Generalized Kummer Surfaces and Differentiable Invariants of Noether–Horikawa Surfaces I

by
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Introduction

The problems which lie behind as motivation to this work are the following:

i) The moduli spaces $\mathcal{M}$ of surfaces of general type with a fixed topological type are known (cf. e.g. [Ca3]) to have several connected components: there is thus a map $\pi_0(\mathcal{M}) \to \text{Diff}$, where Diff is the set of differentiable types of 4-manifolds. An unsolved question is whether is this map injective or not.

ii) One should try to calculate explicitly the Donaldson differentiable invariants of some differentiable 4-manifolds.

Justification for this article, where problem ii) is solved in a very particular case, is also, I hope, the study of some beautiful geometry and algebra related to some surfaces which generalize the so-called Kummer surfaces. Going back to problem i), the simplest surfaces of general type for which the moduli space is known to be disconnected are the so called Noether–Horikawa surfaces.

Noether proved that for any surface of general type, the self intersection K^2 of a canonical divisor on the minimal model S satisfies the inequality $K^2 \geq 2p_g - 4$ (equivalently, $K^2 \geq 2\chi - 6$) and classified explicitly those surfaces for which equality holds, henceforth called N–H surfaces since Horikawa ([Hor]) clarified and extended Noether's work, showing that the moduli space is disconnected, if $16|K^2$.

The two components of the moduli space correspond to two types, that we propose to call respectively of type C (connected branch locus for the canonical map) and of type N (non connected branch locus).

If $K^2 > 8$, then the surfaces of type C are just limits of double covers of $\mathbf{P}^1 \times \mathbf{P}^1$ branched on a divisor of bidegree $(6, 2m)$, whereas the surfaces of type N occur only when $8|K^2$, and they are the double covers of the Segre–Hirzebruch surface F_{2k+2} branched on the negative section Δ_∞ and on a (disjoint) curve B' linearly equivalent to $5\Delta_0$ (Δ_0 being a section disjoint from Δ_∞).

Now, as we shall also see here, for type C the intersection form is odd, whereas for type N, it is odd if and only if $16|K^2$, therefore we get (by Freedman's theorem, [Fr]) homeomorphic surfaces if and only if $16|K^2$.

Notice that for these surfaces the class of K is primitive in cohomology, so one has to look for more refined invariants in order to answer a particular case of problem i), i.e., the following

Question: if $16|K^2$, are N–H surfaces of type C diffeomorphic to N–H surfaces of type N?

The method we are trying to use to attack this problem was introduced by Kronheimer in [Kr] for the case of the Kummer surface, and A. Tjurin, who pointed out this article to us in February '92, asked whether we knew of other algebraic surfaces for which the method could apply.

Kronheimer's idea can be outlined as follows: given S a compact oriented differentiable 4-manifold, a principal SO(3) topological bundle is completely determined, by the theorem of Dold and Whitney ([DW]), by the Pontrjagin number $p_1 = p_1(\mathrm{P})$ and by the Stiefel Whitney class $w = w_2(\mathrm{P})$ in $H^2(S, \mathbf{Z}/2)$ ($w^2 \equiv p_1 \bmod 4\mathbf{Z}$).

If one fixes P, for a generic Riemannian metric g, the moduli space $\mathcal{M}(\mathrm{P}, g)$ of ASD connections modulo gauge equivalence has the expected dimension (the so called virtual dimension) $-2p_1 - 3(b^+ - b^1 + 1)$, which in the case of an algebraic surface equals $-2p_1 - 6\chi$, where χ is the holomorphic Euler Poincaré characteristic.

The interesting case is when the virtual dimension equals zero: then a differentiable invariant $q(\mathrm{S}, \mathrm{P})$ is well defined as the number of points of $\mathcal{M}(\mathrm{P}, g)$ counted with multiplicity (according to the orientation of the moduli space).

Kronheimer's idea was to calculate it, in the case e.g. where S is an algebraic surface, from the topology of some normal singular surface X with quotient singularities (an orbifold) of which S is a minimal resolution of singularities.

He described the procedure when X is nodal, i.e., its singularities are locally isomorphic to $\mathbb{C}^2/\pm 1$.

On X one can consider orbifold bundles P^* instead of bundles (cf. [Kr], [Ka] and section 5 for a sketchy review of his theory), define an analogous moduli space $\mathcal{M}(P^*, g)$ whose virtual dimension equals $-2p_1^* - 6\chi + \delta^-$, where p_1^* is the orbifold Pontrjagin number, and δ^- is the number of "twisted" nodes, those around which P^* is not trivial.

Kronheimer shows that to a bundle P on S one associates a unique orbifold bundle P^* on X such that the virtual dimensions of the respective moduli spaces are the same, and a so called marking (a local lift of the SO(3) local monodromy to U(2)): this marking is essential to reverse the correspondence, since it determines the Stiefel Whitney class of the associated extension to S of the bundle (i.e., of the bundle $P^\#$ which is the restriction of P^* to the nonsingular locus $X^\#$ of X). The crucial aspect of this correspondence is that not only it preserves the virtual dimension of the respective moduli spaces but also, when the above is zero, the respective numbers $q(S, P)$ and $q(X, P^*)$ are the same.

The way to gain something from this construction is to have a situation where moreover $p_1^* = 0$, since then for a Hodge metric we get an orbifold bundle associated to an SO(3) - representation of the fundamental group $\pi_1(X^\#)$, and the study of the moduli space can hopefully be reduced to a completely algebraic problem, of determining certain representations.

A first difficulty, though, is that if the virtual dimension is zero and also $p_1^* = 0$, then X must have many singular points (in the nodal case, by Miyaoka's inequality [Mi] one must then have $K^2 \leq (9/4)\chi$), but certainly a more difficult problem, once we find in the moduli space a surface S whose canonical model has many singular points, is to determine $\pi_1(X^\#)$.

We are lucky for N–H surfaces of type C, because we can let the branch locus degenerate to the union of 6 vertical and $2m$ horizontal lines, and we get $12m$ nodes.

More generally, we define generalized Kummer surfaces of type $(2n, 2m)$ as the double covers of $\mathbf{P}^1 \times \mathbf{P}^1$ branched on a divisor which is the union of $2n$ vertical and $2m$ horizontal lines.

For these $\pi_1(X^\#)$ can be explicitly described (section1) and also all its representations in SO(3) (section 2).

In spite of the fact that in some case the moduli space of marked representations has bigger dimension than the virtual dimension, we are able to prove the following (cf. 6.3 for a more precise statement).

Main Theorem Let X be a generalized Kummer surface of type $(6, 2m)$, and let S be the Noether–Horikawa surface which is the minimal resolution of singularities of X.

Then there are certain classes w in $H^2(S, \mathbf{Z}/2)$, with $w^2 \equiv -1 + 2m \pmod 4$, such that for a bundle P with $w^2(P) = w$ and $p_1(P) = -3\chi = -6m + 3$, the value of the invariant $q(S, P)$ equals $2^{2(m-2)}$.

Some remarks are in order: Kronheimer got in [Kr] a similar result for the Kummer surface, finding the value 1 for $q(S, P)$, whereas Kametani (whose preprint [Ka] was given to me through courtesy of W. Ebeling in Jan. 94) got the same result for the case $m = 3$.

Both Kronheimer and after him Kametani used this calculation to show that the topological 4-manifold underlying S admits infinitely many non equivalent differentiable structures. It looks likely that the same result should also hold here with a similar method.

In order to use Kronheimer's method for N–H surfaces of type N, since a canonical model which is nodal and has sufficiently many nodes can't be found, we need to consider other types of singularities, since then we can find a surface with many singular points (a construction of U. Persson in [Per]).

There the calculations are more heavy, but the numeral I in the title is given in the hope that we may show first the independence of the above number $q(S, P)$ upon w, and then calculate some similar number for the surfaces of type N.

Beyond the main result, proven in the last section 6, we devote section 3 to a more general study of the so called codes associated to the singular set of a nodal surface, determining the strict code K′, and the large code K″.

This study is important here since it relates the global Stiefel Whitney classes on S to the local Stiefel Whitney classes around the nodes.

I would like also, in view of the dedication of these Proceedings volume to Prof. Calabi, to point out how the article [Kr2] shows a thread linking this research with the ideas introduced in [Cal]. Finally, I would like to apologize to the reader since for lack of time I was not able to polish or simplify many arguments.

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1 Generalized Kummer surfaces and their "orbifold" fundamental group

Definition 1.1 A generalized Kummer surface is a singular surface X which is the double cover of $\mathbf{P}^1 \times \mathbf{P}^1$ branched over the union of $2n$ vertical lines and $2m$ horizontal lines, where n, m are integers ≥ 2. In more algebraic terms, X is a divisor inside the total space of the line bundle L over $\mathbf{P}^1 \times \mathbf{P}^1$ whose sheaf of sections is $\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(n, m)$ and it is defined by the equation

$$z^2 = F(x_0, x_1)G(y_0, y_1),$$

where F, G are homogeneous polynomials or respective degrees $2n, 2m$ (both F, G split as a product of linear factors which we assume to be distinct).

The singular points of X are precisely the $\delta = 4nm$ points where $F = G = 0$, and there the singularity is a node (an ordinary quadratic singularity $z^2 = FG$, since z, F, G are local coordinates in L).

Definition 1.2 Given a normal compact complex surface X with as singularities only Rational Double Points $p_1, \ldots, p_\delta$, let $X^\#$ be the nonsingular locus of X, i.e., $X - \{p_1, \ldots, p_\delta\}$. The "orbifold" fundamental group of X is simply the fundamental group $\pi_1(X^\#)$, and an orbifold G-representation is simply a representation $\rho : \pi_1(X^\#) \to G$.

We consider now another way of presenting the generalized Kummer surfaces:

(1.3) X= $(C_1 \times C_2)/i$, where

(1.3′) C_1 is the double cover of $\mathbf{P}^1$ defined by the equation $z_1^2 = F(x_0, x_1)$ (in the total space of the line bundle L_1 over $\mathbf{P}^1$ whose sheaf of sections is $\mathcal{O}_{\mathbf{P}^1}(n)$), similarly C_2 is the double cover of $\mathbf{P}^1$ defined by the equation $z_2^2 = G(y_0, y_1)$

(1.3″) i is the product involution $i_1 \times i_2$, where i_j on C_j is the involution associated to the given double cover (z_j gets multiplied by -1)

(1.3‴) the quotient map $q : (C_1 \times C_2) \to X$ is simply given by setting $z = z_1 \cdot z_2$.

Remark 1.4 For $n = m = 2$ we obtain that C_1, C_2, are elliptic curves and therefore we recover exactly the special Kummer surfaces (cf [Shaf]).

Remark 1.5 In general, our standard notation, if X is as in 1.2, will be to denote by $\pi : S \to X$ the minimal resolution of the singularities of X. Assuming now that X is a generalized Kummer surface, we want to locate S in the surface geography, using standard formulae for double covers (cf. e.g. [BPV]).

Let $f : S \to \mathbf{P}^1 \times \mathbf{P}^1$ be the generically finite double cover. Since the canonical bundle K_S of S equals

(1.5′) $K_S = (n-2)F_1 + (m-2)F_2$,

F_1 being the pull back of the divisor of a vertical line (resp. : F_2 of a horizontal one), it follows immediately that for $n, m \geq 2$ K_S is nef, in particular S is a minimal surface, and

(1.5″) $K_S^2 = 4(n-2)(m-2)$.

Moreover $f_* \mathcal{O}_S = \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(-n, -m)$, whence

(1.5‴) $\chi = \chi(\mathcal{O}_S) = 1 + (n-1)(m-1)$.

From this and from e.g. Noether's formula we get for the second Betti number

(1.5⁗) $b^2(S) = 8nm - 4n - 4m + 6$.

Notice firstly that all our surfaces S, for $n, m \geq 1$, are simply connected (cf. e.g. [Ca2], anyhow we shall reprove this result in cor. 1.11). Secondly that for $n = 2$, we get simply connected elliptic surfaces with arbitrary positive value of the geometric genus $p_g = (n-1)(m-1)$.

Thirdly that for $n = 3$ we have surfaces on the Noether line, i.e., surfaces which, in view of Noether's inequality $K_S^2 \geq 2p_g - 4$, yield the lowest possible value for K_S^2, i.e., $K_S^2 = 8(m-2) = 2p_g - 4$.

The surfaces in the Noether line have been thoroughly investigated by Horikawa ([Hor]), who also determined the intersection form for these surfaces. For these the canonical class K_S is primitive, as we are going to see.

Proposition 1.6 Let S be a minimal resolution of a generalized Kummer surface. Then the intersection form q_S on $H^2(S, \mathbf{Z})$ is even if and only if both n and m are even numbers. More generally, the divisibility index of K_S in $H^2(S, \mathbf{Z})$ equals $r = \text{G.C.D.}(n-2, m-2)$.

Proof If both n, m are even, by formula (1.5′) then K_S is 2-divisible and thus q_S is even (since by Wu's formula ([HNK]), for any class c in $H^2(S, \mathbf{Z})$, $q_S(c) \equiv K_S \cdot c \pmod 2$).

Conversely, let $\xi_1, \ldots, \xi_i, \ldots \xi_{2n}$ be the linear factors of the polynomial $F(x_0, x_1)$, and let F_i' be the unique irreducible divisor on S mapping isomorphically to the divisor $(\mathrm{div}(\xi_i))$.

Let similarly $\eta_1, \ldots, \eta_j, \ldots, \eta_{2m}$ be the linear factors of the polynomial $G(y_0, y_1)$, define similarly F_j'' and let A_{ij} be the inverse image under π of the singular point p_{ij} of X defined by the vanishing of ξ_i and η_j. A_{ij} is smooth rational with self intersection $(A_{ij})^2 = -2$ (whereas $A_{ij} \cdot K_S = 0$).

We have the following linear equivalence: $F_1 \equiv 2F'_i + \sum_{j=1,\dots,2m} A_{ij}$, from which we obtain

$$(2F'_i)^2 = (F_1 - \sum_{j=1,\dots,2m} A_{ij})^2 = -4m,$$

whence $F_i'^2 = -m$.

Similarly $F_j''^2 = -n$, and if either n or m is odd, we get some odd selfintersection number.

For the last statement, since $K_S = (n-2)F_1 + (m-2)F_2$, if $r =$ G.C.D.$(n-2, m-2)$, then it is clear that K_S is r-divisible in $H^2(S, \mathbf{Z})$. But on the other hand $K_S \cdot F_i' = (m-2)$, $K_S \cdot F_j'' = (n-2)$, whence if $K_S = r'D$, then r' divides r.

QED

In order to calculate the orbifold fundamental group of generalized Kummer surfaces, we consider the following diagram (cf. (1.3)):

$$C_1 \times C_2 \to g \to \mathrm{X} \to h \to \mathbf{P}^1 \times \mathbf{P}^1$$

of double covers. Let

$$p'_{ij} = g^{-1}(p_{ij}), \quad p''_{ij} = h(p_{ij}),$$

and let

$$\mathrm{X}^\# = \mathrm{X} - \{p_{ij} | i = 1, \dots, 2n,\ j = 1, \dots, 2m\},$$
$$(C_1 \times C_2)^\# = (C_1 \times C_2) - \{p'_{ij}\},$$
$$(\mathbf{P}^1 \times \mathbf{P}^1)^\# = (\mathbf{P}^1 \times \mathbf{P}^1) - \{p''_{ij}\}.$$

Then

$$(C_1 \times C_2)^\# \to g \to \mathrm{X}^\#$$

is an unramified double cover, and since

$$\Gamma = \pi_1((C_1 \times C_2)^\#) = \pi_1((C_1 \times C_2)) = \pi_1(C_1) \times \pi_1(C_2),$$

we have an exact sequence (where we define $\Pi = \pi_1(\mathrm{X}^\#)$)

$$(\mathbf{1.7}) \qquad 1 \to \Gamma = \pi_1(C_1) \times \pi_1(C_2) \to \Pi = \pi_1(\mathrm{X}^\#) \to \mathbf{Z}/2 \to 1.$$

We are going to calculate the nature of the extension (1.7) using a non-standard presentation of the fundamental groups of the curves C_1, C_2, and then applying van Kampen's theorem.

To this purpose, let $C_1^\# = C_1 - \xi_1 \dots \xi_i \dots \xi_{2n} = 0$, so that

$$C_1^\# \to \mathbf{P}^1 - \xi_1 \dots \xi_i \dots \xi_{2n} = 0$$

is an unramified double covering. Since $\pi_1(\mathbf{P}^1 - \xi_1 \dots \xi_i \dots \xi_{2n} = 0)$ is the group with generators and relations

$$\Gamma_1''^\# = \langle \gamma_1, \dots, \gamma_i, \dots, \gamma_{2n} | \Pi_{i=1,\dots,2n} \gamma_i = 1 \rangle$$

(a free group on $2n-1$ generators) the above double covering is associated to the subgroup kernel of the homomorphism $\Gamma_1''^\# = \mathbf{Z}/2$ sending each γ_i to 1.

