k-dimensional subspace W of V, one should have $\Lambda^{2k}W\oplus W^*\neq 0$. Therefore, for every $2k$-dimensional self conjugate subspace F of $U\oplus U^*$ one should have $\Lambda^{2k}F\neq 0$.

As a matter of fact, any such F occurs in two ways:

(i) either $\dim(F\cap U)=k$, and then, if $U'=F\cap U$, then $F=U'\oplus U'^*$, or

(ii) $\dim(F\cap U)=k-1$, and then, if $U'=F\cap U$, and $U''=U/U'$, then F is the pull-back of a subspace F'' of $U''\oplus U''^*$, given as the graph of the conjugate of a linear map $g\colon U''\to U''$ with $g^2=id_{U''}$ (i.e., F'' is the set of pairs $\{x, g(x)^*\}$).

In the second case, one can take a bais $\varphi_1, \varphi_2, \ldots, \varphi_{k+1}$ of U such that F is the span of $\varphi_1, \varphi_2, \ldots, \varphi_{k-1}, \varphi_1^*, \varphi_2^* \ldots, \varphi_{k-1}^*, \varphi_k+\varepsilon\varphi_k^*, \varphi_{k+1}+\varepsilon'\varphi_{k+1}^*$, $\varepsilon, \varepsilon'$ being the eigenvalues of g, i.e., $+1$ or -1.

In local coordinates, assuming $\varphi_1=dz_1, \ldots, \varphi_k=dz_k$, $\varphi_{k+1}=d(f(z_1,\ldots,z_k))=\sum_{i=1,\ldots,k} f_i\varphi_i$, $\Lambda^{2k}F$ is represented (up to constant) by the form $\psi(dz_1\wedge\ldots\wedge dz_k)\wedge(dz_1\wedge\ldots\wedge dz_k)^*$, where ψ is either the real or the imaginary part of the holomorphic function f_k.

This form can unfortunately be zero in cohomology, as is shown by the following

(2.13) *Example.* Let X be the hyperelliptic Riemann surface of equation $w^2=P(z)$, where P is a polynomial with real roots, hence such that $|P(z)|=|P(z^*)|$. We let $\varphi_1=w^{-1}dz$, $\varphi_2=zw^{-1}dz$, so that $\varphi_1\wedge\varphi_2^*+\varphi_1^*\wedge\varphi_2$ is zero in cohomology, its integral on X being given by twice the integral over the Riemann sphere of the form $-2i\,\mathrm{Im}(z)|P(z)|^{-1}\,dz\wedge dz^*$, which vanishes being antisymmetrical for the involution exchanging z with z^*.

In spite of 2.12, 2.13, a weaker result holds, which is relevant to our purposes.

(2.14) *Remark.* Let V be a k-wedge subspace of $H^1(X,\mathbb{C})$ and assume that $\{\varphi_1,\varphi_2,\ldots,\varphi_r\}$ is a basis of V.

Let V' be the subspace generated by $\varphi_1^*, \varphi_2, \ldots, \varphi_r$.

If $V\cap V^*=0$, then also $V'\cap V'^*=0$, and $\dim V'=r$, since $V\oplus V^*=V'\oplus V'^*$.

We claim that also V' is a k-wedge subspace of $H^1(X,\mathbb{C})$. By the symmetrical role of V, V', it suffices to show that for any $k+1$-dimensional subspace W' of V', $\Lambda^{k+1}W'=0$ (i.e., V' is a h-wedge for some $h\leq k$). Now, W' has a basis of the form $c\varphi_1^*+\psi, \psi_2, \ldots, \psi_{k+1}$, for a suitable constant c. Also, V is a k-wedge, hence $(c\varphi_1^*+\psi)\wedge\psi_2\wedge\ldots\wedge\psi_{k+1}=c\varphi_1^*\wedge\psi_2\wedge\ldots\wedge\psi_{k+1}$ and it is zero by the usual trick (1.5), since $\varphi_1^*\wedge\psi_2\wedge\ldots\wedge\psi_{k+1}\wedge(\varphi_1^*\wedge\psi_2\wedge\ldots\wedge\psi_{k+1})^*$ is zero, again because V is a k-wedge.

(2.15) **Definition.** *A* ($2k$*-wedge*) *self conjugate subspace F of $H^1(X,\mathbb{C})$ is said to be a real k-wedge if there exists a k-wedge V such that $V\oplus V^*=F$. F is said to be a good real $2k$-wedge if moreover one can find a subspace V as above which is good.*

(2.16) **Proposition.** *Let V be a $(k+1)$-dimensional k-wedge subspace of $H^1(X,\mathbb{C})$, and assume that V is the span of independent 1-forms $\omega_1+\eta_1^*, \omega_2+\eta_2^*, \ldots, \omega_{k+1}+\eta_{k+1}^*$, where $*$ denotes complex conjugation.*

Set U to be the span of $\omega_1, \omega_2, \ldots, \omega_{k+1}$, and W to be the span of $\eta_1, \eta_2, \ldots, \eta_{k+1}$. Then either U, or W, is a wedge subspace, or else U can be completed, via a subspace W' of W, to a wedge subspace of $H^0(\Omega_X^1)$ of rank $\leqq k+1$.

Proof. Assume that neither U nor W are wedge subspaces: then, up to a change of basis in V, we can assume $\omega_{h+1} = \ldots = \omega_{k+1} = 0$, and $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_h \neq 0$, with $h \leqq k$.

Note that $\eta_{h+1}, \ldots, \eta_{k+1}$ are linearly independent, and we can assume, again by a change of basis, that $\eta_1 = \ldots = \eta_t = 0$, with $h \geqq t \geqq 1$.

By assumption, then, $\eta_{t+1} \wedge \eta_{t+2} \wedge \ldots \wedge \eta_{k+1} \neq 0$.

