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Generic lemniscates of algebraic functions

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0 Introduction

The purpose of this article is to extend the main theorem of [C-P], i.e., the classification of generic lemniscates of polynomials, to the case of algebraic functions.

An algebraic function is a pair $f: C \to \mathbb{P}^1$, where C is a compact connected curve (Riemann surface) of genus g and f is a holomorphic map of degree $d > 0$. Riemann's existence theorem says that the datum of the isomorphism class of f is equivalent to the datum of the branch locus B of f and of the isomorphism class of the induced unramified covering of $\mathbb{P}^1 - B$.

In turn, once a base point x_0 is fixed in $\mathbb{P}^1 - B$, the unramified covering is determined by the conjugacy class of the monodromy homomorphism μ: $\pi_1(\mathbb{P}^1 - B, x_0) \to \mathscr{S}_d$.

In this way, since the number of choices for μ is finite, one sees that the f's as above form a space, called hereafter Hurwitz space, $\mathscr{H}_{g,d} = \mathscr{H}_{g,d}(\mathbb{P}^1)$, which is a finite cover of an open set of the projective space of divisors B' on $\mathbb{P}^1$ of degree $2g + 2d - 2$.[1]

$\mathscr{H}_{g,d}(\mathbb{P}^1)$ contains an open dense set $\mathscr{H}^{\text{gen}}_{g,d}(\mathbb{P}^1) = \mathscr{H}^{\text{gen}}_{g,d}$, which corresponds to the "generic" f's, those for which the branch divisor B' consists of $2g + 2d - 2$ distinct points. $\mathscr{H}^{\text{gen}}_{g,d}$, whence also $\mathscr{H}_{g,d}$, is connected by the classical theorem of Lüroth-Clebsch (cf. [C1], [Hu], [K1]), for which we give here a "geometric" proof.

The importance of $\mathscr{H}^{\text{gen}}_{g,d}$ lies in the fact that $\mathscr{H}_{g,d}$ is noncompact, and for calculations on moduli spaces of curves one can find a compactification

[1] We should warn the reader that we do not take the quotient by the action of PGL(2) on the target, as done in [Fu]. Moreover, in the existing literature (cf. [B-Fr], [Fu]) the name of Hurwitz spaces has been often used for the individual strata of the natural stratification of the Hurwitz space into locally closed subsets in the Zariski topology; finally, the generic algebraic functions are sometimes called "simple", but we will stick to the notation of [C-W], [C-P])

of $\mathscr{H}_{g,d}^{\text{gen}}$ where the target can be replaced by a tree of $\mathbb{P}^1$'s, and the map f remains "generic" except possibly over the singular points (cf. [H-M]).

On the other hand the space $\mathscr{H}_{g,d}^{\text{gen}}$ has a natural cellularization in which the open cells are homeomorphic to $(S^1)^n \times (\mathbb{R}^+)^n$, where $n = 2g + 2d - 2$. The union of the open cells equals the open (dense) set $\mathscr{L}_{g,d}$ formed by the so called "lemniscate generic algebraic functions"; i.e., those f's for which the n distinct branch points have different absolute values, and are not equal to 0 or ∞.

Using this cellularization we are able to calculate the topological Euler-Poincaré characteristic of $\mathscr{H}_{g,d}^{\text{gen}}$ (Theorem 2.21).

The main result we achieve in this paper is a precise combinatorial description of the connected components of the space $\mathscr{L}_{0,d}$, of lemniscate generic rational functions (in the higher genus case our description becomes more complicated).

The idea employed, which works more generally for algebraic functions, runs essentially as in [C-P]: to f we associate the (weak) Morse function $|f|$, and the topological "lemniscate configuration" of f. The above configuration is given by the pair of C and of the union of the singular level sets of $|f|$ (the so called lemniscates), where, though, we distinguish the two sets $f^{-1}(0)$, $f^{-1}(\infty)$.

This configuration is completely described by a certain "biended" graph $\mathscr{g}$, belonging to the class of graphs that we call "admissible".

There is a main difference of the case of algebraic functions with respect to the case of polynomials: here the topological configuration is not sufficient to determine the topological behaviour of the map (whence, a fortiori the connected component of $\mathscr{L}_{g,d}$).

We need further to assign to each of the edges of $\mathscr{g}$ certain integers called "weights" satisfying Kirchoff's rule of electrical circuits: thus we get an "admissible" biended weighted graph $\mathscr{g}$.

Again here, as in [C-P], we find the occurrence of the group $\bigwedge_n$ of "circular" braids (where the n branch points move in circles around the origin), which acts on the set $\mathscr{E}_{n,d}$ of "generic monodromy" edge labelled graphs.

We are ready to state the

Main Theorem (2.17). *Consider the following sets:*
(a) *the set* $\pi_0(\mathscr{L}_{g,d})$ *of connected components of* $\mathscr{L}_{g,d}$,
(b) *the set of* $\bigwedge_{2g+2d-2}$ – *orbits in* $\mathscr{E}_{2g+2d-2,d}$,
(c) *the set of* "*admissible*", *bi-ended*, *edge weighted graphs of type* (g,d).

There is a natural bijection between the sets (a) *and* (b), *and a natural surjection* ν *of* (a) *onto* (c).

In the case $g = 0$ *of rational functions* ν *is bijective.*

Instead, if $g \geqq 1$ *and* $d \geqq 4$, ν *is not bijective.*

For the benefit of the reader, we mingle together two arguments for the proof of the main theorem, a rigorous algebraic one using braids, and a heuristic one using a paste and glue approach to the problem.

As we mentioned above, the bare topological configuration of the lemniscates does not in general determine the topological type of the map, but this only occurs in positive genus.

To be more precise, we prove (Theorem 3.9) that for the case of rational functions the admissible, bi-ended graphs occurring admit at most one weight-labelling.

It is also a striking contrast to the case of polynomials that not all the abstract configurations can occur (this is shown by exhibiting bi-ended graphs not admitting a weight-labelling by positive integers).

Our results do in particular provide a complete description of the dynamical systems on the Riemann sphere associated to O.D.E.'s integrable by rational functions (cf. [Pa]), and more generally of dynamical systems on Riemann surfaces integrable by algebraic functions.

Here are the contents of the different sections of the paper:

In Sect. 1, which is mainly instructional, and can be skipped by the expert reader, we define more generally the Hurwitz space of maps to a fixed Riemann surface M, we recall basic facts about monodromies, give the description of the covering space $\mathscr{H}^{\text{gen}}_{g,d}(M) \to F_n(M)$, where $F_n(M)$ denotes the space of n-tuples of distinct points of M, and prove the theorem of Lüroth-Clebsch.

Section 2 is devoted to the basic definitions about lemniscates, circular braids, and to a description of the properties defining the admissible graphs. We end this section by calculating the Euler-Poincaré characteristic of $\mathscr{H}^{\text{gen}}_{g,d}$.

The main theorem is then proved in Sect. 3, where also the quoted result about unicity of weights in the rational function case is proved.

Finally, the application to O.D.E.'s can be found in Sect. 4, which is devoted to a brief description of the so called small lemniscate configurations (cf. [C-P]).

1 Algebraic functions and Hurwitz spaces

We begin by fixing the notation and giving some basic definitions.

(1.1) Definition. 1) *Let $f : C \to M$ be a (non constant) holomorphic map between compact Riemann surfaces.*

The **branch points** *of f are exactly the points $w \in M$ such that, writing the inverse image divisor $f^{-1}(w) = \sum_{i=1}^{k}(m_i P_i)$, at least one of the multiplicities m_i is $\geqq 2$.*

The **multiplicity of w** *as a branch point is defined as $m_w = \sum_{i=1}^{k}(m_i - 1)$; then the* **branch divisor B_f of f** *is defined by the sum $\sum_w (m_w)w$.*

(2) *An* **algebraic function** *is a (non constant) holomorphic mapping $f : C \to \mathbb{P}^1$ of a (connected) compact Riemann surface C to the complex projective line.*

(1.2) Remark. Topologically, the decomposition of the inverse image divisor $f^{-1}(w) = \sum_{i=1}^{k}(m_i P_i)$ means that the local monodromy of f around w consists of a product of k disjoint cycles of respective lengths m_i.

A branch point w is said to be *simple* iff (assuming $m_1 \geqq \ldots \geqq m_k$) $m_2 = \ldots = m_k = 1$.

If moreover $m_1 = 2$, then w is called *generic*.

(1.3) Definition. *A holomorphic map $f\colon C \to M$ between Riemann surfaces is called* **generic** *iff f has only generic branch points (i.e., if the local monodromies are transpositions).*

Let C, M be compact Riemann surfaces of respective genera g, g' and let $f\colon C \to M$ be a holomorphic map of degree d.

Then Hurwitz's formula states that the degree $\sum_w m_w$ of the branch divisor is equal to $2g - 2 + d(2 - 2g')$.

In particular, a generic algebraic function of degree d and genus g has n distinct branch points, where $n = 2g + 2d - 2$.

(1.4) Definition. *Let M be a compact Riemann surface of genus g' and let g, d be fixed natural numbers.*

Define $f\colon C \to M$ and $f'\colon C' \to M$ to be equivalent ($f \sim f'$) iff there is an isomorphism $\varphi\colon C \to C'$ such that $f = f' \circ \varphi$.

Then the **Hurwitz space** *of maps of degree d and genus g to M, is defined as the quotient*

$\mathscr{H}_{g,d}(M) := \{f\colon C \to M | C$ is a compact Riemann surface of genus g, f is a holomorphic map of degree $d\}/\sim$, where $\sim$ is the above equivalence relation.

By Hurwitz's formula and Riemann's existence theorem (cf. (1.5)), $\mathscr{H}_{g,d}(M)$ is non-empty if and only if $0 \leqq g \geqq d(g' - 1) + 1$.

We recall:

(1.5) Riemann's existence theorem. *Fix a compact Riemann surface M of genus g', a finite set $B \subset M$ and a point $x_0 \in M - B$.*

Then there is a one to one correspondence between

i) *the equivalence classes of holomorphic maps $f\colon C \to M$ of degree d with branch set B_f contained in B (where C is a connected compact Riemann surface), and*

ii) *the conjugacy classes $[\mu]$ of homomorphisms*

$$\mu\colon \pi_1(M - B, x_0) \to \mathscr{S}_d,$$

such that the image of μ is a transitive subgroup (here, two homomorphisms μ and μ' as above are said to be in the same conjugacy class iff there exists an inner automorphism φ of $\mathscr{S}_d$, such that $\mu = \varphi \circ \mu' \circ \varphi^{-1}$).

Hence we get the following:

(1.6) Proposition. *Let M be a compact Riemann surface of genus g' and let d, g be two natural numbers. Then there is a one to one correspondence between the following two sets*:
1) $\mathscr{H}_{g,d}(M)$,
and
2) *the set of pairs (B, φ) of a divisor $B = \sum m_b b$ on M of degree $n = 2g - 2 + d(2 - 2g')$, and an unramified covering φ of $M - B$ whose associated conjugacy class $[\mu]$ of homomorphisms $\mu\colon \pi_1(M - B, x_0) \to \mathscr{S}_d$ satisfies*
2a) *μ has transitive image*,
2b) *if $b \in B, \gamma_b$ is a conjugate of a small circle around b, and $\mu(\gamma_b)$ is the product of k_i disjoint cycles, then $\sum(k_i - 1) = m_b$.*

(1.7) Remark. Let $\mathscr{H}^{\mathrm{gen}}_{g,d}(M)$ be the subset of $\mathscr{H}_{g,d}(M)$ given by the classes of generic holomorphic maps $f\colon C \to M$.

