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Some remarks on the universal cover of an open K3 surface

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Abstract. We shall give, in an optimal form, a sufficient numerical condition for the finiteness of the fundamental group of the smooth locus of a normal K3 surface. We shall moreover prove that, if the normal K3 surface is elliptic and the above fundamental group is not finite, then there is a finite covering which is a complex torus.

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Introduction

Throughout this note, we work in the category of separated complex analytic spaces endowed with the classical topology.

The first aim of this short note is to add some evidence to the following interesting Conjecture posed by De-Qi Zhang (See [KT], [SZ], [KZ] and [Ca2, 3] for related work):

Conjecture. The universal cover of the smooth locus of a normal K3 surface is a big open set of either a normal K3 surface or of $\mathbf{C}^2$. Furthermore, in the latter case, the universal cover factors through a finite étale cover by a big open set of a torus.

To explain our notation, here and hereafter, a *normal K3 surface* means a normal surface whose minimal resolution is a K3 surface and a *torus* is a 2-dimensional complex torus. By a *big open set* we mean the complement of a discrete subset.

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Note that, by a result of Siu [Si], a K3 surface is always Kähler.

Since the canonical divisor of a smooth K3 surface is trivial, it follows that the singularities of a normal K3 surface can only be the so called Du Val singularities (also called Rational Double Points): these are also the Kleinian singularities obtained as quotients $\mathbf{C}^2/G$ with $G \subset SL(2, \mathbf{C})$.

The property that the singularities x_i are of the form $\mathbf{C}^2/G_i$ allows to define the orbifold Euler number of the normal K3 surface. For each singular point x_i we take a neighbourhood U_i of x_i which is the quotient of a ball in $\mathbf{C}^2$ and decree that its orbifold Euler number equals $1/|G_i|$: using a Mayer Vietoris sequence we can then extend the definition to the whole of the normal K3 surface. One has that the orbifold Euler number is non negative, and indeed R. Kobayashi and A. Todorov [KT] showed that the second case in Zhang's conjecture happens if and only if the orbifold Euler number of the normal K3 surface is zero.

Let $\overline{S}$ be a normal K3 surface, let $S^0 := \overline{S} - \operatorname{Sing} \overline{S}$ and let $\nu : S \to \overline{S}$ the minimal resolution.

Our first result is

Theorem A. *If the normal K3 surface $\overline{S}$ admits an elliptic fibration then either $\pi_1(S^0)$ is finite or there is a finite covering of $\overline{S}$, ramified only on a finite set, which yields a complex torus.*

Our second aim in this note is to establish a sharp sufficient condition for the validity of the first alternative in Zhang's conjecture.

We set E to be the reduced exceptional divisor $E := \nu^{-1}(\operatorname{Sing} \overline{S})$ and decompose E into irreducible components $E := \sum_{i=1}^{r} E_i$. An important invariant is the number $r := r(\overline{S})$ of irreducible components of the exceptional divisor. Clearly, $S^0 = S - E$.

Our observation is as follows:

Theorem B. *If $r = r(\overline{S}) \leq 15$, then $\pi_1(S^0)$ is finite. In particular, if $r \leq 15$, then the universal cover of S^0 is a big open set of a normal K3 surface.*

This easy remark however gives us the best possible uniform bound on r in order that $\pi_1(S^0)$ be finite, in view of the following facts [KT], [KZ]:

(1) A normal Kummer surface $A/-1$ satisfies $r = 16$ and $|\pi_1((A/-1)^0)| = \infty$.
(2) There is a normal K3 surface $\overline{S}$ such that $r(\overline{S}) \leq 15$ but $\pi_1(S^0) \neq \{1\}$.

In our actual proof, the numerical condition $r \leq 15$ will be used twice: in Lemmas 1 and 2. These two Lemmas allow us to establish the existence of an elliptic pencil, whence to reduce the proof of Theorem B to Theorem A. For the last statement of Theorem B, we recall that the category of Du Val singularities is closed under the operation of taking the normalization of a finite covering which is unramified outside a finite set.