The above kernel is itself a free group $\Gamma_1'^\#$ with $(4n-3)$ generators, and is generated by

$$\gamma_1^2, \dots, \gamma_i^2, \dots, \gamma_{2n}^2,\ \gamma_1\gamma_2, \gamma_2\gamma_3, \dots, \gamma_{2n-2}\gamma_{2n-1},$$

subject only to the following relation, (following from the relation $\gamma_{2n}^{-2} = (\Pi_{i=1,\dots,2n-1}\gamma_i)^2$ holding in $\Gamma_1''^\#$):

$$\begin{aligned}(\gamma_{2n}^2)^{-1} &= (\gamma_1\gamma_2)(\gamma_3\gamma_4)\dots(\gamma_{2n-3}\gamma_{2n-2})(\gamma_{2n-1}^2)(\gamma_{2n-2}\gamma_{2n-1})^{-1} \\ &\cdot (\gamma_{2n-2}^2)(\gamma_{2n-3}\gamma_{2n-2})^{-1}\dots(\gamma_1\gamma_2)^{-1}(\gamma_1^2)(\gamma_2\gamma_3)\dots(\gamma_{2n-2}\gamma_{2n-1})\end{aligned}$$

Whence $\Gamma_1 = \pi_1(C_1)$, which is the quotient of $\Gamma_1'^\#$ by the minimal normal subgroup containing $\gamma_1^2, \dots, \gamma_i^2, \dots, \gamma_{2n}^2$, is presented as follows

$$(\mathbf{1.8'}) \qquad \begin{aligned}\Gamma_1 = \langle &(\gamma_1\gamma_2), (\gamma_2\gamma_3), \dots, (\gamma_{2n-2}\gamma_{2n-1}) | (\gamma_1\gamma_2)(\gamma_3\gamma_4) \\ &\dots(\gamma_{2n-3}\gamma_{2n-2})(\gamma_{2n-2}\gamma_{2n-1})^{-1}(\gamma_{2n-3}\gamma_{2n-2})^{-1} \\ &\dots (\gamma_1\gamma_2)^{-1}(\gamma_2\gamma_3)\dots(\gamma_{2n-2}\gamma_{2n-1}) = 1\rangle.\end{aligned}$$

Alternatively, we can take $(\gamma_1\gamma_2), (\gamma_1\gamma_3), \dots, (\gamma_1\gamma_{2n-1})$ as generators: they are subject to the relation

$$(\mathbf{1.8''}) \qquad \begin{aligned}&(\gamma_1\gamma_2)\cdot(\gamma_1\gamma_3)^{-1}\cdot(\gamma_1\gamma_4)\cdot(\gamma_1\gamma_5)^{-1}\cdot\dots\cdot(\gamma_1\gamma_{2n-1})^{-1}(\gamma_1\gamma_2)^{-1} \\ &\cdot(\gamma_1\gamma_3)\cdot(\gamma_1\gamma_4)^{-1}\cdot(\gamma_1\gamma_5)\cdot\dots\cdot(\gamma_1\gamma_{2n-1}) = 1.\end{aligned}$$

We define similarly

$$\Gamma_2''^\# = \langle \delta_1, \dots, \delta_j, \dots, \delta_{2m} | \Pi_{j=1,\dots,2m}\delta_j = 1\rangle, \Gamma_2'^\#, \Gamma_2.$$

Let us consider the following open set:

$$A = h^{-1}((\mathbf{P}^1 - \{\xi_1 \dots \xi_{2n} = 0\}) \times (\mathbf{P}^1 - \{\eta_1 \dots \eta_{2m} = 0\})).$$

We may remark that $\pi_1(A)$ is the kernel of the homomorphism

$$\phi : (\Gamma_1''^{\#}) \times (\Gamma_2''^{\#}) \to \mathbb{Z}/2$$

sending each γ_i and each δ_j to 1.

Since A is obtained from the smooth manifold $X^{\#}$ by removing some connected submanifolds of complex codimension 1, namely $X^{\#} \cap \{\xi_i = 0\}$, $X^{\#} \cap \eta_j = 0$, $\pi_1(A)$ surjects onto $\pi_1(X^{\#})$ with kernel generated by the conjugates, for each such connected submanifold Y, of a small loop γ_Y around Y. (γ_Y is the generator of the fundamental group of the fibre in the punctured normal bundle $(N_Y - Y) \to Y$).

Thus:

(1.9) $\Pi = \pi_1(X^{\#}) = \pi_1(A)$/the minimal normal subgroup containing

$$\gamma_1^2, \ldots, \gamma_i^2, \ldots, \gamma_{2n}^2, \quad \delta_1^2, \ldots, \delta_j^2, \ldots, \delta_{2m}^2.$$

Observe that $\pi_1(A)$ contains the subgroup of index 2 given by $(\Gamma_1'^{\#}) \times (\Gamma_2'^{\#})$ and is generated by this subgroup and by the element $\gamma_1\delta_1$, which acts on the subgroup by conjugation inducing conjugation by γ_1 on the first factor, and conjugation by δ_1 on the second.

Proposition 1.10 Let X be a generalized Kummer surface: then its orbifold fundamental group $\Pi = \pi_1(X^{\#})$ is a semidirect product of $\Gamma = \Gamma_1 \times \Gamma_2$ and $\mathbb{Z}/2$, where the nonnormal subgroup $\mathbb{Z}/2$ is generated by $\gamma_1\delta_1$.

In terms of the presentation (1.8), conjugation by $\gamma_1\delta_1$ acts on Γ_1 by sending the generator $(\gamma_i\gamma_{i+1})$ to

$$(\gamma_1\gamma_2)(\gamma_3\gamma_4)\ldots(\gamma_{i-1}\gamma_i)\cdot(\gamma_i\gamma_{i+1})^{-1}(\gamma_{i-1}\gamma_i)^{-1}\ldots(\gamma_1\gamma_2)^{-1},$$

and similarly for Γ_2.

The action is simpler in terms of the generators $(\gamma_1\gamma_i)$, since conjugation by $\gamma_1\delta_1$ acts on Γ_1 by sending $(\gamma_1\gamma_i)$ to its inverse $(\gamma_i\gamma_1) = (\gamma_1\gamma_i)^{-1}$.

Proof The first statement follows from (1.7) and the fact that $(\gamma_1\delta_1)^2$, inside $\pi_1(A) \subset (\Gamma_1''^{\#}) \times (\Gamma_1''^{\#})$, equals $(\gamma_1^2)(\delta_1^2)$, an element which is trivial in Γ by (1.9).

The second statement follows from the consideration made after (1.9) and from a direct computation.

QED

Corollary 1.11 Let X be a generalized Kummer surface: then its orbifold fundamental group $\Pi = \pi_1(X^{\#})$ is generated by elements of order 2, $(\gamma_i\delta_j)$, for $i = 1, \ldots, 2n$, $j = 1, \ldots, 2m$, whereas $\pi_1(X)$ is trivial.

Proof $(\gamma_i\delta_j)\cdot(\gamma_h\delta_k)$ = the element $(\gamma_i\gamma_h)\cdot(\delta_j\delta_k)$ of Γ, therefore all the generators of Γ belong to the subgroup generated by the $(\gamma_i\delta_j)$'s, and also $(\gamma_1\delta_1)$ does so.

Since we have a surjection $\Pi = \pi_1(X^{\#}) \to \pi_1(X)$, it suffices to observe that $(\gamma_i\delta_j)$ is conjugate to a generator of the local fundamental group around the nodal point p'_{ij} of X, whence it maps to 0 in $\pi_1(X)$.

QED

Corollary 1.12 Let X be a generalized Kummer surface: then the "orbifold" first homology group $H_1(X^{\#}, \mathbb{Z})$ is isomorphic to $(\mathbb{Z}/2)^{2n+2m-3}$.

Proof $H_1(X^{\#}, \mathbb{Z})$ is the abelianization of $\pi_1(X^{\#})$. Thus, let ε be the class of $(\gamma_1\delta_1)$, and α_j be the class of $(\gamma_1\delta_j)$, and β_i be the class of $(\gamma_i\delta_1)$. Then, if $i, j \geq 2$, $\beta_i + \varepsilon + \alpha_j$ is the class of $(\gamma_i\delta_1)(\gamma_1\delta_1)(\gamma_1\delta_j) = (\gamma_i\gamma_1)(\gamma_1\delta_j) = (\gamma_i\delta_j)$, whence by (1.11) the above classes generate $H_1(X^{\#}, \mathbb{Z})$.

We clearly have the two relations:

(1.12′) $\varepsilon + \alpha_2 + \ldots + \alpha_{2m} = 0, \quad \varepsilon + \beta_2 + \ldots + \beta_{2n} = 0.$

Let H be the abstract abelian group generated by elements

$$\varepsilon, \alpha_2, \ldots, \alpha_{2m}, \quad \beta_2, \ldots, \beta_{2n}$$

of period 2, and subject to the two relations (1.12′). We have just seen that H maps onto the abelianization of Π, thus it suffices to show that there is a homomorphism of Π into H compatible with the two given homomorphisms to the abelianization of Π.

In terms of the presentation given in prop. 1.10, it suffices to show e.g. that, $(\gamma_i\gamma_{i+1})$ and

$$(\gamma_1\gamma_2)(\gamma_3\gamma_4)\ldots(\gamma_{i-1}\gamma_i)(\gamma_i\gamma_{i+1})^{-1}(\gamma_{i-1}\gamma_i)^{-1}\ldots(\gamma_1\gamma_2)^{-1}$$

have the same image in H. Since $(\gamma_i\delta_j)$ goes then to $\beta_i + \varepsilon + \alpha_j$, then $(\gamma_i\gamma_h) = (\gamma_i\delta_1)(\gamma_h\delta_1)$ is mapped to $\beta_i + \beta_h$, and our assertion is immediately verified.

QED

Remark 1.13 Clearly the respective groups of diffeomorphisms of the pairs $(\mathbb{P}^1, \{\xi_1 \ldots \xi_{2n} = 0\}), (\mathbb{P}^1, \{\eta_1 \ldots \eta_{2m} = 0\})$: they both act on X preserving $X^{\#}$. In particular , this yields all possible permutations of the indices i, respectively of the indices j.

Whence we obtain other presentations of the orbifold group of $X^{\#}$.

2 "Orbifold" SO(3)-representations of generalized Kummer surfaces

In this section we want to determine, up to conjugation in $G = \mathrm{SO}(3)$, all the possible homomorphisms $\rho : \pi_1(\mathrm{X}^{\#}) \to G = \mathrm{SO}(3)$. In order to treat the case where ρ factors through the abelianization of Π, and since Π is generated by elements of order 2, we start with an easy lemma of linear algebra.

For a nontrivial element a in SO(3), we denote by $Ax(a)$ the axis of the rotation a in $\mathbf{R}^3$.

Lemma 2.1 Let a, c be distinct nontrivial commuting elements of SO(3). Then either:
i') $Ax(a) = Ax(b)$ and they cannot both have order 2, or
i'') a, c have order 2, and, up to simultaneous conjugation, we can assume that a, c are diagonal, in particular $a = \mathrm{diag}\{1, -1, -1\}$, $c = \mathrm{diag}\{-1, 1, -1\}$. They generate thus a conjugate of the so called Klein group $\mathcal{K} \subset \mathrm{SO}(3)$, $\mathcal{K} \cong (\mathbf{Z}/2)^2$.
ii) The homomorphism of $\mathbf{Z}^2$ to SO(3) mapping the two standard generators to a, c respectively, lifts to SU(2) if and only if a, c do not generate a conjugate of the Klein group.
iii) If a, c are as in *i'')* and b commutes with both a, c then also b belongs to the Klein group. b is different from a iff the homomorphism of $\mathbf{Z}^2$ to SO(3) mapping the two standard generators to a, b respectively does not lift to SU(2) (similarly for c).

Proof. Choose a reference frame such that a is a rotation in the orthogonal plane to the vector e_1.

Since c commutes with a, e_1 is also an eigenvector for c, and c sends the orthogonal plane to e_1 into itself.

If $c(e_1) = e_1$, then a, c have the same axis and belong to a group SO(2) of rotations. Moreover $a \neq c$ implies that they cannot both have order 2.

Otherwise, $c(e_1) = -e_1$,and c has order 2. Similarly, a has also order 2 and we can complete e_1 to an orthonormal positively oriented frame such that a, c are as required.

Let b be as in *iii)* : then e_1 and e_2 are eigenvectors for b, which is therefore represented by a diagonal matrix.

Since then the diagonal entries of b are ± 1, b belongs to the conjugate of the Klein group generated by a, c.

Let a, c be as in *ii)* and assume firstly that they have the same axis, i.e., $a(e_1) = e_1$, $c(e_1) = e_1$. Then a, c belong to a group SO(2) of rotations whose inverse image in SU(2) is the group of unit quaternions commuting with i, whence $\cong SO(2)$. Therefore the given homomorphism of $\mathbf{Z}^2$ to SO(3) lifts to SU(2).

Assume instead that a, c generate the Klein group $\mathcal{K}$. Then, as it is well known, the inverse image of $\mathcal{K}$ in SU(2) is the group $\mathcal{Q} = \{\pm 1, \pm i, \pm j, \pm k\}$: since any lift of a, c has order 4 and $\mathcal{Q}$ is not commutative, the given homomorphism of $\mathbf{Z}^2$ to SO(3) does not lift to SU(2).

The last assertion of *iii)* follows right away from *ii)*.

QED

Definition 2.2 Let Π be the orbifold fundamental group $\pi_1(\mathrm{X}^{\#})$ of a generalized Kummer surface, and let $\rho : \Pi \to G$ be a representation. Set for convenience of notation

$$c_{ij} = \rho((\gamma_i \delta_j)),$$
$$a_{ij} = \rho((\gamma_i \gamma_j)),$$
$$b_{ij} = \rho((\delta_i \delta_j)).$$

We shall say that the representation ρ is **hooked** if there exist indices i, h in $1, \dots, 2n$, and j, k in $1, \dots, 2m$ such that
(**2.2'**) $c = c_{ij} = 1, \quad c'' = c_{ik} \neq 1, \quad c' = c_{hj} \neq 1$ (recall that all 3 of them are elements of order 2 in G).

We shall say that ρ is **monohooked** if furthermore either all the nontrivial c_{it}'s are equal to c'', or all the nontrivial c_{sj}'s are equal to c'.

Proposition 2.3 Let Π be the orbifold fundamental group of a generalized Kummer surface, and let $\rho : \Pi \to \mathrm{SO}(3)$ be a hooked representation.

Then either
i) ρ factors through the abelianization of Π and maps to a conjugate of the Klein group in SO(3) (in this case we shall say that ρ is **Kleinian**) or
ii) ρ is **monohooked-nonabelian** (i.e., monohooked and with non abelian image).

Proof In view of remark 1.13, we can assume without loss of generality $i = j = 1$. We consider now the presentation of Π given by 1.10, and we set for convenience of notation $a_t = a_{1t}$: since $c = c_{11} = 1$ we obtain $a_t = c_{t1}$: thus for each index t a_t has order 2. Similarly we set $b_s = b_{s1}$: it has order 2 since it equals c_{1s}.

Since $(\gamma_1 \delta_1)$ is in the kernel of ρ, we have seen that ρ factors through a representation of $\Gamma = \Gamma_1 \times \Gamma_2$. Conversely, since $(\gamma_1 \delta_1)$ conjugates $(\gamma_1 \gamma_i)$ to its inverse $(\gamma_i \gamma_1)$, any representation of $\Gamma_1 \times \Gamma_2$ extends to Π if the a_t's and b_s's have order 2.

Notice that clearly every a_t commutes with every b_s.

In this context, the property of ρ of being hooked implies that a_h and b_k are nontrivial.

I) if all the nontrivial a_t's are equal, and all the nontrivial b_s's are also equal, by lemma 2.1 the image of ρ is contained in a conjugate of the Klein group. From now on we shall only consider the nontrivial a_t's, and b_s's.

Observe that in any case, by *i)* of lemma 2.1, either $a_t = b_s$ (if they have the same axis), or a_t and b_s have orthogonal axes.

II) we can assume that there are at least two distinct axes among the $Ax(a_t)$'s (a similar argument if there are at least two distinct axes among the $Ax(b_s)$'s).

II1) if there are two non orthogonal axes $Ax(a_t)$, by our previous remark all the b_s's are equal and ρ is monohooked.