Then $\mathscr{H}^{\mathrm{gen}}_{g,d}(M)$ corresponds to the divisors B of M consisting of n distinct points together with conjugacy classes $[\mu]$ as above, such that $\mu(\gamma_b)$ is a transposition for each $b \in B$.

In this case $B \in (M^n - \Delta)/\mathscr{S}_n =: F_n(M)$, where Δ is the big diagonal, i.e. $M^n - \Delta := \{(x_1, \dots, x_n) \in M^n\colon x_i = x_j \text{ iff } i = j\}$.

We have a natural map

$$\pi\colon \mathscr{H}^{\mathrm{gen}}_{g,d}(M) \to F_n(M)\,, \tag{1.8}$$

by assigning to a generic map $f\colon C \to M$ its branch locus.

Up to now we only defined the Hurwitz space as a set. By the following result we will see that there is a natural topology on $\mathscr{H}^{\mathrm{gen}}_{g,d}(M)$.

(1.9) Proposition. *There exists a topology on $\mathscr{H}^{\mathrm{gen}}_{g,d}(M)$ such that*

$$\pi\colon \mathscr{H}^{\mathrm{gen}}_{g,d}(M) \to F_n(M)$$

is an unramified covering.

Proof. We consider the universal family $\mathscr{U} := (F_n(M) \times M) - \{(B, x)\colon x \in B\}$ over $F_n(M)$. Let $p\colon \mathscr{U} \to F_n(M)$ be the natural projection. Since p is a differential fibre bundle, fixing an element B_0 of $F_n(M)$, there exists a neighbourhood U of B_0 in $F_n(M)$ such that $p^{-1}(U)$ is diffeomorphic to $U \times (M - B_0)$. We denote by $E_{d,g,g'}$ the set of isomorphism classes of unramified coverings of $M - B_0$ of degree d and genus g as in (1.7). Since we may choose U contractible, we see that there are natural bijections between coverings of $p^{-1}(U)$ and coverings of $(M - B)$ for $B \in U$, whence $\pi^{-1}(U) \cong U \times E_{d,g,g'}$ and we are done. $\square$

An immediate consequence of the proposition is the following:

(1.10) Corollary. *$\mathscr{H}^{\mathrm{gen}}_{g,d}(M)$ has a natural structure of a smooth complex manifold.*

More generally, it holds that $\mathscr{H}_{g,d}(M)$ has a natural structure of an algebraic variety. This result is well known, but since we did not find an appropriate reference (or at least an elementary proof), we shall prove it here.

(1.11) Theorem. *$\mathscr{H}_{g,d}(M)$ has a natural structure of a quasi-projective variety.*

Proof. First of all we recall one of the generalizations of Riemann's existence theorem due to Grauert and Remmert (cf. [G-R]):
Let Y be an irreducible, normal projective variety and let $\mathscr{B}$ be an analytic subset of Y. Then there is an equivalence of categories between unramified topological coverings of $Y - \mathscr{B}$ and isomorphism classes of pairs (X, ψ), where X is a normal projective variety and $\psi : X \to Y$ is a finite morphism with branch locus contained in $\mathscr{B}$.

From this it follows easily that, if we have a normal complex space Z and a finite morphism g from Z to a (Zariski-) open set U in Y, then Z has a natural structure of a quasi-projective variety. (In fact, Z determines an unramified covering of $Y - B$ for a suitable B, and if $\psi : X \to Y$ is the extension to Y, one sees easily by normality that $Z \cong \psi^{-1}(U)$).

Since $\mathscr{H}_{g,d}(M)$ admits a finite map π to the space $\mathscr{D} = M^{(n)}$ of effective divisors of degree n ($n = 2g - 2 + d(2 - 2g')$) on M, it suffices to give to $\mathscr{H}_{g,d}(M)$ the structure of a normal complex space "locally on the base".

I.e., for each branch divisor B we take a neighbourhood U in the Hausdorff topology and we endow the inverse image of U with the structure of a normal complex space in such a way that the map to U is a finite holomorphic map. By the same argument as above, using normality, we do not have to bother about compatibility of the different local structures.

Therefore, let $B = \sum m_i w_i$ be an element of $\mathscr{D}$.

We will choose a (connected) neighbourhood U of B in such a way that the connected components of $\pi^{-1}(U)$ correspond bijectively to the maps in the inverse image of the divisor B.

We thus fix f in $\pi^{-1}(\{B\})$ and moreover, we fix:
1) disjoint neighbourhoods D_i of w_i taken together with a biholomorphism α_i of D_i onto the unit disc $D := \{|z| \leqq 1\}$,
2) a base point $x_0 \in M - \cup D_i$,
3) simple paths connecting x_0 to the point P_i, where $P_i = \alpha_i^{-1}(1)$, and we denote then by $\gamma_1, \ldots, \gamma_k$ the corresponding geometric loops turning around $P_1, \ldots, P_k$
4) a representative μ_f for the monodromy of f,
5) the ramification points $y_{i,j}$ of $f : C \to M$,
6) for each $y_{i,j}$ a biholomorphism $\beta_{i,j}$ of the component of $f^{-1}(D_i)$ containing $y_{i,j}$ with the unit disc D such that in these coordinates f is given by $z \mapsto z^{h_{i,j}}$.

Set for convenience of notation $Q_{i,j} := \beta_{i,j}^{-1}(1)$.

We can now construct explicitly the universal deformation of f. For each $y_{i,j}$ we consider the family of polynomial maps $f_a(z) = z^h + a_{h-2} z^{h-2} + \ldots + a_0$, where for simplicity we set $h := h_{i,j}$. We will consider in the sequel only the restriction $f_a | f_a^{-1}(\{|z| \leqq 1\})$, which will still be denoted by f_a.

Let A_{h-1} be a small disc in $\mathbb{C}^{h-1}$ around zero, so that for each element a of A_{h-1} the branch locus B_{f_a} of f_a is contained in $\{|z| < 1/2\}$.

We remark the following:

a) $\{(a,b) | b \in f_a^{-1}(1)\}$ is a covering of A_{h-1}, which is contractible: hence for each $a \in A_{h-1}$ is determined a unique point $P_a \in f_a^{-1}(1)$ (namely by the condition to be in the same connected component of $(0,1)$).

b) $f_a \sim f_{a'}$ if and only if there is a $\zeta \in \mu_h := \{\xi \in \mathbb{C} : \xi^h = 1\}$, such that $a'_i = \zeta^i a_i$ (in fact $f_a \sim f_{a'}$ on $D = \{|z| \leqq 1\}$ iff $f_a \sim f_{a'}$ over $\mathbb{C}$, which is the case if and only if f_a and $f_{a'}$ are equivalent by an affine transformation in the source (cf. [C-W], [Lo])).

c) We see, more precisely, that

c1) the branch divisors of the f_a's for $a \in A_{h-1}$ fill a neighbourhood U_{h-1} of the divisor $(h-1) \cdot 0$ in $D^{(h-1)}$ and

c2) giving f_a is equivalent to giving the pair consisting of a covering of D with branch divisor in U_{h-1} and of a point in the inverse image of 1.

Hence we set the deformation space Def(f) to be equal to

$$(*) \qquad \mathrm{Def}\,(f) = \prod_{y_{i,j}} (A_{h_{i,j}-1}) \, .$$

For each $a^* = (a(i,j))_{i,j} \in \prod_{y_{i,j}} A_{h_{i,j}-1}$ we obtain a corresponding deformation f_{a^*} which is constructed by glueing $f^{-1}(M - \cup(\mathrm{int}\, D_i))$ to $f^{-1}_{a(i,j)}(D_i)$ in such a way that the mapping to M is preserved and $Q_{i,j}$ is identified with $P_{a(i,j)}$.

Two maps f_{a^*} and f_{b^*} are now equivalent iff there exist isomorphisms of the local pieces which are compatible with the glueing. As it is easy to see any such isomorphism is uniquely determined by an arbitrary automorphism of $f | (M - \cup(\mathrm{int}\, D_i))$, which is naturally identified with the centralizer $C(\mu_f)$ of μ_f inside $\mathscr{S}_d$.

Hence, setting $U := \prod U_{h_{i,j}-1}$, $\pi^{-1}(U)$ is isomorphic to the quotient of Def(f) by the finite group $C(\mu_f)$. □

Let $\gamma \colon I := [0,1] \to F_n(M)$ be a path with $\gamma(0) = \gamma(1) = B_0$ and let $\gamma^*(\mathscr{U})$ be the pull-back of the universal family $\mathscr{U}$ to I.

Since I is contractible, $\gamma^*(\mathscr{U})$ is a trivial fibre bundle, hence there exists a bundle diffeomorphism $\psi \colon \gamma^*(\mathscr{U}) \to I \times (M - B_0)$, such that ψ_0 is the identity.

The image of γ under the monodromy of the fibration $p \colon \mathscr{U} \to F_n(M)$ is the monodromy transformation given by $\psi_1 \colon M - B_0 \to M - B_0$.

The covering of $M - B_0$ obtained by lifting γ to $\mathscr{H}^{\mathrm{gen}}_{g,d}(M)$ with initial point $\alpha : C' \to M - B_0$ is $\psi_1 \circ \alpha$.

We fix $x_0 \in M - B_0$ and define $E_{d,g,g'}$ to be the set of conjugacy classes of homomorphisms $\mu \colon \pi_1(M - B_0, x_0) \to \mathscr{S}_d$ with transitive image and such that $\mu(\gamma_b)$ is a transposition for each $b \in B_0$.

If $\mu \colon \pi_1(M - B_0, x_0) \to \mathscr{S}_d$ is the monodromy associated to the covering $\alpha \colon C' \to M - B_0$, then $\mu' := \mu \circ (\psi_1)_*^{-1} \colon \pi_1(M - B_0, \psi_1(x_0)) \to \mathscr{S}_d$ is the monodromy associated to the covering $\psi_1 \circ \alpha$, (note that the base point of $M - B_0$ is not left invariant).

Each path from x_0 to $\psi_1(x_0)$ ($M - B_0$ is connected) gives an isomorphism $\pi_1(M - B_0, \psi_1(x_0)) \to \pi_1(M - B_0, x_0)$. Let σ be an element in $\pi_1(M - B_0, \psi_1(x_0))$, then $\mu' \circ \mathrm{int}(\sigma) = \mathrm{int}(\mu'(\sigma)) \circ \mu'$, hence the conjugacy class of μ' does not depend on the choice of the path σ from x_0 to $\psi_1(x_0)$.