By the assumption that V is a k-wedge, looking at the component of type $(h, k+1-h)$ of

$$(\omega_1 + \eta_1^*) \wedge (\omega_2 + \eta_2^*) \wedge \ldots \wedge (\omega_h + \eta_h^*) \wedge \eta_{h+1}^* \ldots \wedge \eta_{k+1}^*$$

we infer that

$$\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_h \wedge \eta_{h+1}^* \wedge \ldots \wedge \eta_{k+1}^* = 0,$$

hence either $\omega_1, \omega_2, \ldots, \omega_h, \eta_{h+1}, \ldots, \eta_{k+1}$ generate a wedge subspace, or we can assume $\omega_h = \eta_{h+1}$,

$$\omega_{h-1} = \eta_{h+2}, \ldots, \omega_{h-u+1} = \eta_{h+u},$$

$\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_h \wedge \eta_{h+u+1} \wedge \ldots \wedge \eta_{k+1} \neq 0$, again by a change of basis in V.

But this in turn contradicts $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_h \wedge \eta_{h+1}^* \wedge \ldots \wedge \eta_{k+1}^* = 0$. Q.E.D.

Assume now k to be smallest for the existence of some wedge subspace. We subdivide into two cases keeping the notation of 2.16.

Case II. If U, W are not wedge, $\omega_1, \omega_2, \ldots, \omega_h, \eta_{h+1}, \ldots, \eta_{k+1}$ generate a strict k-wedge subspace, and we can assume (cf. proof of Prop. 2.16)

$$\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_h \wedge \eta_{h+1} \ldots \wedge \eta_k \neq 0.$$

We shall treat this case later.

Case I. If instead U is a wedge, then it is a strict k-wedge, and we can assume $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_k \neq 0$.

Working locally, we can assume to have local coordinates $(z_1, z_2, \ldots, z_n)$ such that $\omega_1 = dz_1$, $\omega_2 = dz_2, \ldots, \omega_k = dz_k$, whereas $\omega_{k+1} = df$, with $f = f(z_1, z_2, \ldots, z_k)$, hence $\omega_{k+1} = \sum_{i=1,\ldots,k} f_i \omega_i$.

By looking at the component of type $(k, 1)$ of

$$(\omega_1 + \eta_1^*) \wedge (\omega_2 + \eta_2^*) \wedge \ldots \wedge (\omega_{k+1} + \eta_{k+1}^*) = 0,$$

we infer

$$\sum_{i=1,\ldots,k+1} \omega_1 \wedge \omega_2 \wedge \ldots \wedge \eta_i^* \wedge \ldots \wedge \omega_{k+1} = 0 \qquad \text{(in cohomology).}$$

Taking the wedge product of the latter form with its conjugate, we again get 0 in cohomology; we can express the form in local coordinates, obtaining

$$\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_k \wedge (\eta_{k+1}^* - \sum_{i=1,\dots,k} f_i \eta_i^*) \wedge \omega_1^* \wedge \omega_2^* \wedge \dots$$
$$\wedge \omega_k^* \wedge (\eta_{k+1} - \sum_{i=1,\dots,k} f_i^* \eta_i).$$

Wedging with the $(n-k)$ wedge power of the Kaehler form, we get, up to a constant, a semipositive integrand (this can be seen pointwise, replacing the functions f_i by constants), which must be zero, therefore the conclusion is that $\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_k \wedge (\eta_{k+1} - \sum_{i=1,\dots,k} f_i^* \eta_i) = 0$.

Since the function f is holomorphic, evaluating the conjugate of the partial derivatives, the f_i^*'s, at any point, we get constants c_i such that $\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_k \wedge (\eta_{k+1} - \sum_{i=1,\dots,k} c_i \eta_i) = 0$.

Since U is strict, the forms $\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_{j-1} \wedge \omega_{j+1} \wedge \dots \wedge \omega_k \wedge \omega_{k+1}$, for $j=1, \dots, k+1$, are $\mathbb{C}$-linearly independent, hence the functions $1, f_1, f_2, \dots, f_k$, are $\mathbb{C}$-linearly independent, and evaluating their conjugates on an open set we obtain forms $(\eta_{k+1} - \sum_{i=1,\dots,k} c_i \eta_i)$ which generate the space W.

We conclude therefore that $\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_k \wedge \eta_j = 0$ for $j=1, \dots, k+1$, i.e., $U+W$ is a k-wedge subspace.

It is clear that the case where V is a wedge subspace is entirely analogous.

Let's consider now Case II, where U and W are not wedge.

This case can be treated as follows: if V is a wedge subspace generated by $\varphi_1, \varphi_2, \dots, \varphi_{k+1}$, if we replace some φ_i by its conjugate, we still get a wedge subspace, since, as we saw in Remark 3, if $\varphi = \varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_j$, $\varphi \wedge \varphi^*$ changes only up to sign via this operation, and is 0 in cohomology if and only if $\varphi = 0$ in cohomology. Using therefore our previous notation, we can replace V by the span of $(\omega_1 + \eta_1^*)$, $(\omega_2 + \eta_2^*)$, $\dots$, $(\omega_h + \eta_h^*)$, $\eta_{h+1}, \dots, \eta_{k+1}$, and now U is the span of $\omega_1, \omega_2, \dots, \omega_h, \eta_{h+1}, \dots, \eta_{k+1}$ and is thus (as we showed) a strict k-wedge subspace.

We can summarize the above discussion with the following

(2.17) **Theorem.** *Let k be the smallest integer for which there exists a k-wedge subspace V. Then V uniquely determines a honest primitive k-wedge subspace of $H^0(\Omega_X^1) = H^{1,0}(X)$.*

Turning now to the more general case (i.e., when k is not smallest) where we have a real $2(k+1)$ dimensional $2k$-wedge F of $H^1(X, \mathbb{C})$, we can always assume, by Remark 2.14, the proof of Prop. 2.16 and the previous considerations, to deal with the case when V is a k-wedge and the associated subspace U of $H^0(\Omega_X^1)$ is a h-wedge subspace, with $h \leqq k$ maximal among the subspaces V such that $V \oplus V^* = F$. Assume that $h < k$. We can of course assume again that $\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_h \neq 0$.

By the maximality of h, we claim that $\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_h \wedge \eta_j = 0$ for all j.

This is clear if $j \leqq h$: in fact, otherwise we can assume e.g. ω_{h+1} to be linearly independent from $\omega_1, \omega_2, \dots, \omega_h$, so that $\omega_1, \omega_2, \dots, \omega_h, \omega_{h+1}, \eta_j$ generate a

$(h+1)$-wedge, and one can first add to $(\omega_{k+1}+\eta^*_{k+1})$ a suitable multiple of $(\omega_j+\eta^*_j)$, then replace the form thus obtained by its conjugate. If instead, $j \geqq h+1$, one can add to $(\omega_1+\eta^*_1)$ a sufficiently general multiple of $(\omega_j+\eta^*_j)$, so that the property $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_h \neq 0$ is preserved, and then apply the previous step.