II2) if all the axes $Ax(a_t)$ are mutually orthogonal, then the a_t's commute and generate a conjugate of the Klein group in SO(3), by *iii)* of lemma 2.1, and the same applies for Π since also the b_s's commute with the a_t's, thus ρ is Kleinian. We need only to remark that in case II1) the image of ρ is non abelian.

QED

Corollary 2.4 Let Π be the orbifold fundamental group of a generalized Kummer surface. Then the set of conjugacy classes of Kleinian representations $\rho : \Pi \to SO(3)$ is the quotient of the vector space $\mathrm{Hom}(\Pi/[\Pi,\Pi] \cong (\mathbf{Z}/2)^{2n+2m-3}, (\mathbf{Z}/2)^2)$ by the action of the symmetric group $\mathcal{S}^3 = \mathrm{GL}((\mathbf{Z}/2)^2)$ acting by composition.

Proof By 2.3, we can assume that ρ factors through ρ' mapping to the standard Klein group $\mathcal{K} \cong (\mathbf{Z}/2)^2$ inside SO(3). It suffices then to apply corollary 1.12 and to observe that moreover, by lemma 2.1. *i)*, every automorphism of $(\mathbf{Z}/2)^2$ is induced by an inner automorphism of SO(3).

QED

We turn now to the non hooked representations.

If you think in terms of a chessboard, where the square with indices (i,j) is white if $c_{ij} = 1$ and black otherwise, we shall show immediately that ρ is not hooked iff either all the rows have the same colour, or if all columns do.

In fact, either all squares are black, or, given a white square, if ρ is not hooked, then either the row or the column containing the square is completely made out of white squares.

We simply claim now that if a row is white, all rows are made out of squares of the same colour (similarly for columns).

This follows from the fact that $c_{ij}\ c_{ik} = b_{jk}$, which is independent of i : thus if a row is white, then $b_{jk} = 1$ for each j,k and therefore for each $i,j,k \quad c_{ij} = c_{ik}$ (recall that these elements have order 2). Thus, non hooked representations are either horizontal or vertical, in the first case there exist

elements c_i of order 2 such that $c_{ik} = c_i$ for each k. Such a ρ (cf. 1.9), once lifted to $\pi_1(A) = \ker\psi$, can be extended to $(\Gamma_1''^{\#}) \times (\Gamma_2''^{\#})$ in such a way that it is trivial on the second factor, and since the c_i's have order 2 and $\Pi_{i=1,\dots,2n} c_i = 1$, we see that the non hooked representations are in a 1-1 correspondence with the representations of the following group

$$\textbf{(2.5)}\qquad \Gamma_1^{\#}(2) = \langle \gamma_1, \dots, \gamma_i, \dots, \gamma_{2n} | \Pi_{i=1,\dots,2n}\gamma_i = 1, \quad \gamma_1^2 = \dots = \gamma_i^2 = \dots = \gamma_{2n}^2 = 1\rangle.$$

We define similarly

$$\Gamma_2^{\#}(2) = \langle \delta_1, \dots, \delta_j, \dots, \delta_{2m} | \Pi_{j=1,\dots,2m}\delta_j = 1, \delta_1^2 = \dots = \delta_j^2 = \dots = \delta_{2m}^2 = 1\rangle$$

and we can state what we have just shown in the following

Corollary 2.6 Let Π be the orbifold fundamental group of a generalized Kummer surface, and let $\rho : \Pi \to G$ be a non hooked representation. Then either ρ is induced by a representation $\rho_1 : \Gamma_1^{\#}(2) \to G$ (under the rule $\rho((\gamma_i\delta_j)) = \rho_1(\gamma_i))$, or ρ is induced by a representation $\rho_2 : \Gamma_2^{\#}(2) \to G$.

Remark 2.7 Let G =SO(3). Then the representations ρ_1 are parametrized by a real algebraic variety as follows. Each c_i is a point of $\mathbf{P} = \mathbf{P}^2_{\mathbf{R}}$ (the axis of rotation), and our algebraic variety is the fibre over 1 of the morphism $R : \mathbf{P}^{2n} \to SO(3)$ such that $R(c_1, \dots, c_{2n}) = c_1 \cdot \dots \cdot c_{2n}$.

Whence in this case the dimension of the variety of conjugacy classes has positive dimension $\geq 4n - 3$ if n is at least 2, since e.g. at the points $c_1 = c_2, \quad c_3 = c_4, \dots c_{2n-1} = c_{2n}, \quad R$ is a submersion (in fact, any rotation can be written as a product c_1c_2 of two reflections).

Finally, we describe in some detail the monohooked nonabelian representations.

Proposition 2.8 Let ρ be a monohooked representation such that, in the notation of proposition 2.3, all the nontrivial b_s are equal to a single reflection b. Their number must be even. By remark 1.13, we may assume that $a_t = 1$, for each $t = 2, \dots, r-1$, and that $a_t = b$ for $t = r, \dots, p-1$, whereas $a_t \neq 1, b$ for $t \geq p$.

Set $a = a_p$: then the axis of a is orthogonal to $Ax(b)$ and every a_t, for $t \geq p$, can be written as

(2.9) $a_t = R_{\theta(t)} a R_{-\theta(t)}$, where $R_{\theta(t)}$ is a rotation of angle $\theta(t)$ in the orthogonal plane to the axis $Ax(b)$.

Moreover p is an even number and (recall that $b = R_\pi$)

(2.10) $2(\theta(p) - \theta(p+1) + \theta(p+2) - \theta(p+3) \dots - \theta(2m))$ is $\equiv 0 \pmod{2\pi}$ for r even, and $\equiv \pi (mod 2\pi)$ for r odd.

Conversely, the data of b, a and angles $\theta(t)$ not all equal to 0 or π and satisfying (2.10) determine a monohooked nonabelian representation.

Proof In view of the presentation (1.8) of Γ_1 and since $\rho(\gamma_i\gamma_i+1) = a_i a_{i+1}$, ($a_i$ being equal to its inverse) we see that the a_i's determine a representation of Γ_1 if and only if $\Pi_{j=2,\dots,2m} a_i = 1$. Similarly for the b_j's, whence the number of b_j's equal to b is even. Similarly, we obtain that p is even by applying the above formula to a vector in $Ax(b)$.

The axes of the a_t's for $t \geq p$ are orthogonal to $Ax(b)$ by lemma 2.1, whence (2.9).

We shall show that (2.10) is simply a reformulation of $\Pi_{j=2,\dots,2m} a_i = 1$.

In fact, we have the simple formula

(2.11) $a_t R_\theta a = R_{-\theta}$, by which follows

(2.12) $R_\theta a R_{-\theta} R_\psi a R_{-\psi} = R_{2(\theta-\psi)}$, and thus we get (2.10).

QED

3 Codes of nodal surfaces and of generalized Kummer surfaces

In this section again X will be as in (1.2) a normal compact complex surface X with as singularities only Rational Double Points $p_1, \dots, p_\delta$, $X^\#$ is the nonsingular locus of X, and $\pi : S \to X$ is the minimal resolution of the singularities of X.

Let U_i be a contractible neighbourhood of the point p_i, and let A_i be the fundamental cycle over p_i (a π-exceptional divisor equal to the pull back of the maximal ideal of the point p_i).

Set $U_i^\# = U_i \cap X^\#$, and observe that $U_i' = \pi^{-1}(U_i)$ is homotopically equivalent to the support of A_i, a bouquet of 2-spheres which by abuse of notation we shall still denote by A_i, while $U_i^\#$ is homotopically equivalent to a lens space S^3/Γ, where Γ is a finite subgroup of SL(2,**C**) such that the germ of singularity of X at p_i is analytically isomorphic to $\mathbf{C}^2/\Gamma$.

Since the fundamental group of $U_i^\#$ is finite $H^1(U_i^\#, \mathbf{Z}) = 0$ and we have a commutative diagram of Mayer-Vietoris exact sequences :

(3.1)

$$\begin{array}{ccccccc} 0 \to & H^2(S,\mathbf{Z}) & \to & (\oplus_{i=1,\dots,\delta} H^2(A_i,\mathbf{Z})) \oplus H^2(X^\#,\mathbf{Z}) & & \to \oplus_{i=1,\dots,\delta} H^2(U_i^\#,\mathbf{Z}) \\ & \uparrow & & \uparrow & & \uparrow \\ 0 \to & H^2(X,\mathbf{Z}) & \to & H^2(X^\#,\mathbf{Z}) & \to \lambda & \to \oplus_{i=1,\dots,\delta} H^2(U_i^\#,\mathbf{Z}) \end{array}$$

where we observe that the last arrow in the upper line is always surjective since the cokernel of $H^2(A_i,\mathbf{Z}) \cong H^2(U_i',\mathbf{Z}) \to H^2(U_i^\#,\mathbf{Z})$ is contained in $H^3(U_i', U_i^\#, \mathbf{Z})$ which by Lefschetz duality (cf. [G-H], 28.18) is isomorphic to $H_1(U_i', \mathbf{Z}) \cong H_1(A_i, \mathbf{Z}) = 0$.

We have in fact, again since $H^1(U_i^\#, \mathbf{Z}) = 0$, an exact sequence

$$0 \to H_2(A_i,\mathbf{Z}) \cong H^2(U_i', U_i^\#, \mathbf{Z}) \to H^2(A_i,\mathbf{Z}) \to H^2(U_i^\#,\mathbf{Z}) \to 0.$$

The considerations just made illustrate the importance of understanding the map

$$\lambda : H^2(X^\#,\mathbf{Z}) \to \lambda \to \oplus_{i=1,\dots,\delta} H^2(U_i^\#,\mathbf{Z})$$

(knowing it, one has in particular an immediate description of the group $H^2(S,\mathbf{Z})$).

Definition 3.2 Let $K = \oplus_{i=1,\dots,\delta} H^2(U_i^\#,\mathbf{Z})$, let K'' be the image of λ, let K' be the image of the torsion subgroup T of $H^2(X^\#,\mathbf{Z})$. K' is said to be the **strict code** of (the singular set of) X, K'' is said to be the **large code** of X.

Remark-definition 3.3 We easily see that K'' is the image of $H^2(S,\mathbf{Z})$ inside K. One can define therefore K^{alg} as the image of the algebraic classes of $H^2(S,\mathbf{Z})$. Thus $K'' \supset K^{\text{alg}} \supset K'$, since if a class in $H^2(S,\mathbf{Z})$ has its component in $H^2(X^\#,\mathbf{Z})$ which is torsion, then a multiple of it is in $(\oplus_{i=1,\dots,\delta} H^2(A_i,\mathbf{Z}))$, and is therefore algebraic. K^{alg} is called the **algebraic code** of X.

We want now to compare the above definitions with those of **even** sets of nodes given in [Ca1].

To this purpose, assume that all the singularities are nodes, therefore (3.1) can be rewritten as

(3.4)

$$0 \to H^2(S,\mathbf{Z}) \to (\oplus_{i=1,\dots,\delta} \mathbf{Z}E_i) \oplus H^2(X^\#,\mathbf{Z}) \to (\mathbf{Z}/2)^\delta \to 0,$$

since $H^2(A_i,\mathbf{Z}) \cong H^2(U_i',\mathbf{Z}) \cong H^2(U_i', U_i^\#, \mathbf{Z})$ is $\cong \mathbf{Z}$ and generated by a class, called $E_i = (1/2)A_i$, whose double is the fundamental class of A_i. Similarly we denote by e_i the generator of $(\mathbf{Z}/2) \cong H^2(U_i^\#,\mathbf{Z})$, which is induced by E_i.

In this case, our codes are honest binary codes, and an element k of K corresponds to a set $\mathcal{N}$ of nodes, $k = \sum_{i\in\mathcal{N}} e_i$.

Proposition 3.5 Given a nodal surface X, let K_2' be the image of the torsion elements of order 2.

An element $k = \sum_{i\in\mathcal{N}} e_i$ of K is in K_2' iff $\mathcal{N}$ is **strictly even**, i.e., $k = \sum_{i\in\mathcal{N}} A_i$ is 2-divisible in $H^2(S,\mathbf{Z})$.

K' and K_2' coincide if $H^2(S,\mathbf{Z})$ is torsion free.

k is instead in K'' iff $\mathcal{N}$ is **topologically-half-even**, i.e., there exists a class D in $H^2(X,\mathbf{Z})$ (i.e., in $H^2(S,\mathbf{Z})$ and orthogonal to the A_i's) such that $D + \sum_{i\in\mathcal{N}} A_i$ is 2-divisible in $H^2(S,\mathbf{Z})$.

Similarly, k is in K^{alg} iff $\mathcal{N}$ is **algebraically-half-even** (same as before except that D is the class of a divisor). Finally, K'' is orthogonal to K_2' (in 3.14 i) it will be shown that if $K' = K_2'$, then K'' is the orthogonal of K').

Proof There exists $L = (\sum \mu_i E_i, c)$ in $H^2(S, \mathbf{Z})$ such that $2L = \sum_{i \in \mathcal{N}} A_i$ iff $\mu_i = 1$ for $i \in \mathcal{N}$, and 0 otherwise and moreover $2c = 0$, and $\lambda(c) = \sum_{i \in \mathcal{N}} e_i$. Since the intersection of T with $\ker \lambda$ yields a torsion subgroup of $H^2(S, \mathbf{Z})$ by (3.1), we obtain that if $H^2(S, \mathbf{Z})$ is torsion free then T injects in K, thus all the elements of T have order 2 : thus the first assertion is proven.

For the second, observe preliminarly that the image of $H^2(X, \mathbf{Z})$ is, by the projection formula, orthogonal to the A_i's.

Indeed, $H^2(X, \mathbf{Z})$ consists of the elements in $\ker(\lambda)$ and we claim that it is just the orthogonal complement B to the A_i' s.

In fact, if D belongs to B, $D = (a, c)$, then $2D$ belongs to B, but since $(0, 2c)$ belongs to B, $(2a, 0)$ belongs to B, implying that $a = 0$, and that $c \in \ker(\lambda)$.

We are now done, since there is a class $D = (0, c)$ with c in $\ker(\lambda)$ such that $(\sum_{i \in \mathcal{N}} 2E_i, c) = 2L$, for L in $H^2(S, \mathbf{Z})$, iff there is a c' with $c = 2c'$ and $\lambda(c') = \sum_{i \in \mathcal{N}} e_i$.

The orthogonality of K'' and K_2' follows since

$$2L = \sum_{i \in \mathcal{N}} A_i, \quad 2L' = \sum_{i \in \mathcal{N}'} A_i + D,$$

with $D \cdot A_i = 0$ forces $(\sum_{i \in \mathcal{N}} A_i) \cdot (\sum_{i \in \mathcal{N}'} A_i) \equiv 0 \pmod{4\mathbf{Z}}$, whence $k' \cdot k'' \equiv 0 \pmod{2\mathbf{Z}}$.

QED

It will not be so easy in general to calculate the codes of a nodal surface (except very special cases, cf. [Bea]), but the situation improves if we make the following

Assumption 3.6 X is the quotient M/i of a smooth algebraic surface M by an involution having only a finite number of fixed points $p_1', \ldots, p_\delta'$.

We define here similarly $M^\#$, and we have $p : M \to X$ which induces an unramified double covering $p^\# : M^\# \to X^\#$.

We shall first show how a simple spectral sequence argument can be used in order to calculate the above code K'. Later we shall specialize to the case of generalized Kummer surfaces, where in fact everything can be done directly by using the results of the previous paragraphs in order to calculate K' and K'' (indeed, knowing $\pi_1(X^\#)$, one also knows K_2' and K'').

We have in our case a commutative diagram of maps

(3.7)

$$\begin{array}{ccc} M & \to & X = M/i \\ \uparrow & & \uparrow \\ N & \to & S \end{array}$$

where N is the blow up of M at the isolated fixed points of i, and S is as before the minimal resolution of X.

Let E_i be the exceptional divisor in N lying over the point p_i', so that $H^j(M, \mathbf{Z}) \cong H^j(N, \mathbf{Z})$ for $j \neq 2$, while $H^2(N, \mathbf{Z}) \cong (\oplus_{i=1,\ldots,\delta} \mathbf{Z} E_i) \oplus H^2(M, \mathbf{Z})$.