Therefore we have the following result:

(1.12) Proposition. *The monodromy of the cover* (1.9) *is described as follows*: $\pi_1(F_n(M))$ *acts via* $\mu \mapsto \mu \circ (\psi_1)_*^{-1}$ *on the conjugacy classes of monodromy homomorphisms* $\{[\mu] \,|\, \mu : \pi_1(M - B_0, x_0) \to \mathscr{S}_d$ *has transitive image and is such that* $\mu(\gamma_b)$ *is a transposition for each geometric loop around a point* $b \in B_0\}$.

(1.13) Remark. Let $V \subset F_n(M)$ be a subset such that there exists a section $s\colon V \to \mathscr{U}, B \mapsto x_B \in M - B$ of $p\colon \mathscr{U} \to F_n(M)$. In this case we can assume that $\psi(x_B) = x_B$ for all B, in particular we can take x_B as the variable base point and have $\psi_1(x_0) = x_0$.

Thus $\pi_1(V)$ acts via $\mu \mapsto \mu \circ (\psi_1)_*^{-1}$ on the monodromy homomorphisms $\{\mu \,|\, \mu : \pi_1(M - B_0, x_0) \to \mathscr{S}_d$ has a transitive image and is such that $\mu(\gamma_b)$ is a transposition for each $b \in B_0\}$ (and not only on the coverings, resp. on the conjugacy classes of monodromy homomorphisms as we had before).

We want to point out that the action on the μ's heavily depends on the choice of the section s.

(1.14) Definition-notation. *The group* $\mathscr{B}_n := \pi_1(F_n(\mathbb{C}), \{1, \ldots, n\})$ *is called* **Artin's braid group.**

(1.15) Example. Let $V \subset \mathscr{H}_{0,d}^{\mathrm{gen}}$ be the subset given by the rational functions not ramified over ∞. Then there is a section

$$s\colon V \to \mathscr{U},\ B \mapsto \infty \ \text{of}\ p\colon \mathscr{U} \to F_{2d-2}(\mathbb{C})\,.$$

Hence, if γ_i is the standard geometric loop around i (cf. [C-P], p. 632), $\mathscr{B}_{2d-2}$ acts on the monodromy homomorphisms $\{\mu \,|\, \mu(\gamma_i) = \tau_i$ is a transposition, $\mathrm{im}(\mu)$ is transitive and $\prod \tau_i = \mathrm{id}\}$.

Let $\mathscr{Z} := ((\mathbb{P}^1)^d - \Delta) \times W_d - \Sigma$, where W_d is the space of polynomials of degree d and Σ is the complex hypersurface s.th. $\mathscr{Z} := \{(P_1, \ldots, P_d, Q) \in ((\mathbb{P}^1)^d - \Delta) \times W_d : Q(P_i) \neq 0$ and $f := \dfrac{Q(z)}{\prod(z - P_i)}$ is a generic rational function$\}$.

We remark that Σ has real codimension 2 in $((\mathbb{P}^1)^d - \Delta) \times W_d$, whence $\mathscr{Z}$ is connected. Moreover, V is the image of $\mathscr{Z}$ under the holomorphic map g such that $g((P_1, \ldots, P_d, Q)) = f := \dfrac{Q(z)}{\prod(z - P_i)}$. Therefore V is connected, hence $\mathscr{B}_{2d-2}$ operates transitively on $\{\mu \,|\, \mu(\gamma_i) = \tau_i$ is a transposition, im μ is transitive and $\prod \tau_i = \mathrm{id}\}$.

We will now recall the classical theorem of Lűroth-Clebsch, which we will prove using essentially two connectedness arguments.

(1.16) Theorem (Lűroth-Clebsch). *Let g, d be two natural numbers and set $n = 2d + 2g - 2$. Then*:
1) $\mathscr{H}^{\mathrm{gen}}_{g,d} := \mathscr{H}^{\mathrm{gen}}_{g,d}(\mathbb{P}^1)$ *is connected.*
2) *More precisely, let B be an element of $F_n(\mathbb{P}^1 - \infty) = F_n(\mathbb{C})$; then $\mathscr{B}_n$ acts transitively on $\{\mu\colon\ \pi_1(\mathbb{P}^1 - B, \infty) \to \mathscr{S}_d \,|\, \mu(\gamma_i) = \tau_i$ is a transposition, μ has transitive image and $\prod \tau_i = \mathrm{id}\}$ (here, as above, $\gamma_1, \ldots, \gamma_n$ is a geometric basis of $\pi_1(\mathbb{P}^1 - B, \infty)$).*

(1.17) Remark. Let $\mathscr{H}'^{\mathrm{gen}}_{g,d}$ be the subspace of $\mathscr{H}^{\mathrm{gen}}_{g,d}$ of algebraic functions not branched over ∞; then, analogously to the situation in (1.15), we see that the monodromy of $\pi : \mathscr{H}'^{\mathrm{gen}}_{g,d} \to F_n(\mathbb{P}^1)$ is given by the action of $\pi_1(F_n(\mathbb{P}^1), \{1, \ldots, n\})$ on $\{\mu\colon\ \pi_1(\mathbb{P}^1 - B, \infty) \to \mathscr{S}_d \,|\, \mu(\gamma_i) = \tau_i$ is a transposition, μ has transitive image and $\prod \tau_i = \mathrm{id}\}$. Since the natural homomorphism $\mathscr{B}_n = \pi_1(F_n(\mathbb{C}), \{1, \ldots, n\}) \to \pi_1(F_n(\mathbb{P}^1), \{1, \ldots, n\})$ is surjective, 2) implies that the monodromy subgroup of π is transitive, hence $\mathscr{H}'^{\mathrm{gen}}_{g,d}$ is connected and therefore also $\mathscr{H}^{\mathrm{gen}}_{g,d}$ is connected, thus part (2) of (1.16) implies (1) of (1.16).

(1.18) Corollary. *The Hurwitz space $\mathscr{H}_{g,d}(\mathbb{P}^1)$ is connected.*

Proof (of (1.16)). 2) We prove the assertion by induction on g. The case $g = 0$ has already been treated in (1.15).

We will show that by acting with $\mathscr{B}_n$ we can bring $\tau_1, \ldots, \tau_n$ in the normal form: $(1,2), \ldots, (1,2), (1,2), (2,3), \ldots, (d-1, d), (d, d-1), \ldots, (2,1)$. For this it suffices to show that after acting with $\mathscr{B}_n$ we can assume $\tau_1 = \tau_2 = (1,2)$. In fact: $\mathscr{B}_n \supset \mathscr{B}'_{n-2} = \langle \sigma_3, \ldots, \sigma_{n-1} \rangle$ and $\mathscr{B}'_{n-2}$ leaves τ_1, τ_2 invariant and brings (by induction hypothesis) $\tau_3, \ldots, \tau_n$ in the normal form (note that $\tau_3 \cdot \ldots \cdot \tau_n = \mathrm{id}$).

Step 1: We can assume (after acting with $\mathscr{B}_n$) that $\bigcup_{1 \leqq i \leqq d-1} \operatorname{supp}(\tau_i) = \{1, \ldots, d\}$, where $\operatorname{supp}(\tau) := \{x\colon\ \tau(x) \neq x\}$.

In fact we prove by induction on j that (after acting with $\mathscr{B}_n$) the cardinality of $T_j := \bigcup_{1 \leqq i \leqq j} \operatorname{supp}(\tau_i)$ equals $j+1$. For $j = 0$ the assertion is trivial. Assume that the cardinality of T_j is equal to $j+1$. Then there exists a minimal h such that $\operatorname{supp}(\tau_h)$ intersects T_j and its complement. After acting with the standard braids $\sigma^{-1}_{j+h-1}, \ldots, \sigma^{-1}_{j+1}$ we obtain that the new τ_{j+1} is the old τ_{j+h} and so we are done.

Therefore $\langle \tau_1, \ldots, \tau_{d-1} \rangle$ is a transitive subgroup of $\mathscr{S}_d$, so $\tau_1, \ldots, \tau_{d-1}$ give the monodromy of a generic polynomial and in particular $\tau_1 \cdot \ldots \cdot \tau_{d-1}$ is a cyclical permutation.

Step 2:

Lemma. *Let $\tau_1, \ldots, \tau_{d-1}$ be transpositions such that $\prod \tau_i$ equals a cyclical permutation $(1, r_2, \ldots, r_d)$. Then there exist transpositions $\tau'_1, \tau'_2, \ldots, \tau'_{d-1}$ such that $\tau'_1 = (1,2)$ and $\prod \tau_i = \prod \tau'_i$.*

Proof. Let j be such that $2 = r_j$: then $(1,2) \cdot (r_j, r_{j+1}) \cdot \ldots \cdot (r_{d-1}, r_d) \cdot (r_2, r_3) \cdot \ldots \cdot (r_{j-1}, r_j) = (1, r_2, \ldots, r_d)$. □

Moreover, the assumption of the lemma implies that the classes $[\mu]$ and $[\mu']$ (given by $\tau_1, \ldots, \tau_{d-1}$ resp. $\tau'_1, \ldots, \tau'_{d-1}$) are in the same $\mathscr{B}_{d-1}$-orbit.

Step 3: $\mathscr{B}_{d-1}$ acts transitively on the homomorphisms $\mu\colon \mathbb{F}_{d-1} \to \mathscr{S}_d$ such that $\mu(\gamma_i) = \tau_i$ is a transposition, im μ is transitive and $\prod \tau_i = (1, \ldots, d)$ (or $\prod \tau_i$ is any fixed d-cycle).

Proof I. It will suffice to show that μ is $\mathscr{B}_{d-1}$-equivalent to μ'', where the transpositions corresponding to μ'' are $\tau''_i = (i, i+1)$. By the remark after Step 2 we know that $\mathscr{B}_{d-1}$ is transitive on the classes $[\mu]$ as above, hence μ is $\mathscr{B}_{d-1}$-equivalent to a μ' such that $[\mu'] = [\mu'']$.

This means that there is a permutation $s \in \mathscr{S}_d$ such that $\tau'_i = (s(i), s(i+1))$ and since $\prod \tau_i = \prod \tau'_i = (1, \ldots, d)$ we get $(s(1), \ldots, s(d)) = (1, \ldots, d)$.

This implies that there exists a j such that $s(1) = j+1$, $s(2) = j+2$, $\ldots, s(i) = j + i \pmod{(d)}$.

Define $\delta_{d-1} := \sigma_{d-2}\sigma_{d-3}\ldots\sigma_1$, where $\sigma_1, \ldots, \sigma_{d-2}$ are the standard generators of $\mathscr{B}_{d-1}$ and let it act on μ'. We obtain:

$$\delta_{d-1}(\tau'_i) = \tau'_{i-1} \quad \text{for } i \geqq 2 .$$

Setting $\tau'_d := (d, 1)$, then $\delta_{d-1}(\tau'_1) = \tau'_d$.

Therefore we finally get $(\delta_{d-1})^j(\tau'_i) = (i, i+1) = \tau''_i$, and we are done.

Proof II. (By geometry, details left to the reader). As we saw in the proof of (1.11) the space of polynomials f of the type $f(z) = z^d + a_{d-2}z^{d-2} + \ldots + a_0$ is connected.

