Canonical rings of surfaces whose canonical system has base points

I. C. Bauer, F. Catanese, R. Pignatelli[*]

August 5, 2013

Contents

0   Introduction

In Enriques' book on algebraic surfaces ([Enr]), culminating a research's lifespan of over 50 years, much emphasis was set on the effective construction of

---

1

surfaces, for instance of surfaces with $p_g = 4$ and whose canonical map is a birational map onto a singular surface Σ in $\mathbb{P}^3$.

The problem of the effective construction of such surfaces for the first open case $K^2 = 7$ has attracted the attention of several mathematicians, and special constructions have been obtained by Enriques ([Enr]), Franchetta ([Fran]), Maxwell ([Max]), Kodaira ([Kod]). Until Ciliberto ([Cil1]) was able to construct an irreducible Zariski open set of the moduli space of (minimal algebraic) surfaces with $K^2 = 7$, $p_g = 4$, and constituted by surfaces with a birational canonical morphism whose image Σ has ordinary singularities.

Later on, through work of the first two named authors and of Zucconi ([Ba], [Cat1], [Zuc]), the complete classification of surfaces with $K^2 = 7$, $p_g = 4$ was achieved, and it was shown in [Ba] that the moduli space consists of three ireducible components (two of them consist of surfaces with non birational canonical map). But, as in the previous work of Horikawa ([HorI-V], [HorQ]) who classified surfaces with $K^2 = 5, 6$, $p_g = 4$, a complete picture of the moduli space is missing (for instance, it is still an open question whether the moduli space for $K^2 = 7$, $p_g = 4$ has one or two connected components).

Usually, classifying surfaces with given invariants K^2, p_g, is achieved by writing a finite number of families such that every such surface occurs in precisely one of those families. Each family yields a locally closed stratum of the moduli space, and the basic question is how are these strata patched together.

Abstract deformation theory is very useful since Kuranishi's theorem ([Kur]) gives a lower bound for the local dimension of the moduli space, thus it helps to decide which strata are dominating a component of the moduli space.

In principle, the local structure of the moduli space ([Pal]) is completely described by a sequence of Massey products on the tangent cohomology of the surface, and Horikawa clarified the structure of the moduli space in the "easy" case of numerical 5-ics ($K^2 = 5, p_g = 4$) by using the Lie bracket $H^1(S, \Theta_S) \times H^1(S, \Theta_S) \to H^2(S, \Theta_S)$.

However, the analytic approach does not make us see concretely how do surfaces belonging to one family deform to surfaces in another family, and therefore Miles Reid, in the Montreal Conference of 1980 proposed to look at the deformations of the canonical rings for numerical 5-ics (cf.[Rei0]).

His program was carried out by E. Griffin ([Gri]) in this case, later on D. Dicks found an interesting approach to the question and applied it to the case of surfaces with $K^2 = 4, p_g = 3$ ([Dic1],[Dic2]). His method was clearly exposed in the article ([Rei2]) by Miles Reid, where he set as a challenge the problem to apply these methods to the hitherto still partially unexplored

2

case of surfaces with $K^2 = 7$, $p_g = 4$.

In [Cat1] was given a method (of the so called quasi generic canonical projections) allowing in principle to describe the canonical rings of surfaces of general type. The method works under the assumption that the surface admits a morphism to a 3-dimensional projective space which is a projection of the canonical map, and is birational to its image Σ.

What happens when the canonical system has base points, in particular in our case of surfaces with $K^2 = 7$, $p_g = 4$?

Thus the first aim of this paper is to introduce a general method to calculate the canonical ring of minimal surfaces of general type whose canonical system has base points but yields a birational canonical map.

We will then apply this method in the case of the surfaces S with $K^2 = 7$, $p_g = 4$. We will compute the canonical ring of those minimal smooth algebraic surfaces S with $K^2 = 7$, $p_g = 4$, whose canonical system has just one simple base point and gives a birational map from S onto a sextic in $\mathbb{P}^3$: this is the only case, for these values of K^2, p_g, where the canonical system has base points, but yields a birational map.

What does it mean to compute a ring? As a matter of fact, using the computer algebra program Macaulay II, we will give three different descriptions of the above canonical rings. These presentations will allow us to deform explicitly the canonical ring of such a minimal surface (with $K^2 = 7$, $p_g = 4$ and with birational canonical map onto a sextic in $\mathbb{P}^3$) to the canonical ring of a surface with the same invariants but with base point free canonical system.

That these deformations should exist was already seen in [Ba], since it was proven there that the surfaces with $K^2 = 7$, $p_g = 4$ such that the canonical system gives a birational map from S onto a sextic in $\mathbb{P}^3$ form an irreducible family of dimension 35 in the moduli space $\mathfrak{M}_{K^2=7,p_g=4}$ and therefore they cannot dominate an irreducible component of the moduli space (by Kuranishi's theorem the dimension of $\mathfrak{M}_{K^2=7,p_g=4}$ in any point has to be at least $10\chi - 2K^2 = 36$).

Therefore it was clear from the classification given in [Ba] that this family has to be contained in the irreducible component of the moduli space whose general point corresponds to a surface with base point free canonical system (obviously then with birational canonical morphism).

Enriques ([Enr]) proposed to obtain this deformation starting by the surface of degree seven, union of the sextic surface (the canonical image) together with the plane containing the double curve: in our case, however, we see that the canonical images of degree seven do indeed degenerate to the union of the sextic canonical surface together with another plane, namely the tacnodal plane (cf. section 3).

Now, although our method applies in a much more general setting, the complexity of the computations which are needed in every specific case grows incredibly fast.

We consider therefore a real challenge for our present days computer algebra programs to make it possible to treat surfaces with higher values of the invariant K^2.

We would however like to remark, that all our explicit computations are more "computer assisted computations" than computer algebra programs. That is: it would be almost impossible to do them without a computer algebra program, but on the other hand there are always several steps which have to be done by hand, because looking carefully with a mathematical eye we can see tricks that the computer alone cannot detect.

Our paper is organized as follows.

In the first chapter we introduce under quite general conditions a naturally defined graded subring $\tilde{\mathcal{R}}$ of the canonical ring $\mathcal{R}$, such that there is an exact sequence

$$0 \longrightarrow \tilde{\mathcal{R}}_m \longrightarrow \mathcal{R}_m \longrightarrow H^0(\tilde{S}, \mathcal{O}_{m\mathcal{E}}) \longrightarrow \mathbb{C}^r \longrightarrow 0,$$

where $\mathcal{E}$ is an exceptional divisor on the surface $\tilde{S}$ obtained from S blowing up the base points of the canonical system. Then we introduce the "dual" module $M = Ext^1(\tilde{\mathcal{R}}, \Gamma_*(\omega_{\mathbb{P}^3}))$.

The rough idea is now to calculate the subring $\tilde{\mathcal{R}}$ (and the dual module M) using the geometry of the canonical image of S. We proceed in each degree "enlarging" $\tilde{\mathcal{R}}$ to $\mathcal{R}$: we will see how the module M provides automatically a certain number of the "missing" generators and relations; the few remaining generators and relations have to be computed "by hand" by the above exact

sequence.

In chapter 2 we will run this program in the special case of surfaces with $K^2 = 7$, $p_g = 4$, whose canonical map has exactly one base point and is birational. The main result of this section leads to the following

Theorem 0.1 *The canonical ring of a surface with $p_g = 4$, $K^2 = 7$, such that $|K|$ has one simple base point and φ_K is birational, is of the form $\mathcal{R} :=$ $\mathbb{C}[y_0, y_1, y_2, y_3, w_0, w_1, u]/I$ where the respective degrees of the generators of $\mathcal{R}$ are $(1, 1, 1, 1, 2, 2, 3)$.*
There exist a quadratic polynomial Q and a polynomial B of degree 4 such that I is generated by the 2×2 minors of the matrix

$$A := \begin{pmatrix} y_1 & y_2 & y_3 & w_1 \\ w_0 & w_1 & Q(y_i) & u \end{pmatrix},$$

and by three more polynomials (of respective degrees $4, 5, 6$), the first one of the form $-w_1^2 + B(y_i) + \sum \mu_{ijk} y_i y_j w_k$, and the other 2 obtained from the first via the method of rolling (cf. section 2, in particular they have the form $-w_1 u + \ldots$, $-u^2 + \ldots$).

In chapter 3 we will describe our canonical ring $\mathcal{R}$ in three different formats introduced by D. Dicks and M. Reid. It turns out that in order to deform the surfaces with $K^2 = 7$, $p_g = 4$ whose canonical system has one base point and is birational the third format (= antisymmetric and extrasymmetric format) is the most suitable one.
The result for this case is

Theorem 0.2 *Let S be a minimal (smooth, connected) surface with $K^2 = 7$ and $p_g(S) = 4$, whose canonical system $|K_S|$ has one (simple) base point $x \in S$ and yields a birational canonical map. Then the canonical ring of S can be presented as $\mathcal{R} = \mathbb{C}[y_0, y_1, y_2, y_3, w_0, w_1, u]/\mathcal{I}$, where deg $(y_i, w_j, u) = (1, 2, 3)$ and the ideal of relations $\mathcal{I}$ of $\mathcal{R}$ is generated by the 4×4 - pfaffians of the following antisymmetric and extrasymmetric matrix*

$$P = \begin{pmatrix} 0 & 0 & w_0 & Q(y_0,y_1,y_2) & w_1 & u \\ & 0 & y_1 & y_3 & y_2 & w_1 \\ & & 0 & -u+\overline{C} & y_3\overline{Q}_3(w_0,y_0,y_1,y_3) & Q\overline{Q}_3 \\ & & & 0 & y_1\overline{Q}_1(w_0,w_1,y_0,y_1,y_3) & w_0\overline{Q}_1 \\ & & & & 0 & 0 \\ -sym & & & & & 0 \end{pmatrix},$$

where $Q,\overline{Q}_1,\overline{Q}_3$ are quadratic forms of a subset of the given variables as indicated, and $\overline{C}$ is a cubic form. Moreover $\overline{C}$ does not depend on u and on the w_iy_j's for $j \le 1$, $\overline{Q}_1$ does not depend on y_3^2.