Therefore, by the classification of smooth compact Kähler surfaces due to Castelnuovo, Enriques, Kodaira, the normalization of a finite cover of a normal

K3 surface is either a normal K3 surface or a 2-dimensional torus if the covering is unramified outside a finite set . This fact will be frequently used in our proof, too.

The proof of theorem A is on one side based on an exact sequence for open fibred surfaces, and which relies on the notion of orbifold fundamental group (cf. e.g. [Ca2]).

On the other hand, it is based on the existence of finite branched covers of given branching type provided by finite index subgroups of the orbifold fundamental group.

Of course, the most interesting part of the Conjecture concerns the case where the fundamental group is infinite. As such, the question seems to belong more to the transcendental theory and we hope to return on the question using the current technologies on uniformization problems.

In the last paragraph of this note (Remark 4) we will just comment on a more or less obvious reduction process.

Proofs

Lemma 1. *If $r = r(\overline{S}) \leq 15$ then $\overline{S}$ does not admit any finite covering by a complex torus which is unramified outside a finite set.*

Proof. Assuming the contrary, we shall show that $r \geq 16$. Let Q_k $(1 \leq k \leq m)$ be the singular points of $\overline{S}$ and let n_k be the number of the irreducible components of $\nu^{-1}(Q_k)$. Note that the contribution of Q_k to the orbifold Euler number of S is $(n_k + 1) - 1/\delta_k$, where δ_k is the order of the local fundamental group around Q_k. Since the orbifold Euler number of $\overline{S}$ equals 0 (the obvious direction of [KT]) one has

$$e(S) = 24 = \sum_{k=1}^{m} \left(n_k + 1 - \frac{1}{\delta_k} \right).$$

Note that $\delta_k = 2$ if $n_k = 1$. Then, one has also

$$n_k + 1 - \frac{1}{\delta_k} \leq \frac{3}{2} n_k.$$

Now, substituting the second inequality into the first equality, one obtains

$$24 \leq \frac{3}{2} \sum_{k=1}^{m} n_k = \frac{3}{2} r.$$

This implies $r \geq 16$. □

Lemma 2. *If $r = r(\overline{S}) \leq 15$ then there exist an elliptic K3 surface $f' : S' \to \mathbf{P}^1$ and an effective divisor E' supported in fibers of f' such that S^0 is diffeomorphic to $S' - E'$ and such that E and E' are of the same Dynkin type.*

Proof. We argue by descending induction on $\rho(S)$. If $\rho(S) = 20$, then S is algebraic and the orthogonal complement of $\mathbf{Z}\langle E_i | 1 \le i \le r\rangle$ in $\mathrm{Pic}(S)$ is a hyperbolic lattice of rank $20 - r(S) \ge 20 - 15 = 5$.

Then, by Meyer's theorem (cf. e.g. [Se], Chapter IV, 3, Cor. 2), there exists $X \in \mathrm{Pic}(S) - \{0\}$ such that $(X.E_i) = 0$ for all E_i and such that $(X^2) = 0$. (*However, in general this X does not lie in the nef cone of S.*)

Let $u : \mathcal{U} \to \mathcal{K}$ be the Kuranishi family of S and choose a trivialization $R^2 u_* \mathbf{Z}_{\mathcal{U}} \simeq \Lambda$ over $\mathcal{K}$. Here Λ is an even unimodular lattice of index (3, 19) called the K3 lattice. Then one can define the period map $p : \mathcal{K} \to \mathcal{P}$, where $\mathcal{P}$ is the period domain. This p is a local isomorphism by the local Torelli Theorem and allows us to identify $\mathcal{K}$ with a small open set of the period domain $\mathcal{P}$ (denoted again by $\mathcal{P}$).

Let's denote by e_i, x the elements in Λ corresponding to E_i and X. Let us take the sublocus $\mathcal{B} \subset \mathcal{K}$ defined by the equations

$$\{\omega \in \mathcal{P} | (e_1.\omega) = (e_2.\omega) = \cdots = (e_r.\omega) = (x.\omega) = 0\},$$

and consider the induced family $\mathcal{T} \to \mathcal{B}$. Here $\mathcal{B}$ is of dimension $20 - 1 - r > 0$. Note that $h^0(\mathcal{O}_S(E_i)) = 1$ and $h^1(\mathcal{O}_S(E_i)) = 0$.