Therefore, $U+W$ is a h-wedge. Since $V \oplus V^*$ is contained in $(U+W) \oplus (U+W)^*$, and $V \oplus V^*$ is a $2k$-wedge, then necessarily we have $h=k$ and also $U+W$ is a k-wedge.

Again, we summarize the above arguments in the following

(2.18) **Theorem.** *Let V be a k-wedge subspace. Then there are subspaces U, W of $H^0(\Omega^1_X)=H^{1,0}(X)$ such that V is contained in $U \oplus W^*$, and $U+W$ is a k-wedge subspace.*

In particular, every real $2k$-wedge is contained in a unique maximal real $2k$-wedge subspace F of $H^1(X, \mathbb{C})$ (cf. 2.15).

Such an F can be uniquely written in the form $Y \oplus Y^*$, with Y a maximal k-wedge subspace of $H^0(\Omega^1_X)$.

Proof. We have proven the first assertion when $\dim V=(k+1)$.

In general, let, as usual, U be the projection of V into $H^0(\Omega^1_{X'})$, and let W^* be the projection into $H^0(\Omega^1_X)^*$. It suffices, by the previous case, to show that, setting $Z=U+W$, then $\Lambda^{k+1} Z=0$. But for any $(k+1)$ dimensional subspace S of $Z=U+W$, we clearly can find a $(k+1)$ dimensional subspace V' of V such that $S \subset U'+W'$, and we are done, by the first case.

There remains thus to prove the second assertion. We obviously may assume $\dim V=(k+1)$, and take Y to be the subspace of $H^0(\Omega^1_X)$ consisting of the elements which annihilate $\Lambda^k(U+W)$ under the natural map into $H^0(\Omega^k_X)$. Then it is clear that $Y \oplus Y^*$ is a maximal $2k$-wedge, by the usual trick (1.5). Let F be as in the statement of the theorem and assume $F=V' \oplus V'^*$. Then, by the first part of the theorem applied to V', V' is contained in $U' \oplus W'^*$, and $U'+W'$ is a k-wedge subspace. Since F contains $V \oplus V^*$, the projection of F into $H^0(\Omega^1_X)$, which is contained in $U'+W'$, must contain $U+W$. Hence $U'+W'$ is a k-wedge containing $U+W$, and is thus contained in Y. Hence, F is contained inside $Y \oplus Y^*$, and is therefore equal to it by maximality. Q.E.D.

(2.19) **Definition.** *A maximal real $2k$-wedge subspace F ($F=Y \oplus Y^*$ as in Theorem 2.18) shall also be called a saturated real $2k$-wedge.*

In order to continue our analysis, let's try now to describe how a $(k+1)$-dimensional k-wedge subspace U can fail to be a strict k-wedge.

Define then K to be the kernel of the natural epimorphism of $\Lambda^k(U)$ into $\Lambda^k U$, and assume that K is non zero. To each ξ in K one associates a hyperplane H_ξ in U, consisting of the vectors u such that $u \wedge \xi=0$ in $\Lambda^{k+1}(U)$, hence ξ is a generator of $\Lambda^k(H_\xi)$, hence also $\Lambda^k H_\xi=0$. Conversely, all hyperplanes H such that $\Lambda^k H=0$ arise in this way.

Moreover, $K^\perp$ is clearly equal to $\bigcap_{\xi \in K} H_\xi$.

We make now an important remark; assume we write U as a direct sum $U=A \oplus B$, and $a=\dim A$, $b=\dim B$: then $k+1=a+b$, and $\Lambda^k(U)=(\Lambda^{a-1}(A) \otimes \Lambda^b(B)) \oplus (\Lambda^a(A) \otimes \Lambda^{b-1}(B))$.

Therefore, if $\Lambda^a(A)=0$, then $\Lambda^k U=\Lambda^{a-1} A \wedge \Lambda^b B$, and it has thus dimension $\leqq a$, and equal to a if A is a strict $(a-1)$-wedge.

We saw that, if U is not strict, i.e., $K \neq 0$, then there is some strict $(a-1)$-wedge A, which must clearly have dimension a, since $\Lambda^k U \neq 0$. Then $a = \dim \Lambda^k U$, and $K = (\Lambda^a(A) \otimes \Lambda^{b-1}(B))$. But then also $A = K^{\perp}$, thus A is uniquely determined.

We have thus proven the following

(2.20) **Lemma.** *Let U be a $(k+1)$-dimensional k-wedge. Then there is a unique subspace A of U which is a strict wedge subspace; moreover, if* $\dim A = a$, *then A is a strict $(a-1)$ wedge.*

We proceed our discussion, analyzing the structure of non-honest k-wedges in $H^{1,0}(X)$.

We need the following preliminary result

(2.21) **Lemma.** *$H^{1,0}(X)$ contains only a countable number of maximal honest k-wedges.*

Proof. Let U be a maximal honest k-wedge in $H^{1,0}(X)$. Then, by the generalized Castelnuovo de Franchis theorem, U determines and is uniquely determined by a factor Abelian variety of the Albanese variety of X. But these factors are at most a countable number. Q.E.D.

(2.22) *Remark.* One can prove indeed the sharper statement with "finite" instead of "countable".

(2.23) **Proposition.** *Let U be a k-wedge in $H^{1,0}(X)$.*

Then U is not a honest k-wedge if and only if there is a honest h-wedge $U' \subset U$, with $h<k$, and such that $\dim U - \dim U' = k-h$.

Proof. The "if" part is obvious, since if D is any $(k+1)$-dimensional subspace, $D \cap U'$ has dimension $\geqq h+1$, hence $D \cap U'$ is a h-wedge, with $h<k$, hence D cannot be a strict k-wedge.