In the fourth chapter we will finally show how the above presentation of the canonical ring allows a deformation to the canonical ring $\mathcal{S}$ of a surface with $K^2 = 7$, $p_g = 4$ and free canonical system.

These last surfaces and their canonical rings are described in [Cat1], and it follows also from ([B-E]) that in this presentation the relations can be given by the 4×4 - Pfaffians of a skew-symmetric 5×5 matrix.

Our final result is

Theorem 0.3 *Let P be an antisymmetric and extrasymmetric matrix as in the previous theorem. Consider the 1-parameter family of rings $\mathcal{S}_t = \mathbb{C}[y_0,y_1,y_2,y_3,w_0.w_1,u]/\mathcal{I}_t$ where the ideal $\mathcal{I}_t$ is given by the 4×4 pfaffians of the antisymmetric and extrasymmetric matrix*

$$P_t = P + \begin{pmatrix} 0 & t & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & t\overline{Q}_1\overline{Q}_3 \\ -sym & & & & & 0 \end{pmatrix}.$$

This is a flat family and describes a flat deformation of the surface corresponding to the matrix P to surfaces with $p_g = 4$, $K^2 = 7$ and with $|K|$ base point free. For $t \ne 0$, $\mathcal{S}_t$ is isomorphic to $\mathbb{C}[y_0,y_1,y_2,y_3,w_0.w_1]/J_t$ where J_t is the ideal generated by the 4×4 pfaffians of the matrix

$$\begin{pmatrix} 0 & y_1 & -y_3 & -t^2w_0 & -tQ \\ & 0 & y_2 & t^3\overline{Q}_3 & tw_1 \\ & & 0 & t^2w_1 & t^2\overline{Q}_1 \\ & & & 0 & -t^3c-t^2y_1Q+t^2y_3w_0 \\ -sym & & & & 0 \end{pmatrix}.$$

6

Then we compare the above family with the one predicted by Enriques showing (as mentioned above) that it is a completely different type of degeneration.

Finally we have two appendices. In fact, although theoretically all the computation can be done by hands, it is better to use a computer program (as we did with Macaulay 2) to shorten the time needed and be sure that no computation's mistakes occured. We have put in the appendix the two Macaulay 2 scripts (without output) that we needed. The second appendix is interesting, because it shows how the computer suggested to us the 5×5 matrix appearing in Theorem 0.3.

1   Canonical systems with base points

Let S be a minimal surface of general type defined over the complex numbers and let $|K_S|$ be its canonical system. If $H^0(S, \mathcal{O}_S(K_S)) \neq 0$, then $|K_S|$ defines a rational map
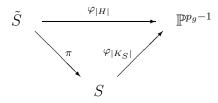
$$\varphi_{|K|} : S -- \rightarrow \mathbb{P}^{p_g - 1},$$

where $p_g = p_g(S) := \dim H^0(S, \mathcal{O}_S(K_S))$ is the geometric genus of S.

Throughout this paper we make the following

Assumption. $|K_S|$ has no fixed part.

Let $\pi : \tilde{S} \longrightarrow S$ be a (minimal) sequence of blow - ups such that the movable part $|H|$ of $|\pi^* K_S|$ has no base points. Then we have a commutative diagram:

$$
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\varphi_{|H|}} & \mathbb{P}^{p_g - 1} \\
& \pi \searrow \quad \nearrow \varphi_{|K_S|} & \\
& S &
\end{array}
$$

Since π is a sequence of blow ups $\pi_i : S_i \to S_{i-1}$ with centre a point $p_i \in S_{i-1}$, we denote by E_i the (-1)-divisor in $\tilde{S}$ given by the full transform of p_i, and we denote by m_i the multiplicity in p_i of the proper transform of a general divisor in K_S, so that

7

Remark 1.1 a) $K_{\tilde{S}} \equiv H + \sum_i (m_i + 1)E_i$.
To simplify the notation, we set $\mathcal{E} := \sum_i m_i E_i$, $E := \sum_i E_i$.

b) If $\varphi_{|K_S|}$ is not composed with a pencil, then $\varphi_{|H|} : \tilde{S} \longrightarrow \Sigma_1 \subset \mathbb{P}^{p_g-1}$ is a generically finite morphism from $\tilde{S}$ onto a surface Σ_1 in $\mathbb{P}^{p_g-1}$ of degree $d = H^2 = K_S^2 - \sum_i m_i^2$ and

$$\varphi_{|H|}^*(\mathcal{O}_{\Sigma_1}(1)) = \mathcal{O}_{\tilde{S}}(H) = \mathcal{O}_{\tilde{S}}(\pi^* K_S - \mathcal{E}) = \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} - \mathcal{E} - E).$$

Definition 1.2
a) Let us denote by $\mathcal{F}_1$ the coherent sheaf of $\mathcal{O}_{\Sigma_1}$ - modules $(\varphi_H)_* \mathcal{O}_{\tilde{S}}$.

b) We define $\tilde{\mathcal{R}}(S)$ as the graded ring associated to the divisor H on $\tilde{S}$, thus

$$\tilde{\mathcal{R}}(S) := \bigoplus_{m=0}^{\infty} H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(mH)) = \bigoplus_{m=0}^{\infty} H^0(\Sigma, \mathcal{F}_1(m)).$$

We make the following easy observation:

Remark 1.3 $\tilde{\mathcal{R}}(S)$ is a (graded) subring of the canonical ring $\mathcal{R}(S) := \bigoplus_{m=0}^{\infty} H^0(S, \mathcal{O}_S(mK_S))$.

Proof. The claim follows from the fact that the natural homomorphism

$$H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(mH)) \longrightarrow H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(mK_{\tilde{S}})) \cong H^0(S, \mathcal{O}_S(mK_S))$$
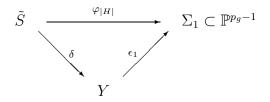
is injective for all $m \geq 0$.

$$\underline{Q.E.D.}$$

Remark 1.4 We have by our assumption

$$\tilde{\mathcal{R}}_1 = H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(H)) = H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}})) = \mathcal{R}_1.$$

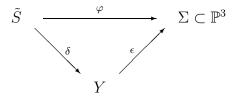Consider first the Stein factorization of $\varphi_{|H|}$:

$$\tilde{S} \xrightarrow{\quad \varphi_{|H|} \quad} \Sigma_1 \subset \mathbb{P}^{p_g - 1}$$

with δ and ϵ_1 to Y.

where in general Y is a normal algebraic surface, δ has connected fibres and ϵ_1 is a finite morphism.

We shall moreover from now on make the following assumption

B) $\varphi_{|H|}$ *is a birational morphism onto its image*, whence in particular $p_g(S) \geq 4$.

Under the above assumption we shall moreover consider a general projection of Σ_1 to a surface Σ in $\mathbb{P}^3$.
We have therefore the following diagram

$$\tilde{S} \xrightarrow{\quad \varphi \quad} \Sigma \subset \mathbb{P}^3$$

with δ and ϵ to Y.

and we may therefore assume
B') $\varphi : \tilde{S} \to \Sigma$ *is a birational morphism.*

We shall write the singular locus $Sing(\Sigma)$ of Σ as $\Gamma \cup Z$, where Γ is the subscheme of Σ corresponding to the conductor ideal $\mathcal{C}$ of the normalization morphism ϵ and where Z is the finite set $\epsilon(Sing(Y)) \subset \Sigma$ (note that if the support of Γ is disjoint from Z, also Z has a natural subscheme structure given by the adjunction ideal).

Σ is Cohen-Macaulay, whence Γ is a pure subscheme of codimension 1.

We remark that our methods apply also if the degree of φ is equal to two, but we have then to make more complicated technical assumptions.

Defining $\mathcal{F} := (\varphi)_* \mathcal{O}_{\tilde{S}}$, we have

$$\tilde{\mathcal{R}}(S) := \bigoplus_{m=0}^{\infty} H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(mH)) = \bigoplus_{m=0}^{\infty} H^0(\Sigma, \mathcal{F}(m)),$$

whence we may observe that the graded ring $\tilde{\mathcal{R}}$ is a module over the polynomial ring $\mathcal{A} := \mathbb{C}[y_0, y_1, y_2, y_3]$ (the homogeneous coordinate ring of $\mathbb{P}^3$). Since this module has a support of codimension 1, it has a graded free resolution of length equal to 1 if and only if it is a Cohen-Macaulay module. The following result is essentially the same result as theorem 2.5 of [Cil1].

Proposition 1.5 $\tilde{\mathcal{R}}$ *is a Cohen Macaulay* $\mathcal{A}$-*module if and only if the subscheme* $\Gamma \subset \mathbb{P}^3$ *is projectively normal.*

Proof. It is well known that the Cohen Macaulay property is equivalent to the vanishing of the cohomology groups

$$H^1(\Sigma, \mathcal{F}(m)) = H^1(\Sigma, \epsilon_* \mathcal{O}_Y \otimes \mathcal{O}_\Sigma(m)) = H^1(Y, \mathcal{O}_Y(mH)),$$

for all m.
By Serre's theorem B(m) these groups obviously vanish for $m >> 0$.
Serre-Grothendieck duality tells us that these are the dual vector spaces of $Ext^1(\epsilon_* \mathcal{O}_Y(m), \omega_\Sigma)$.
By the local-to-global spectral sequence of the Ext groups, there is an exact sequence

$$0 \to H^1(\mathcal{H}om(\epsilon_* \mathcal{O}_Y(m), \omega_\Sigma)) \to Ext^1(\epsilon_* \mathcal{O}_Y(m), \omega_\Sigma) \to$$
$$H^0(\mathcal{E}xt^1(\epsilon_* \mathcal{O}_Y(m), \omega_\Sigma)) \to H^2(\mathcal{H}om(\epsilon_* \mathcal{O}_Y(m), \omega_\Sigma)).$$

But $\mathcal{E}xt^1(\epsilon_* \mathcal{O}_Y(m), \omega_\Sigma)$ is zero because $\epsilon_* \mathcal{O}_Y$ is a Cohen-Macaulay $\mathcal{O}_\Sigma$−module and ω_Σ is invertible.
Therefore it follows that $Ext^1(\epsilon_* \mathcal{O}_Y(m), \omega_\Sigma) = 0$ if and only if $H^1(\mathcal{H}om(\epsilon_* \mathcal{O}_Y(m), \omega_\Sigma)) = 0$.
In turn, since $\omega_\Sigma = \mathcal{O}_\Sigma(d-4)$, $H^1(\mathcal{H}om(\epsilon_* \mathcal{O}_Y(m), \omega_\Sigma)) = H^1(\mathcal{C}(m+d-4))$, where $\mathcal{C}$ is the conductor ideal of ϵ, and this last group, in view of the exact sequence

$$0 \to \mathcal{C} \to \mathcal{O}_\Sigma \to \mathcal{O}_\Gamma \to 0$$

is the cokernel of the map $H^0(\mathcal{O}_\Sigma(m+d-4)) \to H^0(\mathcal{O}_\Gamma(m+d-4))$.
Since however $H^0(\mathcal{O}_\Sigma(n))$ is a quotient of $H^0(\mathcal{O}_{\mathbb{P}^3}(n))$ we conclude that our desired vanishing is equivalent to the projective normality of Γ.