We also have $H_j(M, \mathbf{Z}) \cong H_j(M^\#, \mathbf{Z})$ for $j \leq 2$, while there is an exact sequence

$$0 \to H^3(M, \mathbf{Z}) \to H^3(M^\#, \mathbf{Z}) \to (\oplus_{i=1,\ldots,\delta} \mathbf{Z}) \to H^4(M, \mathbf{Z}) \cong \mathbf{Z} \to 0.$$

The commutative diagram (3.7) gives rise to a diagram whose rows, as we already saw, are exact

(3.8)

$$\begin{array}{ccccccc} 0 \to & H^2(S, \mathbf{Z}) & \to & (\oplus_{i=1,\ldots,\delta} \mathbf{Z} E_i) \oplus H^2(X^\#, \mathbf{Z}) & \to & \oplus_{i=1,\ldots,\delta} \mathbf{Z}/2(e_i) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^2(N, \mathbf{Z}) & \to & (\oplus_{i=1,\ldots,\delta} \mathbf{Z} E_i) \oplus H^2(M^\#, \mathbf{Z}) & \to & 0 & \end{array}$$

Since the upper row is exact, it follows that

$$0 \to H^3(S, \mathbf{Z}) \to H^3(X^\#, \mathbf{Z}) \to \oplus_{i=1,\ldots,\delta} H^3(U_i^\#, \mathbf{Z}) \to H^4(S, \mathbf{Z}) \cong \mathbf{Z} \to 0$$

is also exact, since $H^4(X^\#, \mathbf{Z}) = 0$. In particular, if $H^3(S, \mathbf{Z})$ is free, then also $H^3(X^\#, \mathbf{Z})$ is free and therefore we obtain that

(3.9) $H^2(X^\#, \mathbf{Z}/2) \cong H^2(X^\#, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}/2$

(in fact the cokernel of the natural inclusion is isomorphic, by the Universal Coefficients Formula, to $\mathrm{Tor}^1_{\mathbf{Z}}(H^3(X^\#, \mathbf{Z}), \mathbf{Z}/2)$). The similar Mayer–Vietoris sequence for X implies that

(3.10) If $H^3(S, \mathbf{Z}) = 0$, then also the following sequence is exact

$$0 \to H^2(X, \mathbf{Z}) \to H^2(X^\#, \mathbf{Z}) \to \lambda \to \oplus_{i=1,\ldots,\delta} H^2(U_i^\#, \mathbf{Z}) \to H^3(X, \mathbf{Z}) \to 0$$

(so that $H^3(X, \mathbf{Z}) \cong K/K''$).

In general, one can obtain information on $H^r(X^\#, \mathbf{Z})$ using (3.10). There is a spectral sequence with E_2 term equal to $H^i(\mathbf{Z}/2, H^j(M^\#, \mathbf{Z}))$ and converging to a suitable graded quotient of $H^{i+j}(X^\#, \mathbf{Z})$.

In order to use the spectral sequence, we recall a well known result (cf. [Ja], ex.7-8, page 363, [Ser], pages 140-145).

Lemma 3.11 Let R be a finitely generated abelian group which we assume to be a $\mathbf{Z}/2$ module under the action of an isomorphism

$$g : R \to R \quad (\text{thus } g^2 - 1 = (g+1)(g-1) = 0).$$

If $R^+ = ker(g-1)$, $\quad R^{++} = \text{im}(g+1)$, $\quad R^- = \ker(g+1), R^{--} = \text{im}(g-1)$,
a) $H^0(\mathbf{Z}/2, R) = R^+$, $\quad H^{2n}(\mathbf{Z}/2, R) = R^+/R^{++}$ for $n \geq 1$,
$H^{2n+1}(\mathbf{Z}/2, R) = R^-/R^{--}$ for $n \geq 0$, so that
b) if multiplication by 2 is invertible in R, then R splits as $R^+ \oplus R^-$ and $H^i(\mathbf{Z}/2, R) = 0$ for $i \geq 1$.
c) if R is free, then R is the direct sum of copies of 3 modules, $\mathbf{Z} = \mathbf{Z}^+$, $\quad \mathbf{Z}^-$, and the rank 2 module equal to the group algebra $U(\mathbf{Z}) = \mathbf{Z}[\mathbf{Z}/2]$.

We have $H^i(\mathbf{Z}/2, \mathbf{Z}) = 0$ for i odd and $= \mathbf{Z}/2$ for i even, $H^i(\mathbf{Z}/2, U) = 0$ for each $i \geq 1$, and $H^i(\mathbf{Z}/2, \mathbf{Z}^-) = 0$ for i even and $= \mathbf{Z}/2$ for i odd.
d) $H^i(\mathbf{Z}/2, \mathbf{Z}/2) = \mathbf{Z}/2$ for each i.

Proof The first assertion a) can be found in [Ja], ex.7-8, page 363, or [Ser], pages 140-145.

b) follows from a) and the high school algebra calculation $x = 1/2(x + g(x)) - 1/2(x - g(x))$.

For c), $R \supset R^+ \oplus R^-$, and the quotient $N = R/(R^+ \oplus R^-)$ is a 2 torsion group on which the induced $\mathbf{Z}/2$-action is trivial. A basis $x_1, \ldots, x_r$ of N as a $\mathbf{Z}/2$ vector space can be lifted to a set $y_1, \ldots, y_r$ of vectors in R, such that $g(y_i) - y_i \in R^-$, $\quad g(y_i) + y_i \in R^+$ are both non zero. We claim that the vectors $u_i = (g(y_i) + y_i)$ are linearly independent: else one would have a linear relation $\sum_i \lambda_i u_i = 0$, with the λ_i's coprime integers, but then $\sum_i \lambda_i y_i$ would be a vector in R^-, its image in N would be zero, and $\sum_i \lambda_i x_i = 0$, whence all the λ_i's are even, absurd.

Similarly for the vectors $v_i = (g(y_i) - y_i)$ in R^-.

Thus, the vectors $g(y_i)$ and y_i generate a submodule V of R with $V \cong U^r$. Now, we are done if we can achieve, by a suitable choice of the vectors y_i, that R/V is torsion free, since then we can find a supplementary submodule to V which is a direct sum $W^+ \oplus W^-$, with W^+ a supplementary subspace in R^+ to V^+ (resp. W^- for V^-).

We can thus replace R by the largest submodule V' such that V'/V is torsion, so that we may then assume that R^+, R^- are both free of rank r, and R is contained in $1/2(R^+ \oplus R^-) \subset (R^+ \oplus R^-) \otimes \mathbf{Q}$.

By induction on r, we may also assume that the vectors u_i, v_i for $i \leq r-1$, are the first $(r-1)$ vectors of respective bases of R^+, R^-. Consider $y_r = 1/2(u_r + v_r)$: we are done if the last coordinate of u_r (resp. : v_r) is $+1$, but notice that we may always alter y_r by adding any vector of the form $(u+v)$, whence either this is possible or we may assume that e.g. the last coordinate of u_r is 0, contradicting the previously established independence of the vectors u_i.

QED

By the previous lemma, we can say something about the cohomology of $H^{i+j}(X^{\#}, \mathbf{Z})$ under special assumptions on the groups $H^j(M^{\#}, \mathbf{Z})$, namely when it is torsion with no 2-torsion, when it is torsion of exponent 2, and also in the particular case we are interested in when it is free over $\mathbf{Z}$.

Lemma 3.12 Assume $H^j = H^j(M^{\#}, \mathbf{Z})$ is a free $\mathbf{Z}$-module and set $H^{j+} = H^j(M^{\#}, \mathbf{Z})^+, b^{j+} = \text{rank} H^j(M^{\#}, \mathbf{Z})^+$, and define similarly b^{j-}. Then, if $H^j(M^{\#}, \mathbf{Z}) \cong (c^j)U \oplus (\beta^{j-})\mathbf{Z} \oplus (\beta^{j-})\mathbf{Z}^-$, there is a spectral sequence, converging to the graded quotient of a suitable filtration of $H^{p+q}(X^{\#}, \mathbf{Z})$, whose $E_2^{p,q}$ terms are as follows

$$\begin{array}{ccccccc}
H^{3+} & \to & (\beta^{3-})\mathbf{Z}/2 & \to & (\beta^{3+})\mathbf{Z}/2 & \to & (\beta^{3-})\mathbf{Z}/2 \\
H^{2+} & \to & (\beta^{2-})\mathbf{Z}/2 & \to & (\beta^{2-})\mathbf{Z}/2 & \to & (\beta^{2-})\mathbf{Z}/2 \\
H^{1+} & \to & (\beta^{1-})\mathbf{Z}/2 & \to & (\beta^{1-})\mathbf{Z}/2 & \to & (\beta^{1-})\mathbf{Z}/2 \\
\mathbf{Z} & \longrightarrow & 0 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 0
\end{array}$$

and in particular, if $H^{1+} = 0$, then $H^1(X^{\#}, \mathbf{Z}) = 0$, and $H^2(X^{\#}, \mathbf{Z})$ has rank equal to b^{2+} and torsion subgroup T fitting into an exact sequence

$$0 \to \mathbf{Z}/2 \to T \to (b^{1-})\mathbf{Z}/2 \to 0.$$

In particular, if $H^2(S, \mathbf{Z})$ is torsion free, then λ yields an isomorphism of the torsion group T with the strict code K', and they are both $\cong (1 + b^{1-})\mathbf{Z}/2$. Finally, if $H^{1+} = 0$, $\quad (\beta^{2-}) = 0$, $\quad H^3$ is free, then also $H^3(X^{\#}, \mathbf{Z})$ is free.

Proof Since $H^1 = H^{1-}, b^{1-} = \beta^{1-}$ and thus everything follows from the previous lemma and the displayed spectral sequence.

Just for the last assertion, remark that by 3.8 λ is then injective on the torsion subgroup T of $H^2(X^{\#}, \mathbf{Z})$ whence in particular T has exponent 2 and rank $(1 + b^{1-})$ over $\mathbf{Z}/2$.

For $H^3(X^{\#}, \mathbf{Z})$, all the terms with $p+q = 3$ are then zero, except possibly H^{3+}. $H^3(X^{\#}, \mathbf{Z})$ is then a subgroup of H^{3+}, which is a subgroup of H^3, which is free. Whence $H^3(X^{\#}, \mathbf{Z})$ is free also.

QED

Example 3.13 Assume X is a generalized Kummer surface.

Then S is simply connected and in particular $H^2(S, \mathbf{Z})$ is torsion free, moreover $H^1(M, \mathbf{Z}) \cong H^1(M^{\#}, \mathbf{Z}) = H^1(M^{\#}, \mathbf{Z})^-$, whence the previous lemma applies in this case and the code K' is $(\mathbf{Z}/2)^{2n+2m-3}$. Since moreover $H^2(M,\mathbf{Z}) \cong H^2(M^\#,\mathbf{Z}) = H^2(M^\#,\mathbf{Z})^+$, there is (cf. 3.8) a splitting $H^2(X^\#,\mathbf{Z}) = H^2(M^\#,\mathbf{Z}) \oplus K'$, and also $H^2(S,\mathbf{Z})$ embeds into $H^2(N,\mathbf{Z})$ with cokernel K/K'.

In this case we want also to determine explicitly the embedding of K' into K.

Before doing that let us go back to the general case of a nodal surface X for which $H^2(S,\mathbf{Z})$ is torsion free and let us consider the following homology Mayer–Vietoris sequence: (since $\oplus_{i=1,\dots,\delta} H_2(U_i^\#,\mathbf{Z}) = 0$)

$$0 \to H_2(X^\#,\mathbf{Z}) \oplus (\oplus_{i=1,\dots,\delta} H_2(A_i,\mathbf{Z})) \to H_2(S,\mathbf{Z}) \to \oplus_{i=1,\dots,\delta} H_1(U_i^\#,\mathbf{Z}).$$

From it we infer that $H_2(X^\#,\mathbf{Z})$ is free and actually that

$$H_2(X^\#,\mathbf{Z}) \cong \mathrm{Hom}_{\mathbf{Z}}(H^2(X^\#,\mathbf{Z}),\mathbf{Z}) \cong \mathrm{Hom}_{\mathbf{Z}}(H^2(M,\mathbf{Z}),\mathbf{Z}) \cong H_2(M,\mathbf{Z}).$$

The torsion part of $H^2(X^\#,\mathbf{Z})$ is exactly $\mathrm{Ext}^1_{\mathbf{Z}}(H_1(X^\#,\mathbf{Z}),\mathbf{Z})$, and the map

$$H^2(X^\#,\mathbf{Z}) \to \lambda \to \oplus_{i=1,\dots,\delta} H^2(U_i^\#,\mathbf{Z})$$

induces an injective map

$$\mathrm{Ext}^1_{\mathbf{Z}}(H_1(X^\#,\mathbf{Z}),\mathbf{Z}) \to \oplus_{i=1,\dots,\delta} H^2(U_i^\#,\mathbf{Z}).$$

Since $H_1(X^\#,\mathbf{Z})$ and $H_1(U_i^\#,\mathbf{Z})$ are vector spaces over $\mathbf{Z}/2$, there are natural isomorphisms

$$\mathrm{Ext}^1_{\mathbf{Z}}(H_1(X^\#,\mathbf{Z}),\mathbf{Z}) \cong \mathrm{Hom}(H_1(X^\#,\mathbf{Z}),\mathbf{Z}/2)$$

and

$$(\oplus_{i=1,\dots,\delta} H^2(U_i^\#,\mathbf{Z})) \cong (\oplus_{i=1,\dots,\delta} H^2(U_i^\#,\mathbf{Z}/2)) \cong (\text{Poincare' duality})$$

$$\cong \mathrm{Hom}((\oplus_{i=1,\dots,\delta} H_1(U_i^\#,\mathbf{Z}/2)),\mathbf{Z}/2) \cong \mathrm{Hom}((\oplus_{i=1,\dots,\delta} H_1(U_i^\#,\mathbf{Z})),\mathbf{Z}/2).$$

To verify that thus the above injective map which gives the inclusion of K' into K is the $\mathbf{Z}/2$-dual of the surjective map of $\mathbf{Z}/2$-vector spaces

$$(\oplus_{i=1,\dots,\delta} H_1(U_i^\#,\mathbf{Z})) \to H_1(X^\#,\mathbf{Z})$$

we simply recall that by the theory of covering spaces every double unramified covering of a space Y is canonically corresponding to an element of $\mathrm{Hom}(H_1(Y,\mathbf{Z}),\mathbf{Z}/2)$, and the elements of K' correspond naturally to double covers $X^\#$, while their image in K corresponds to the restriction of the double covering to the union of the $U_i^\#$'s.

We can summarize the previous discussion in the following result. (3.14 ii) was established in the case of Kummer surfaces by Nikulin in [Ni].)

Corollary 3.14

i) Let X be a nodal surface such that $H^2(S,\mathbf{Z})$ is torsion free: then the codes K'' and K' are dual to each other.

ii) Assume X is a generalized Kummer surface.

Then the code K' is the subspace of $K \cong (\mathbf{Z}/2)^{4nm}$, with coordinates (y_{ij}), consisting of the vectors such that $y_{ij} = y_{11} + y_{1j} + y_{i1}$, $\sum_{i=1,\dots,2n} y_{i1} = 0 = \sum_{i=1,\dots,2n} y_{1j}$.

The minimum weight (number of non zero coordinates) of a vector in K' equals $W = \min(4n, 4m)$.

Proof i) by 3.5, since K'' is contained in the orthogonal of K', there only remains to prove that these two $\mathbf{Z}/2$-vector spaces have the same rank. Since K'' is the image of

$$\lambda : H^2(X^\#,\mathbf{Z}) \to \oplus_{i=1,\dots,\delta} H^2(U_i^\#,\mathbf{Z}) = H^2(\partial X^\#,\mathbf{Z})$$

we seek to calculate rank λ by a suitable exact sequence.

By Lefschetz duality, λ is isomorphic to the homomorphism

$$H_2(X^\#,\partial X^\#,\mathbf{Z}) \to H_1(\partial X^\#,\mathbf{Z})$$

which however fits into the exact sequence of the pair $(X^\#,\partial X^\#)$.

Therefore its image equals the kernel of $H_1(\partial X^\#,\mathbf{Z}) \to H_1(X^\#,\mathbf{Z})$.

By what we have just seen the rank of the latter linear map equals the rank of K' : thus rank (K') = corank (K'') and i) is proven.

ii) in the notation of 1.12, and using the above statement, we get that, * denoting duality of $\mathbf{Z}/2$-vector spaces, the map dual to the inclusion $K' \subset K$ is the map λ^* from $(\mathbf{Z}/2)^{4nm}$, with basis e_{ij} $i = 1,\dots,2n$, $j = 1,\dots,2m$, to the quotient of $(\mathbf{Z}/2)^{2n+2m-1}$ (with basis ε, $\alpha_2,\dots,\alpha_{2m}$, $\beta_2,\dots,\beta_{2n}$) by the subgroup generated by the two vectors $\varepsilon + \alpha_2 + \dots + \alpha_{2m}$, $\varepsilon + \beta_2 + \dots + \beta_{2n}$, such that $\lambda^*(e_{ij}) = \varepsilon + \alpha_j + \beta_i$.