Now we have proven that after acting with the braid group we can obtain that τ_1 equals $(1,2)$. Applying again the Steps 1–3, we can also assume $\tau_2 = (1,2)$ and we have proven the theorem. □

Let $\mathbb{F}_r$ be the free group with generators $\gamma_1, \ldots, \gamma_r$. Then the set

$$\mathscr{M}'_{r,d} := \{[\mu] \mid \mu\colon \mathbb{F}_r \to \mathscr{S}_d \text{ has transitive image and, for each } i, \; \mu(\gamma_i) \text{ is a transposition } \tau_i\}$$

is in a natural bijection with the set $\mathscr{E}'_{r,d}$ of isomorphism classes of connected graphs with d vertices and r edges numbered from one to r.

Obviously, the map from $\mathscr{M}'_{r,d}$ to $\mathscr{E}'_{r,d}$ is given by associating to the transposition $\tau_i = \langle a_i, b_i \rangle$ the edge joining a_i and b_i labelled by the index i (and forgetting about the labelling of the vertices).

Let $\mathscr{M}_{r,d}$ be the set of all $[\mu]$'s in $\mathscr{M}'_{r,d}$ such that $\prod \mu(\gamma_i) = \mathrm{id}$ and let $\mathscr{E}_{r,d}$ be the set of corresponding edge labelled graphs (obviously, then r has to be an even integer).

Now, given a finite set $B = \{w_1, \dots, w_{2g+2d-2}\} \subset \mathbb{C}$, once we have fixed a geometric basis $\gamma_1, \dots, \gamma_{2g+2d-2}$ of the fundamental group of $\mathbb{C} - B$ with base point ∞ to any edge labelled graph $E \in \mathscr{E}_{2g+2d-2,d}$ we can associate a connected compact Riemann surface C of genus g together with a generic algebraic function $f\colon C \to \mathbb{P}^1$ of degree d and with B as branch locus.

Hence we have the following result:

(1.19) Proposition. *Let g, d be two integers and $B := \{w_1, \dots, w_{2g+2d-2}\} \subset \mathbb{P}^1$, not containing $0, \infty$. Moreover let $\gamma_1, \dots, \gamma_{2g+2d-2}$ be a geometric basis of the fundamental group of $\mathbb{C} - B$.*
A) *Then there is a natural bijection between*:

1) *the set $\mathscr{E}_{2g+2d-2,d}$ of isomorphism classes of edge labelled graphs with d vertices and $2g + 2d - 2$ edges numbered from 1 to $2d + 2g - 2$ such that the product $\tau_1 \cdot \ldots \cdot \tau_{2g+2d-2}$ of the transpositions corresponding to the edges equals the identity*,
2) *the set $\mathscr{M}_{2g+2d-2,d}$ of conjugacy classes of μ: $\mathbb{F}_{2g+2d-2} \to \mathscr{S}_d$ with transitive image, such that each generator γ_i is mapped to a transposition τ_i, and such that $\prod \tau_i = \mathrm{id}$*,
and
3) *the set of equivalence classes of connected, compact Riemann surfaces C of genus g together with a generic map $f\colon C \to \mathbb{P}^1$ of degree d and branch locus B.*

B) *Moreover, the standard action of $\mathscr{B}_{2g+2d-2} \subset \mathrm{Aut}(\mathbb{F}_{2g+2d-2})$ induces an action on $\mathscr{M}_{2g+2d-2,d}$ by $[\mu] \mapsto [\mu \circ \sigma^{-1}]$, which is the monodromy action of the covering $\pi\colon \mathscr{H}^{\mathrm{gen}}_{g,d}(\mathbb{P}^1) \to F_n(\mathbb{P}^1)$.*

2 Lemniscates of algebraic functions

(2.1) Definition. *Fix $0, \infty \in \mathbb{P}^1$. The* **big lemniscate configuration** *Γ_f of an algebraic function $f\colon C \to \mathbb{P}^1$ is the pair consisting of*
2.1.a) *the subset $f^{-1}(0) \cup f^{-1}(\infty) \cup \Gamma$, where Γ is the singular level set of $|f|$*,
2.1.b) *a coloring of the discrete set $f^{-1}(0) \cup f^{-1}(\infty)$ according to the (ordered) partition into the two subsets $f^{-1}(0), f^{-1}(\infty)$.*

(2.2) Remark. If $w_1, \dots, w_n$ are the critical values (or branch points) of f different from $0, \infty$, we can clearly assume from now on that

$$|w_1| \leqq \ldots \leqq |w_n| \, .$$

Then the critical values of $|f|$ different from $0, \infty$, are equal to $c_1 < \ldots < c_k$ where $c_1 := |w_1|$, $c_{i+1} := \min\{|w_h| : |w_h| > c_i\}$.

Letting then in general be $\Delta_c = \{z \in C\colon\ |f(z)| = c\}$, we have

$$\Gamma_f = f^{-1}(0) \cup f^{-1}(\infty) \cup \bigcup_{i=1,\dots,k} \Delta_{c_i}\,. \tag{2.3}$$

At a critical point P of multiplicity $(m-1)$ the lemniscate $\Delta_{|w|}$ $(w = f(P))$ has a singularity consisting of m smooth curves intersecting with angles $\frac{\pi}{m}$.

In the case $m = 2$ this singularity is called a *node*.

(2.4) Definition. *An algebraic function* $f : C \to \mathbb{P}^1$ *of degree* d *is called* **lemniscate generic** *if and only if the following conditions hold*:
a) $f^{-1}(0)$ *as well as* $f^{-1}(\infty)$ *consist exactly of* d *points*,
b) f *has* n *distinct generic branch points* $w_1, \dots, w_n$ *on* $\mathbb{C} = \mathbb{P}^1 - \{\infty\}$ *and* $|w_i| = |w_j|$ *implies* $i = j$.

(2.5) Remark. 1) If f is lemniscate generic, then $(c_1, \dots, c_k)$ (in (2.1)) $= (|w_1|, \dots, |w_n|)$, in particular $n = k$.
2) Let $f : C \to \mathbb{P}^1$ be a lemniscate generic algebraic function of degree d with branch locus $B \subset \mathbb{P}^1$. Then $\pi_1(\mathbb{C} - B)$ is a free group in $2d + 2g - 2$ generators $\langle \gamma_1, \dots, \gamma_{2g+2d-2} \rangle$, where g is the genus of C. Moreover, $\mu(\gamma_i) = \tau_i$ is a transposition and $\prod_i \mu(\gamma_i) = \mathrm{id}$.

(2.6) Definition. *A* **bi-ended edge weighted graph** $\mathcal{G}$ *is the datum of*
2.6.i) *a graph* $\mathcal{G}$,
2.6.ii) *a function assigning to each edge a rational number called* **weight**,
2.6.iii) *a partition of the set* $\mathcal{E}$ *of end vertices into two sets*, $\mathcal{E}^0 =$ **set of lower ends**, *respectively* $\mathcal{E}^\infty =$ *set of* **upper ends**.

To a lemniscate generic algebraic function $f : C \to \mathbb{P}^1$ we associate a **bi-ended edge weighted graph** $\mathcal{G} = \mathcal{G}_f$ in the following way:

Let $w_1, \dots, w_n$ be the critical values of f (with as usual $0 < |w_1| \leqq \dots \leqq |w_n|$). Fix a geometric basis $\gamma_1, \dots, \gamma_n$ of $\pi_1(\mathbb{C} - \{w_1, \dots, w_n\}, 0)$.

Then the vertices of $\mathcal{G}$ correspond to the connected components of Γ_f (cf. (2.3)); two vertices v, v' are connected by an edge iff there exists an i such that $v \subset \Delta_{|w_i|}$, $v' \subset \Delta_{|w_{i+1}|}$ and moreover if starting from a smooth point $x \in v$, and then following the gradient of $|f|$ one gets a curve meeting v'.

(2.7) Remarks-definitions. 1) Since C is connected, $\mathcal{G}$ is a connected graph.
2) Let $\varepsilon > 0$ be a sufficiently small real number such that $\varepsilon < c_1$ and, for each i, $c_i + \varepsilon < c_{i+1}$. Then one can easily see that the edges of $\mathcal{G}$ correspond uniquely to the connected components of $f^{-1}(S_{c_i+\varepsilon})$, where $S_{c_i+\varepsilon} \subset \mathbb{C}$ is the circle around 0 of radius $c_i + \varepsilon$.
3) The *ends* of $\mathcal{G}$ are the vertices corresponding to the points of $f^{-1}(0)$ and $f^{-1}(\infty)$.
4) The vertices corresponding to the elements of $f^{-1}(0)$ are called *lower ends* (resp. those of $f^{-1}(\infty)$ are the *upper ends*), thus $\mathcal{E}^\infty = f^{-1}(\infty)$, $\mathcal{E}^0 = f^{-1}(0)$ as the notation suggests.

(2.8) Definition. 1) *If v, v' are two vertices of $\mathcal{G}$, then there exists a subtree $T \subset \mathcal{G}$ having v and v' as ends and with a minimal number of edges. The number of edges of T is called the geodesic* **distance of v and v'**.
2) *The* **height $h(v)$** *of a vertex of a biended graph $\mathcal{G}$ is the minimal distance of v to a lower end of $\mathcal{G}$.*
3) *An edge e is* **of level i** *iff the minimal height of a vertex contained in e is $i-1$.*

(2.9) Remark. 1) A vertex v of a graph $\mathcal{G}$ associated to an algebraic function f has height i if and only if it corresponds to a connected component of $\Delta_{c_i} = \{z \in \mathbb{C}\colon |f(z)| = c_i\}$.
2) An edge is of level i iff it corresponds to a connected component of $f^{-1}(S_{c_i+\varepsilon})$.

Let $f: C \to \mathbb{P}^1$ be a lemniscate generic algebraic function and let $\mathcal{G}$ be the associated (bi-ended) graph.
We can give weights to the edges of $\mathcal{G}$ in the following way:

(2.10) The weight $w(e)$ of an edge e corresponding to a connected component Δ of $f^{-1}(S_{c_i+\varepsilon})$ is defined to be equal to the degree of the map $f \mid \Delta : \Delta \to S_{c_i+\varepsilon}$.

We have up to now described the graph $\mathcal{G}$ of a lemniscate generic algebraic function in geometric terms, but indeed one can give a purely algebraic description of $\mathcal{G}$ in terms of the monodromy of the unramified covering

$$f \mid C - f^{-1}(B) \to \mathbb{P}^1 - B \,.$$

(2.11) Remark. Let C be a compact Riemann surface and let $f: C \to \mathbb{P}^1$ be a lemniscate generic algebraic function of degree d with branch points $w_1, \ldots, w_n$. Then it is easy to see that the edges of level i of the associated weighted graph $\mathcal{G}$ correspond to the orbits of $\mu(\gamma_1 \ldots \gamma_i)$, where $\{\gamma_1, \ldots, \gamma_n\}$ is the usual fixed geometric basis of $\pi_1(\mathbb{C} - \{w_1, \ldots, w_n\}, 0)$ and $\mu\colon \pi_1(\mathbb{C} - \{w_1, \ldots, w_n\}, 0) \to \mathcal{S}_d$ is the monodromy homomorphism of the unramified covering

$$f^{-1}(\mathbb{P}^1 - \{w_1, \ldots, w_n\}) \to \mathbb{P}^1 - \{w_1, \ldots, w_n\} \,.$$

(2.12) Remark. 1) If f is a lemniscate generic algebraic function of degree d with n distinct simple branch points, then $\mathcal{G}_f$ has exactly d lower ends, d upper ends and at most *nodes* (i.e. the vertices have valence at most three).