<div align="right">*Q.E.D.*</div>

Recalling that $\mathcal{O}_{\tilde{S}}(H) = \mathcal{O}_{\tilde{S}}(\pi^* K_S - \mathcal{E})$, we consider now the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{S}}(mH) \longrightarrow \mathcal{O}_{\tilde{S}}(m\pi^*K_S) \longrightarrow \mathcal{O}_{m\mathcal{E}} \longrightarrow 0.$$

Observe moreover that, since S is minimal and of general type, for $m \geq 2$ we have

$$H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(m\pi^*K_S) = H^1(S, \mathcal{O}_S(mK_S)) = 0.$$

Whence, we arrive for each $m \geq 2$ to the following crucial exact sequence, which will be repeatedly used in the sequel.

$$(ii) \qquad 0 \longrightarrow \tilde{\mathcal{R}}_m \longrightarrow \mathcal{R}_m \longrightarrow H^0(\tilde{S}, \mathcal{O}_{m\mathcal{E}}) \longrightarrow H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(mH)) \longrightarrow 0.$$

The vanishing $H^1(\Sigma, \mathcal{F}(m)) = 0$ for all m, is clearly equivalent to the chain of equalities:

$$dim H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(mH)) = dim H^0(\Sigma, R^1(\varphi_H)_*\mathcal{O}_{\tilde{S}}(m)) = length(R^1(\varphi_H)_*\mathcal{O}_{\tilde{S}}) := l.$$

Putting together the above considerations we obtain the following

Remark 1.6 $\tilde{\mathcal{R}}$ is a Cohen Macaulay module over the polynomial ring $\mathcal{A}$ if and only if the surface S is regular ($H^1(S, \mathcal{O}_S(K_S)) = 0$) and
$dim\mathcal{R}_m - dim\tilde{\mathcal{R}}_m = dim H^0(\tilde{S}, \mathcal{O}_{m\mathcal{E}}) - l = \sum_i \frac{mm_i(mm_i+1)}{2} - l$.

Proof.
Assume $\tilde{\mathcal{R}}$ to be Cohen Macaulay:
since we know that $H^1(\mathcal{O}_{\mathcal{E}}) = 0$, and that the map $\tilde{\mathcal{R}}_1 \to \mathcal{R}_1$ is an isomorphism (from the definition of $\tilde{\mathcal{R}}$), we get an exact sequence

$$0 \longrightarrow H^0(\tilde{S}, \mathcal{O}_{\mathcal{E}}) \longrightarrow H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(H)) \longrightarrow H^1(S, \mathcal{O}_S(K_S)) \longrightarrow 0.$$

which shows that the surface must be regular.
If conversely the surface is regular, $H^1(S, \mathcal{O}_S) = H^1(S, \mathcal{O}_S(K_S)) = 0$, so the sequence $ii)$ is exact also for $m = 0, 1$, then $\forall m \in \mathbb{Z}$.
In particular, $dim\mathcal{R}_m - dim\tilde{\mathcal{R}}_m = dim H^0(\tilde{S}, \mathcal{O}_{m\mathcal{E}}) - dim H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(mH))$, whence $\tilde{\mathcal{R}}$ is a Cohen Macaulay $\mathcal{A}-$module iff $H^1(\Sigma, \mathcal{F}(m)) = 0$, i.e., iff $dim H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(mH)) = l$, equivalently, iff $dim\mathcal{R}_m - dim\tilde{\mathcal{R}}_m = dim H^0(\tilde{S}, \mathcal{O}_{m\mathcal{E}}) - l = \sum_i \frac{mm_i(mm_i+1)}{2} - l$.

$$\underline{Q.E.D.}$$

Remark 1.7 $\tilde{\mathcal{R}} \subset \mathcal{R}$ is a subring, but it is easy to see that $\mathcal{R}$ is not a finitely generated $\tilde{\mathcal{R}}$ - module.

We assume now that $\tilde{\mathcal{R}}$ is a Cohen-Macaulay $\mathcal{A}$-module, and we observe that it contains the coordinate ring of Σ and is contained in $\mathcal{R}$.
Therefore, choosing a minimal system of generators $v_1 = 1, v_2, \ldots v_n$ of $\tilde{\mathcal{R}}$ as an $\mathcal{A}-$module, defining $l_i := \deg v_i$ we find (by Hilbert's syzygy theorem, as in [Cat1]), a resolution of the form

$$(\#) \qquad 0 \longrightarrow \oplus_{j=1}^h \mathcal{A}(-r_j) \overset{\alpha}{\longrightarrow} \oplus_{i=1}^h \mathcal{A}(-l_i) \longrightarrow \tilde{\mathcal{R}} \longrightarrow 0;$$

Under the assumption that φ be birational follows that Σ has equation $f := \det \alpha = 0$.
As in [Cat1], being $\tilde{\mathcal{R}}$ a ring, the matrix has to fulfill the standard Rank Condition, which we will later recall.
In order to describe the ring $\mathcal{R}$, we first look for generators of $\mathcal{R}$ as an $\mathcal{A}$-module.
On the other hand, when looking for generators of $\mathcal{R}$ as a ring, we may restrict ourselves to consider elements of low degree by virtue of the following result by M. Reid (cf. [Rei1], cf. also Ciliberto [Cil2]).

Theorem 1.8 *Let X be a canonical surface (i.e., the canonical model of a surface of general type). We suppose that*
(i) $p_g(X) \geq 2$, $K_X^2 \geq 3$,
(ii) $q(X) = 0$,
(iii) X has an irreducible canonical curve $C \in |K_X|$.
Then the canonical ring $\mathcal{R} = \mathcal{R}(X, K_X)$ of X is generated in degrees ≤ 3 and its relations are generated in degrees ≤ 6.

Now we define the $\tilde{\mathcal{R}}$-module

$$M := \Gamma_*(\mathcal{C}\omega_\Sigma) = \Gamma_*(\omega_Y),$$

where $\mathcal{C}$ is the conductor ideal of ϵ and $\Gamma_*(\mathcal{F})$ denotes as usual $\oplus_{n \in \mathbb{Z}} H^0(\mathcal{F}(n))$. We consider the following chain of inclusions of $\mathcal{A}-$modules

$$\mathcal{A}/(f) \subset \tilde{\mathcal{R}} \subset \Gamma_*(\varphi_* \omega_{\tilde{S}})[-1] \subset M[-1].$$

We observed that $\tilde{\mathcal{R}}$ is Cohen Macaulay if and only if it has a free resolution as an $\mathcal{A}-$module of the form $(\#)$

$$0 \to L_1 \xrightarrow{\alpha} L_0 \to \tilde{\mathcal{R}} \to 0.$$

Dualizing it, we get

$$0 \to L_0^\vee \otimes \Gamma_*(\omega_{\mathbb{P}^3}) \xrightarrow{\alpha^t} L_1^\vee \otimes \Gamma_*(\omega_{\mathbb{P}^3}) \to Ext_{\mathcal{A}}^1(\tilde{\mathcal{R}}, \Gamma_*(\omega_{\mathbb{P}^3})) \to 0.$$

By virtue of the exact sequence

$$0 \to \Gamma_*(\omega_{\mathbb{P}^3}) \xrightarrow{f} \Gamma_*(\omega_{\mathbb{P}^3})(\deg f) \to \Gamma_*(\omega_\Sigma) \to 0,$$

and since $Ext_{\mathcal{A}}^1(\tilde{\mathcal{R}}, f) = 0$, we get

$$Ext_{\mathcal{A}}^1(\tilde{\mathcal{R}}, \Gamma_*(\omega_{\mathbb{P}^3})) = Hom_{\mathcal{A}}(\tilde{\mathcal{R}}, \Gamma_*(\omega_\Sigma)) = \Gamma_*(\mathcal{H}om_{\mathcal{O}_\Sigma}(\mathcal{F}, \omega_\Sigma)) = \Gamma_*(\mathcal{C}\omega_\Sigma) = M.$$

Moreover, M satisfies the Ring Condition (cf. [dJ-vS])

$$\tilde{\mathcal{R}} = Hom(M, \mathcal{A}/(f)) = Hom(M, M)$$

or, in other words, there is a bilinear pairing $\tilde{\mathcal{R}} \times M \to M$ which, in the given bases, is determined by the matrix $\beta = \Lambda^{h-1}(\alpha)$. In turn the Ring Condition is equivalent to the so called Rank Condition for (α) : there exist elements λ_{jh}^k of $\mathcal{A}$ such that $\beta_{1k} = \sum \lambda_{jh}^k \beta_{jh}$.

Since we have the inclusion $\tilde{\mathcal{R}} \subset M[-1]$, we can fix bases $v_1 = 1, v_2, \dots, v_h$, for $\tilde{\mathcal{R}}$, $z_1, z_2, \dots, z_h$, for M, with $v_i z_j = \frac{\beta_{ij} z_1}{\beta_{11}}$ and such that z_1 is the image of $v_1 = 1$.

For the same reason, $v_2, \dots, v_n$ can be written as linear combinations $v_i = \sum \zeta_{ij} z_j / z_1$.