Conversely, if D is a $(k+1)$-dimensional k-wedge of U, D is a strict k-wedge, unless (cf. Lemma 2.20) it contains a strict h-wedge subspace D', such that $h<k$, and D' has codimension $=(k-h)$ in D. In turn, D' is contained in a maximal honest h-wedge U' of U, which is the intersection of U with a maximal honest h-wedge subspace of $H^{1,0}(X)$. Summing up, there is at most a countable number of such subspaces U', and if for all of them the codimension would be $>(k-h)$, it would be possible, by Baire's theorem, to find a $(k+1)$-dimensional k-wedge D of U intersecting all such subspaces U' transversally, hence D would be a strict k-wedge. Q.E.D.

We can finally get to the main result of this section

(2.24) **Theorem.** *Every saturated real $2k$-wedge F of $H^1(X, \mathbb{C})$ uniquely determines a maximal k-wedge subspace Y of $H^{1,0}(X)$.*

Y is a honest k-wedge if and only if F does not contain any saturated real $2h$-wedge subspace F' of codimension $2(k-h)$ in F.

Y is a primitive k-wedge if and only if F does not contain any h-wedge subspace, for all $h<k$.

Proof. Every saturated real $2k$-wedge F of $H^1(X, \mathbb{C})$, is (by Theorem 2.18 and Definition 2.19) of the form $F = Y \oplus Y^*$ with Y a maximal k-wedge subspace of $H^{1,0}(X)$. By Proposition 2.23 if Y were not honest, Y would contain a honest h-wedge U, with $h<k$, of codimension $k-h$.

Since Y is maximal, U is maximal too.

Then $F' = U \oplus U^*$ would be as required in the theorem. Conversely, the existence of such a subspace $F' = U \oplus U^*$ implies that U is a maximal h-wedge of codimension $k-h$ inside Y, thus, again by Proposition 2.23, Y is not honest. The other assertion is clear. Q.E.D.

By Theorems 2.18 and 2.24 we infer as a corollary the following

(2.25) **Theorem.** *An irregular Kaehler manifold X of dimension n is Albanese primitive if and only if $H^1(X, \mathbb{C})$ contains no k-wedge subspace with $k < n$. Moreover, there is a bijection between the following two sets:*
$\{$Albanese general type k-fibrations $f\colon X \to Y\}$
$\{$Saturated real $2k$-wedge subspaces F of $H^1(X, \mathbb{C})\}$.

(2.26) *Remark.* A question that we have for the time being circumvented (cf. the statement of Theorem 2.24) is whether the maximal k-wedge subspace Y of $H^{1,0}(X)$ determined by a saturated (i.e., maximal) real $2k$-wedge F is honest if and only if F contains a good real $2k$-wedge.

3. Moduli of algebraic surfaces with irrational pencils

In this section we shall deal with the problem of giving a good upper bound for the "number of moduli" M of an algebraic surface S fibred over a curve C of genus $b \geqq 2$, and with fibres of genus $g \geqq 2$ (then S is of general type, cf. [B-P-V], Chap. III).

We recall (cf. e.g. [Ca 1], [Ca 2]) that M is the dimension of the moduli space at the point corresponding to S, or alternatively the dimension of the base B of the Kuranishi family of deformations of S.

(3.1) **Theorem.** *Let S be a complex surface admitting a (nonconstant) holomorphic map $f\colon S \to C$, with genus $(C) = b \geqq 2$, and with a general fibre F of f of genus $= g \geqq 2$.*

Then we have the following upper bounds for the number M of moduli of S (c_2 stands for $c_2(S)$, which equals the topological Euler-Poincarè characteristic of S)

(i) if the fibres of f have nonconstant moduli, then $M \leqq c_2 - (b-1)(4g-7) \leqq c_2 - (b-1)(4(q-b)-3)$,

(ii) if the fibres of f have constant moduli, then $M \leqq c_2 - (2b-3)((2g-4)+5$,

(iii) if f is a holomorphic bundle, but not a product, then

$$M \leqq 3(b-1) + 2(g-1) + 4 \leqq 3/4\, c_2 - 3(b-2)(g-2) + 6,$$

(iv) if f is a product projection, then

$$M = 3(b-1) + 3(g-1) \leqq 3/4\, c_2 - 3(b-2)(g-2) + 3.$$

Proof. Let $\Sigma \subset C$ be the set of critical values of f, and σ its cardinality. If $x \in C$, we set $F_x = f^{-1}(x)$, and we denote by $e(F_x)$ the topological Euler-Poincaré characteristic of the fibre F_x, and we observe that it equals $-2(g-1)$ unless $x \in \Sigma$.

We denote by $\delta(x)$ the defect $e(F_x) + 2(g-1)$, and we recall that, if we partition Σ as $\Sigma' \cup \Sigma''$, where $\Sigma' = \{x \mid (F_x)_{\mathrm{red}}$ is singular$\}$, $\Sigma'' = \{x \mid (F_x)_{\mathrm{red}}$ is non singular$\}$

(and thus for $x \in \Sigma''$ there is an integer $m_x \geqq 2$ such that $F_x = m_x(F_x)_{\mathrm{red}}$), then we have (cf. [B-P-V] or [Be 1])

$$(3.2) \qquad \delta(x) \geqq 1 \quad \text{for } x \in \Sigma', \qquad \delta(x) \geqq 2(1-1/m_x)(g-1) \geqq (g-1) \quad \text{for } x \in \Sigma''.$$

We can apply the classical Zeuthen-Segre formula, asserting that

$$(3.3) \qquad c_2 = 4(b-1)(g-1) + \sum_{x \in \Sigma} \delta(x).$$

We derive from (3.2) and (3.3) the following inequality for the respective cardinalities σ', σ'' of Σ', Σ'':

$$(3.4) \qquad \sigma' + (g-1)\,\sigma'' \leqq c_2 - 4(b-1)(g-1).$$

We let now $C^* = C - \Sigma$ (so that f is a differentiable fibre bundle over C^*) and we can now apply Arakelov's theorem ([Ara], cf. also [B-P-V], and [Sz] for the case of positive characteristic): given C^*, there are only a finite number of fibrations f': $S' \to C$, with genus of a general fibre F' of $f' = g$, and with critical values of f' contained in Σ, or else f has constant moduli (i.e., after a base change u: $C'' \to C$, which we may assume to be Galois with group G, we obtain a product fibration f'').