Whence K' is the subspace of $(\mathbf{Z}/2)^{2n+2m-1}$ given by the vectors with coordinates $(x, a_2, \dots a_{2m},\ b_2,\dots,b_{2n})$ and such that

$$x = a_2 + \dots + a_{2m} = b_2 + \dots + b_{2n},$$

or also the subspace of $(\mathbf{Z}/2)^{2n+2m}$ given by the vectors with coordinates $(a_1, a_2, \dots, a_{2m},\ b_1, b_2, \dots, b_{2n})$ with $a_1 = b_1$, $a_1 + \dots + a_{2m} = b_1 + \dots + b_{2n} = 0$.

Then λ is the linear map taking the above vector to the vector with coordinates $y_{ij} = x + a_j + b_i = a_1 + b_i + a_j$.

Therefore K' is also the subspace of those vectors (y_{ij}) such that $y_{ij} = y_{11} + y_{1j} + y_{i1}$, $\sum_{i=1,\dots,2n} y_{i1} = 0 = \sum_{j=1,\dots,2n} y_{1j}$.

We would now like to calculate the weight of a vector in K', that is, the number of non zero coordinates in the given embedding into K. We use to this purpose the possibility of permuting the indices i and j different from 1.

An easy calculation (where one may distinguish the cases $x = 0$ and $x = 1$, even if this can be avoided) shows that if $2k$ is the number of a_j's equal to 1, $2h$ is the number of b_i's equal to 1, then the weight w of our vector is just $4(kh + (n-h)(m-k))$.

We want to find now what is the minimum weight of a non zero vector in K'.

In any case, replacing k by $k+1$, w is replaced by $w + 4(2h - n)$, and we get something smaller if $2h \leq n$, bigger if $2h > n$.

Thus the minima have to be searched (since e.g. if $k = 0$ then $h = 0$ and also $h = n$ are not allowed) among the cases $0 < 2h \leq n$, $k = m$, $h = 0$, $k = m - 1$, or $2h > n > h$, $k = 0$, or $n = h$, $k = 1$.

In these cases we get $w = 4mh$, $w = 4n$, $w = 4m(n-h)$, $w = 4n$ respectively. The minimum weight is then $W = \min(4n, 4m)$.

QED

We end this section by describing explicitly the homomorphism λ in the case of generalized Kummer surfaces.

Let us denote by $M\hat{}$ the universal cover of $M = C_1 \times C_2$, so that $M\hat{}$ is homeomorphic to a ball and let $M\hat{}^{\#}$ be the inverse image of $M^{\#}$: then $M\hat{}^{\#}$ has trivial cohomology groups $H^j(M\hat{}^{\#}, \mathbf{Z})$ for $j = 1, 2$.

Let c be any of the order 2 elements $c_{ij} = (\gamma_i \delta_j)$ in Π: then the following diagram

(3.15)

$$\begin{array}{ccccc} M^{\#} = & (M\hat{}^{\#})/\Gamma \to X^{\#} = (M\hat{}^{\#})/\Pi & \supset U^{\#}_{ij} = U^{\#} \\ & \uparrow \qquad\qquad\qquad \uparrow & \uparrow \\ & M\hat{}^{\#} \qquad \longrightarrow \qquad (M\hat{}^{\#})/\langle c \rangle & \supset (U^{\#}) \times (\Pi/\langle c \rangle) \end{array}$$

and the usual spectral sequence argument shows that

$$H^2(X^{\#}, \mathbf{Z}) = H^2(\Pi, \mathbf{Z}) \to H^2(M^{\#}, \mathbf{Z}) = H^2(\Gamma, \mathbf{Z})$$

and

$$H^2(X^{\#}, \mathbf{Z}) = H^2(\Pi, \mathbf{Z}) \to H^2(U^{\#}_{ij}, \mathbf{Z}) = H^2(\langle c_{ij} \rangle, \mathbf{Z})$$

are given by restriction of group cocycles $f : \Pi \times \Pi \to \mathbf{Z}$ to the subsets $\Gamma \times \Gamma$ and $\langle c_{ij} \rangle \times \langle c_{ij} \rangle$ respectively.

In particular, assume that all the group cocycles f are normalized by the condition

(3.16) $f(g, 1) = f(1, g) = 0$.

Then our map

$$\lambda : H^2(X^{\#}, \mathbf{Z}) = H^2(\Pi, \mathbf{Z}) \to \oplus_{i,j} H^2(U^{\#}_{ij}, \mathbf{Z}) = \oplus_{i,j} H^2(\langle c_{ij} \rangle, \mathbf{Z})$$

is completely determined once we know, for each cocycle f, the value of $f(c_{ij}, c_{ij})$ modulo $2\mathbf{Z}$. Let us recall the basic formulae for group cohomology (cf. e.g. [Ser]): f is a 2 cocycle iff

(3.17) $f(g_1, g_2 g_3) - f(g_1, g_2) = f(g_1 g_2, g_3) - f(g_2, g_3)$.

Whereas a coboundary is obtained by considering, for each function $\varphi : \Pi \to \mathbf{Z}$, the associated 2-cocycle

(3.18) $\delta\varphi(g_1, g_2) = -\varphi(g_1 g_2) + \varphi(g_1) + \varphi(g_2)$.

By (3.17), setting $g_2 = 1$, we get

(3.19) $f(g_1, 1) = f(1, g_3)$ for each choice of g_1, g_3.

Thus $f(g_1, 1) = f(1, g_3) = f(1, 1)$ and we can make this equal to zero simply adding the coboundary of a constant function. From now on we may assume that all cocycles we consider are normalized (cf. 3.16).

Observe moreover that bilinear functions obviously satisfy the cocycle condition (3.17), and the next lemma shows that at least for Γ these functions are enough.

Lemma 3.20 The cohomology group $H^2(\Gamma, \mathbf{Z}) = H^2(M, \mathbf{Z}) = H^2(M^{\#}, \mathbf{Z})$ can be represented by bilinear functions $f : \Gamma \times \Gamma \to \mathbf{Z}$.

Proof The cohomology group $H^2(\Gamma, \mathbf{Z}) = H^2(M, \mathbf{Z})$ is represented by bilinear functions since in general the cohomology group $H^1(\Gamma, \mathbf{Z}) = \mathrm{Hom}(\Gamma, \mathbf{Z})$, and in our particular case we claim that the cup product map: $H^1(\Gamma, \mathbf{Z}) \times H^1(\Gamma, \mathbf{Z}) \to H^2(\Gamma, \mathbf{Z})$ is surjective.

Let us verify first that if $\varphi, \psi \in \mathrm{Hom}(\Gamma, \mathbf{Z})$ their cup product is induced by the cocycle f such that

(3.21) $f(g_1, g_2) = \varphi(g_1)\psi(g_2)$.

In fact, we know that the cohomology groups $H^i(\Gamma, \mathbf{Z})$ are given by the cohomology of the complex of invariant cochains on a simplicial complex whose simplices are the subsets of Γ.

That is, one considers (cf. [God]) the functions

$$F : \Gamma^{n+1} \to \mathbf{Z}, F(x_0, x_1, \dots, x_n)$$

such that for every $g \in \Gamma$

$$F(gx_0, gx_1, \dots, gx_n) = F(x_0, x_1, \dots, x_n)$$

and the classical coboundary

$$\delta F(x_0, x_1, \hat{x}_i, \dots, x_{n+1}) = \sum_{i=0,\dots,n+1} (-1)^i F(x_0, x_1, \dots, x_{n+1}).$$

In this context cup product takes the standard form

$$F \cdot G(x_0, \ldots, x_p, x_{p+1}, \ldots, x_{p+q}) = F(x_0, \ldots, x_p)G(x_p, \ldots, x_{p+q}).$$

As is well known, the correspondence between the former and the latter cocycles is given by

(3.22) $f(g_1 \ldots, g_n) = F(1, g_1, g_1g_2, \ldots, g_1 \cdot \ldots \cdot g_n)$.

Therefore, to $\varphi, \psi \in \mathrm{Hom}(\Gamma, \mathbf{Z})$ correspond cocycles F, G with $F(1, x_1) = \varphi(x_1)$, $G(1, x_1) = \psi(x_1)$, and the corresponding 2-cocycle f satisfies the formula

$$\begin{aligned} f(g_1, g_2) = F \cdot G(1, g_1, g_1g_2) &= F(1, g_1)G(g_1, g_1g_2) \\ &= F(1, g_1)G(1, g_2) = \varphi(g_1)\psi(g_2), \end{aligned}$$

as required.

The surjectivity of the cup product in our particular case follows then directly from the Künneth formula and the fact that the cohomology algebra of a compact complex curve (as well as the one of a torus) is generated in degree 1.

QED

Lemma 3.23 Fixed c, one of the order 2 elements $c_{ij} = (\gamma_i\delta_j)$ in Π, the cohomology group $H^2(\mathrm{X}^\#, \mathbf{Z}) = H^2(\Pi, \mathbf{Z})$ can be represented by normalized cocycles f such that

i) f is bilinear on $\Gamma \times \Gamma$

ii) $f(c, \gamma) = 0$ for each $\gamma \in \Gamma$.

Proof Recall that our group Π is the disjoint union of Γ and of the coset $c\Gamma$. By proposition 3.20 we can achieve i) by adding to f the coboundary of a function φ vanishing on $c\Gamma$.

Recall that f remains normalized provided $\varphi(1) = 0$.

Next, we take a function φ vanishing on Γ and such that $\varphi(c\gamma) = f(c, \gamma)$. The new cocycle $f + \delta\varphi$ vanishes on the pair (c, γ) since

$$(f + \delta\varphi)(c, \gamma) = f(c, \gamma) - \varphi(c\gamma) + \varphi(c) + \varphi(\gamma),$$

and by our choice $\varphi(\gamma) = \varphi(c) = 0$ (since we have set $\varphi(c) = f(c, 1) = 0$).

QED

What we are now going to show sharpens the result of lemma 3.12 on the graded module associated to a suitable filtration of $H^2(\mathrm{X}^\#, \mathbf{Z})$, showing that the cocycles yielding the zero bilinear form on $\Gamma \times \Gamma$ and vanishing on (c, c) correspond to the elements in $\mathrm{Hom}(\Pi, \mathbf{Z})$. More precisely:

Lemma 3.24 Assume f is as in 3.23, i.e., represented by normalized cocycles f such that

i) f is bilinear on $\Gamma \times \Gamma$

ii) $f(c, \gamma) = 0$ for each $\gamma \in \Gamma$. Then, γ_1, γ_2 denoting elements of Γ,

iii) $f(\gamma_1\gamma_2, c) = f(\gamma_1, c) + f(\gamma_2, c) - f(\gamma_1, \gamma_2) + f(c\gamma_1 c, c\gamma_2 c)$

iv) $f(\gamma_1 c, \gamma_2 c) = -f(\gamma_1, c) + f(\gamma_1, c\gamma_2 c) + f(c, c)$.

Proof We are just going now to use the equality $c \cdot c = 1$ and the cocycle condition (3.17) which yields:

$$f(\gamma_1\gamma_2, c) = f(\gamma_2, c) - f(\gamma_1, \gamma_2) + f(\gamma_1, \gamma_2 c).$$

But

$$f(\gamma_1, \gamma_2 c) = f(\gamma_1, c \cdot (c\gamma_2 c)) = f(\gamma_1, c) + f(\gamma_1 c, (c\gamma_2 c))$$

[since in fact $f(c, (c\gamma_2 c)) = 0$].

Finally,

$$f(\gamma_1 c, (c\gamma_2 c)) = f(c \cdot (c\gamma_1 c), (c\gamma_2 c)) = f((c\gamma_1 c), (c\gamma_2 c)).$$

Thus iii) is proven. While we have:

$$f(\gamma_1 c, \gamma_2 c) = f((\gamma_1) \cdot c, \gamma_2 c) = f((\gamma_1), c\gamma_2 c) - f(\gamma_1, c) + f(c, \gamma_2 c);$$

but

$$f(c, \gamma_2 c) = f(c, c \cdot (c\gamma_2 c)) = f(c, c) - f(1, (c\gamma_2 c)) - f(c, (c\gamma_2 c)) = f(c, c).$$

QED

We obtain further simplifications when we apply our formulae in order to calculate $f(c', c')$ for the other order 2 elements $c_{ij} = (\gamma_i\delta_j)$ in Π.

From now on we set

(3.25) $c = c_{11} = (\gamma_1\delta_1)$, $\tau_{ij} = (\gamma_i\gamma_1)(\delta_j\delta_1)$ in Γ, so that $c_{ij} = (\gamma_i\delta_j) = \tau_{ij} \cdot c$ in Π, $(\gamma_1\gamma_i) = a_i$, $(\delta_1\delta_j) = b_j$. Since $\tau_{ij} = a_i^{-1}b_j^{-1}$, by (1.10) we obtain

(3.26) $c\tau_{ij}c = \tau_{ij}^{-1}$.

Thus

$$f(\tau_{ij}c, \tau_{ij}c) = f(c, c) - f(\tau_{ij}, c) + f(\tau_{ij}, -\tau_{ij}),$$

and

$$f(\tau_{ij}, c) = f(a_i^{-1}b_j^{-1}, c) = f(a_i^{-1}, c) + f(b_j^{-1}, c)$$

(the other two terms cancel each other by iv) and the bilinearity of f). Finally,

(3.27) $f(\tau_{ij}c, \tau_{ij}c) = f(c,c) + f(\tau_{ij}, -\tau_{ij}) + f(a_i^{-1}, c) + f(b_j^{-1}, c)$.

Since we are interested only in the value modulo 2, we can rewrite

(3.28)

$$f(c_{ij}, c_{ij}) \equiv f(\tau_{ij}c, \tau_{ij}c) \equiv f(c,c) + f(a_i b_j, a_i b_j) + f(a_i, c) + f(b_j, c) \pmod{2\mathbf{Z}}.$$

We can now explicitly describe K'' and λ in the case of the generalized Kummer surfaces.

Theorem 3.29 Let $K = \oplus_{i,j} H^2(U_{ij}^{\#}, \mathbf{Z})$, and let $K'' \subset K$ be the image of

$$\lambda : H^2(X^{\#}, \mathbf{Z}) \to \oplus_{i,j} H^2(U_{ij}^{\#}, \mathbf{Z}).$$

Identify, through the above direct sum decomposition, K with its dual vector space. Then [as seen in 3.14 i)] K'' is the orthogonal subspace of its subspace $K' = \lambda$ (Torsion subgroup of $H^2(X^{\#}, \mathbf{Z})$) described in 3.14, and can be explicitly described as follows.

Let

$$\sigma : K \to H = (\mathbf{Z}/2)^{2n+2m}$$

be the linear map such that $\sigma(k_{ij}) = (p_i, q_j)$ with

$$p_i = \sum_{j=1,\ldots 2m} k_{ij}, \quad q_j = \sum_{i=1,\ldots,2n} k_{ij}.$$

Then the image of σ equals the hyperplane

$$V = \{(p_i, q_j) | \sum_j q_j = \sum_i p_i\},$$

whereas

$$K'' = \sigma^{-1}\{(p_i, q_j) | \text{ for each pair } j, j', \text{resp.} i, i' \quad q_j = q'_j, \quad p_i = p'_i\}.$$

Proof The map

$$\lambda : H^2(X^{\#}, \mathbf{Z}) = H^2(\Pi, \mathbf{Z}) \to \oplus_{i,j} H^2(U_{ij}^{\#}, \mathbf{Z}) = \oplus_{i,j} H^2(\langle c_{ij} \rangle, \mathbf{Z})$$

is induced by the one associating to a cocycle f (normalized as in lemma 3.24) the vector $k = (k_{i,j})$ with $k_{i,j} = f(c_{ij}, c_{ij}) \pmod{2\mathbf{Z}}$, and we will explicitly exhibit a generating set of cocycles and their images under λ (they generate then by 3.14 the orthogonal of K').

Observe that, in view e.g. of the relation

$$(\gamma_1\gamma_2) \cdot (\gamma_1\gamma_3)^{-1} \cdot (\gamma_1\gamma_4) \cdot (\gamma_1\gamma_5)^{-1} \cdot \ldots \cdot (\gamma_1\gamma_{2n-1})^{-1} \cdot (\gamma_1\gamma_{2n}) = 1,$$

in the abelianization Γ^{ab} of Γ, we have that

(3.30) $a_2 - a_3 + a_4 + \ldots a_{2n} = 0, \quad b_2 - b_3 + b_4 + \ldots b_{2m} = 0$

Observe moreover that, in view of iii) of lemma 3.24 and the action of c by conjugation, we obtain that

$$\sum_{i=1,\ldots,2n} f(a_i, c) \equiv \sum_{j=1,\ldots,2m} f(b_j, c) \equiv 0 \pmod{2\mathbf{Z}}.$$

We obtain thus that

$$(k_{ij}) \equiv (f(c,c) + f(a_i, c) + f(b_j, c) + f(a_i b_j, a_i b_j)) \pmod{2\mathbf{Z}}$$

is the sum of a vector in K' plus the vector

(3.31) $(k''_{ij}) \equiv (f(a_i b_j, a_i b_j)) \pmod{2\mathbf{Z}}$.