Now, as in [Cat1], the ring structure of $\tilde{\mathcal{R}}$ is equivalent to the Rank Condition which can also be phrased as follows:

R.C. the ideal of the $(h-1) \times (h-1)$ minors of α coincide with the ideal of the $(h-1) \times (h-1)$ minors of the matrix α' obtained by deleting the first row of α.

The polynomials λ_{jh}^k with $\beta_{1k} = \sum \lambda_{jh}^k \beta_{jh}$ determine therefore the ring structure of $\tilde{\mathcal{R}}$ by the following multiplication rule :

$$v_i v_h = \sum_{j,k} \zeta_{ij} \lambda_{jh}^k v_k.$$

13

We have now all the general ingredients at our disposal and we can explicitly describe our method in order to compute the canonical ring of the regular surfaces of general type with given values of the invariants K^2, p_g, canonical system with base points (and without fixed part), and birational canonical map.

Under the above assumptions, observe that obviously $p_g \geq 4$, moreover Castelnuovo's inequality $K^2 \geq 3p_g - 7$ holds, in particular the hypotheses of theorem 1.8 are fullfilled.

We need first of all to assume that the subscheme Γ of $\mathbb{P}^3$ given by the conductor ideal of the normalization of Σ be projectively normal.

The last assumption, as we just saw, ensures that the ring $\tilde{\mathcal{R}}$ is a Cohen-Macaulay $\mathcal{A}$−module; whence, argueing as in [Cat1] we can find a length 1 presentation of $\tilde{\mathcal{R}}$ as an $\mathcal{A}$−module, given by a square matrix α fullfilling the Rank Condition .

Let $v_1, \ldots, v_h$ be the generators of $\tilde{\mathcal{R}}$ we used in order to write down α, and let $z_1, \ldots z_h$ be the dual generators of M (i.e., the module M is generated by the z_i's and presented by the matrix α^t).

We have seen that there is an inclusion $\tilde{\mathcal{R}} \subset M[-1]$; assuming by sake of simplicity that ω_Y is Cartier, one can write explicit sections σ_d $(d \in \mathbb{N})$ of suitable line bundles $\mathcal{L}_d$ so that the above inclusion is obtained multiplying every element r_d homogeneous of degree d in $\tilde{\mathcal{R}}$, by σ_d. The σ_d's are of the form $e_d \cdot c$ where c is a section of the dual of the relative canonical bundle of the map $\delta : \tilde{S} \to Y$, and e_d is supported on the exceptional locus of $\pi : \tilde{S} \to S$.

It is not possible to construct a similar inclusion $\mathcal{R} \subset M$, but we can consider the module $\oplus_n H^0(K_{\tilde{S}} + nH)$. This is the submodule of M given by the elements divisible by c (in particular it contains $\tilde{R}$), and clearly it is a submodule of $\mathcal{R}$ so it is completely natural to denote it by $M \cap \mathcal{R}$.

The second step of our method is to study this module: first we compute the subset $\{w_1, \ldots w_r\}$ of a set of generators for $M \cap \mathcal{R}$ as an $\mathcal{A}$-module, consisting of the elements of degree ≤ 3 (with the grading of $\mathcal{R}$). Then wefind the relations holding among them in degree ≤ 6. In fact, by theorem 1.8, generators and relations in higher degrees will not be relevant for the canonical ring.

The elements $y_i's$, w_j's will not in general generate the canonical ring; the third step of our method consists in the research of the missing generators and relations.

It is clear that in every case $\mathcal{R}_1 \subset \mathcal{R} \cap M$. We assume now that the base

14

points are simple (but a similar analysis can be carried out in every case), i.e.

$$(*) \qquad\qquad \mathcal{E} = E.$$

Directly by the definition follows the equality $K_{\tilde{S}} + H = 2\pi^* K_S$, so $H^0(K_{\tilde{S}} + H) = H^0(2K_S)$, i.e. $\mathcal{R}_2 \subset \mathcal{R} \cap M$. Instead, if $\mathcal{E} \neq E$ the equality $h^0(K_{\tilde{S}} + H) = h^0(2K_S)$ cannot hold, otherwise $|2K_S|$ would not be base point free, as it has to be for $p_g > 0$ ([Fra]).
In degree 3 we have the exact sequence

$$0 \rightarrow H^0(K_{\tilde{S}} + 2H) \rightarrow H^0(3K_S) \rightarrow H^0(\mathcal{O}_{\mathcal{E}}) \rightarrow 0$$

since $H^1(K_{\tilde{S}} + 2H) = 0$ (by Mumford's vanishing theorem).
Therefore, $\mathcal{R}_3 \cap M$ has codimension 3 in $\mathcal{R}_3$, i.e. we need l elements $u_1, \dots, u_l$, to complete a basis of $H^0(K_{\tilde{S}} + 2H)$ to a basis of $H^0(3K_S)$.
By theorem 1.8 the generators $y_0, \dots, y_3$ of $\mathcal{A}$ (seen as elements of $H^0(K_{\tilde{S}})$), together with $w_1, \dots, w_r$ and $u_1, \dots, u_l$ are a system of generators of $\mathcal{R}$ as a ring.
The relations as $\mathcal{A}$-module among the v_i's and among the w_j's are determined by the matrix α, and similarly the relations given by the products of type $v_i v_j$, and $v_i w_j$ are also determined by α: these provide automatically a list of relations among the above generators of $\mathcal{R}$.
Some relation is still missing; in particular our method do not produce automatically any relation involving the u_k's. In order to complete the analysis one needs to find a way to espress the u_k's "in terms of M", so that the known relation among the elements of $\tilde{R}$ and M will produce also relations involving them. This should be possible case by case by "ad hoc" arguments, but we do not know a general argument.
We devote the next sections to the application of the above method to finding a description of the canonical ring of the stratum of the moduli space of surfaces of general type with $K^2 = 7$ and $p_g = 4$ corresponding to surfaces with birational canonical map and whose canonical system has base points.

15

2 The canonical ring of surfaces with $K^2 = 7$, $p_g = 4$ birational to a sextic: from algebra to geometry

Let S be a minimal (smooth, connected) surface with $K^2 = 7$ and $p_g(S) = 4$. We remark that S is automatically regular (cf. [Deb]), i.e. $q(S) = 0$.

In [Ba] the first author gave an exact description of minimal surfaces with $K^2 = 7$ and $p_g(S) = 4$, where the canonical system has base points, proving in particular that if moreover the canonical map is birational then the canonical system $|K_S|$ has exactly one simple base point $x \in S$.
Let $\pi : \tilde{S} \longrightarrow S$ be the blow up of S in x and let $E := \pi^{-1}(x)$ be the exceptional curve of π. Thus we have:

$$|K_{\tilde{S}}| = |\pi^* K_S| + E = |H| + 2E,$$

where $|H|$ is base point free.
Thus we will assume in this paragraph that $\varphi_{|H|} : \tilde{S} \longrightarrow \Sigma := \{F_6 = 0\}$ is a birational morphism (from $\tilde{S}$ onto a surface Σ of degree six in $\mathbb{P}^3$).

We recall now the description of the minimal surfaces with $K^2 = 7$ and $p_g(S) = 4$, whose canonical system has exactly one base point and whose canonical map is birational given in [Ba].

First we need the following definition.

Definition 2.1 A *generalized tacnode* is a two dimensional elliptic hypersurface singularity $(X, 0)$, such that the fundamental cycle has self intersection (-2).
In particular, $(X, 0)$ is Gorenstein and by [Lau], theorem (1.3), $(X, 0)$ is a double point singularity, whose local analytic equation is given by

$$z^2 = g(x, y),$$

where g vanishes of order four in 0. The normal cone of the singularity is given by the plane $\{z = 0\}$, called the tacnodal plane.

More precisely, a generalized tacnode $(X, 0)$ is the singularity obtained as the double cover branched along a curve with a quadruple point, which after a blow up decomposes in at most simple triple points or double points.

Theorem 2.2 *([Ba], theorem 3.11, 5.5) 1) Let S be a minimal surface with $K^2 = 7$ and $p_g(S) = 4$, whose canonical system has exactly one base point and whose canonical map is birational. Then the blow-up $\tilde{S}$ of the base point is the minimal desingularization of a surface $\Sigma \subset \mathbb{P}^3$ of degree six with the following properties:*

(a) the double curve $\Gamma \subset \Sigma$ is a plane conic,

(b) if $\gamma \subset \mathbb{P}^3$ is the plane containing Γ, then Σ has a generalized tacnode $o \in \gamma \backslash \Gamma$ with tacnodal plane $\alpha \neq \gamma$,

(c) the image $\varphi_H(E)$ of the exceptional curve equals the line $\alpha \cap \gamma$.

2) The surfaces with $K^2 = 7$ and $p_g = 4$ such that the canonical system has exactly one base point and $\varphi_{|K|}$ is birational form an irreducible set $\mathfrak{M}_{(I.1)}$ of dimension 35 in their moduli space.

Moreover, it was shown (ibidem) that for a general element of $\mathfrak{M}_{(I.1)}$ the canonical image Σ has an equation of the form

$$\alpha^2 Q^2 + \gamma F_5 = 0,$$

where F_5 is an element of the linear subsystem $\Delta \subset |5H - \Gamma - 2o|$ in $\mathbb{P}^3$ consisting of quintics with tangent cone α^2 in o and where $Q \subset \mathbb{P}^3$ is an irreducible quadric containing Γ.

In this section we will study the ring $\tilde{\mathcal{R}}$ and the module M defined in the previous section in the case of the surfaces in the above class.

This study will allow us to compute the canonical ring and to give a purely algebraic proof of part 1) of theorem 2.2, under a few generality assumptions. For the convenience of the reader we will give here a list of notation (partly already introduced in the last section) which will be frequently used in the following.