Moreover, for each $i \in \{1, \ldots, n\}$ there exists exactly one node of height i.
2) The graph $\mathcal{G}$ (also in the non generic case) satisfies *Kirchoff's rule*, i.e., if v is a vertex of $\mathcal{G}$, $e_1, \ldots, e_k$ are the edges of level i containing v and $f_1, \ldots, f_l$ are the edges of level $i+1$ containing v, then the weights have to fulfill the following relation: $w(f_1) + \ldots + w(f_l) = w(e_1) + \ldots + w(e_k)$.

In fact by remark (2.11), if σ is the permutation $\mu(\gamma_1 \ldots \gamma_i)$ and we assume for simplicity that $\mu(\gamma_{i+1})$ is a transposition τ, we have to relate the respective orbits of σ and $\sigma\tau$. The assertion follows then from the following:

Subremark. Let σ be a permutation and τ a transposition.

a) If supp(τ) is not contained in an orbit of σ, we can assume, $O_1, \dots, O_k$ being the orbits of σ, that $\tau = (a, b)$ with $a \in O_1$ and $b \in O_2$. Then the orbits of $\sigma\tau$ are $O_1 \cup O_2, O_3, \dots, O_k$.
b) If supp(τ) is contained in an orbit of σ, then, $O_1, \dots, O_k$ being the orbits of $\sigma\tau$, the orbits of σ are $O_1 \cup O_2, O_3, \dots, O_k$.

3) As we shall see later, for a lemniscate generic rational function the weights of the edges are uniquely determined by the graph (which in this particular case is a *tree*, i.e., it is simply connected) and by Kirchoff's rule, whereas this is no longer true in general.
4) The graph (without the weights of the edges) associated to a lemniscate generic algebraic function only depends upon its big lemniscate configuration.

(2.13) Definition. *Let $Y_n \subset F_n(\mathbb{C})$ be the subset*

$$\{\{w_1, \dots, w_n\}\colon 0 < |w_1| < \dots < |w_n| < \infty\}$$

and let $\bigwedge_n$ be the image of $\pi_1(Y_n, \{1, \dots, n\}) \to \pi_1(F_n(\mathbb{C}), \{1, \dots, n\}) = \mathscr{B}_n$.

(2.14) Remark. Writing $w_i = |w_i| \dfrac{w_i}{|w_i|}$, and $r_1 = |w_1|$, $r_2 = \dfrac{|w_2|}{|w_1|}, \dots, r_n = \dfrac{|w_n|}{|w_{n-1}|}$, we see that Y_n is homeomorphic to $(S^1)^n \times (\mathbb{R}^+)^n$, hence $\pi_1(Y_n) \cong \mathbb{Z}^n$. The images T_j of the generators of $\pi_1(Y_n)$ are the braids, which keep fixed the points $1, \dots, n$ different from j, and move j in a circle around the origin

$$T_j\colon (t \mapsto e^{2\pi i t} j) \,.$$

(2.15) Remarks. 1) Let d, g be two natural numbers. Then the set $\mathscr{L}_{g,d}$ of lemniscate generic algebraic functions is an open subset in $\mathscr{H}^{\text{gen}}_{g,d}$, whose complement $\mathscr{H}^{\text{gen}}_{g,d} - \mathscr{L}_{g,d}$ is a union of real hypersurfaces.
2) Let $\pi : \mathscr{H}'^{\text{gen}}_{g,d}(\mathbb{P}^1) \to F_n(\mathbb{C})$ be the restriction of (1.9) to $\mathscr{H}'^{\text{gen}}_{g,d}(\mathbb{P}^1)$ $(= \{f \in \mathscr{H}^{\text{gen}}_{g,d}(\mathbb{P}^1) | f$ is not branched over $\infty\})$, where again $n = 2g + 2d - 2$. Then $\pi^{-1}(Y_n) = \mathscr{L}_{g,d}$.

Since the connected components of a covering space correspond to the orbits of its monodromy, we obtain the following result.

(2.16) Proposition. *There is a natural bijection between the set $\pi_0(\mathscr{L}_{g,d})$ of connected components of $\mathscr{L}_{g,d}$ and the set of $\bigwedge_n$-orbits ($n = 2g + 2d - 2$) in $\mathscr{M}_{n,d} = \{[\mu] \,|\, \mu\colon \mathbb{F}_n \to \mathscr{S}_d$ s.t. $\mu(\gamma_i) = \tau_i$ is a transposition, μ has transitive image and $\prod \tau_i = \mathrm{id}\}$.*

Moreover, if $\mathscr{L}'$ is a component of $\mathscr{L}_{g,d}$ corresponding to an orbit of cardinality r, then the covering $\pi|\mathscr{L}'$ has degree r.

We recall that by (1.19) the set $\mathscr{M}_{2g+2d-2,d}$ is in a natural bijection with $\mathscr{E}_{2g+2d-2,d}$.

We are now going to state a weaker version of the main theorem. The stronger version stated in the introduction will follow from Theorem (2.17), Proposition (3.3) and Corollary (3.4).

(2.17) Main Theorem (Weaker version). *Let g, d be two natural numbers and let $n = 2g + 2d - 2$. Consider the following sets*:
(a) *the set* $\pi_0(\mathscr{L}_{g,d})$ *of connected components of* $\mathscr{L}_{g,d}$,
(b) *the set of* $\bigwedge_n$*-orbits in* $\mathscr{M}_{n,d} = \{[\mu] \mid \mu : \mathbb{F}_n \to \mathscr{S}_d$ *s.t.* $\mu(\gamma_i) = \tau_i$ *is a transposition*, μ *has transitive image and* $\prod \tau_i = \mathrm{id}\}$
(c) *the set of bi-ended, edge weighted graphs associated to lemniscate generic algebraic functions of degree d and genus g.*
Then there is a natural bijection between (a) *and* (b), *and a natural surjection* υ *of* (a) *onto* (c). *Moreover*, υ *is bijective in the case* $g = 0$ *of rational functions, but not injective as soon as* $g \geqq 1$ *and* $d \geqq 4$.

The correspondence of the sets in (a) and (b) has already been shown (cf. (2.16)). Since, obviously, varying an algebraic function inside one component of $\mathscr{L}_{g,d}$ its associated edge weighted graph does not change, it remains to see whether the contrary is valid, i.e., if two lemniscate generic algebraic functions have the same edge weighted graph, then they are contained in the same connected component of $\mathscr{L}_{g,d}$.

Paragraph 3 will be devoted to this analysis which will complete the proof of the main theorem in its stronger form. In the rest of this section we will devote ourself to the calculation of the Euler Poincaré characteristic of the Hurwitz space.

In order to do so, we recall some basic facts (cf. [B-M], [B-H]) about Borel-Moore homology which, roughly speaking, equals ordinary homology for compact spaces, and reduced homology of the Alexandrov one point-compactification for locally compact spaces:

(2.18) Fundamental properties of Borel-Moore homology

Let X be a locally compact Hausdorff space, then there are defined homology groups $H_i(X, \mathbb{Z})$ such that

i) if Y is a closed subset, one has a long exact sequence

$$\to H_i(Y, \mathbb{Z}) \to H_i(X, \mathbb{Z}) \to H_i(X - Y, \mathbb{Z}) \to H_{i-1}(Y, \mathbb{Z}) \to ,$$

ii) for X compact $H_i(X, \mathbb{Z})$ equals the singular homology group,
iii) $H_i(X, \mathbb{Z})$ is invariant by proper homotopy,
iv) $H_i(X, \mathbb{Z})$ is finitely generated for a finite CW complex.

(2.19) Notation. Let X be a finite CW complex, then we shall denote by $e(X)$ the Euler-Poincare' characteristic in Borel Moore homology,

$$e(X) = \sum (-1)^i \operatorname{rank}(H_i(X, \mathbb{Z})) .$$

(2.20) More properties of Borel-Moore homology

It is immediate to see that

(2.20.i) $e(X) = e(Y) + e(X - Y)$ for Y a closed or open subspace of X,

(2.20.ii) $e(X \times Y) = e(X) \times e(Y)$

and since bundles are locally trivial, again for finite CW complexes we have that

(2.20.iii) if X is a fibre bundle on Y with fibre F, then

$$e(X) = e(F) \times e(Y)$$

and in particular for a finite unramified covering of degree d

(2.20.iv) $e(X) = d \cdot e(Y)$.

We shall now apply the previous remarks.

(2.21) Theorem. *The Euler Poincaré characteristic in Borel Moore homology of the Hurwitz space $\mathscr{H}_{g,d} = \mathscr{H}_{g,d}(\mathbb{P}^1)$ equals zero for $g > 0$, and equals 1 for $g = 0$.*

Proof. We notice that we can stratify the Hurwitz space, $\mathscr{H}_{g,d} = \mathscr{H}_{g,d}(\mathbb{P}^1)$, as a union of locally closed manifolds $\mathscr{H}_{g,d}(k,n)$, indexed as in (2.2) by the number n of the branch points different from $0, \infty$ and the number k of their absolute values $c_1 < \ldots < c_k$.

We claim that, for $k > 0$, each connected component $\mathscr{C}$ of $\mathscr{H}_{g,d}(k,n)$ has Euler Poincaré characteristic $e(\mathscr{C})$ equal to zero. In fact, $\mathscr{C}$ is a finite covering of a manifold

$$((\mathbb{R}^+)^k \times ((S^1)^{r(1)} - \Delta(r(1)) \times \ldots \times ((S^1)^{r(k)} - \Delta(r(k))))$$

where $r(1) + \ldots r(k) = n$, and $\Delta(j)$ is the big diagonal in $(S^1)^j$. Therefore, by 2.20 iv) and i) it suffices to show that $e((S^1)^j - \Delta(j)) = 0$. But since (S^1) is a group, the sum map exhibits this space as a trivial bundle over (S^1), and therefore we are done by (2.20 iii).

Thus the only strata which give a non zero contribution are those for which $k = 0$.

Id est, when 0 and ∞ are the only branch points.

But if 0 and ∞ are the only branch points, we have a cyclic covering of $\mathbb{C}^*$, thus only the algebraic function $z \to z^d$, and we are in genus $g = 0$, with contribution to e equal to 1. □

(2.22) Remark. With an entirely similar proof one can calculate the Euler Poincaré characteristic in Borel Moore homology of the "Harris-Mumford" compactification of our Hurwitz space $\mathscr{H}^{\mathrm{gen}}_{g,d}$. It would be interesting to calculate those numbers also for the quotient of the Hurwitz space by PGL(2) acting on the target $\mathbb{P}^1$.

3 Proof of the main theorem

In this paragraph we first want to give a proof of our main theorem (2.17).