Then, by construction and by the base change Theorem, the smooth rational curves E_i lift uniquely to effective divisors $\mathcal{E}_i$ flat over $\mathcal{B}$ in such a way that the fibers $\mathcal{E}_{i,b}$ are smooth rational curves (of the same Dynkin type as E_i) and $S - \cup_{i=1}^r E_i$ is diffeomorphic to $\mathcal{T}_b - \cup_{i=1}^r \mathcal{E}_{i,b}$ for all $b \in \mathcal{B}$ if $\mathcal{B}$ is chosen to be sufficiently small. Furthermore, for generic $b \in \mathcal{B}$, the Picard group of $\mathcal{T}_b$ is isomorphic to the primitive closure of $\mathbf{Z}\langle x, e_i | 1 \le i \le r\rangle$ in Λ (See for instance [Og]) and is a semi-negative definite lattice. Thus $\mathcal{T}_b$ is of algebraic dimension one. Now, the algebraic reduction map of this $\mathcal{T}_b$ gives an elliptic fibration with the required properties.

Assume that $\rho := \rho(S) < 20$. As before, we let $u : \mathcal{U} \to \mathcal{K}$ the Kuranishi family of S and choose a trivialization $R^2 u_* \mathbf{Z}_{\mathcal{U}} \simeq \Lambda$ over $\mathcal{K}$ and identify $\mathcal{K}$ with a small open set of the period domain $\mathcal{P}$. By x_i $(1 \le i \le \rho)$ we denote the elements in Λ corresponding to some integral basis of $\mathrm{Pic}(S)$. Let us take the sublocus $\mathcal{A} \subset \mathcal{K} = \mathcal{P}$ defined by the equations

$$\{\omega \in \mathcal{P} | (x_1.\omega) = (x_2.\omega) = \cdots = (x_\rho.\omega) = 0\},$$

and consider the induced family $\pi : \mathcal{S} \to \mathcal{A}$. Here $\mathcal{A}$ is of dimension $20 - \rho > 0$.

Then, for the same reason as before, the smooth rational curves E_i lift uniquely to effective divisors $\mathcal{C}_i$ flat over $\mathcal{A}$ in such a way that the fibers $\mathcal{C}_{i,a}$ are smooth rational curves (of the same Dynkin type as E_i) and $S - \cup_{i=1}^r E_i$ is diffeomorphic to $\mathcal{S}_a - \cup_{i=1}^r \mathcal{C}_{i,a}$ for all $a \in \mathcal{A}$ if $\mathcal{A}$ is chosen to be sufficiently small. Therefore the orbifold Euler numbers are also the same for $\overline{S}$ and $\overline{\mathcal{S}}_a$. Here $\overline{\mathcal{S}}_a$ is the normal K3 surface obtained by the contraction of $\cup_{i=1}^r \mathcal{C}_{i,a}$. Furthermore, by construction, π

is not isotrivial and the fibers $\mathcal{S}_a$ satisfy $\rho(\mathcal{S}_a) \geq \rho$. Then by [Og] there is $a \in \mathcal{A}$ such that $\rho(\mathcal{S}_a) > \rho$. Now, we are done by descending induction on $\rho(S)$. □

By Lemmas 1 and 2, it is now clear that Theorem A implies Theorem B. Before we proceed to the proof of theorem A we state a very easy and quite general lemma.

Lemma 3. *Let $f : S \to C$ be a fibration of a complete surface onto a complete curve with general fiber F, and assume that for each singular fibre F_j we are given a divisor Z_j contained in F_j, possibly empty, but with support properly contained in $supp(F_j)$.*

Let S^0 be the Zariski open set of S defined as the complement of the union of the divisors Z_j.

Denote by W_j the maximal divisor $\leq F_j$ which has no common component with Z_j, and denote by m_j the greatest common divisor of the multiplicities of the components of W_j.

Set $P_j = f(F_j)$,where we may assume that the points $\{P_1, \cdots, P_k\}$ are all distinct, and denote by $B := C - \{P_1, \cdots, P_k\}$.