Notation:

We consider:

- $\pi : \tilde{S} \to S$, the blow-up of the base point of $|K_S|$;

- $\varphi : \tilde{S} \twoheadrightarrow \Sigma \subset \mathbb{P}^3$, the morphism induced by the canonical system;

- $\epsilon : Y \to \Sigma$, the normalization of Σ;

- $\delta : \tilde{S} \to Y$, such that $\varphi = \epsilon \circ \delta$;

- e, a generator of $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} - \pi^* K_S))$ and E the corresponding divisor;

- c, a generator of $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\delta^* K_Y - K_{\tilde{S}}))$ and C the corresponding divisor (this makes sense under the assumption 2) below);

- F_6 an equation of $\Sigma \subset \mathbb{P}^3$.

  We have the following list of graded rings respectively modules:

- $\mathcal{A} := \bigoplus_{m=0}^{\infty} S^m(H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(H))) = \mathbb{C}[x_0, x_1, x_2, x_3]$,

- $\tilde{\mathcal{R}} := \bigoplus_{m=0}^{\infty} H^0(\Sigma, (\varphi_{|H|})_* \mathcal{O}_{\tilde{S}}(m)) = \bigoplus_{m=0}^{\infty} H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(mH))$,

- $M := \bigoplus_{m \in \mathbb{Z}} H^0(\Sigma, \mathcal{C}\omega_\Sigma(m)) = \bigoplus_{m=-1}^{\infty} H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}((m+1)H + 2E + C))$, where

- $\mathcal{C} := \mathcal{H}om_{\mathcal{O}_\Sigma}(\epsilon_* \mathcal{O}_Y, \mathcal{O}_\Sigma)$ is the conductor ideal;

- $\mathcal{R} = \bigoplus_{m=0}^{\infty} H^0(S, mK_S) = \bigoplus_{m=0}^{\infty} H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(m(H + E)))$, the canonical ring of S.

The assumptions we will make are the following:
1) The conductor ideal of ϵ defines a projectively normal subscheme of $\mathbb{P}^3$;
2) The singular points of Y do not lie in the preimage of the non normal locus of Σ; in particular it follows that ω_Y is Cartier.
These two assumptions give in fact no restriction, as it can be shown with geometrical arguments ([Ba], remark 3.1. and prop. 3.6.(v)).
First, by the results of the previous section, we give a presentation of the ring $\tilde{\mathcal{R}}$ as $\mathcal{A}$-module.

Theorem 2.3 $\tilde{\mathcal{R}}$ *is a Cohen - Macaulay $\mathcal{A}-$module and has a resolution (as an $\mathcal{A}$-module) as follows*

$$
0 \rightarrow \begin{matrix} \mathcal{A}(-5) \\ \oplus \\ \mathcal{A}(-4) \end{matrix} \overset{\alpha}{\rightarrow} \begin{matrix} \mathcal{A} \\ \oplus \\ \mathcal{A}(-3) \end{matrix} \rightarrow \tilde{\mathcal{R}} \rightarrow 0.
$$

Proof. (cf. [Cat1] for similar computations).
By remark 1.6, $\dim \tilde{\mathcal{R}}_m = \dim \mathcal{R}_m - \frac{m(m+1)}{2} + 1$, therefore by Riemann-Roch's Theorem, for $m \geq 2$, it equals $\chi(\mathcal{O}_S) + \frac{7}{2}m(m-1) - \frac{m(m+1)}{2} + 1 = 3m^2 - 4m + 6$.
$\tilde{\mathcal{R}}$ is Cohen Macaulay by theorem 1.5 (and assumption 1)), whence it has a resolution as $\mathcal{A}-$module of length 1.

By definition the first generator of $\tilde{\mathcal{R}}$ has degree 0 ($\tilde{\mathcal{R}}_0 = H^0(\mathcal{O}_{\tilde{S}})$) and will be denoted by 1.

The above dimension formula gives us immediately that there are no other generators in degrees ≤ 2 and that one more generator (denoted by v) is needed in degree 3. Moreover, the relations live in degrees ≥ 4.

Again by the above dimension formula we get at least one relation in degree 4. If there were two independent relations in degree 4 they would have the form $x_0 \cdot v = f_4(x_i) \cdot 1$; $x_1 \cdot v = g_4(x_i) \cdot 1$ and this would force a non trivial relation of the form $(x_1 f_4 - x_0 g_4) \cdot 1 = 0$ of degree 5, contradicting that obviously for any $f \in \mathcal{A}$ the equality $f \cdot 1 = 0$ implies that f is a multiple of F_6. By the dimension formula therefore there are no new generators in degree 4.

Again counting the dimensions we get a relation in degree 5; a straightforward computation shows that for all m, dim $\tilde{\mathcal{R}}_m$ = dim $\mathcal{A}_m$+ dim $\mathcal{A}_{m-3}$−dim $\mathcal{A}_{m-4}$−dim $\mathcal{A}_{m-5}$; this shows that the resolution has the form

$$0 \to \begin{matrix} \mathcal{A}(-5) \\ \oplus \\ \mathcal{A}(-4) \\ \oplus \\ \mathcal{L} \end{matrix} \xrightarrow{\alpha} \begin{matrix} \mathcal{A} \\ \oplus \\ \mathcal{A}(-3) \\ \oplus \\ \mathcal{L} \end{matrix} \to \tilde{\mathcal{R}} \to 0.$$

where $\mathcal{L}$ is a free module $\oplus \mathcal{A}(-s_i)$, with $s_i \geq 5$ for all i.

The minimality of the resolution ensures that $\mathcal{L} = 0$ (there are no non zero constants as coefficients in α; considering the row of α corresponding to the new generator of maximal degree (≥ 5) we get a row of zeroes, contradicting the injectivity of α).

$$\underline{Q.E.D.}$$

Corollary 2.4 *M has a resolution as $\mathcal{A}$-module of the form*

$$0 \to \begin{matrix} \mathcal{A}(-4) \\ \oplus \\ \mathcal{A}(-1) \end{matrix} \xrightarrow{{}^t\alpha} \begin{matrix} \mathcal{A}(1) \\ \oplus \\ \mathcal{A} \end{matrix} \to M \to 0.$$

We denote by z_{-1}, z_0 the generators of M in the respective degrees -1 and 0;

Remark 2.5 Notice that $z_{-1} = e^2 c$.

Corollary 2.6 *Up to a suitable choice of the generators of the $\mathcal{A}-$modules $\tilde{\mathcal{R}}, M$, we can write*

$$\alpha = \begin{pmatrix} QG + \gamma B & Qq \\ Q & \gamma \end{pmatrix};$$

where deg $(\gamma, Q, q, G, B) = (1, 2, 2, 3, 4)$, moreover

$$vz_{-1} = qz_0$$

$$vz_0 = Bz_{-1} - Gz_0$$

Proof. By the above resolution of M we know the degrees of the entries of α. The Rank Condition for α means that the elements of the first row are in the ideal generated by the elements of the second row. To obtain now the desired form of α it suffices to add a suitable multiple of the second row to the first one.

The two relations can be easily obtained writing explicitly the pairing $\tilde{\mathcal{R}} \times M \to M$.

$$Q.E.D.$$

In order to understand the structure of the canonical ring, we have now to investigate which elements of the module M can be divided by c. In fact, the graded parts of our rings are related by the following (commutative) diagram:

$$(*) \qquad \begin{array}{ccc} \tilde{R}_n & \xrightarrow{e^2} & H^0(\tilde{S}, \mathcal{O}(nH + 2E)) \\ e^n\downarrow & \diagup & \downarrow c \\ R_n & & M_{n-1} \end{array}$$

where the diagonal arrow is multication by e^{n-2} $(n \geq 2)$.

In order to write down explicitely the ring, we will fix a basis x_0, x_1, x_2, x_3 for $\mathcal{A}_1 = \tilde{\mathcal{R}}_1$. We will denote by $y_i := ex_i$ the induced elements in $\mathcal{R}$.

Remark 2.7 First, since $\gamma \neq 0$ (or $\Sigma = \{\gamma F_5 = qQ^2\}$ would be reducible), we set $x_3 := \gamma$. Moreover we shall assume from now on that $G = G(x_0, x_1, x_2)$ (this is clearly possible without loss of generality by corollary 2.6).

By the resolution of M, $\dim_{\mathbb{C}} M_1 = 13$ and a basis of M_1 is provided by the 10 elements of the form $x_i x_j z_{-1}$ and $x_0 z_0, x_1 z_0, x_2 z_0$ (the only relation in degree 1 has the form $x_3 z_0 = -Qz_{-1}$).

By Riemann-Roch, $P_2 = 12$; recalling that $c|z_{-1}$ diagram $(*)$ shows that we can fix the basis $x_0, x_1, x_2, x_3 = \gamma$ of $\mathcal{A}_1$ such that $c|x_i z_0$ for $i \geq 1$ but c does

not divide $x_0 z_0$, and setting $w_0 := -\frac{x_1 z_0}{c}$, $w_1 := -\frac{x_2 z_0}{c}$, $w_0, w_1, y_i y_j$ is a basis of $\mathcal{R}_2$.

Lemma 2.8 $R = \mathbb{C}[y_0, y_1, y_2, y_3, w_0.w_1, u]/I$ where $\deg(y_i, w_j, u) = (1, 2, 3)$, and the generators in degree ≤ 3 of the ideal I are given by the vanishing of the 2×2 minors of the following matrix:

$$\begin{pmatrix} y_1 & y_2 & y_3 \\ w_0 & w_1 & Q(y_i) \end{pmatrix}$$

where $Q(y_i)$ is obtained from Q replacing every x_i by the corresponding y_i.

Proof.
In the previous remark, we saw that in degree ≤ 2, $\mathcal{R}$ is generated by the y_i's and the w_j's.
Every relation in degree ≤ 3 of $\mathcal{R}$ is of the form $w_0 l_0(y_i) + w_1 l_1(y_i) = g_3(y_i)$, where l_0 and l_1 are linear forms, g_3 is a cubic form.
Thus we get from it the following relation in degree 2 for M as $\mathcal{A}-$module:
$z_0(x_1 l_0(x_i) + x_2 l_1(x_i)) = z_{-1} g_3(x_i)$.
By corollaries 2.4 and 2.6, must be a multiple of the relation $Q z_{-1} + x_3 z_0 = 0$.
Thus we immediately see that this relation is a linear combination of the three given by the 2 by 2 minors of the matrix in the statement.
We obtain therefore that the subspace of R_3 generated by the monomials in the y_i, w_i has dimension 25. Since $P_3 = 26$ we need to add a new generator u.
Finally, there are no more generators in degree ≥ 4 by theorem 1.8.

$$Q.E.D.$$

We noticed in fact at the end of the previous section that if $E = \mathcal{E}$ we need exactly $l = h^0(\mathcal{O}_{\mathcal{E}})$ new generators of $\mathcal{R}$ in degree 3 which do not come from elements in M. This gives in our case ($l = 1$) exactly one new generator in degree 3 (the only one not vanishing in E), that is, our generator u.