Since K' is contained in K'' and also in its orthogonal, as it is easy to verify, we shall limit our considerations to the vectors (k''_{ij}) in the particular case where f on $\Gamma \times \Gamma$ is the product of two linear forms (we can take them only mod $2\mathbf{Z}$).

We will choose a basis of those linear forms as follows:

(3.32) $\alpha_h(a_i) = \delta_{ih} + \delta_{i,2n} (h = 2, \ldots, 2n-1)$,
$\beta_k(b_j) = \delta_{kj} + \delta_{2m,j} (k = 2, \ldots, 2m-1)$.

We have 3 possible choices for f on $\Gamma \times \Gamma$:

1) $\alpha_h \otimes \alpha_{h'}$
2) $\beta_k \otimes \beta_{k'}$
3) $\alpha_h \otimes \beta_k$ (since $\beta_k \otimes \alpha_h$ yields the same vector)

and correspondingly, by (3.31), we get a vector (k''_{ij}) equal to:

1) $(\alpha_h(a_i)\alpha_{h'}(a_i)) = (\delta_{ih} + \delta_{i,2n})(\delta_{ih'} + \delta_{i,2n})$
2) $(\beta_k(b_j)\beta_{k'}(b_j)) = (\delta_{kj} + \delta_{2m,j})(\delta_{k'j} + \delta_{2m,j})$
3) $(\alpha_h(a_i)\beta_k(b_j)) = (\delta_{ih} + \delta_{i,2n})(\delta_{kj} + \delta_{2m,j})$.

The vectors of type 1) can be written more simply as

1') $(\delta_{i,2n})$ (for $h \neq h'$), $(\delta_{ih} + \delta_{i,2n})$ (for $h' = h \leq 2n-1$),
and their span equals the one of the vectors

1'') (δ_{ih}) (for $h \leq 2n$).

Similarly those of type 2) have the same span as the vectors

2'') (δ_{kj}) (for $k \leq 2m$).

The corresponding equations for K' are:

1''') $\sum_{j=1,\ldots,2m} y_{hj} = 0$
2''') $\sum_{j=1,\ldots,2n} y_{ik} = 0$
3''') $y_{hk} = y_{2n,2m} + y_{2n,k} + y_{h,2m}$.

These equations are by the way easily seen (arguing as in 3.14, but replacing $2n$ and $2m$ by 1) to generate the equations

4) $y_{hk} + y_{i,j} + y_{i,k} + y_{h,j} = 0$ which are all the equations defining K'.

Finally, it is easy to verify our assertion concerning $\text{im}(\sigma)$, just by looking at the image of the standard basis vectors of K. Moreover, by looking at the images of the basis vectors of K'', we see that we get nonzero vectors only in case 1'') and 2''). In this case we get the two vectors ($p_i = 0, \quad q_j = 1$), resp. ($p_i = 1, q_j = 0$). Thus $\sigma(K'')$ has dimension 2, and we see that $\sigma(K'')$ equals

$$\{(p_i, q_j) | \text{for each pair } j, j', \text{resp. } i, i' \quad q_j = q_{j'}, \quad p_i = p_{i'}\}.$$

Since the codimension of K'' equals the codimension of $\sigma(K'')$ in $V = \mathrm{im}(\sigma)$, we obtain $K'' = \sigma^{-1}(\sigma(K''))$ as desired.

QED

Remark 3.33 Consider the vector k of K given by $(k_{ij}) = (\delta_{ij})$. Assuming as usual $n \leq m$, we get that $p_i = 1$ for each $i = 1, \ldots, 2n$, while $q_j = 1$ for $j \leq 2n$, $q_j = 0$ for $j \geq 2n + 1$. Whence our vector lies in K'' if and only if $n = m$.

We try now to summarize the results obtained so far in this section pertaining to the description of the lattice $H^2(S, \mathbf{Z})$. We define

(3.34)

$$\Lambda = \oplus_{i=1,\ldots,2n\,;\,j=1,\ldots,2m} \mathbf{Z}(E_{ij}), \qquad K = \oplus_{i,j} (\mathbf{Z}/2)(E_{ij}),$$

Λ'' the inverse image of the subspace K'', Γ' the one of K'.

We can thus replace the diagram (3.8) by the following diagram with exact rows and columns

(3.35)

$$\begin{array}{ccc} 0 & \to 0 & \to K' \\ \downarrow & \downarrow & \downarrow \\ 0 \to H^2(S,\mathbf{Z}) & \to \Lambda'' \oplus H^2(X^{\#},\mathbf{Z}) & \to K'' \to 0 \\ \downarrow & \downarrow & \downarrow \\ 0 \to H^2(N,\mathbf{Z}) & \to \Lambda \oplus H^2(M,\mathbf{Z}) & \to 0 \\ \downarrow & \downarrow & \downarrow \\ K/K' & \to K/K'' & \to 0 \\ \downarrow & & \\ 0 & & \end{array}$$

We may observe more, namely that since $H^2(X, \mathbf{Z}) = \ker \lambda$, $\lambda : H^2(X^{\#}, \mathbf{Z}) \to K$, and $H^2(S, \mathbf{Z})$ is torsion free, then also $H^2(X, \mathbf{Z})$ is such, whence it embeds into $H^2(M, \mathbf{Z})$. Moreover, we have the exact sequence

(3.36)

$$0 \to (2\Lambda) \oplus H^2(X, \mathbf{Z}) \to H^2(S, \mathbf{Z}) \to K'' \to 0,$$

and considering the increasing sequence of modules

$$0 \subset (2\Lambda) \oplus H^2(X, \mathbf{Z}) \subset H^2(S, \mathbf{Z}) \subset \Lambda'' \oplus H^2(M, \mathbf{Z}),$$

if we factor out by the first module $[(2\Lambda) \oplus H^2(X, \mathbf{Z})]$, we obtain

$$0 \to K'' \to K'' \oplus (H^2(M, \mathbf{Z}) / H^2(X, \mathbf{Z})) \to K''/K' \to 0,$$

thus

(3.37') $H^2(M, \mathbf{Z}) / H^2(X, \mathbf{Z}) \cong K''/K'$.

Since $H^2(M, \mathbf{Z})$ has rank $4nm - 4n - 4m + 6 = \dim K''/K'$, the previous formula can be rewritten as

(3.37'') $H^2(X, \mathbf{Z}) = 2H^2(M, \mathbf{Z})$.

Finally, in order to describe the intersection form on $H^2(S, \mathbf{Z})$ we use (3.34), representing each class σ in $H^2(S, \mathbf{Z})$ as a pair

(3.38) $\sigma = (a, f)$ with a in Λ, f in $H^2(X^{\#}, \mathbf{Z})$ such that $\lambda(f) \equiv a \pmod 2$ and the embedding in $H^2(N, \mathbf{Z})$, which multiplies by 2 the intersection product.

Therefore, if we let $\mu =$ image of f in $H^2(M, \mathbf{Z})$, then

(3.39) $\sigma^2 = (1/2)[a^2 + \mu^2]$.

We observe that the correspondence $f \to \mu$ has been described *quite* explicitly in lemma 3.24.

We have analogous exact sequences concerning the cohomology with $\mathbf{Z}/2$ coefficients. In fact, tensoring (3.8) with $\mathbf{Z}/2$ gives rise to an exact sequence

$$0 \to \mathrm{Tor}^1(K', \mathbf{Z}/2) \cong K' \to \mathrm{Tor}^1(K, \mathbf{Z}/2) \cong K \to H^2(S, \mathbf{Z}) \otimes \mathbf{Z}/2 \to$$
$$\to (\Lambda \otimes \mathbf{Z}/2) \oplus (H^2(X^{\#}, \mathbf{Z}) \otimes \mathbf{Z}/2) = K \oplus ((H^2(M, \mathbf{Z}) \otimes \mathbf{Z}/2) \oplus K') \to K \to 0.$$

Whence in particular we get the sequence

(3.40)

$$0 \to Tor^1(K', \mathbf{Z}/2) \cong K' \to Tor^1(K, \mathbf{Z}/2) \cong K \to$$
$$\to H^2(S, \mathbf{Z}/2) \to H^2(X^{\#}, \mathbf{Z}/2) \to 0$$

which will be interpreted in section 5 via Kronheimer's correspondence, whereas from (3.36) we get the sequence

(3.41)

$$0 \to K'' \to \oplus_{i,j} (\mathbf{Z}/2)(A_{ij}) \oplus (H^2(X, \mathbf{Z}) \otimes \mathbf{Z}/2) \to H^2(S, \mathbf{Z}/2) \to K'' \to 0$$

whose left initial part was already interpreted in prop. 3.5.

4 Stiefel Whitney classes of orbifold representations of generalized Kummer surfaces

We shall now use some of the results collected in the previous sections in order to describe the orbifold representations of a generalized Kummer surface in terms of their Stiefel Whitney classes.

As in section 2, let Π be the orbifold fundamental group of a generalized Kummer surface , and let $\rho : \Pi \to SO(3)$ be a representation.

Then ρ induces a flat SO(3)-bundle $P^{\#}$ on $X^{\#}$, which as a topological bundle is determined by its second Stiefel Whitney class $w_2(\rho) = w_2(P^{\#})$.

Proposition 4.1 Assume X is a generalized Kummer surface, and let ρ be a orbifold representation. Then first of all the property that ρ is hooked depends only upon its Stiefel Whitney class $w_2(\rho)$; moreover if ρ is Kleinian, then the conjugacy class of ρ (the isomorphism class of the flat bundle) is completely determined by $w_2(\rho)$.

Proof Notice that, since Π is a semi direct product of the normal subgroup $\Gamma = \Gamma_1 \times \Gamma_2 = \pi_1(C_1) \times \pi_1(C_2)$ by $\mathbf{Z}/2$, and since ρ factors through the embedding of $(\mathbf{Z}/2)^2$ to the Klein group $\mathcal{K}$ inside SO(3), ρ induces a homomorphism of the Abelianization Γ^{ab} of Γ to $\mathcal{K}$. We have thus

$$\Gamma^{ab} \cong \mathbf{Z}^{2n+2m-4} \to \Pi^{ab} \cong (\mathbf{Z}/2)^{2n+2m-3} \to \mathcal{K}.$$

On the other hand we may observe that by 3.13, $H^2(X^{\#}, \mathbf{Z}) = H^2(M, \mathbf{Z}) \oplus K'$, where $K' \cong (\mathbf{Z}/2)^{2n+2m-3}$.

Since moreover (cf. 3.12) $H^3(X^{\#}, \mathbf{Z})$ is free, then (cf. 3.8-3.9)

$$H^2(X^{\#}, \mathbf{Z}/2) \cong (H^2(M, \mathbf{Z}) \otimes \mathbf{Z}/2) \oplus K'.$$

Thus we may decompose

(4.2) $w_2(\rho) = W \oplus w'$, with $W \in H^2(M, \mathbf{Z}/2)$, $\quad w' \in K'$.

W is the second Stiefel Whitney class of the pull back of $P^{\#}$ to $M^{\#}$ (notice that this pull back extends to $M = C_1 \times C_2$), while under the natural map

$$H^2(X^{\#}, \mathbf{Z}/2) \to \oplus_{i,j} H^2(U^{\#}_{i,j}, \mathbf{Z}/2) = \oplus_{i,j} H^2(U^{\#}_{i,j}, \mathbf{Z}) = K$$

w' in $K' \subset K$ is mapped to itself.

The image $w_{\text{loc}}(\rho)$ of $w_2(\rho)$ inside K lies therefore in K'' and is called the local Stiefel Whitney class, and we shall say that a node $p_{i,j}$ is **twisted** if the image of $w_{\text{loc}}(\rho)$ in $H^2(U^{\#}_{i,j}, \mathbf{Z}/2)$ is non zero (i.e., the bundle is non trivial in $U^{\#}_{i,j}$).

As in 2.2 , we set

$$c_{ij} = \rho((\gamma_i \delta_j)), \quad a_{ij} = \rho((\gamma_i \gamma_j)), \quad b_{ij} = \rho((\delta_i \delta_j)),$$

so that $c_{ih} = c_{ij} + b_{jh} = c_{kh} + a_{ik}$. The information carried by $w_{\text{loc}}(\rho)$ is exactly which one of the c_{ij}'s is trivial or not, what proves the first assertion.

Assume now that ρ is hooked: then first of all there exists a c_{ij} which is trivial, and, by the above equations, not all the b_{jh}'s are trivial, and similarly not all the a_{ik}'s are trivial.

There are now two cases:

I) All the non trivial b_{jh}'s and a_{ik}'s are equal, so the image of ρ is a cyclic group of order 2.

II) there exist two elements b_{jh} and a_{ik} which are different.

We are going now to apply lemma 2.1, recalling that $W \in H^2(M, \mathbf{Z}/2) = \Lambda^2(\text{Hom}(H_1(M, \mathbf{Z}), \mathbf{Z}/2)$, and that the bilinear form W gives, for each x, y in $H_1(M, \mathbf{Z}) = \Gamma^{ab}$, the obstruction $W(x, y)$ to lifting to SU(2) the homomorphism of $\mathbf{Z}^2 \to SO(3)$ mapping the two standard generators to $\rho(x)$, resp. $\rho(y)$.

Therefore, in case I $W = 0$, and since c_{kh} is either trivial or is an element of order 2 in $\mathcal{K}$ (these are all conjugates), ρ is completely determined by w'. In case II, by 2.1 ii) the restriction of ρ to Γ^{ab} does not lift (in particular $W \neq 0$) , and by 2.1. iii) the image of any x in Γ^{ab} is completely determined by $W(x, (\delta_j \delta_h))$ and $W(x, (\gamma_i \gamma_k))$ (e.g. if both are non zero then $\rho(x) = b_{jh} + a_{ik}$).

Since $w_{\text{loc}}(\rho)$ forces a fixed c_{ij} to be trivial, and the restriction to $\Gamma_a b$ is completely determined up to conjugation, the same occurs for ρ, as we had to show.

QED

We are now particularly interested in the representations ρ such that their virtual dimension is 0: recall (cf.[Kron])

Remark 4.3 The **virtual dimension** $\nu(\rho)$ of a representation ρ is equal to $\delta^- - 6\chi$, where δ^- is the number of twisted nodes. In the case of a generalized Kummer surface, $4nm = \delta \geq \delta^-$, while $\chi = 1 + (n-1)(m-1)$. Recall moreover that the number δ^- equals the weight of the vector $w_{\text{loc}}(\rho) \in K''$, which gives the local Stiefel Whitney class of ρ.

It is easy arithmetic to see that $\nu(\rho) \geq 0$ forces

(4.4) $(n-3)(m-3) \leq 3$, which has solutions, for $n, m \geq 2$,

i) $n = 2$, $m \geq 2$ arbitrary (elliptic surfaces)

ii) $n = 3$, $m \geq 3$ arbitrary (surfaces on the Noether line)

iii) $n = 4$, $m = 4, 5, 6$ (surfaces over the Noether line)

Letting $\delta^+ = \delta - \delta^- = 4nm - \delta^-$, we obtain that

(4.5) the virtual dimension $\nu(\rho)$ is equal to 0 iff δ^+ equals $2[(3-n)(m-3)+3]$ which yields in case i): $2m$, case ii): 6, case iii): 4, 2, 0 resp.

We want now to study the representations which have virtual dimension $\nu(\rho) = 0$.

There are several possibilities:

A) ρ is non hooked

B) ρ is hooked and with image $\mathbb{Z}/2$
C) ρ is hooked and its image is the full Klein group $\mathcal{K}$ in SO(3) ($\mathcal{K} \cong (\mathbb{Z}/2)^2$, with elements $0, f_1, f_2, f_3$).
D) ρ is monohooked nonabelian

Case A): Since then δ^+ is either divisible by $2n$ or by $2m$, case iii) cannot occur, whereas in cases i), ii) we have to consider only the case where ρ is induced by a representation $\rho_2 : \Gamma_2^{\#}(2) \to SO(3)$ mapping only one δ_j, say δ_1, to the identity (in case $n = m = 2$, we need to switch the roles of n, m). In case $n = m = 2$, we have three elements $c_j = \rho_2(\delta_j)$, of order 2 and with product equal to the identity, whence it follows easily (cf. 2.1.) that they are precisely, up to conjugation, the three non zero elements of the Klein group $\mathcal{K}$.