In the latter case S is birationally a quotient $F \times C''/G$, where we can assume that G is a subgroup of $\mathrm{Aut}(F)$. $\mathrm{Aut}(F)$ has, by Hurwitz's theorem, order at most $84(g-1)$: whence there is only a finite number of choices for G, once g is fixed. Moreover, then, the irregularity q of S equals the dimension $h^0(\Omega^1_{F \times C''})^G$ of the space of G-invariant 1-forms on $F \times C''$: since $C''/G = C$, we obtain that $q = b + h^0(\Omega^1_F)^G$, and thus F/G has genus $= (q-b)$.

The number δ of branch points for the projection of F onto F/G, letting $d = |G|$, can be bounded by Hurwitz's formula, since, letting Δ be the branch set, and, for $y \in \Delta$, v_y the local multiplicity of the quotient map, we get

$$(3.5) \qquad \delta = \sum_{y \in \Delta} 1 \leqq \sum_{y \in \Delta} 2(1 - 1/v_y) \leqq (4/d)(g-1) - 4(q-b-1).$$

We notice now, that any surface S' homeomorphic to S carries, by the results of the previous paragraph, a fibration over a curve of genus g, and if there is a diffeomorphism carrying one saturated subspace to the corresponding one, then also the genus g of the fibres is the same. Moreover, observe that if f has nonconstant moduli, the same occurs for any small deformation of S. Hence in this former case, we can apply Arakelov's theorem, implying that there is a quasi-finite map of the (local) moduli space onto the (local) moduli space for the pair $(\Sigma \subset C)$, which has dimension $\sigma + 3(b-1) \leqq$ (by (3.4)$\leqq c_2 - 4(b-1)(g-1) + 3(b-1)$, whence our first assertion (since $g + b \geqq q$, equality holding if and only if f has constant moduli, cf. [Be 3]).

If f is a differentiable bundle, then this property is also deformation invariant, and $M \leqq 3(b-1)$ if f has nonconstant moduli, whereas in the contrary case, there is now a Galois unramified covering C'' of C, and the moduli space has a quasi finite morphism to the moduli space for pairs $(C, F/G, \Delta)$, whose dimension is (by (3.5))$\leqq 3(b-1) - (q-b) + 4/d(g-1) + 4 \leqq 3(b-1) + 2(g-1) + 4$. In this case, though, one has to observe that $c_2 = 4(b-1)(g-1)$, and thus $M \leqq 3(b-1) + 3(g-1) + 3 \leqq 3/4 c_2 - 3(b-2)(g-2) + 6$.

If f is not a bundle, but has constant moduli, we have to add the number σ of critical values of f, for which we have the estimate (3.4), yielding the final upper bound $M \leqq c_2 - 4(b-1)(g-1) + 3(b-1) + 2(g-1) + 4 = c_2 - (4bg - 7b - 6g + 5) \leqq c_2 - (2b-3)(2g-4) + 5$. Q.E.D.

(3.6) *Question.* Can the above bounds in Thm. 3.1 be significantly improved upon?

(3.7) *Remark.* The inequalities become worse when b, or g, are low, say $=2$. For instance, the bound can be improved when $g=2$, as it was suggested by Rick Miranda, with whom we carried out the following computation. Let S be a double cover of $C \times \mathbb{P}^1$, branched on a smooth irreducible curve B. Then, if γ is the genus of B, then by Hurwitz's formula, since B is a $6-1$ cover of $\mathbb{P}^1$, we have $2\gamma - 2 = \mu + 12$, where μ is the number of ramification points of the projection of B onto $\mathbb{P}^1$. On the other hand, since $f\colon S \to C$ has nonconstant moduli, the Zeuthen-Segre formula yields $\mu \leqq c_2 - 4(q-1)$, and moreover the genus b of C is q or $q-1$.

To bound the number of moduli M of S, it suffices, by the considerations made previously, to add to $3(q-1)$ the dimension $h^0(\mathcal{O}_B(B))$ of the characteristic series of B. We know that the canonical divisor on $C \times \mathbb{P}^1$ is algebraically equivalent to $2(b-1)F_1 - 2F_2$, where F_1, F_2 are the fibres of the two projections of $C \times \mathbb{P}^1$; we can bound as follows: $\gamma + 12 \geqq h^0(\omega_B(2F_2)) \geqq$ (by Clifford's inequality) $\geqq h^0(\mathcal{O}_B(B)) + (q-1)\,d$, where d is the degree of the projection of B onto C.

Hence $h^0(\mathcal{O}_B(B)) \leqq \mu/2 + 19 - (q-1)\,d \leqq 1/2(c_2) - (2+d)(q-1) + 19$, and finally $M \leqq 1/2(c_2) - (d-1)(q-1) + 19$.

(3.8) *Remark.* Since $c_2 = 12\chi - K^2$, and the classical Enriques-Kodaira-Kuranishi lower bound by the "expected number of moduli" (cf. [Ca 1–3] for some results and a discussion about the number M of moduli of surfaces of general type) is given by $M \geqq 10\chi - 2K^2$, we see that the theorem gives numerical obstructions to the existence of fibrations to curves of genus $b \geqq 2$.

4. Algebraic surfaces with irrational pencils in positive characteristic and the higher dimensional case

In order to treat the case of an algebraic surface defined over an algebraically closed field of positive characteristic $= p$, we cannot use topological considerations any longer, and we need to understand the deformation theoretic analogues of the arguments we employed.

Similarly, if we deal with a variety, or a Kaehler manifold of dimension $\geqq 3$ fibred over a curve, we do not yet have, so far as I know, an Arakelov type theorem, and so we must again translate the finiteness statement into a vanishing result for some term in some exact sequence of deformation functors.

Therefore, we very briefly recall the basic results about deformation of maps that we are going to use (cf. [Il 1–2], [Fl], [Ho 1–3], [Pal], [Ran 2]).

Let as usual $f\colon X \to Y$ be either a holomorphic map of compact complex spaces or a morphism of canonically polarized smooth complete algebraic varieties of general type.

In this situation we can consider 4 deformation functors:

Def(X) = Deformations of X, Def(Y), Def$(f\colon X\to Y)$ = Deformations of the morphism (i.e., of the whole triple, X, Y, and f), and Def(X/Y), that is, the subfunctor of deformations of the morphism, but where the target Y is fixed.