Remark 2.9 For later use we observe that c does not divide $x_0^2 z_0$.
This holds since otherwise $\frac{x_0^2 z_0}{c}$ would yield (cf. diagram (*)) an element in $H^0(3H + 2E)$, whence $\frac{e x_0^2 z_0}{c}$ would be an element of R_3 linearly dependent upon the monomials in the y_i's, w_i's. This however contradicts corollary 2.4.

21

As pointed out at the end of the previous section, the main problem in computing the canonical ring is given by the "additional" generators in degree 3 (in our case there is only one, namely u).
In the next lemma (part 5) we manage to express u "in terms of M"; this will allow us to compute the canonical ring.

Lemma 2.10 *Choosing suitable coordinates in $\mathbb{P}^3$ and suitable generators of M and $\tilde{\mathcal{R}}$ as $\mathcal{A}$ - modules we can assume:*
1) Q does not depend on the variable x_3;
2) $q = x_2^2$;
3) $(1, 0, 0, 0)$ is a double point, which is locally a double cover of the plane branched along a curve with a singularity of order at least 4.
4) $c^2|x_2 z_0^2$;
5) $u = -\frac{w_1 z_0}{ec}$.
Moreover $\varphi(E)$ is the line $x_2 = x_3 = 0$.

We would like to point out that the coordinates and generators in the previous lemma can be chosen such that remark 2.5, lemma 2.6, remark 2.7 and lemma 2.8 will remain valid as it can be traced in the following proof.
Proof.
First, taking suitable linear combinations of rows and columns of α, we can assume that both q and Q do not depend on the variable x_3, and part 1) is proved.

In corollary 2.6 we have seen that $v z_{-1} = q z_0$.
As a matter of fact, for every quadric $q'(x_0, x_1, x_2)$ with the property

$$(**) \qquad z_{-1}|q' z_0,$$

$\frac{q' z_0}{z_{-1}}$ is an element in $\tilde{R}_3$; so if there were two independent quadrics with this property, we would get two independent elements of $\tilde{R}$ in degree 3, and by theorem 2.3, for a suitable quadric q'' in the pencil generated by them, we would get a relation $g_3(x_i) z_{-1} = q''(x_0, x_1, x_2) z_0$, contradicting corollary 2.4. Therefore there can be only one quadric with this property.
We already noticed (cf. remark 2.9) that c does not divide $x_0^2 z_0$ but divides $x_i z_0$ for every $i \geq 1$; so q is the only quadric of the form $x_0 l(x_1, x_2) + l_1(x_1, x_2) l_2(x_1, x_2)$ such that $q z_0$ vanishes twice on E.
By definition $HE = 1$ whence $\varphi(E)$ is a line and there are two independent linear forms in $\mathbb{P}^3$ vanishing on E; in particular there is at least one linear

form in the span of x_0, x_1, x_2 vanishing on E. Note that e does not divide z_0 (or we could easily find two different quadrics with the required property, a contradiction).

One of these linear forms belongs to the span of x_1, x_2: otherwise we could assume (up to a change of coordinates) that x_0 vanishes on E, and since q is divisible by e^2 we get $l_1 = l_2 = 0$ and $e^2 | x_0$, contradicting again the unicity of q (we can take $x_0 x_1$ and $x_0 x_2$).

Up to a change of coordinates we can then assume $e | x_2$; x_2^2 fulfills $(**)$, hence $q = x_2^2$, and part 2) is proved.

Note that since $x_3 z_0 = -Q z_{-1}$, and $e \nmid z_0$ (else the base point of the canonical system would also be a base point of the bicanonical system), $e^2 | x_3$, and $\varphi(E) = \{x_2 = x_3 = 0\}$.

Let us write $C = C_1 + C_2$ (and accordingly $c = c_1 c_2$) where C_2 is the greatest common divisor of C and the divisor of z_0, hence obviously $C_1 \neq 0$. Since c divides $x_i z_0$ for all $i \geq 1$, and not $x_0 z_0$, so C_1 maps to the point $(1, 0, 0, 0)$.

By assumption 2) $(1, 0, 0, 0)$ is an isolated singular point of Σ, and since the equation of Σ is given by the determinant of α, i.e. by $Q^2 x_2^2 = x_3 Q G + x_3^2 B$, Q is invertible in a neighbourhood of it. Therefore Σ has a double point in $(1, 0, 0, 0)$, which is not a rational double point (otherwise $C = 0$). By the form of the equation of Σ we see immediately that the tangent cone has then an equation of the form $(x_2 + a x_3)^2 = 0$.

After a linear change of coordinates we can assume that the tangent cone is given by $x_2^2 = 0$. Notice that this coordinate change "corrupts" the previous choices, i.e. statement 1) and 2) do not hold anymore, but we can easily act on the rows and columns of α in order to "recover" them.

We can now consider the double point as a (local analytic) double cover of the plane branched on a singular curve with a singularity of order at least 3. Assume that the singularity is a triple point. Then by [B-P-V] it has to be at least a $(3, 3)$ - point (since otherwise we would have a rational double point). By our equations the tangent direction is $\{x_3 = 0\}$ and after a blow - up there is again a triple point exactly on the intersection of the exceptional divisor with the strict transform of the "tangent" line $\{x_3 = 0\}$.

In particular the strict trasform of x_3 with respect to this blow up pulls back on $\tilde{S}$ to a divisor with some common component with C_1, while the one of x_1 has no common component with C_1. This however contradicts the equality $x_3 z_0 = -Q e^2 c$ since c divides $x_1 z_0$ and Q is invertible at C. This shows that the branch curve has a singular point of order at least 4, and part 3) is proven.

23

This in particular implies (looking at the equation of Σ) that $c_1^4|x_2^2$, so $c_1^2|x_2$ and $c^2|x_2 z_0^2$; this proves part 4) of the statement.

It is now clear that $\frac{w_1 z_0}{ec}$ is a holomorphic section in $3(H+E)$. Moreover, as we already observed, z_0 does not vanish on E (or S would have base points for the bicanonical system), and w_1 vanishes on E with multiplicity 1 (it vanishes there because it is multiple of x_2, with multiplicity 1 or w_1 would induce an element of $\tilde{R}$). So $\frac{w_1 z_0}{ec}$ does not vanish on E; we can therefore choose $u = \frac{w_1 z_0}{ec}$, and part 5) is proven.

$$Q.E.D.$$

The choice of u allows us immediately to write the matrix

$$A := \left(\begin{array}{cccc} y_1 & y_2 & y_3 & w_1 \\ w_0 & w_1 & Q(y_i) & u \end{array} \right),$$

and to notice that the 2×2 minors of A are relations in $\mathcal{R}$.

The ring is in fact the canonical ring of a surface with 7 generators, so it has codimension 4. There is no structure theorem for rings of this codimension, but Reid noticed that most of them have 9 relations (joked by 16 syzygies) that can be expressed in some "formats" (cf. [Rei1], [Rei2]) that help in the study of the deformations.

An important format introduced by Reid is the "rolling factor" format; we try now to recall shortly how it is defined, referring to the above quoted papers by Reid for a more detailed treatement and other examples.

Definition 2.11 One says that a sequence of 9 equations $f_1, \ldots, f_9$ (usually joked by 16 syzygies and defining a Gorenstein ring of codimension 4, but we do not need this here) is in the "rolling factor" format if:

1) $f_1, \ldots f_6$ can be written as the (determinants of the) 2×2 minors of a 2×4 matrix A;

2) f_7 is in the ideal generated by the entries of the first row of A (for the matrix A above, it means that f_7 can be written as a linear combination $ay_1 + by_2 + cy_3 + dw_1 = 0$);

3) f_8 is obtained "rolling" f_7, i.e. taking a linear combination with the same coefficients, but of the entries of the second row of A (in our case f_8 can be chosen as $aw_0 + bw_1 + cQ + du = 0$).

4) f_8 is in the ideal generated by the entries of the first rows of A and f_9 is obtained "rolling" f_8.

24

Notice that there can be different ways to "roll" the same equation, but all equivalent up to the equations given by the minors of A.

Remark 2.12 If $\mathcal{R} = \mathbb{C}[t_0, \ldots t_n]/I$ is an integral domain, I contains the 2×2 minors of a $2 \times n$ matrix, $n \geq 2$, and contains one equation f in the ideal generated by the entries of the first row of A, I must contain also the equation obtained "rolling" f.

This remark will be useful for the next theorem, where we will compute f_7 and prove that f_7 can be "rolled" twice, obtaining f_8 and f_9.
Philosophically, all the three relations come from a single relation in a bigger ring, that is, the last equation in corollary 2.6.