Case B): Here, cf. 3.13, 3.14, ρ can be viewed as a linear form on the $\mathbb{Z}/2$ vector space $H_1(X^{\#}, \mathbb{Z})$, so ρ is just an element k' of the code $K' \subset K$ and then δ^- is exactly the weight of k'. Then, cf. the notation of 3.14, $\delta^+ = 4[h(n-k) + k(m-h)]$, whence case ii) is manifestly impossible, as well as iii) with $m > 4$; if $n = m = 4$, we should have $1 = h(4-k) + k(4-h)$, which is also clearly impossible.

In case i), we should have $m = 2(h(m-k) + k(2-h))$: for h=1, this implies $m = 2m$, absurd, for $h = 0$ then ρ is not hooked. Thus
(4.6) in case B), where ρ is hooked and with image $\mathbb{Z}/2$, there is no ρ of virtual dimension 0.

Case C): Here to ρ we attach 8 numbers, since our group $H_1(X^{\#}, \mathbb{Z})$ is generated by $\alpha_1, \ldots, \alpha_{2m}$, $\beta_1, \ldots, \beta_{2n}$, with $\beta_1 + \ldots + \beta_{2n} = 0$, $\alpha_1 + \ldots + \alpha_{2m} = 0$, $\alpha_1 = \beta_1$, obtained by considering the number A_h of α_j's mapping to f_h (we have set for convenience $f_0 = 0$), and similarly the number B_h of β_i's mapping to f_h.

We clearly have $A_0 + \ldots + A_3 = 2m$, $\quad B_0 + \ldots + B_3 = 2n$.

Remark 4.7 In classifying the representations ρ of virtual dimension 0, we may do so up to conjugation in SO(3) (this induces any permutation of the non zero elements of the Klein group $\mathcal{K}$) and up to diffeomorphisms of $X^{\#}$. Among these last ones, we have those induced by the action of the direct product $\mathcal{B}_{2n} \times \mathcal{B}_{2m}$ of the braid groups of motions of the branch points (roots of $F(x_0, x_1)$, respectively $G(y_0, y_1)$), (as we already pointed out in 1.13, they clearly allow any permutation whatsoever of the indices i, resp. j).

In view of the above remark, if ρ is hooked, we easily see that we may assume that $\alpha_1 = \beta_1$ maps to 0 in $\mathcal{K}$.

In this case the number δ^+ equals $A_0B_0 + \ldots + A_3B_3$, whence we must simply solve $A_0B_0 + \ldots + A_3B_3 = 2m$, resp. 6, resp. 4,2,0 according to the respective cases i), ii), iii), and assuming $A_0, B_0 \geq 1$, and moreover, again by the hook property, that $A_0 < 2m, B_0 < 2n$.

We observe that e.g. the property $\alpha_1 + \ldots + \alpha_2 m = 0$ implies that if A_0 is odd, then necessarily each A_h is ≥ 1 (similarly for the B_i's).

Clearly then in any case $A_0B_0 + \ldots + A_3B_3 \geq 4$, but if A_0 is odd, then $A_0B_0 + \ldots + A_3B_3 \geq B_0 + \ldots + B_3 = 2n$, (and if equality holds $A_0 = 1$) similarly if B_0 is odd, then $A_0B_0 + \ldots + A_3B_3 \geq A_0 + \ldots + A_3 = 2m$ (and if equality holds $B_0 = 1$).

In case iii) the only possibility to consider is $A_0B_0 + \ldots + A_3B_3 = 4$: then $A_0 = B_0 = 2$, and we may assume $A_2 = A_3 = 0$, $\quad B_1 = B_2 = 0$, which is the only solution.

In case i) there are two possibilities:
i') $B_0 = \ldots = B_3 = 1$,
i'') $B_0 = B_1 = 2$, $\quad B_2 = B_3 = 0$, $\quad A_0 + A_1 = A_2 + A_3 = m$.

Finally, in case ii), by the above, if $m \geq 4$, then B_0 is even, thus either
ii') $B_0 = 4$, $\quad B_1 = 2$, $\quad B_2 = B_3 = 0$, and $A_0 = A_1 = 1$
ii'') $B_0 = B_1 = B_2 = 2$, $\quad B_3 = 0$, and $A_0 = A_1 = A_2 = 1$.

If instead B_0 is odd and $m = 3$, either we are in a symmetrical case of ii'), ii''), or then $B_0 = 1, A_0 = 1$, but this absurd since then each $A_h, B_h \geq 1$, and $A_0B_0 + \ldots + A_3B_3 \geq 8$.

(4.8) Table for Case C):
i') $B_0 = \ldots = B_3 = 1$.
i'') $B_0 = B_1 = 2$, $\quad B_2 = B_3 = 0$, $\quad A_0 + A_1 = A_2 + A_3 = m$.
ii') $B_0 = 4$, $\quad B_1 = 2$, $\quad B_2 = B_3 = 0$, and $A_0 = A_1 = 1$.
ii'') $B_0 = B_1 = B_2 = 2$, $\quad B_3 = 0$, and$A_0 = A_1 = A_2 = 1, A_3 = 2m - 3$.
iii) $B_0 = 2, B_1 = B_2 = 0$, $\quad B_3 = 6$, $\quad A_0 = 2$, $\quad A_1 = 6$, $\quad A_2 = A_3 = 0$.

Observe that in the case $n = 3$ of Noether Horikawa surfaces there are only two types of representations, which are obviously non equivalent, since if N_j denotes the number of nodal elements $\alpha_j + \alpha_1 + \beta_i$ mapped to f_j, we have
in case ii'): $N_0 = 6$, $\quad N_1 = 6, N_2 = 4m - 2$, $\quad N_3 = 8m - 10$,
in case ii''): $N_0 = 6$, $\quad N_1 = N_2 = N_3 = 4m - 2$,
and the respective sets N_1, N_2, N_3 are different.

We turn now to the case of monohooked nonabelian representations.

Case D): In the notation of proposition 2.8, the number δ^+ of untwisted nodes is equal to a number either of the form
D1) $2\{(n-k)r + (p-r)k\}, n - 1 \geq k \geq 1$ or of the form
D2) $2\{(m-k)r + (p-r)k\}$.
Since δ^+ is at least $2n$ in case D1), and at least $2m$ in case D2), we see immediately that case iii) cannot occur.

In case ii) only case D1) is possible, with $r = 1$, $p = 2$.

In case i) D1) is possible (here $k = n - k = 1$) with $2p = m$, whereas D2) is possible with $r = 1, p = 2$.

Corollary 4.9 If $\rho : \Pi \to \mathcal{K}$ is a Kleinian representation of virtual dimension 0, with $n = 3$, case ii''), then the set of its untwisted nodes cannot coincide with the set of untwisted nodes of a monohooked nonabelian representation.

Proof In the former case the set of untwisted nodes is contained in three different rows, in the latter in only two rows.

QED

Remark 4.10 If $\rho : \Pi \to \mathcal{K}$ is a Kleinian representation of virtual dimension 0, with $n = 3$, case i''), then ρ can be deformed to a monohooked nonabelian representation. Therefore also in this case the moduli space of representations is positive dimensional.

We show, by the way, that in case C) the Kleinian (hooked) representations do not lift to U(2).

Proposition 4.11 Let $\rho : \Pi \to \mathcal{K}$ be a Kleinian surjective orbifold representation of a generalized Kummer surface. Then ρ does not lift to $\sigma : \Pi \to U(2)$.

Proof Let us use the notation of 2.2, 2.3, and , assuming that σ exists, denote by $d_{ts} = \sigma((\gamma_t\delta_s))$, which must be an element of order 2.

By the hook property, $d_{ij} = \pm 1$, thus again as in 2.3 $\alpha_{it} = \sigma((\gamma_i\gamma_t))$ is an element of order 2, as well as $\beta_{sj} = \sigma((\delta_s\delta_j))$, and moreover $d'' = d_{ik} = \beta_{jk}$ commutes with every α_{it}, while $d' = d_{hj} = \alpha_{ih}$ commutes with every β_{sj}.

In particular if $d' = d''$ the image of σ is commutative, and also if they are distinct, they can be simultaneously diagonalized and generate a conjugate of the diagonal group $\cong (\mathbf{Z}/2)^2$ of diagonal elements with ± 1 on the diagonal.

Every element commuting with either of them (since they have a non trivial image in PU(2), whence distinct eigenvalues) is also then diagonal, whence we conclude in each case that the image is a conjugate of the above $(\mathbf{Z}/2)^2$, which however has only $\mathbf{Z}/2$ as its image in SO(3) = PU(2).

QED

In the end of this section we restrict ourselves to the case n=3 of Noether–Horikawa surfaces, and to the case where $\delta^- = 6$.

We want to classify first the possible vectors in K'' with $\delta^- = 6$, or equivalently, since K'' contains the vector $\mathbf{1}$ such that $\mathbf{1}_{i,j} = 1$ for each i, j, the ones with $\delta^+ = 6$, i.e. only 6 coordinates equal to 1.

Proposition 4.12 Assume $n = 3$, and let k be a vector of weight 6 (i.e., with exactly 6 coordinates equal to 1). Then, if $m \geq 4$, $k + \mathbf{1} = w_{\text{loc}}(\rho)$ for some orbifold SO(3)-representation ρ. If $m = 3$, there is also the vector (δ_{ij}) considered in remark (3.33).

Proof Let us consider the weight of vectors modulo 4. This equals zero for all vectors of K' (cf. the proof of cor. 3.14) , and also for all basis vectors of K'' of type 3), (cf. proof of theorem 3.29). The only contribution (mod 4) comes from the (δ_{ih})'s and (δ_{kj})'s, the formers contributing 2, the latters contributing $2m \pmod 4$.

Therefore the image in H is nonzero. In case $m \geq 4$, it is not possible to have $q_j = 1$ for each $j = 1, \ldots, 2m$, the weight being 6. Thus $p_i = 1$ for each i, and for the row indexed by i there is exactly one coordinate equal to 1. Each column contains an even number, thus the associated partitions of 6 are just 6, 4+2, 2+2+2, which respectively correspond to representations of type A), C) - ii'), C) - ii'').

If $n = m = 3$, either we are in the previous case (possibly by exchanging the two factors) or $p_i = q_j = 1$ for each i, j: thus (up to permutation of the indices) we have the vector δ.

QED

5 Kronheimer's correspondence and marked orbifold representations

Again in this section, although in our main application we only need the case of a surface with nodes, we let X as in (1.2) be a normal compact complex surface X with as singularities only Rational Double Points $p_1, \ldots, p_\delta$, $X^\#$ be the nonsingular locus of X; $\pi : S \to X$ is the minimal resolution of the singularities of X, U_i is a a small neighbourhood of p_i, $\mathcal{A}_i$ is the fundamental cycle over p_i, $U_i^\# = U_i \cap X^\#$. There is a finite subgroup G_i of SL(2,$\mathbb{C}$) such that the germ of singularity of X at p_i is analytically isomorphic to $\mathbb{C}^2/G_i$.

Definition 5.1 (cf. [Kr])

1) An **orbifold SO(3)-bundle** P^* on X given by the following data:

$$(P^\#, \rho_i, \varphi_i)_{i=1,\ldots,\delta},$$

where $P^\#$ is a principal SO(3) = $\mathbf{P}$U(2) bundle on $X^\#$, $\rho_i : G_i \to \mathbf{P}U(2)$ is a representation, and φ_i is an isomorphism between the restriction of $P^\#$ to $U_i^\#$ and the flat $\mathbf{P}$U(2) bundle P_{ρ_i} associated to $\rho_i : G_i \to \mathbf{P}U(2)$,

$$P_{\rho_i} = \mathbb{C}^2 \times \mathbf{P}U(2)/G_i,$$

where the group G_i acts by $g(z,p) = (gz, \rho_i(g)p)$.
2) Two such data $(P'^{\#}, \rho'_i, \varphi'_i)$, $(P^{\#}, \rho_i, \varphi_i)$, are equivalent if there is a bundle isomorphism between $P'^{\#}$ and $P^{\#}$, there are bundle isomorphisms between P'_{ρ_i} and P_{ρ_i} which are compatible with the φ'_i's and the φ_i's.

One can of course view the same situation from two points of view, of $\mathbb{CP}^1$-bundles, respectively principal SO(3)-bundles.

The main point here is that, once we have fixed a canonical embedding of the local fundamental group $G_i \to \pi_1(X^{\#})$, an orbifold representation $\rho : \pi_1(X^{\#}) \to SO(3)$ automatically determines the data $(P^{\#}, \rho_i, \varphi_i)_{i=1,\dots,\delta}$, whence an orbifold bundle.

Kronheimer's correspondence associates (in the nodal case) to an SO(3)-bundle P on S the pair of an orbifold bundle P^* on X and of a so called marking: we attempt to describe some aspects of this correspondence in the case of A_n-singularities, where each G_i is $\cong \mathbb{Z}/(n+1)\mathbb{Z}$, and is the group of diagonal matrices with diagonal entries $\{\zeta, \zeta^{-1}\}$, where ζ is a $(n+1)^{\text{th}}$-root of 1. Then every $\rho' : G_i \to U(2)$ is of the form $\mathbb{P}\tau_{k,h}$, where $\tau_{k,h} : G_i \to U(2)$ is the diagonal representation with characters ζ^k, ζ^h. We need to give the following definition:

Definition 5.2 (cf. [Kr])
1) Given an orbifold SO(3)-bundle P^* on X wich has only A_n- singularities, a **marking**of P^* is the datum, for each non trivial ρ_i, of a choice of a lifting $\rho' i$ of ρ_i to U(2).

Via this lift we construct a coherent sheaf on $U_i \cong \mathbb{C}^2/G_i$ as follows. Let $\pi : \mathbb{C}^2 \to U \cong \mathbb{C}^2/G$ be the quotient projection, let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}^2$, and let

(5.3) $\pi_*(\mathcal{O}) = \oplus_{k=0,\dots,n}\mathcal{L}_k$ be the eigensheaf decomposition $(\mathcal{L}_0 = \mathcal{O}_U)$.
Then, in case this lift is of type $\tau_{k,h}$, we get a coherent sheaf $\mathcal{F}_{k,h}$, on $U_i \cong \mathbb{C}^2/G_i$ as

(5.4) $\mathcal{F}_{k,h} = \mathcal{L}_k \oplus \mathcal{L}_h$.

We associate to $\mathcal{L}_k$ a rank 1 locally free sheaf $\mathcal{E}_k$ on the resolution V of U. $\mathcal{E}_k$ is defined to be the **strict transform** of $\mathcal{L}_k$ in the sense of divisorial sheaves on normal singularities, but we can give a very concrete description.

In fact, the singularity U has a natural embedding in $\mathbb{C}^3$, given as

$$\operatorname{Spec}(\mathbb{C}[z]^G) = \operatorname{Spec}(\mathbb{C}[x,y,w]/(xy - w^{n+1})),$$

where $x = z_1^{n+1}, y = z_2^{n+1}, w = z_1 z_2$.

Let $\mathcal{O}'$ be the sheaf of holomorphic functions on $\mathbb{C}^3$, then $\mathcal{L}_k$ has a length 1 free resolution, corresponding to the choice of generators z_1^k, z_2^{n+1-k} for $\mathcal{L}_k$ as a $\mathcal{O}'$-module:

(5.5) $0 \to \mathcal{O}'^2 \to \alpha \to \mathcal{O}'^2 \to \mathcal{L}_k \to 0$, where the matrix α is given by

$$\begin{pmatrix} y & -w^k \\ -w^{n+1-k} & x \end{pmatrix}.$$

We obtain the first blow up as follows: choose homogeneous variables x_1, y_1, w_1, t_1, of respective weights $(1,1,1,-1)$ so that the blow up $(\mathbb{C}^3)^{(1)}$ of the origin in $\mathbb{C}^3$ is viewed as $(\mathbb{C}^3)^{(1)} = ((\mathbb{C}^3 - 0) \times \mathbb{C})/\mathbb{C}^*$, with the projection down to $\mathbb{C}^3$ obtained by setting $x = x_1 t_1$, $y = y_1 t_1$, $w = w_1 t_1$.

The first blow-up y_1 of X is the hypersurface inside $(\mathbb{C}^3)^{(1)}$ defined by the equation:

(5.6) $x_1\, y_1 = w_1^{n+1} t_1^{n-1}$,
with as the only singular point a singularity of type A_{n-2}, at the point $x_1 = y_1 = t_1 = 0$, $w_1 = 1$.