There are natural morphisms of functors

(4.1) $\mathrm{Def}(f\colon X\to Y)\to\mathrm{Def}(X)$

(4.2) $\mathrm{Def}(f\colon X\to Y)\to\mathrm{Def}(Y)$, and Def$(X/Y)$ is the kernel of the last morphism.

Correspondingly, we have tangential functors $T^i(X)$, $T^i(Y)$, $T^i(f)$, $T^i(X/Y)$, where $T^1(\dots)$ is the Zariski tangent space to the (representable) Deformation functor, and corresponds to First Order Deformations, whereas $T^2(\dots)$ is the Obstruction space (and therefore Def is locally formally isomorphic to the fibre of some map of $T^1(\dots)$ to $T^1(\dots)$ given by power series).

Associated to (4.2) is an exact sequence of tangential functors (cf. [Fl], p. 50, Thm. 3.4)

(4.3) $$\to T^1(X/Y)\to T^1(f)\to T^1(Y)\to T^2(X/Y)\to T^2(f).$$

The tangential functors are effectively computable in terms of Ext functors, and we have

(4.4) $$T^i(X)\cong \mathrm{Ext}^i\mathcal{O}_X(\Omega^1_X,\mathcal{O}_X)$$

(4.5) $$T^i(X/Y)\cong \mathrm{Ext}^i\mathcal{O}_X(\Omega^1_{X/Y},\mathcal{O}_X),$$

where, as usual, $\Omega^1_{X/Y}$ is the cokernel in the standard exact sequence

(4.6) $$0\to f^*\Omega^1_Y\to\Omega^1_X\to\Omega^1_{X/Y}\to 0.$$

Furthermore, e.g. as in [Ran2], pp. 249–251, the $T^i(f)$'s also fit in into another (essentially equivalent) exact sequence

(4.7) $$\begin{aligned}\dots&\to T^1(f)\to T^1(X)\oplus T^1(Y)\to \mathrm{Ext}^1_f(\Omega^1_Y,\mathcal{O}_X)\to T^2(f)\\ &\to T^2(X)\oplus T^2(Y)\to\mathrm{Ext}^2_f(\Omega^1_Y,\mathcal{O}_X),\end{aligned}$$

where the $\mathrm{Ext}^i_f(^1_Y,\mathcal{O}_X)$'s can be computed by means of two spectral sequences, respectively $\mathrm{Ext}^p\mathcal{O}_X(L_qf^*\Omega^1_Y,\mathcal{O}_X)$, and $\mathrm{Ext}^p\mathcal{O}_Y(\Omega^1_Y,\mathcal{R}^qf_*\mathcal{O}_X)$, where the L_qf^*'s are the left derived functors, and the $\mathcal{R}^qf_*$'s the right derived functors. One may observe that the first spectral sequence degenerates to $\mathrm{Ext}^i\mathcal{O}_X(f^*\Omega^1_Y,\mathcal{O}_X)$ in case the morphism f is flat, and if moreover Y is smooth, then Ω^1_Y is locally free, hence $\mathrm{Ext}^i_f(\Omega^1_Y,\mathcal{O}_X)$ reduces to $H^i(f^*\Theta_Y)$, where Θ_Y is the tangent sheaf of Y.

We recover in this way Horikawa's Theorem 8.1 in [Ho III], whose assumption

(4.8) $H^1(\Theta_Y)$ surjects onto $H^1(f^*\Theta_Y)$, $\quad H^2(\Theta_Y)$ injects into $H^2(f^*\Theta_Y)$,

is equivalent, via the Leray spectral sequence for f, to

(4.8′) $$H^0(\mathcal{R}^1f_*\mathcal{O}_X\otimes\Theta_Y)=0.$$

More generally, using the first spectral sequence, we see that $T^1(Y)\to\mathrm{Ext}^1_f(\Omega^1_Y,\mathcal{O}_X)$ is surjective provided $\mathrm{Ext}^0\mathcal{O}_Y(\Omega^1_Y,\mathcal{R}^1f_*\mathcal{O}_X)=0$ (if Y is smooth, this amounts to (4.8′)), and $f_*\mathcal{O}_X\cong\mathcal{O}_Y$.

Let's assume f to be flat and Y to be smooth, and (4.8′) to hold: then $T^1(Y)\to\mathrm{Ext}^1_f(\Omega^1_Y,\mathcal{O}_X)$ is an isomorphism, while $T^2(Y)\to\mathrm{Ext}^2_f(\Omega^1_Y,\mathcal{O}_X)$ is injective, hence, by diagram chasing, $T^1(f)\to T^1(X)$ is onto, while $T^2(f)\to T^2(X)$

is injective. This is a standard criterion to imply that the morphism $\mathrm{Def}(f\colon X \to Y) \to \mathrm{Def}(X)$ is smooth, and in particular locally surjective.

We can apply the above results obtaining immediately the following

(4.9) **Theorem.** *Let X, Y be compact complex manifolds and let $f\colon X \to Y$ be a surjective holomorphic map with connected fibres. Then* $\mathrm{Def}(f\colon X \to Y)$ *maps onto the Kuranishi family of X,* $\mathrm{Def}(X)$, *provided* (4.8′) *holds,* i.e., $H^0(\mathscr{R}^1 f_* \mathcal{O}_X \otimes \Theta_Y) = 0$.

(4.10) *Remark.* $H^0(\mathscr{R}^1 f_* \mathcal{O}_X \otimes \Theta_Y) = 0$ holds, in particular, by virtue of a theorem of Kollar [Kol] generalizing an earlier result of Fujita [Fu], if Y is a curve of genus $\geqq 2$, and X is projective. In fact, by relative duality for f, the dual vector space equals $H^1(\mathscr{R}^{n-2} f_* \omega_{X/Y} \otimes \Omega^1_Y) = 0$, Ω^1_Y being an ample invertible sheaf. So our result is stronger than the combination of Horikawa's and Kollar's theorems in the situation above, but it is an interesting question whether one can dispense with the Kaehler assumption on X.

(4.11) **Theorem.** *Let X, Y be complete smooth varieties of general type (of arbitrary characteristic), and let $f\colon X \to Y$ be a surjective morphism. Then* $\mathrm{Def}(f\colon X \to Y)$ *maps onto* $\mathrm{Def}(X)$ *provided* (4.8′) *holds,* i.e., $H^0(\mathscr{R}^1 f_* \mathcal{O}_X \otimes \Theta_Y) = 0$.