We define the orbifold fundamental group $\pi_1^{orb}(f^0)$ of the open fibration $f^0 : S^0 \to C$ as the quotient of $\pi_1(B)$ by the normal subgroup generated by the elements $\gamma_j^{m_j}$, where γ_j is a simple loop going around the point P_j. Then the fundamental group $\pi_1(S^0)$ fits into an exact sequence

$$\pi_1(F) \to \pi_1(S^0) \to \pi_1^{orb}(f^0) \to 1.$$

Proof. Let S' be the complement of the given fibres, i.e., $S' = S - \{F_1, \cdots, F_k\}$, so that $f' : S' \to B$ is a fibre bundle with fibre F. Then we have the homotopy exact sequence

$$\pi_1(F) \to \pi_1(S') \to \pi_1(B) \to 1.$$

However, $S' = S^0 - \{D_1, \cdots, D_k\}$ where $D_j := F_j|S^0$: therefore the kernel K of the surjection $\pi_1(S') \to \pi_1(S^0)$ is normally generated by simple loops $\delta_{j,i}$ going around the components $D_{j,i}$ of the D_j's.

Each $\delta_{j,i}$ maps into $\pi_1(B)$ to the $m_{j,i}$-th power of a conjugate of γ_j, where $m_{j,i}$ is the multiplicity of $D_{j,i}$ in D_j. Whence the image K' in $\pi_1(B)$ of K is normally generated by the elements $\gamma_j^{m_{j,i}}$. Since $D_j = W_j|S^0$ we see that K' is normally generated by the elements $\gamma_j^{m_j}$, thus we have the desired surjection $\pi_1(S^0) \to \pi_1^{orb}(f^0) \to 1$ with kernel generated by $\pi_1(F)$.

One may more precisely observe that, unless S' is complete, then B is homotopically equivalent to a bouquet of circles. Since the fibre is connected, it follows that $S' \to B$ has a section. Whence our group $\pi_1(S')$ is indeed a semidirect product of the normal subgroup $\pi_1(F)$ with the free subgroup $\pi_1(B)$ generated by the γ_j's.

The structure of the semidirect product is provided by the so-called monodromy homomorphism $T : \pi_1(B) \to Aut(\pi_1(F))$.

In other words, if we set $T_j := T(\gamma_j)$, $\pi_1(S')$ is the quotient of the free product $\pi_1(F) * \pi_1(B)$ by the relations $\gamma_j^{-1}\phi\gamma_j = T_j(\phi)$, $\forall \phi \in \pi_1(F)$ and $\forall j$.

The kernel K mentioned above is generated by the elements which are conjugates of some $\delta_{j,i}$, thus it is normally generated by elements which can be written in the form $\phi_{j,i}\gamma_j^{m_{j,i}}$, for some suitable element $\phi_{j,i}$. □

Remark. In the special case where the G.C.D. m_j is 1, $\forall j$, we can replace the orbifold fundamental group by the fundamental group of the base.

A fortiori, this holds if $\forall j$, $\exists i$ s.t. $m_{j,i} = 1$: this was precisely the assumption made in [No], Lemma 1.5 (as observed by the referee).

Theorem A. *Let $\overline{S}$ be a normal K3 surface such that $\overline{S}$ admits an elliptic fibration $\overline{f} : \overline{S} \to \mathbf{P}^1$. Then either*

(1) $\pi_1(S^0)$ is finite or
(2) $\overline{S}$ admits a finite covering by a torus, ramified only on a finite set.

Proof. Note by the way that all the assumptions and assertions made in the theorem are stable under replacement of S^0 by a finite unramified covering.

Let $f : S \to \mathbf{P}^1$ be the elliptic fibration induced by $\overline{f}$. We denote by F a general fiber of f and use the same notation as in lemma 3, so that $f(E) \subset \{P_1, \cdots, P_k\}$, $B := \mathbf{P}^1 - \{P_1, \cdots, P_k\}$.

Let $F_j := f^{-1}(P_j)$ be the scheme theoretic fiber, and $D_j := F_j|S^0$. Note that Supp D_j is not empty for each j because the intersection product on the fibre is not negative definite.