The matrix α becomes divisible by t_1, and if we set $\alpha_1 = \alpha/t_1$, then α_1 defines a sheaf $\mathcal{L}_k^{(1)}$, which we call the first strict transform of $\mathcal{L}_k$, and is given as a cokernel

(5.7) $0 \to \mathcal{O}_{(1)}(A'_1)^2 \to \alpha_1 \to \mathcal{O}^2_{(1)} \to \mathcal{L}_k^{(1)} \to 0$,
$\mathcal{O}^{(1)}$ being the sheaf of holomorphic functions on $(\mathbb{C}^3)^{(1)}$, and where A'_1 is the exceptional divisor defined by $t_1 = 0$.

We observe that α_1 is given by the matrix

$$\begin{pmatrix} y_1 & -t_1^{k-1} w_1^k \\ -t_1^{n-k} w_1^{n-k+1} & x_1 \end{pmatrix}$$

therefore at the new singular point we have a sheaf $\mathcal{L}_{k-1}$.

Remark that the blow-up procedure is the same for $\mathcal{L}_k$ and for $\mathcal{L}_{n+1-k}$, whence we temporarily assume $2k \le n+1$.

Then after k steps we obtain a locally free sheaf of rank 1,

(5.7) $0 \to \mathcal{O}_{(k)}(A'_1 + A'_2 + \ldots + A'_k)^2 \to \alpha_k \to \mathcal{O}^2_{(k)} \to \mathcal{L}_k^{(k)} \to 0$,
where A'_1 is the total transform of the exceptional divisor defined by $t_1 = 0$, A'_2 is the total transform of the divisor $t_2 = 0$, and so on.

An easy Chern class calculation then shows that $c(\mathcal{L}_k^{(k)})$ on $(\mathbb{C}^3)^{(k)}$ equals $1 - 2(A'_1 + A'_2 + \ldots + A'_k) + 3(A'_1 + A'_2 + \ldots + A'_k)^2$, while the class of $Y(k)$ on $(\mathbb{C}^3)^{(k)}$ equals $-2(A'_1 + A'_2 + \ldots + A'_k)$.
We let $\mathcal{E}_k$ be the pull back of $\mathcal{L}_k^{(k)}$ to the resolution V of U.

Finally, the construction of P on the resolution S of X is achieved by glueing each $\mathbb{P}(\mathcal{E}_k \oplus \mathcal{E}_h)$, which restricts canonically to $\mathbb{P}(\mathcal{L}_k \oplus \mathcal{L}_h)$ on $U^{\#}$, with the restriction of $P^{\#}$ to $U_i^{\#}$ via the chosen isomorphism φ_i. We will show now that for the bundles $P_{k,h} = \mathbb{P}(\mathcal{E}_k \oplus \mathcal{E}_h)$, $P_{k',h'}$ and $P_{k,h}$ are topologically equivalent if and only if the unordered pairs k, h, k', h' are the same.

Indeed this follows from the following known result (cf. [MK]).

Lemma 5.8 Let U be an A_n-singularity, and let V be its minimal resolution of singularities (obtained by a sequence of $h = [(n+1)/2]$ point blow ups). Then the invertible sheaves $\mathcal{E}_k$ form a basis of $H_2(V, \mathbf{Z})$ which is dual to the basis for $H^2(V, \mathbf{Z})$ given by the irreducible components of the exceptional divisor.

Proof Let us recall the notational set up:

$$x_1 = x_2 t_2, \quad y_1 = y_2 t_2, t_1 = w_2 t_2, \ldots$$

$$x_{h-1} = x_h t_h, \quad y_{h-1} = y_h t_h, \quad t_{h-1} = w_h t_h.$$

After the h^{th} blow up, we obtain V which lives in a manifold $(\mathbf{C}^3)^{(h)}$, containing the irreducible exceptional divisors

$$D_2 = \{w_2 = 0\}, \quad D_3 = \{w_3 = 0\}, \ldots, D_h = \{w_h = 0\},$$

which are all isomorphic to the non minimal ruled surface Σ_1, and $A'_h = \{t_h = 0\}$, which is isomorphic to the projective plane.

But, on V, D_2 cuts the two lines

$$D'_2 = \{w_2 = x_1 = 0\}, \quad D''_2 = \{w_2 = y_1 = 0\},$$

and similarly for $D_3, \ldots$.

In order to prove our statement it suffices to calculate the degree of the restriction of $\mathcal{E}_k$ to each of these lines.

In order to do this, using right exactness of the tensor product we represent this restriction as the cokernel of $\alpha_k = (t_1 t_2 \ldots t_k)^{-1} \alpha$.

We can rewrite α_k as

$$\begin{pmatrix} y_k & -w_1^k w_2^{k-1} \ldots w_k \\ -t_k^{n+1-2k} w_1^{n-k+1} \ldots w_k^{n+2-2k} & x_k \end{pmatrix}$$

Therefore on a D_s with $s \geq k+2$, all entries vanish except the one on the upper right corner which does not vanish, whence $\mathcal{E}_k$ is trivial on these lines. Instead, on a D_s with $s \leq k$, all non diagonal entries vanish; the ones on the diagonal are constants if $s \leq k$, whence $\mathcal{E}_k$ is again trivial there.

Finally, on $D'_{k+1} = \{w_{k+1} = x_k = 0\}$, the lower row vanishes, while the first row gives a basis of $H^0(\mathcal{O}_{\mathbf{P}_1}(1))$, therefore the image of $\mathcal{O}_{\mathbf{P}_1}(-1)^2$ generates the first copy of $\mathcal{O}_{\mathbf{P}_1}$ and the cokernel is just the second copy, whence again $\mathcal{E}_k$ is trivial.

On the other hand, on $D''_{k+1} = w_{k+1} = y_k = 0$, the first column is vanishing, and the second column contains a basis of $H^0(\mathcal{O}_{\mathbf{P}_1}(1))$; whence $\mathcal{E}_k$ is there isomorphic to $\mathcal{O}_{\mathbf{P}^1}(1)$.

The rest of the verifications are quite similar. **QED**

Kronheimer ([Kr]) gives the converse correspondence, in the case where the singularities are just nodes (A_1-singularities).

Given any bundle P on S, let V_i be the inverse image of the neighbourhood U_i of p_i. Then the restriction $P|V_i$ is either trivial or isomorphic to the bundle $P_{0,1}$.

One obtains an orbifold bundle just by letting $P^{\#}$ = the restriction of P to $X^{\#}$, ρ_i = the trivial representation if $P|V_i$ is trivial, or else $\mathbf{P}_{\tau_{0,1}}$.

Moreover φ_i is chosen in such a way (cf. [Kr], lemma 1), when blowing down the exceptional curves one at the time, that each blow down increases the Pontrjagin number exactly by 1/2.

Moreover (cf. [Kr], remark page 232) changing the marking at the point p_i alters the second Stiefel Whitney class of P by adding the class modulo 2 of the exceptional curve A_i (intuitively speaking, switching the line bundle summands in $(\mathcal{O} \oplus \mathcal{E}_i)$ one replaces $\mathbf{P}(\mathcal{O} \oplus \mathcal{E}_i)$ by $\mathbf{P}(\mathcal{O} \oplus \mathcal{E}_i^{-1})$, whence w_2 changes by A_i): this is the interpretation of the exact sequence (3.40), in fact adding a sum of A_i's corresponding to a vector in K' we get a class which is 2-divisible, whence $w_2(P)$ does not change then.

6 Differentiable invariants of Noether--Horikawa surfaces of connected type

As a first step in this section we deal with the moduli spaces of marked orbifold representations of virtual dimension zero.

Theorem 6.1 Let X be a generalized Kummer surface of type $(6, 2m)$. Let S be the Noether–Horikawa surface which is the minimal resolution of singularities of X.

Consider the moduli space $\mathcal{M}''(w)$ of marked orbifold representations of virtual dimension zero yielding, via Kronheimer's correspondence, a fixed topological SO(3)-bundle P on S with $w_2(P) = w$ (in particular then $p_1(P) = -3\chi = -6m + 3$, and $w^2 \equiv -1 + 2m \pmod{4\mathbf{Z}}$).

Then, for different choices of w, with $w^2 \equiv -1 + 2m \pmod 4$ the following are all the occurring possibilities if $m \geq 4$: either

1) $\mathcal{M}''(w)$ is positive dimensional or
2) $\mathcal{M}''(w)$ consists of $2^{2(m-2)}$ points. Moreover,
3) If $m \geq 4$, any w with $w_{\text{loc}} = k''$ having exactly 6 coordinates equal to 0 is the Stiefel Whitney class of some bundle P associated to a marking of an orbifold representation.

This representation is unique in case 2) (when $k'' = w_{loc}(\rho)$ for ρ of type C) , ii")).

Proof Let us recall that we already classified the orbifold representations of virtual dimension zero, dividing them into several cases:
1') Case A), where ρ is non hooked and the moduli space is positive dimensional
1") Case C) ii'), where ρ is Kleinian, by (4.10) deforms to the case D2) of a monohooked representation: again here the moduli space is positive dimensional
2) ρ is Kleinian, of type C) ii"), there is exactly one ρ with given Stiefel Whitney class in $H^2(X^{\#}, \mathbf{Z}/2)$.

Therefore to count the isomorphism classes of marked orbifold representations when ρ is type C) ii") it suffices to count all the isomorphism classes of the pairs (ρ, μ) where μ is a marking yielding the fixed class w in $H^2(S, \mathbf{Z}/2)$. These markings, in view of (3.40) can be put in a 1-1 correspondence with vectors of K', moreover, by the definition of marking (it has support only at the twisted nodes) these markings will correspond bijectively with the vectors of K' such that the coordinate corresponding to an untwisted node must be zero.

Recalling 4.8, in this case we must therefore count the vectors in K' such that

$$k_{11} = k_{21} = 0, \quad k_{32} = k_{42} = 0, \quad k_{53} = k_{63} = 0.$$

i.e. such that

$$k_{11} = k_{21} = 0, \quad k_{31} + k_{12} = 0, \quad k_{31} + k_{41} = 0,$$

(the previous conditions automatically imply then $k_{51} + k_{61} = 0$), $k_{51} + k_{13} = 0$.

We obtain thus a subspace of K' of codimension 5, whence with 2^{2m-2} elements.

Finally, we have to determine when these marked orbifold representations are isomorphic.

Any such will induce an element g in SO(3) conjugating the Kleinian representation ρ. Since then g must preserve the 3 axes of the elements of $\mathcal{K}$, g must belong to $\mathcal{K}$.

Our assertion will be then proved if we show that $\mathcal{K}$ acts freely on our set of markings. To analyze the effect of $\mathcal{K}$ on the possible lifts to U(2) , we need to describe the subgroup of PU(2) generated by the liftings of order 2 of the nonzero elements of $\mathcal{K}$.

Lemma 6.2 Let $\mathcal{K}'$ be the subgroup of PU(2) generated by the liftings of order 2 of the nonzero elements of $\mathcal{K}$. Namely, $\mathcal{K}'$ is generated by

(6.3)

$$\pm x \quad , \pm y, \quad \pm z,$$

with

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Then

$$yxy = -x, \quad yzy = -z, \quad xyx = -x, \quad xzx = -z, \quad zxz = -x, \quad zyz = -y.$$

More conceptually, let $t = xyz$. Then t equals $-i$ (identity), t is an element of order 4 which generates the centre C of $\mathcal{K}'$, $\mathcal{K}'/C = \mathcal{K}$.

$\mathcal{K}'$ is the semidirect product of the normal subgroup $(\mathbf{Z}/4) \times (\mathbf{Z}/2)$, generated by t, x with the subgroup $\mathbf{Z}/2$ generated by y, $(yxy = ttx)$.

Proof Straightforward calculation.

QED

Concerning the first assertion of 3), it follows directly from prop. 4.12 that $k'' = w_{loc}(\rho)$ for some ρ.
Let us take a Kleinian representation ρ of type C), ii"), and let us fix its local Stiefel Whitney class yielding a vector k'' in K'' which we assume to be the vector with all coordinates equal to 1 except

$$k_{11} = k_{21} = k_{32} = k_{42} = k_{53} = k_{63} = 0.$$

Then, if we denote by $c_{ij} = \rho((\gamma_i \delta_j))$, then

$$c_{11} = c_{21} = c_{32} = c_{42} = c_{53} = c_{63} = 0$$

in the Klein group $\mathcal{K}$, and the others c_{ij} are non zero. But then

$$c_{31} = c_{41} = c_{12}, c_{51} = c_{61} = c_{13},$$

and these two elements are non zero in $\mathcal{K}$ and distinct. Up to conjugation we may assume the former to be e_1, the latter to be e_2, and then each c_{1j} with $j \geq 4$ is then equal to e_3, else some coordinate k_{ij} with $j \geq 4$ would be equal to zero, a contradiction.

We have thus shown that in this case there is only one representation with the given local Stiefel Whitney class k''.

Let now ρ be arbitrary, let P be the bundle on S obtained by choosing a marking of ρ, and let $w' = w_2(P)$ be its Stiefel Whitney class; by the exact sequence (3.41) we can obtain any other class $w = w' + a' + u$ in $H^2(S, \mathbf{Z}/2)$ with the same local Stiefel Whitney class k'' by adding to w' the image of a class A' in $\oplus_{i,j}(\mathbf{Z}/2)(A_{ij})$ and of a class U in $H^2(X, \mathbf{Z}) \otimes \mathbf{Z}/2$.

Varying A', we obtain all the classes $w = w' + a'$ corresponding to different markings of ρ.

But by (3.9) since we have

$$0 \to H^2(X, \mathbf{Z}) \to H^2(X^{\#}, \mathbf{Z}) \to \lambda \to K'' \to 0,$$

tensoring with $\mathbf{Z}/2$ gives

$$0 \to \mathrm{Tor}^1(K', \mathbf{Z}/2) \cong K' \to \mathrm{Tor}^1(K'', \mathbf{Z}/2) \cong K'' \to$$

$$\to H^2(X, \mathbf{Z}) \otimes \mathbf{Z}/2 \to H^2(X^{\#}, \mathbf{Z}/2) \to \lambda \to K'' \to 0,$$

therefore by (3.37) the map

$$H^2(X, \mathbf{Z}) \otimes \mathbf{Z}/2 \to H^2(X^{\#}, \mathbf{Z}/2)$$

is zero, whence $u = 0$.

To show 4), cf. prop. 4.12, we consider the vector $k'' = \mathbf{1} + (\delta_{ij})$.

QED

Main Theorem 6.3 Let X be a generalized Kummer surface of type $(6, 2m)$, and let S be the Noether–Horikawa surface which is the minimal resolution of singularities of X.

Let w in $H^2(S, \mathbf{Z}/2)$ be a class with $w^2 \equiv -1 + 2m \pmod 4$ of particular type, i.e. with a $w_{\mathrm{loc}} = (k''_{ij})$ having exactly 6 coordinates k''_{ij} equal to 0, this occurring for precisely 3 values of j.

Then, for a bundle P with $w_2(P) = w$ and $p_1(P) = -3\chi = -6m + 3$, the value of the invariant $q(S, P)$ equals $2^{2(m-2)}$.

Proof By 3) of theorem 6.1 and cor. 4.9 w is the Stiefel Whitney class of some bundle P associated to a marking of a unique orbifold representation of type C), ii'').

By Kronheimer's theorem 2 in [Kr] it suffices to calculate $q(X, P^*)$ for the associated marked orbifold bundle.

Since the corresponding moduli space $\mathcal{M}''(w)$ of marked orbifold representations of virtual dimension zero consists of $2^{2(m-2)}$ points, it suffices to show that the corresponding flat connections D are regular.

We follow here almost verbatim Kronheimer's argument: since the index of the Atiyah–Hitchin–Singer complex

$$\Omega^0_X(\mathrm{Ad}P^*) \to d_D \to \Omega^1_x(\mathrm{Ad}P^*) \to d^+_D \to \Omega^+_X(\mathrm{Ad}P^*)$$

is zero, and $H^0_D = 0$ since the centralizer of ρ is finite, it suffices to show that also $H^1_D = 0$.

Letting D^+ be the pull back of D to M, and since H^1_D is isomorphic to the invariant part of $H^1_{D^+}$, it suffices to show that the last cohomology group is zero.

Since ρ lands to $\mathcal{K}$, our flat SO(3) bundle is the direct sum of 3 real flat line bundles L_τ (corresponding to the 3 possible compositions τ of ρ with a linear form from $\mathcal{K}$ to $\mathbf{Z}/2$).

Since $M = C_1 \times C_2$, by the Künneth formula it suffices to show that for each τ the pull back of L_τ to C_i has $H^0 = 0$.

But in fact the pull back of L_τ to C_i has $H^0 = 0$, else this pull back would be trivial, i.e. the restriction of τ to $\pi_1(C_i)$ would be trivial, whereas, ρ being hooked and since at least 2 of the 3 non zero vectors of $\mathcal{K}$ appear both in the first row and in the first column, this does not occur.

QED

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