(4.12) *Remark.* $H^0(\mathscr{R}^1 f_* \mathcal{O}_X \otimes \Theta_Y)$ does not need to vanish in positive characteristic, even when Y is a curve of genus $\geqq 2$. In fact, there are examples of quasi elliptic fibrations, due to Lang [La], where $\mathscr{R}^1 f_* \mathcal{O}_X$ has some nonzero torsion. If we think in terms of the Castelnuovo-De Franchis theorem, the possible trouble can be the existence of non-closed regular 1-forms (cf. [La], [Il 3], [Ny], [Fos]), so that one may ask whether in positive characteristic there are algebraic surfaces with an irrational pencil of genus $\geqq 2$, and whose deformations do not have irrational pencils.

We turn now to the problem of bounding the number of moduli, and the strategy for this will simply be of giving an upper bound for the dimension of $T^1(X)$.

In order to obtain effective estimates, we shall assume that Y is a smooth curve of genus b. Hence, it will suffice to bound the dimension of $T^1(X/Y) \cong \mathrm{Ext}^1 \mathcal{O}_X(\Omega^1_{X/Y}, \mathcal{O}_X)$.

By the local to global spectral sequence for Ext, it suffices to give upper bounds for $H^1(\mathscr{H}om\, \mathcal{O}_X(\Omega^1_{X/Y}, \mathcal{O}_X))$, and for $H^0(\mathscr{E}xt^1\, \mathcal{O}_X(\Omega^1_{X/Y}, \mathcal{O}_X))$.

By applying $\mathscr{H}om\, \mathcal{O}_X(-, \mathcal{O}_X)$ to the exact sequence (4.6), we get

$$(4.13) \qquad 0 \to \mathscr{H}om\, \mathcal{O}_X(\Omega^1_{X/Y}, \mathcal{O}_X) \to \Theta_X \to f^* \Theta_Y \to \mathscr{E}xt^1\, \mathcal{O}_X(\Omega^1_{X/Y}, \mathcal{O}_X) \to 0.$$

Now, the Arakelov type property is the vanishing of $H^1(\mathscr{H}om\, \mathcal{O}_X(\Omega^1_{X/Y}, \mathcal{O}_X))$, whereas the Zeuthen-Segre formula is, in modern terms, a computation of Chern classes via (4.13).

(4.14) **Theorem.** *Let X be a complete smooth surface of general type (of arbitrary characteristic), and let $f\colon X \to C$ be a surjective morphism, with C a curve of genus b, and where f has only a finite number of critical points, and a general fibre of genus g.*

Then we have the following inequalities for the tangent dimension M' of the space of moduli of X ($M' = h_1(\Theta_X)$) if the fibres of f have nonconstant moduli:

$$M' \leqq c_2 - (b-1)(4g-7) \leqq c_2 - (b-1)(4(q-b)-3).$$

Proof. By the above discussion, M', the dimension of $T^1(X) = H_1(\Theta_X)$, is bounded by $\dim T^1(X/C) + \dim T^1(C)$, this last being bounded by $3b(3b-3$ if $b \geqq 2)$.

In turn, $\dim T^1(X/C)$, as we already remarked, is bounded by $h^1(\mathcal{H}om\,\mathcal{O}_X(\Omega^1_{X/C}, \mathcal{O}_X)) + h^0(\mathcal{E}xt^1\,\mathcal{O}_X(\Omega^1_{X/C}, \mathcal{O}_X))$, thus we have to identify more concretely the terms appearing in the exact sequence (4.13).

In general, the relative dualizing sheaf $\omega_{X/C}$ is obtained by taking determinants of (4.6), thus $\omega_{X/C} \cong \mathcal{O}_X(K_X - f^* K_C)$, K denoting as usual a canonical divisor.

There is a natural homomorphism

(4.15) $\xi\colon\ \Omega^1_{X/C} \to \omega_{X/C}$, induced by the homomorphism
$\xi'\colon\ \Omega^1_X \to \omega_{X/C}$, which, if t is the pull back of a local parameter on C,

associates to a 1-form η the relative differential $(\eta \wedge dt) \otimes (dt)^{-1}$. Clearly $\mathcal{C} = \operatorname{coker} \xi$ is supported on the critical set $\mathcal{S}$ of f, and actually

$$\operatorname{coker} \xi = \mathcal{C} \cong \omega_{X/C} \otimes \mathcal{O}_{\mathcal{S}}. \tag{4.16}$$

If we denote by $\mathcal{T} = \ker \xi$, then clearly $\mathcal{T} = 0$ under the assumption that the set $\mathcal{S}$ of critical points is finite. More generally, if S is the divisorial part of the scheme $\mathcal{S}$ (i.e., S is defined by $\sigma = 0$, where σ is the greatest common divisor of the components of the section dt), we have an exact sequence

$$0 \to \mathcal{T} \cong \mathcal{O}_S(f^* K_c + S) \to \Omega^1_{X/C} \to \omega_{X/C} \to \mathcal{C} \to 0 \tag{4.17}$$

(in fact $\mathcal{T}$ is the quotient of $\mathcal{O}_X(f^* K_c + S)$ by $\mathcal{O}_X(f^* K_c)$).

By 4.17, since both $\mathcal{T}$ and $\mathcal{C}$ are torsion, we obtain the exact sequence

$$0 \to \omega^{-1}_{X/C} \to \mathcal{H}om\,\mathcal{O}_X(\Omega^1_{X/Y}, \mathcal{O}_X) \to \mathcal{E}xt^1\,\mathcal{O}_X(\mathcal{C}, \mathcal{O}_X) \to 0.$$

Since the middle term is locally free, by virtue of 4.13, we finally get

$$\mathcal{H}om\,\mathcal{O}_X(\Omega^1_{X/Y}, \mathcal{O}_X) \cong \omega^{-1}_{X/C}(S).$$

Also, we get the following exact sequence

$$0 \to \mathcal{E}xt^2\,\mathcal{O}_X(\mathcal{C}, \mathcal{O}_X) \to \mathcal{E}xt^1\,\mathcal{O}_X(\Omega^1_{X/Y}, \mathcal{O}_X) \to \mathcal{O}_S(-f^* K_c) \to 0$$

(since $\mathcal{E}xt^1\,\mathcal{O}_X(\mathcal{T}, \mathcal{O}_X) \cong \mathcal{O}_S(-f^* K_c)$), which clearly reduces, when $S = 0$, to $\mathcal{E}xt^1\,\mathcal{O}_X(\Omega^1_{X/Y}, \mathcal{O}_X) \cong \mathcal{C}$ (by 4.13, or, equivalently, by local duality given by the Koszul complex).