Moreover, we will show that the weights of the edges of a graph associated to a lemniscate generic rational function are determined uniquely by Kirchoff's rule (cf. (2.12), 2)).

For algebraic functions of Riemann surfaces of genus $\geqq 1$ this is in general not true and we will give a counterexample.

For the proof of (2.17) we will essentially proceed as in [C-P], whereas for the statement for $g \geqq 1$, $d \geqq 4$ we shall consider a special subgraph and we shall make some explicit computations.

Let $\mathscr{G}$ be a connected biended graph, so that it is given a decomposition $\mathscr{E} = \mathscr{E}^{\infty} \cup \mathscr{E}^{0}$ of the set $\mathscr{E}$ of ends of $\mathscr{G}$ in two disjoint subsets. Then recall that the height $h(v)$ of a vertex v is defined to be its geodesic distance from $\mathscr{E}^{0}$, while the *coheight* $\mathrm{ch}(v)$ of v is defined to be the distance from $\mathscr{E}^{\infty}$.

(3.1) Definition. *Let g, d be two natural numbers and let $\mathscr{G}$ be a bi-ended connected graph. Then $\mathscr{G}$ is called* **admissible bi-ended, edge weighted of type (g, d)**, *for abbreviation we shall also say "admissible of type (g, d)", if and only if the following conditions hold*:
1) *$\mathscr{G}$ has d upper ends and d lower ends.*
2) *For all vertices v of $\mathscr{G}$ it holds*:

$$h(v) + \mathrm{ch}(v) = 2d + 2g - 1 .$$

3) *For each i $(1 \leqq i < 2g + 2d - 1)$ there is exactly one node of height i and the other vertices of height i are smooth points (i.e., have valency $\leqq 2$).*
4) *The edges are given positive integer weights and the edges containing an end have weight one.*
5) *The graph fulfills Kirchoff's rule, i.e. if v is a vertex of $\mathscr{G}$ of height $i, e_1, \ldots, e_k$ are the edges of level $i - 1$ containing v and $f_1, \ldots, f_l$ are the edges of level i containing v, then the weights have to fulfill the following relation*: $w(f_1) + \ldots + w(f_l) = w(e_1) + \ldots + w(e_k)$.

(3.2) Remark. 1) If $\mathscr{G}$ is a bi-ended, edge weighted graph fulfilling 1)–4) above, e is an edge of $\mathscr{G}$ and v, v' are the vertices joined by e, then $h(v) \neq h(v')$. More precisely, $|h(v) - h(v')| = 1$. Therefore Kirchoff' s rule makes sense, since a vertex of height i is only contained in edges of level $i - 1$ and i.
2) For an admissible graph $\mathscr{G}$ of type (g, d) we have:

$$\chi(\mathscr{G}) = 1 - g .$$

3) If $\mathscr{G}_f$ is a bi-ended, edge weighted graph associated to a lemniscate generic algebraic function $f : C \to \mathbb{P}^1$ of degree d, then $\mathscr{G}$ is admissible of type $(\mathrm{genus}(C), d)$.

(3.3) Proposition. *Let $\mathscr{G}$ be an admissible (bi-ended, edge weighted) graph of type (g,d). Then there is a Riemann surface C of genus g, and a lemniscate generic algebraic function $f : C \to \mathbb{P}^1$ such that $\mathscr{G}$ is the bi-ended edge weighted graph associated to f.*

Proof. We shall construct all the ramified coverings of degree d of $\mathbb{P}^1$ with branch points $1,\dots,n$, where $n = 2g + 2d - 2$, and associated to the given graph $\mathscr{G}$ by gluing together the respective coverings of A_i, $0 \leqq i \leqq n+1$, where $A_0 := \{z \in \mathbb{P}^1 : |z| \leqq 1/2\}$, $A_i := \{z \in \mathbb{P}^1 : i - 1/2 \leqq |z| \leqq i + 1/2\}$, $(1 \leqq i \leqq n)$ and $A_{n+1} := \{z \in \mathbb{P}^1 : |z| \geqq n+1\}$.

For $1 \leqq i \leqq n$, we let $\varphi_i : R_i \to A_i$ be the unique covering of degree d, simply branched over i such that

1) the vertices of $\mathscr{G}$ of height i correspond to the connected components of R_i,

2) if the vertex v corresponding to a component Δ has valency 2, then Δ is the standard connected unramified covering of degree equal to the weight of the two edges meeting in v,

3) if the vertex v corresponding to Δ has valency 3, then Δ is homeomorphic to a "pair of pants" whose 3 boundary S^1's correspond to the three edges meeting in v, and map with degree equal to their weight to $(S_{i-1/2})$ if they are at level i, and they map to $(S_{i+1/2})$ if their level is $(i+1)$. In this case the only critical value is i, and its local monodromy is a transposition.

To verify that such a cover as in 3) exists, assume for simplicity that among the three edges there is exactly one at level $(i+1)$. Remark that the fundamental group of $A_i - \{i\}$ is free on two generators, say corresponding to $(S_{i-1/2})$ and the small circle around i. If the weights of the lower edges are m, n respectively, consider then the monodromy assigning to the first generator the product of two disjoint cycles $\{1,\dots,m\}$ $\{m+1,\dots,m+n\}$, and to the second a transposition (a,b) with $1 \leqq a \leqq m$, $m+1 \leqq b \leqq m+n$. Clearly then the monodromy of $(S_{i+1/2})$ is a cyclical permutation of length $m+n$.

We give therefore φ_i together with respective bijections of the set of edges of level i of $\mathscr{G}$ (resp. $i+1$) with the set of connected components of $\varphi_i^{-1}(S_{i-1/2})$ (resp. $\varphi_i^{-1}(S_{i+1/2})$) (s.t. the weight of an edge e corresponds to the degree of the restriction of φ_i to the corresponding component).

Moreover, let $\varphi_0 : R_0 \to A_0$ (resp. $\varphi_{n+1} : R_{n+1} \to A_{n+1}$) be the trivial covering of degree d together with a bijection between the connected components of R_0 (resp. R_{n+1}) with the set of lower (resp. upper) ends of $\mathscr{G}$.

Taking the disjoint union of the R_i's an edge of weight w corresponds to the two coverings of S^1 of the same degree w. There exist exactly w isomorphism of these two covers, and using for each edge an arbitrary one of these isomorphisms, one can glue together the coverings φ_i and obtain a connected ramified covering $f : C \to \mathbb{P}^1$ simply branched over $1,\dots,n$.

By Hurwitz's formula the Riemann surface C (without boundary) got in this way has genus g. □

(3.4) Corollary. *There is a bijection between the set of isomorphism classes of bi-ended, edge weighted graphs associated to lemniscate generic algebraic* *functions of degree d of a Riemann surface of genus g and the set of isomorphism classes of admissible bi-ended, edge weighted graphs of type* (g, d).

Let $\mathfrak{A}\mathscr{G}_{g,d}$ be the set of (isomorphism classes of) admissible (bi-ended, edge weighted) graphs of type (g, d). Then (cf. (1.19)) we have a map υ : $\mathscr{M}_{2g+2d-2,d} \to \mathfrak{A}\mathscr{G}_{g,d}$.

(3.5) Proposition. *The map* $\upsilon\colon \mathscr{M}_{2g+2d-2,d} \to \mathfrak{A}\mathscr{G}_{g,d}$ *is surjective.*

If $g = 0$ *and* μ, μ' *are such that* $\upsilon(\mu) = \upsilon(\mu') = \mathscr{g}$, *then* μ *and* μ' *are in the same* $\bigwedge_{2g+2d-2}$*-orbit.*

If instead $g \geqq 1, d \geqq 4$ *there do exist* μ, μ' *with* $\upsilon(\mu) = \upsilon(\mu') = \mathscr{g}$, *which are not in the same* $\bigwedge_{2g+2d-2}$*-orbit.*

With this result we will have proven our main theorem.

Proof of (3.5). The surjectivity of υ follows from (3.3) and the second assertion essentially follows from a more careful analysis of the proof of (3.3).

Namely, we want to find out to which extent the graph determines the monodromy homomorphism, once we have established a bijection of the set $\{1, \dots, d\}$ with the lower ends of the graph $\mathscr{g}$.

Assume inductively, that $\tau_1, \dots, \tau_i$ have been determined and $\tau_j = \tau'_j$ for $1 \leqq j \leqq i$, in such a way that the identification is compatible with υ.

Let $O_1, \dots, O_k$ be the orbits of $\tau_1 \cdot \ldots \cdot \tau_i$: these have respective cardinalities $m_1, \dots, m_k$ (where $m_1, \dots, m_k$ are the weights of the edges of level i).

We have two possibilities, namely:

a) there are $k - 1$ edges at level $i + 1$,

b) there are $k + 1$ edges at level $i + 1$.

In case a) τ_{i+1} transposes two elements belonging to two different orbits, which we can assume without loss of generality to be O_1 and O_2. The same property holds for τ'_{i+1}, since $\upsilon(\mu) = \upsilon(\mu')$.

In case b) τ_{i+1} permutes two elements belonging to the same orbit, say O_1; idem for τ'_{i+1}. The orbits of $\tau_1 \cdot \ldots \cdot \tau_{i+1}$ are given by the orbits $O_2, \dots, O_k$ and by $O_{1,1}$, $O_{1,2}$ (where O_1 is the disjoint union of $O_{1,1}$ and $O_{1,2}$). Similarly we define $O'_{1,1}$, $O'_{1,2}$ and we observe that their cardinalities, being given by weights of $\mathscr{g}$, equal the cardinalities of $O_{1,1}$ and $O_{1,2}$.

We set for convenience $\tau_{i+1} = (x, y)$ and $\tau'_{i+1} = (x', y')$.

It suffices to show that we can act with $\bigwedge_n$ on μ' in such a way that $\tau'_1, \dots, \tau'_i$ are left invariant and τ'_{i+1} is transformed to τ_{i+1}.

We firstly treat case b).

In case b) the situation is simpler. In fact $\tau_1 \cdot \ldots \cdot \tau_i$ acts by a cyclical permutation σ on O_1 and, setting $T := T_1 \cdot \ldots \cdot T_i$, we let a suitable power T^{-k} of T^{-1} act on μ'. Using the formulae from [C-P], pages 632–633 we see that $\tau'_1, \dots, \tau'_i$ are left invariant, whereas τ'_{i+1} is conjugated by the k^{th}-power σ^k of σ.

We use now the essential information that τ_{i+1} and τ'_{i+1} multiplied by σ yield two disjoint cycles of the same length. We shall use the following elementary

(3.6) Remark. Let σ be the cyclical permutation $(1, \dots, m)$ and let τ be the transposition $(1, z)$, where $1 \leqq z \leqq m/2$.

Then $\sigma\tau = (2, \dots, z)(1, z+1, \dots, m)$. In particular, if τ is any transposition such that $\sigma\tau$ has two orbits of respective cardinalities $(z-1)$, $(m-z+1)$, then τ is conjugate to $(1, z)$ by a suitable power of σ.

It follows now immediately that τ'_{i+1} is obtained from τ_{i+1} by conjugating with a suitable power of σ and therefore a suitable power of T^{-1} sends τ'_{i+1} to τ_{i+1}.