Theorem 2.13 *The canonical ring of a surface with $p_g = 4$, $K^2 = 7$, such that $|K|$ has one simple base point and φ_K is birational, is of the form $\mathcal{R} := \mathbb{C}[y_0, y_1, y_2, y_3, w_0, w_1, u]/I$ where I is generated by the 2×2 minors of the matrix A above, and three more polynomials; one of degree 4 of the form $-w_1^2 + B(y_i) + \sum \mu_{ijk} y_i y_j w_k$, and the other 2 (of respective degrees $5, 6$) obtained rolling it twice (so they have the form $-w_1 u + \ldots, -u^2 + \ldots$).*

In the next section we will write explicitly these equations.
Proof.
We know already all the generators and the relations in degree ≤ 3; moreover we know that all the minors of A are relations of $\mathcal{R}$. An easy dimension count shows that there is one relation missing in degree 4; this relation is in fact induced by the rank condition, as follows:
in corollary 2.6 we have seen that

$$v z_0 = B z_{-1} - G z_0.$$

Using the fact that $v = \frac{q z_0}{z_{-1}} = \frac{x_2^2 z_0}{e^2 c}$ (cf. corollary 2.6 and lemma 2.10)

$$x_2^2 \frac{z_0^2}{e^2 c} = B z_{-1} - G z_0.$$

Multipliying by $\frac{e^2}{c}$ we get the equality

$$w_1^2 = B(y_i) - \frac{G(y_i)}{ec} z_0.$$

Using that $Q^2 x_2^2 = x_3 QG + x_3^2 B$ is singular in $(1, 0, 0, 0)$ (cf. proof of lemma 2.10) and recalling that $G = G(x_0, x_1, x_2)$ (cf. remark 2.7), we see that the coefficient of x_0^3 in G has to be zero and therefore $\frac{G(y_i)}{ec} z_0$ can be written as $-\sum \mu_{ijk} y_i y_j w_k$ for suitable coefficients μ_{ijk}: this provides a non trivial relation in degree 4 of the form

$$(\#\#) \quad w_1^2 = B(y_i) + \sum \mu_{ijk} y_i y_j w_k$$

which obviously is not in the ideal generated by the minors of A.

There are no further relations in degree 4 because they would force a new generator of $\mathcal{R}$ in degree 4, which is excluded by theorem 1.8.

We showed in the proof of the last lemma that the singular point $(1, 0, 0, 0)$ is locally a double cover of the plane branched along a curve with a singularity of order at least 4 and has tangent cone x_2^2.

It is easy to verify that this implies that $QG + x_3 B$ is contained in the ideal $(x_2^2, x_2 x_1^2, x_2 x_1 x_3, x_2 x_3^3, x_1^4, x_1 x_3^3, x_1^2 x_3^2, x_1 x_3^3, x_3^4)$, i.e. the monomials $x_0^3, x_0^2 x_1, x_0^2 x_2, x_0 x_1^2$ do not appear in G, whence B is a quartic in $\mathbb{P}^3$ such that the monomials $x_0^4, x_0^3 x_1, x_0^3 x_3$ have coefficient zero.

Implementing this in $(\#\#)$ we find that the right side can be chosen (up to adding some element of the ideal generated by the minors of A) to be in the square of the ideal generated by the first row of A, hence it can be rolled twice.

By theorem 1.8, we know that the ideal I of relations is generated in degree ≤ 6. We have three elements f_1, f_2, f_3 in I_3, f_4, f_5, f_6, f_7 in I_4, f_8 in I_5, f_9 in I_6.

Let I' be the ideal $(f_1, \ldots, f_9)$, R' be the quotient ring $\mathbb{C}[y_0, y_1, y_2, y_3, w_0, w_1, u]/I'$. To show $I' = I$ it suffices to show $I'_k = I_k$ $\forall k \leq 6$, or equivalently, $\dim R'_k \leq \dim R_k$ for $k \leq 6$. This is a calculation done by Macaulay 2 (cf. Appendix 1, where the verification is done using the equations in theorem 3.7).

$$\underline{Q.E.D.}$$

3 The canonical ring of surfaces with $K^2 = 7$, $p_g = 4$ birational to a sextic: explicit computations

Let S be as in the previous section, i.e., a minimal surface with $K^2 = 7$, $p_g = 4$, such that the canonical system has one simple base point and the canonical map is birational.

In the last section we have shown that the image $\varphi(S) = \Sigma$ of the canonical map has an equation of the form

$$-x_2^2 Q^2 + x_3 QG + x_3^2 B$$

where Q is a quadric, G is a cubic, and both do not depend on x_3.

Moreover, we have seen in the proof of theorem 2.13 that the monomials $x_0^3, x_0^2 x_1, x_0^2 x_2, x_0 x_1^2$ do not appear in G, and that B is a quartic in $\mathbb{P}^3$ such that the monomials $x_0^4, x_0^3 x_1, x_0^3 x_3$ have coefficient zero.

In the following three subsections we write the canonical ring in three explicit ways, using different formats.

3.1 Rolling factors format

The proof of theorem 2.13 provides immediately the equations in the "rolling factors" format that we have introduced in the previous section.

We have defined

$$A := \left(\begin{array}{cccc} y_1 & y_2 & y_3 & w_1 \\ w_0 & w_1 & Q(y_0, y_1, y_2) & u \end{array} \right).$$

We write

$$G = k x_1^3 + x_2 Q_0(x_1, x_2) + x_0 x_2 l(x_1, x_2),$$

$$B = x_2 C(x_0, x_1, x_2, x_3) + x_3^2 Q_3(x_0, x_1, x_3) + x_3 x_1 Q_2(x_0, x_1) + x_1^2 Q_1(x_0, x_1),$$

where $k \in \mathbb{C}$, l is linear, the Q_i's are quadratic and C is a cubic.

With the above notation, the proof of theorem 2.13 gives the following description for the canonical ring:

Theorem 3.1 *Let S be a minimal (smooth, connected) surface with $K^2 = 7$ and $p_g(S) = 4$, such that the canonical system $|K_S|$ has one (simple) base point $x \in S$ and the canonical map is birational. Then the canonical ring $\mathcal{R}(S)$ of S can be written as*

$$\mathbb{C}[y_0, y_1, y_2, y_3, w_0, w_1, u]/\mathcal{I},$$

where $\deg(y_i, w_j, u) = (1, 2, 3)$ and $\mathcal{I}$ is generated by the 2×2 - minors of the matrix

$$\begin{pmatrix} y_1 & y_2 & y_3 & w_1 \\ w_0 & w_1 & Q(y_0, y_1, y_2) & u \end{pmatrix},$$

and by

$$- w_1^2 + kw_0 y_1^2 + w_1 Q_0(y_1, y_2) + w_1 y_0 l(y_1, y_2) + y_2 C(y_0, y_1, y_2, y_3) + \\ + y_3^2 Q_3(y_0, y_1, y_3) + y_1 y_3 Q_2(y_0, y_1) + y_1^2 Q_1(y_0, y_1),$$

$$- w_1 u + kw_0^2 y_1 + u Q_0(y_1, y_2) + u y_0 l(y_1, y_2) + w_1 C(y_0, y_1, y_2, y_3) + \\ + y_3 Q(y_0, y_1, y_2) Q_3(y_0, y_1, y_3) + w_0 y_3 Q_2(y_0, y_1) + y_1 w_0 Q_1(y_0, y_1),$$

$$- u^2 + kw_0^3 + w_1 Q_0(w_0, w_1) + u y_0 l(w_0, w_1) + u C(y_0, y_1, y_2, y_3) + \\ + Q^2(y_0, y_1, y_2) Q_3(y_0, y_1, y_3) + w_0 Q(y_0, y_1, y_2) Q_2(y_0, y_1) + w_0^2 Q_1(y_0, y_1).$$

Proof. This is just the explicit expression of the computation in theorem 2.13.

$$\underline{Q.E.D.}$$

Remark 3.2 1) We remark that the above equations are not exactly "rolled", but only up to changing f_8 by a suitable combination of the 2×2 - minors of A. Nevertheless we prefer to leave the equations in the above form because they are more readable.

2) Our goal is to show that this canonical ring can be deformed to a canonical ring of surfaces with the same invariants but with base point free canonical system.

Other ways of writing the same canonical ring might under this aspect be more convenient.

3.2 The $AM(^tA)$ -format.

Following the notation of M. Reid in [Rei2], the $AM(^tA)$ - format is another way to describe the canonical ring of a Gorenstein variety of codimension 4 (defined by 9 relations).It was introduced by D. Dicks and M. Reid.

We briefly recall the definition of the $AM(^tA)$ - format, referring for details to [Rei1], [Rei2], [Rei3].

Definition 3.3 Assume we are given a commutative ring $R = \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_9)$. Then we call this presentation in $AM(^tA)$ - format iff there is a 2×4 - matrix A such that $f_1, \dots f_6$ are the 2×2 - minors of A and there is a symmetric 4×4 - matrix M such that

$$AM(^tA) = \begin{pmatrix} f_7 & f_8 \\ f_8 & f_9 \end{pmatrix}.$$

This format is slightly more restricted than the previous one. In fact, the equations given in the $AM(^tA)$ format are automatically in the "rolling factor" format, whereas equations given in the "rolling factor" format can be expressed in an $AM(^tA)$ format if and only if the 3 supplementary relations are "quadratic forms in the rows of A". This is equivalent to saying that the first one (the one we denoted by f_7) is in the square of the ideal generated by the entries of the first row of A, what happens to occur in our specific situation.

We are now ready to state our result.

Theorem 3.4 *Let S be a minimal (smooth, connected) surface with $K^2 = 7$ and $p_g(S) = 4$, such that the canonical system $|K_S|$ has one (simple) base point $x \in S$ and the canonical map is birational. Then the canonical ring $\mathcal{R}$ of S is of the form $\mathbb{C}[y_0, y_1, y_2, y_3, w_0, w_1, u]/\mathcal{I}$ (deg $(y_i, w_j, u) = (1, 2, 3)$) and can be presented in the $AM(^tA)$ - format with*

$$A = \begin{pmatrix} y_1 & y_2 & y_3 & w_1 \\ w_0 & w_1 & Q(y_0, y_1, y_2) & u \end{pmatrix},$$

and

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & k_1 y_0 \\ & m_{22} & m_{23} & k_2 y_0 \\ & & m_{33} & L \\ sym & & & -1 \end{pmatrix}.$$

Here $k_1, k_2 \in \mathbb{C}$, $L = L(y_0, y_1, y_2, y_3)$ is linear and m_{ij} are quadratic forms depending on the variables $y_0, y_1, y_2, y_3, w_0, w_1$ as follows:

$$
\begin{aligned}
m_{11} &= m_{11}(y_0, y_1, w_0, w_1), \\
m_{12} &= a_1 w_1 + a_2 y_0 y_1 + a_3 y_0 y_2 + a_4 y_1^2, \\
m_{13} &= m_{13}(y_0, y_1), \\
m_{22} &= b_1 w_1 + b_2 y_0 y_2 + b_3 y_1^2 + b_4 y_1 y_2 + b_5 y_2^2, \\
m_{23} &= m_{23}(y_0, y_1, y_2, y_3), \\
m_{33} &= m_{33}(y_0, y_1, y_3),
\end{aligned}
$$

where $a_i, b_j \in \mathbb{C}$.