From now on, we stop analyzing the general case, where the Arakelov type property $H^1(\mathcal{H}om\,\mathcal{O}_X(\Omega^1_{X/Y}, \mathcal{O}_X)) = 0$ does not need to hold.

The assumption that the set $\mathcal{S}$ of critical points is finite reduces the problem to the vanishing of $H^1(\omega^{-1}_{X/C})$, and we use the assumption that the general fibres are not isomorphic to ensure that $\omega_{X/C}$ is numerically positive (Thm. 2′ of [Sz], p. 56); notice then that all the fibres are reduced, hence the vanishing follows from [Me], p. 35.

Since $h^0(\mathcal{E}xt^1\,\mathcal{O}_X(\Omega^1_{X/C}, \mathcal{O}_X)) = \text{length}\ (\mathcal{C})$, which equals to $-c_2(\mathcal{C})$, and can be computed by 4.17, being $\mathcal{T} = 0$, and 4.6.

An easy calculation gives $-c_2(\mathcal{C}) = c_2 - K_X \cdot f^* K_C = c_2 - 4(b-1)(g-1)$.

Q.E.D.

(4.18) *Remark.* The main difficulty in extending the previous result to a more general case is the following: it is known (cf. [Des] 5.18) that, by taking the pull back of $f\colon X \to C$ via a covering C' of C, unramified outside of the set of critical values of f, one reduces to the case of a semistable fibration (it suffices that the subgroup of l-torsion points of the relative Jacobian, where l is a fixed integer relatively prime to the characteristic p of the field, has points which are rational over the function field of C').

The only trouble is that not much can be said in general on the nature of the covering, and in characteristic p the number of coverings with prescribed ramification is not finite (as shown by the unramified Artin-Schreyer extensions $y_p - y = g(x)$ of the affine line).

Of course, in extending the above proof, the only point where characteristic zero would be used, would be in the vanishing for the first cohomology group $H^1(\omega_{X/C}^{-1}(S))$.

Furthermore, even in treating the "isotrivial" case where two general fibres are isomorphic, more care would be needed, in analyzing again the needed base change, and in treating separately the case where the Hurwitz bound for the group of automorphism does not hold.

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Appendix – From the letter sent by Arnaud Beauville to the author on November 22, 1988

Arnaud Beauville
Mathématiques, Bât. 425, Université Paris-Sud, F-91405 Orsay Cedex, France

Theorem. *Let X be a compact Kähler manifold, of dimension $\geqq 2$. Let g be an integer $\geqq 2$, and let Γ_g denote the fundamental group of a Riemann surface of genus g. Then X has a pencil of genus $\geqq g$ if and only if there exists a surjective homomorphism from $\Pi_1(X)$ onto Γ_g.*

The "if" part is clear (observe that there are plenty of surjective homomorphisms $\Gamma_h \to \Gamma_g$ for $h \geqq g$). Let us prove the "only if" part.

(1) Let us say that an element L of $\mathrm{Pic}^0(X)$ is *special* if $H^1(X, L)$ is nonzero. Recall that the set of special elements of $\mathrm{Pic}^0(X)$ is the union of a finite set and of the Abelian subvarieties $p^* \mathrm{Pic}^0(B)$, for all irrational pencils p: $X \to B$. It follows from this that if X has no irrational pencil of genus $\geqq g$, the number of elements of order n in $\mathrm{Pic}^0(X)$ which are special is bounded by $C n^{2g-2}$, where C is a constant.

(2) Let ϕ: $\Pi_1(X) \to \Gamma_g$ be a surjective homomorphism. For each surjective homomorphism η: $\Gamma_g \to \mathbb{Z}/(n)$, the kernel of $\phi \circ \eta$ maps onto the kernel of η, which is isomorphic to $\Gamma_{n(g-1)+1}$. In particular, the irregularity of the covering $\tilde{X}$ of X corresponding to $\phi \circ \eta$ is greater than the irregularity of X. In terms of the line bundle L on X corresponding to $\phi \circ \eta$, this means that some nontrivial power of L is special (one has $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^1(X, \mathcal{O}_X) \oplus H^1(X, L) \oplus H^1(X, L^2) \oplus \ldots \oplus H^1(X, L^{n-1})$). If n is prime, this implies that $\mathrm{Pic}^0(X)$ contains at least n^{2g-1} special elements of order n. In view of part (1), we conclude that X has an irrational pencil of genus $\geqq g$.

Corollary. *Let g be an integer $\geqq 2$. The property of having an irrational pencil of genus g can be read off the fundamental group; in particular, it is homotopy invariant.*

References for the Appendix

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Notes added in proof

Kollàr (letter, Feb. 6 1990) pointed out that the vanishing of $H^0(\mathscr{R}^1 f_* \mathcal{O}_X \oplus \Theta_Y)$ needed in Theorem (4.9.) holds indeed in greater generality.

Since $\mathscr{R}^1 f_* \mathcal{O}_X$ is torsion free, it suffices to prove vanishing after restriction to a general curve section $C < Y$.

In order to achieve this, one can apply Tsuji's theorem to the effect that Θ_Y is semitstable if Y is of general type and K_Y is nef, and then by a result of Miyaoka Θ_Y is seminegative, as well as $\mathscr{R}^1 f_* \mathcal{O}_X$.

As suspected, if one drops the Kaehler assumption, the existence of a non constant holomorphic map to a curve C of genus $\geqq 2$ is not dictated by topology. In fact, M. Kato, in a preprint dated August 26, answered my question showing the existence of a (non Kähler) complex structure on $C \times \mathbb{P}^1 \times \mathbb{P}^1$ with algebraic dimension 0.

Recently, Beauville was able to prove that Problem 1.25 has an affirmative answer for $i = 1$ and under the assumption that the commutator subgroup of $\Pi_1(X)$ is of finite type.