Let us proceed to consider case a) when $g = 0$. Then, as we noticed in (3.2), the graph $\mathscr{G}$ is a tree.

Therefore, if we remove from the graph the unique node v of height $(i+1)$, we obtain exactly three connected components, of which two are stemming down from v, i.e., are the ones containing the respective two edges corresponding to O_1 and O_2.

Consider now only the part $\mathscr{H}$ of the graph formed by the edges and vertices of height up to i: then the two edges corresponding to O_1 and O_2 belong to distinct connected components of $\mathscr{H}$, call them $\mathscr{C}$, resp. $\mathscr{D}$.

We can interpret then O_1 as the set of lower ends in the subgraph $\mathscr{C}$, and similarly for O_2.

Since $\tau_1 \cdot \ldots \cdot \tau_i$ acts as a cyclical permutation on O_1, if we define the subset $C \subset \{1, \dots, i\}$ as the one corresponding to the nodes of $\mathscr{C}$, then C has $m_1 - 1$ elements and $\tau_C := \prod_{c \in C} \tau_c$ gives on O_1 the same cyclical permutation as $\tau_1 \cdot \ldots \cdot \tau_i$, whereas each τ_c acts trivially on O_2.

Similarly we define $D \subset \{1, \dots, i\}$ and $\tau_D := \prod_{d \in D} \tau_d$ is a cyclical permutation on O_2, each τ_d acts trivially on O_1.

As in [C-P], page 632, we define $T_C := \prod_{c \in C} T_c, T_D := \prod_{d \in D} T_d$ and we notice (ibidem, page 633) that

$$T_C(\tau_j) = \tau_j, \quad \text{for } j \leqq i\,,$$

$$T_D(\tau_j) = \tau_j, \quad \text{for } j \leqq i\,;$$

whereas the action of T_C^{-1} on τ'_{i+1} is given by conjugation by τ_C, respectively the action of T_D^{-1} on τ'_{i+1} is given by conjugation by τ_D.

It is now obvious that conjugating $\tau'_{i+1} = (x', y')$ with appropriate powers of τ_C and τ_D we obtain $\tau_{i+1} = (x, y)$.

There remains to prove the last assertion.

In order to do this, it will be sufficient to give an example of such μ, μ' in the case $g = 1$, $d = 4$. In fact, let $\mathscr{G}$ be the graph corresponding to μ, μ'. For each $d \geqq 4$, $g \geqq 2$, we set $d' = d - 2$, $g' = g - 2$, whereas for $g = 1$, $d \geqq 5$, we set $d' = d - 3$, $g' = 0$. Thus in any case we obtain $d' \geqq d - 3$, $g' \geqq 0$, and $2g + 2d - 2 = 2g' + 2d' - 2 + 8$. 8 is the number of nodes of $\mathscr{g}$.

Pick any admissible graph $\mathscr{H}$ in $\mathfrak{A}_{\mathscr{G}_{g',d'}}$, and construct an admissible graph $\mathscr{K}$ in $\mathfrak{A}_{\mathscr{G}_{g,d}}$ from $\mathscr{g}$ and $\mathscr{H}$ in the following way:

1) add to $\mathscr{g}$ $(d - 4)$ strings consisting of 9 edges and 10 vertices each, thus getting $\mathscr{g}'$ with d lower ends and d upper ends,

2) add to $\mathscr{H}(d - d')$ strings made of $(2g' + 2d' - 1)$ edges and $(2g' + 2d')$ vertices, thus getting $\mathscr{H}'$ with d lower, resp. upper ends,

3) glue the upper ends of $\mathscr{g}'$ with the lower ends of $\mathscr{H}'$ in such a way that we get a connected graph $\mathscr{K}'$ (this is possible, simply glue the lower strings to ends of $\mathscr{H}$).

4) $\mathscr{K}'$ has no node at height 9, therefore to obtain $\mathscr{K}$ simply delete the vertices at height 9, and for each such vertex collapse to one edge the two edges meeting it.

As in the surjectivity statement we see easily that we can extend the two monodromies μ, μ' mapping to the symmetric group $\mathscr{S}_4$ to monodromies M, M' mapping to the symmetric group $\mathscr{S}_d$.

Moreover, the first 8 transpositions will act only on the given 4 elements. Assume that M, M' are in the same $\bigwedge_{2g+2d-2}$-orbit.

Then notice that the circular braids T_j, for $j \geqq 9$, act trivially on the first 8 transpositions $\tau_1, \ldots, \tau_8$ (resp. $\tau'_1, \ldots, \tau'_8$). Therefore it follows that μ, μ' have to be in the same $\bigwedge_8$-orbit, a contradiction.

The case $g = 1$, $d = 4$ will follow from proposition (3.8) which will deal with the example forthcoming in (3.7). □

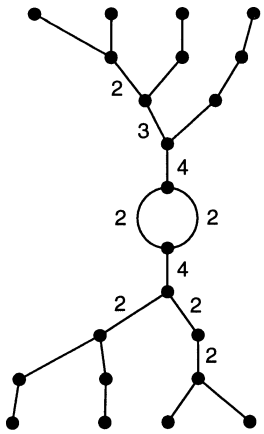
(3.7) Example ($g = 1$, $d = 4$). We shall now consider a set of 4 elements $\{a, b, c, d\}$ and three monodromies μ, μ', μ'' yielding three classes in $\mathscr{M}_{8,4}$ (as it is easy to verify).

We shall give the monodromies by writing three sequences of 8 transpositions $((\tau_1, \ldots, \tau_8)(\tau'_1, \ldots) \ldots)$. Namely:

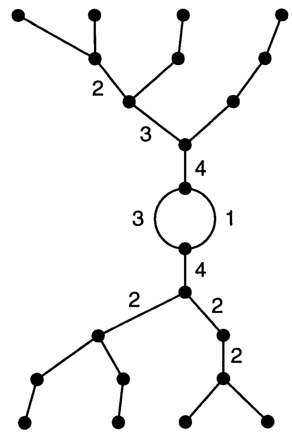
1	2	3	4	5	6	7	8
(c,d),	(a,b),	(b,c),	(a,d),	(a,d),	(c,d),	(a,b),	(a,d)
(c,d),	(a,b),	(b,c),	(a,d),	(a,b),	(a,d),	(b,c),	(a,b)
(c,d),	(a,b),	(b,c),	(a,c),	(a,c),	(a,c),	(b,c),	(c,d) .

If we associate to μ, μ', μ'' the respective admissible edge weighted biended graphs in $\mathfrak{A}_{\mathscr{G}_{1,4}}$, we obtain the same one for μ, μ', whereas for μ'' the graph is the same except for the weighting.

Here is the picture of the two elements of $\mathfrak{A}_{\mathscr{G}_{1,4}}$:



1) graph associated to μ, μ'



2) graph associated to μ''

(3.8) Proposition. *The above monodromy classes* μ, μ' *given in* (3.7) *are not in the same* $\bigwedge_8$*-orbit.*

Proof. The statement can of course be shown by brute force calculation, but we prefer to give a simple argument.

First of all, consider the sequence of the first 5 transpositions of any given monodromy, and call them $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$. The braids T_6, T_7, T_8 act trivially on them.

By the usual formula ([C-P], page 633), $T_1 + T_2 + T_3 + T_4 + T_5$ also acts trivially on them. T_5 acts trivially on $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, while it conjugates σ_5 by the product $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_4$, (here we mean, σ_1 acts first, then $\sigma_2, \dots$).

If we assume, as we do, that the associated weighted graph $\mathscr{G}$ is the same as for μ, μ', then $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_4$ is a double cycle, and therefore we conclude that $2\ T_5 = 0$ (we mean that the image of $2\ T_5$ into the group of permutations of this subset of 5-tuples of transpositions is trivial).

Although not strictly indispensable, we may observe more. Since σ_1 is always left unchanged, and T_1 acts by conjugation by σ_1, we infer that also $2\,T_1 = 0$.

Again if the sequence of σ_j's is in our special subset having $\mathscr{G}$ as associated graph, then σ_1, σ_2 have disjoint support, whence also σ_2 is always left unchanged (T_1 infact conjugates σ_2 by σ_1, which commutes with it, $T_1 + T_2$ is trivial on it as well as the T_j's with $j \geqq 3$).

It follows then immediately that also $2\,T_2 = 0$.

Let us now look at the orbit of the 5-tuple $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$. Its orbit corresponds bijectively to a quotient of the (Abelian!) group generated by

T_1, T_2, T_3, T_4, T_5 with the relations $T_1 + T_2 + T_3 + T_4 + T_5 = 0 = 2\,T_1 = 2\,T_2 = 2\,T_5$.

Since moreover τ_3 and $\tau_4 = \tau_5$ have disjoint support, T_3 acts trivially on τ_4, τ_5 and by our previous relations we then also get $2\,T_3 = 0$.

In fact we can show that the orbit of $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ has precisely 16 elements (although only 4 conjugacy classes), but we can conclude rather quickly as follows.

Consider the mutually exclusive properties
i) $\sigma_4 = \sigma_5$,
ii) σ_4 and σ_5 have disjoint support,
iii) σ_4 and σ_5 do not commute.
These properties are clearly left unchanged by the braids T_1, T_2, T_3 (since they act on σ_4, σ_5 by conjugation).

The conclusion is that in the orbit of $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ only two of the above can hold. Since i) holds for $\sigma = \tau$, and ii) holds for $T_5(\tau_4) = \tau_4 = (a, d)$ and $T_5(\tau_5) = (b, c)$, we conclude that iii) does not hold true in the above orbit.

Since iii) holds for $(\tau'_1, \tau'_2, \tau'_3, \tau'_4, \tau'_5)$, and i)–iii) are properties which depend only upon the conjugacy class of the monodromy, it follows that the classes of μ, μ' are not in the same $\bigwedge_8$-orbit. $\square$

In the case $g = 0$ the situation is different.

(3.9) Theorem. *Let $\mathscr{T}$ be the bi-ended graph* (*without labelling of the edges*) *associated to a lemniscate generic rational function of degree d.*
1) *$\mathscr{T}$ is a tree* (*i.e. simply connected*).
2) *If we impose that the weight of the edges containing the ends is one, then there is a unique way to label the edges of $\mathscr{T}$ so that Kirchoff's rule holds.*

Therefore, in the case of rational functions, our main result can be reformulated in the following way.

(3.10) Corollary. *There are natural bijections between the following sets*:
(a) *the set $\pi_0(\mathscr{L}_{0,d})$ of connected components of $\mathscr{L}_{0,d}$,*
(b) *the set of $\bigwedge_{2d-2}$-orbits in $\mathscr{M}_{2d-2,d} = \{[\mu]:\ \pi_1(\mathbb{P}^1 - B, 0) \to \mathscr{S}_d :\ \mu(\gamma_i) = \tau_i$ is a transposition, μ has transitive image and $\prod \tau_i = \mathrm{id}\}$,*
(c) *the isomorphism classes of biended admissible graphs* (*i.e., such that* 1), 2) *and* 3) *of* (3.1) *hold*) *which admit an admissible weighting of the edges* (*i.e., such that* 4) *and* 5) *of* (3.1) *hold*).