Lemma 3 gives us an exact sequence

$$\pi_1(F) \simeq \mathbf{Z}^{\oplus 2} \to \pi_1(S^0) \to \pi_1^{orb}(f^0) \to 1.$$

We subdivide our analysis into two cases:

(1) $\pi_1^{orb}(f^0)$ is finite.
(2) $\pi_1^{orb}(f^0)$ is infinite.

In case (1), we take the unramified covering of S^0 associated with the epimorphism onto $\pi_1^{orb}(f^0)$. We thus get another elliptic normal K3 surface since the minimal resolution has the canonical divisor trivial and the first Betti number $b_1 \leq 2$ (by the classification theorem, either $b_1 = 4$ and we have a torus, or $b_1 = 0$ and we have a K3 surface).

Moreover, we are reduced to the case where $\pi_1^{orb}(f^0)$ is trivial.

In fact, the unramified covering of S^0 associated to the finite group $G := \pi_1^{orb}(f^0)$ is the normalization of the fibre product of the given fibration with the ramified Galois cover associated to G. Therefore, in the new fibration, for each fibre the G.C.D. of the multiplicities of the components equals 1.

In this case we can see that, the fundamental group $\pi_1(F) \simeq \mathbf{Z}^{\oplus 2}$ being abelian, the fundamental group $\pi_1(S^0)$, which is a quotient of $\pi_1(F) \simeq \mathbf{Z}^{\oplus 2}$ by the subgroup generated by the images of $I_2 - T_j$ (I_2 being the identity matrix and T_j being the local monodromy matrix around the point P_j), is a finite group.

As pointed out by the referee, this statement is sufficient, and follows in a more simple way by the cohomology long exact sequence of the pair (S, E), as in prop. 1.8. of [Ca1]. Thus the fundamental group $\pi_1(S^0)$ is the cokernel of $H^2(S, \mathbf{Z}) \to H^2(E, \mathbf{Z})$, since $S^0 = S - E$ (E is the union of the previously defined Z_j's). But this cokernel is finite since the intersection matrix for E is negative definite.

In case (2), we use that $\pi_1^{orb}(f^0)$ is the covering group of a non compact simply connected Riemann surface Σ branched over $\mathbf{P}^1$ with branching locus equal to $\{P_1, \cdots, P_k\}$ and branching multiplicities $\{m_1, \cdots, m_k\}$ (i.e., $\pi_1^{orb}(f^0)$ acts on Σ with quotient $\mathbf{P}^1$).

If the Riemann surface Σ were the disk then we would get a Fuchsian group and there is a normal subgroup $\Gamma \subset \pi_1^{orb}(f^0)$ of finite index acting freely on Σ, whence we get a finite Galois covering $C \to \mathbf{P}^1$ where C has genus at least 2.

The epimorphism $\pi_1(S^0) \to \pi_1^{orb}(f^0)/\Gamma$ yields an unramified covering of S^0 which compactifies to a smooth surface with trivial canonical bundle. But such a surface admits no nontrivial holomorphic map to a curve of genus ≥ 2, thus we conclude that Σ is the affine line.

By considering again a normal subgroup $\Gamma \subset \pi_1^{orb}(f^0)$ of finite index acting freely on Σ, we get an elliptic curve C and a Galois covering $C \to \mathbf{P}^1$. Taking the normalization of the fibre product we obtain again an unramified covering of S^0 which compactifies to a surface X with trivial canonical bundle. Since we get a map of X to an elliptic curve, X is a torus (observe that X is minimal), and this proves our claims. □

Remark 4. If $\pi_1(S^0)$ is residually finite for all $\overline{S}$ with $e_{\mathrm{orb}}(\overline{S}) > 0$, then the Conjecture has an affirmative answer.

Proof. The case $e_{\mathrm{orb}}(\overline{S}) = 0$ is covered by [KT]. The remaining case follows from the assumption together with the uniform boundedness of the order of finite automorphism groups of K3 surfaces. Here the uniform boundedness is a consequence of the global Torelli Theorem together with the Burnside property of finite groups of $GL_n(\mathbf{Z})$. One may use the result of Mukai [Mu]. □

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