Recall only that we pass from (a) to (c) by considering the big lemniscate configuration of a function f.

We shall prove (3.9) with a more general argument than the one we gave in the first version of the paper, following a suggestion by Rick Miranda.

Namely, we shall consider a wider class of graphs than our admissible graphs.

(3.11) Definition. a) *A* **graph** $\mathcal{G}$ *is said to be* **oriented** *if for each edge is assigned an orientation, i.e., a boundary vertex where the edges enters and one where the edges goes out.*

b) *Given an oriented edge weighted graph* $\mathcal{G}$, *the* **weight** *of a vertex is defined to be the sum of the weights of the incoming edges minus the sum of the outgoing edges.*

c) *Given an oriented edge weighted graph* $\mathcal{G}$, $\mathcal{G}$ *is said to satisfy* **Kirchoff's law** *if each vertex which is not an end (i.e., which has valency* $\geqq 2$*) has weight zero.*

d) *An edge incident to an end vertex is said to be an* **end edge**.

e) *The* **charge** *of an oriented edge weighted graph is the sum of the weights of the end vertices.*

(3.12) Remark. If $\mathcal{G}$ is an oriented edge weighted graph satisfying Kirchoff's law, then the charge of $\mathcal{G}$ is equal to zero.

Proof. The charge equals then the sum of the charge with the sum of the weights of the non end vertices. In turn the latter is the sum over the incoming and over the outgoing vertices, therefore we get the sum of the weight edges minus itself. □

(3.13) Proposition. *Let* $\mathcal{G}$ *be an oriented tree. Then, for each choice of weights of the end edges such that the charge of* $\mathcal{G}$ *is thus zero, there is a unique way of extending the choice of weights to all the edges in such a way that Kirchoff's law holds true.*

Proof. The proof goes by induction on the number of non end edges. If this number is zero, then $\mathcal{G}$ is a star, i.e., there is a unique non end vertex where all edges are incident. Clearly the weight of this vertex is minus the charge, thus Kirchoff's law holds.

Otherwise, let e be a non end edge. Since $\mathcal{G}$ is a tree, there are exactly two vertices v^+, v^- incident to e (as the notation suggests, e is outgoing from v^-).

Let $\mathcal{G}'$ be the tree obtained by deleting the edge e and collapsing v^+, v^- to a single vertex v. By induction, $\mathcal{G}'$ admits a unique edge weighting satisfying Kirchoff's law and extending the given weighting of the end edges.

The weight of v equals $w^+ + w^-$, where w^+ is the sum (counted with sign) of the weights of the edges incident to v^+ and different from e (Similarly for w^-).

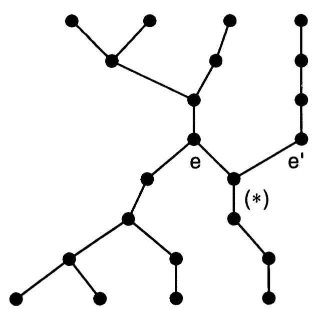
Since this weight is zero, the only possibility is to define the weight of e as $w^- = -w^+$ and Kirchoff's law is thus satisfied. □

Proof of Theorem 3.9. We have a tree $\mathcal{T}$, which is naturally oriented going up from the lower end vertices to the upper end vertices.

The weights of the end vertices have to be equal to 1, but the number of upper ends equals the number of lower ends, therefore the charge is zero. □

(3.14) Remarks. 1) We were not so precise when specifying the group where the weights should take values. In fact, as the proof of 3.13 shows, if the weights of the end edges takes values in a group G, say $\mathbb{Z}$, then also the other weights w take value in G.
2) If on the other hand the weights of the end edges take values in $\mathbb{N}^+$, not necessarily the other weights take values in $\mathbb{N}^+$: this is the reason why Theorem 3.9, contrary to proposition 3.13, is only a unicity result, and not an existence result. The possibility of non positive weights creates obstructions for graphs in order to be associated to algebraic functions, as Example 3.15 will show.
3) More generally, if the graph in 3.13 is not a tree, the edge weightings satisfying Kirchoff's rule and extending a given end-edge weighting with charge zero are seen to be an affine space of dimension equal to the first Betti number of the graph.

(3.15) Example. However we choose a partition $\mathscr{E} = \mathscr{E}^\infty \cup \mathscr{E}^0$ (into two sets of 4 elements) of the set $\mathscr{E}$ of ends, the following graph does not admit a weighting of the edges by positive integers, such that 4), 5) of (3.1) is fulfilled.



In fact, the edge $(*)$ would necessarily have weight 1, but the other two edges e, e' are of different level than $(*)$. Hence the graph cannot satisfy Kirchoff's rule with strictly positive edge weights.

We end this section by drawing a corollary of the proof of Proposition (3.5).

(3.16) Remark. Let $\mathscr{G}$ be an admissible bi-ended, edge weighted graph of genus g and degree d. Define a *loop upward node* v to be a node such that the following holds:
1) if the height of v is $(i+1)$, there are two edges e, e' of level i incident to v,
2) consider the part $\mathscr{H}$ of the graph formed by the edges and vertices of height up to i: then the two edges e, e' belong to the same connected component of $\mathscr{H}$.
For each loop upward node, consider the G.C.D. of the weights of the incident edges, finally take the product D of these G.C.D.'s over all the loop upward nodes. Then
3) the number of $\bigwedge_{2g+2d-2}$-orbits mapping to $\mathscr{G}$ is bounded by D.

4 Small lemniscate configurations

Let $f : C \to \mathbb{P}^1$ be an algebraic function. Then (cf. [Pa]) the level sets of the real valued function $|f|$ can be characterized as the integral curves of the following O.D.E.

In fact, $|f|^2 =$ constant iff $d\log(f)$ pulls back to an imaginary form on the curve. Since $d\log(f)$ is a closed form, given a local coordinate z on C, there exists a real parameter t on the integral curve such that $\frac{f'}{f}dz = i(dt)$, or, in other words, the following O.D.E. is satified

$$\frac{dz}{dt} = i\frac{f}{f'} \ . \tag{4.1}$$

(4.2) Definition. *An O.D.E. on a Riemann surface is said to be integrable by algebraic functions iff there exists a diffeomorphism with an algebraic curve C such that the O.D.E. becomes*
(4.2a) $\mathrm{Re}(\omega) = 0$, *where*
(4.2b) ω *is a meromorphic* 1*-form with simple poles and with integer residues* m_i *at the poles* P_i, *and*
(4.2c) *more generally the periods of* ω (*integrals over closed paths in* $C - \{$*poles of* $\omega\}$) *are integer multiples of* $2\pi i$.

(4.3) Remark. 1) If ω is as in (4.2), then $\int \omega$ is multivalued, but c) implies that $f = \exp(\int \omega)$ is single valued in $C - \{\text{poles of } \omega\}$, and by b) f is meromorphic on C, whence algebraic.
2) Thus, if ω is as in (4.2), the integral curves are the connected components of the level sets of $|f|$; in particular, the singular solutions correspond to the poles and zeros of f, and to the singular connected components of the level sets of $|f|$.
We have therefore a subset of the big lemniscate configuration Γ_f (cf. 2.3).

(4.4) Definition. *Given an algebraic function f, its small lemniscate configuration is the subset*

$$\textstyle\sum_f = f^{-1}(0) \cup f^{-1}(\infty) \cup \bigcup_{i=1,\dots,k} \bigwedge_{c_i} , \tag{4.4a}$$

where $\bigwedge_{c_i}$ *is the union of the connected components of* Δ_{c_i} *passing through the critical points of f where $|f|$ takes the value c_i.*

Therefore, the topological classification of the O.D.E.'s integrable by algebraic functions on a Riemann surface is equivalent to the problem of the topological classification of the small lemniscate configurations of algebraic functions.

In view of our surjectivity statement (3.3)–(3.5), we obtain the following corollary

(4.5) Corollary. *The topological classification of small lemniscate configurations of lemniscate generic algebraic functions of genus g and degree d is given by the isomorphism classes of connected graphs $\mathscr{H}$ such that*
i) *$\mathscr{H}$ has $2d$ ends, $2g + 2d - 2$ nodes and no other vertices,*
ii) *$\mathscr{H}$ is obtained from an admissible bi-ended edge weighted graph $\mathscr{g}$ by the following forgetful map:*
iii) *forget the colouring of the end vertices of $\mathscr{g}$, forget the weights of the edges, remove the vertices of valencies 2, and collapse to an edge the strings of edges connecting an end to a node.*

We omit the proof, since it is straightforward.

Observe only that ii) implies i), but in this light example 3.15 shows that the converse is not true even for $g = 0$.

In any case from $\mathscr{H}$ one can as usual reconstruct the Riemann surface, by glueing disks for each end vertex, and pairs of pants for each node.

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References

[B-C] I. Bauer, F. Catanese: Generic lemniscates of algebraic functions. Math. Gott. Heft **8** (1992) 1–32

[B-Fr]$_1$ R. Biggers, M. Fried: Moduli spaces of covers and the Hurwitz monodromy group. J. Reine Angew. Math. **335**, 87–121 (1982)

[B-Fr]$_2$ R. Biggers, M. Fried: Irreducibility of moduli spaces of cyclic unramified covers of genus g curves. Trans. Am. Math. Soc. **295** (n.1) 59–70 (1986)

[B-H] A. Borel, A. Haefliger: La classe d'homologie fondamentale d'un espace analytique. Bull. Soc. Math. Fr. **89**, 461–513 (1961)

[B-M] A. Borel, J.C. Moore: Homology theory for locally compact spaces. Mich. Math. J. **7**, 137–159 (1960)

[C-P] F. Catanese, M. Paluszny: Polynomial-lemniscates, trees and braids. Topology **30** (n. 4) 623–640 (1991)

[C-W] F. Catanese, B. Wajnryb: The fundamental group of generic polynomials. Topology **30** (n. 4) 641–651 (1991)

[Cl] A. Clebsch: Zur Theorie der Riemann'schen Fläche. Math. Ann. **6**, 216–230 (1872)

[Fu] W. Fulton: Hurwitz schemes and the irreducibility of moduli of algebraic curves. Ann. Math. **90**, 542 (1969)

[Fu2] W. Fulton: On the irreducibility of the Moduli space of curves. Invent. Math. **67**, 87–88 (1982)

[G-R] H. Grauert, R. Remmert: Komplexe Råume. Math. Ann. **136**, 245–318 (1958)
[Hu] A. Hurwitz: Über Riemann'sche Flachen mit gegebenen Verzweigungspunkten. Math. Ann. **39**, 1–61 (1891)
[H-M] J. Harris, D. Mumford: On the Kodaira dimension of the moduli space of curves. Invent. Math. **67**, 23–86 (1982)
[Kl] P. Kluitmann: Hurwitz action and finite quotients of braid groups. Contemp. Math. **78**, 299–325 (1988)
[Lo] E. Looijenga: The complement of the bifurcation variety of a simple singularity. Invent. Math. **23**, 105–116 (1974)
[Pa] M. Paluszny: On periodic solutions of polynomial ODE's in the plane. J. Diff. Equat. **53**, 24–29 (1984)