As we will see in the proof, the coefficients of the entries of M can be explicitly determined from the equation of Σ.

Proof. First we note that A was already found in theorem 3.1. In order to write down the matrix M, we have to set up some more notation.

We write explicitly (keeping the terminology which was introduced earlier)

$$Q_0(y_1, y_2) = q_{11} y_1^2 + q_{12} y_1 y_2 + q_{22} y_2^2,$$

$$l(y_1, y_2) = l_1 y_1 + l_2 y_2,$$

where $q_{ij}, l_k \in \mathbb{C}$.

We remark that the coefficient of y_0^2 in Q has to be different from zero because of the assumption that the generalized tacnode o is not contained in Γ. Therefore we can write

$$
\begin{aligned}
C(y_0, y_1, y_2, y_3) = {}& l'(y_0, y_1, y_2, y_3)Q + y_0(Q'_{11} y_1^2 + Q'_{12} y_1 y_2 + Q'_{22} y_2^2) + \\
& y_0 y_3 l''(y_1, y_2, y_3) + s y_1^3 + y_2 Q''(y_1, y_2) + y_3 Q'''(y_1, y_2, y_3),
\end{aligned}
$$

30

where $s, Q'_{ij} \in \mathbb{C}$, l', l'' are linear forms and Q'', Q''' are quadratic forms.

Now, a rather lengthy, but straightforward calculation shows that for $M =$

$$
\begin{pmatrix}
kw_0 + Q_1 + q_{11}w_1 & \frac{1}{2}(q_{12}w_1 + y_0(Q'_{11}y_1 + Q'_{12}y_2) + sy_1^2) & \frac{1}{2}Q_2 & \frac{1}{2}l_1y_0 \\
 & q_{22}w_1 + Q'_{22}y_0y_2 + Q'' & \frac{1}{2}(y_0l'' + Q''') & \frac{1}{2}l_2y_0 \\
 & & Q_3 & \frac{1}{2}l' \\
sym & & & -1
\end{pmatrix},
$$

the conditions rank $A \leq 1$ and $AM(^tA) = 0$ define the ideal $\mathcal{I}$ of theorem 3.1.

<div align="right"><u>Q.E.D.</u></div>

Remark 3.5 In [Rei2] M. Reid shows that, given a polynomial ring $\mathbb{C}[x_0, \dots, x_n]$ and a Gorenstein $\mathbb{C}[x_0, \dots, x_n]$−algebra $\mathcal{R}$ of codimension 4 and presented in $AM(^tA)$ - format, all the syzygies are induced by A and M. In particular in this case the $AM(^tA)$-format is *flexible*, i.e. every deformation of the matrices A and M preserving the symmetry of M induces a flat deformation of $\mathcal{R}$.
In order to find a flat family $\mathcal{S}/T$ of surfaces such that S_{t_0} is a surface with $K^2 = 7$, $p_g = 4$ such that the canonical system has one base point and the canonical map is birational, whereas S_t is a surface such that the canonical system has no base points for every $t \neq t_0$, it would be sufficient to find a deformation of the above matrices, which induces the right ring.
Unfortunately this does not work, since by [Cat1] the canonical ring of S_t (for $t \neq t_0$) is generated in degrees 1 and 2. In particular, if $\mathcal{R}_t$ is such a deformation of $\mathcal{R}$, for $t \neq 0$, one of the three relations in degree three has to eliminate the generator u. But, considering only deformations of $\mathcal{R}$ induced by deformations of A and M, the relations in degree three of $\mathcal{R}_t$ are given by the two by two minors of a matrix A_t with the following degrees in the entries:

$$
\begin{pmatrix}
1 & 1 & 1 \\
2 & 2 & 2
\end{pmatrix},
$$

which obviously cannot eliminate an element u having degree 3.

3.3 The antisymmetric and extrasymmetric format.

The aim of this section is to give a third description of our canonical ring $\mathcal{R} = \mathbb{C}[y_0, y_1, y_2, y_3, w_0, w_1, u]/\mathcal{I}$.

Definition 3.6 1) A 6×6 - matrix P is called *antisymmetric and extrasymmetric* if and only if it has the form

$$
P = \begin{pmatrix}
0 & a & b & c & d & e \\
 & 0 & f & g & h & d \\
 & & 0 & i & pg & pc \\
 & & & 0 & qf & qb \\
 & & & & 0 & pqa \\
-sym & & & & & 0
\end{pmatrix}.
$$

2) Assume we are given a commutative ring $\mathcal{R} = \mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}$. If there is an antisymmetric and extrasymmetric matrix P whose 4×4 pfaffians generate the ideal $\mathcal{I}$, we say that the ring has an *antisymmetric and extrasymmetric format given by P*.

This format was first introduced by D. Dicks and M. Reid in a less general form [Rei1]. The above more general form appeared in [Rei3].
An easy computation shows that the ideal of the 15 (4×4) pfaffians of an antysimmetric and extrasymmetric matrix is in fact generated by nine of them.
Under suitable generality assumptions (e.g. that $\mathcal{R}$ be Gorenstein of codimension 4 and that the 9 equations be independent), it should be easy (but rather lengthy) to show, following the lines of Reid's argument used in [Rei1] to prove a special case, that the format is flexible (i.e., that a deformation of the entries of P induces a flat deformation of the ring).
Since however we are only interested in our particular case, it does not pay off here to perform this calculation. Hence we will proceed as follows:
we shall put our canonical ring in an antisymmetric and extrasymmetric format, deform the associated 6×6 antisymmetric and extrasymmetric matrix (preserving the extrasymmetry) and then verify the flatness of the induced family by the constancy of the Hilbert polynomial.
Using the same notation as in the previous section we obtain

Theorem 3.7 *Let S be a minimal (smooth, connected) surface with $K^2 = 7$ and $p_g(S) = 4$, whose canonical system $|K_S|$ has one (simple) base point $x \in$*

S and yields a birational canonical map. Then the canonical ring of S can be presented as $\mathcal{R} = \mathbb{C}[y_0, y_1, y_2, y_3, w_0, w_1, u]/\mathcal{I}$, where $\deg(y_i, w_j, u) = (1, 2, 3)$ and the ideal of relations $\mathcal{I}$ of $\mathcal{R}$ is generated by the 4×4 - pfaffians of the following antysimmetric and extrasymmetric matrix

$$
P = \begin{pmatrix}
0 & 0 & w_0 & Q(y_0, y_1, y_2) & w_1 & u \\
 & 0 & y_1 & y_3 & y_2 & w_1 \\
 & & 0 & -u + \overline{C} & y_3\overline{Q}_3(w_0, y_0, y_1, y_3) & Q\overline{Q}_3 \\
 & & & 0 & y_1\overline{Q}_1(w_0, w_1, y_0, y_1, y_3) & w_0\overline{Q}_1 \\
 & & & & 0 & 0 \\
 & -sym & & & & 0
\end{pmatrix},
$$

where $Q, \overline{Q}_1, \overline{Q}_3$ are quadratic forms of a subset of the given variables as indicated, and $\overline{C}$ is a cubic form. Moreover $\overline{C}$ does not depend on u and the $w_i y_j$'s for $j \le 1$, $\overline{Q}_1$ does not depend on y_3^2.

Proof.
Recall that the coefficient of x_0^2 in Q is different from zero. This allows us to choose $s \in \mathbb{C}$ and linear forms $\overline{l}_1(y_0, y_1)$, $\overline{l}_2(y_0, y_1, y_2)$ such that

$$
Q_2(y_0, y_1) = sQ(y_0, y_1, y_2) + y_1\overline{l}_1(y_0, y_1) + y_2\overline{l}_2(y_0, y_1, y_2).
$$

Therefore we can write

$$
\overline{C} = q_{12}w_0y_2 + q_{22}w_1y_2 + y_0l(w_0, w_1) + C(y_0, y_1, y_2, y_3) + y_1y_3\overline{l}_2(y_0, y_1, y_2);
$$

$$
\overline{Q}_1 = kw_0 + q_{11}w_1 + Q_1(y_0, y_1) + y_3\overline{l}_1(y_0, y_1)
$$

$$
\overline{Q}_3 = -Q_3 - sw_0.
$$

Now the rest of the proof is a straightforward calculation.

$$\underline{Q.E.D.}$$

4   An explicit family.

In this section we will find an explicit deformation of the canonical ring $\mathcal{R}$ to the canonical ring $\mathcal{S}$ of a surface with $K^2 = 7$, $p_g = 4$, such that the canonical system has no base points.

Before doing this we have to recall some of the results ([Cat1]) on surfaces with $K^2 = 7$, $p_g = 4$, whose canonical system is base point free.

Let X be a nonsingular surface with $K^2 = 7$, $p_g = 4$, such that the canonical system is base point free, hence the canonical map is a birational morphism onto a septic surface in $\mathbb{P}^3$. We denote by $\mathcal{S}$ the canonical ring of X. As in the previous case, we denote by y_i an appropriate basis of $H^0(X, \mathcal{O}_X(K))$. We set $\mathcal{A} := \mathbb{C}[y_0, y_1, y_2, y_3]$.

Theorem 4.1 *1) $\mathcal{S}$ has a minimal resolution as $\mathcal{A}$ - module given by the matrix*

$$\alpha = \begin{pmatrix} d_1 d_2 y_0 + (d_3 d_4 + d_2^2) y_1 + (d_2 d_3 + d_1 d_4) y_2 & d_4 y_1 & d_1 y_0 + d_2 y_1 + d_3 y_2 \\ d_4 y_1 & y_0 & y_2 \\ d_1 y_0 + d_2 y_1 + d_3 y_2 & y_2 & y_1 \end{pmatrix},$$

where d_1, d_2, d_3, d_4 are arbitrary quadratic forms in y_i.
2) α satisfies the rank condition $\Lambda^2(\alpha) = \Lambda^2(\alpha')$, where α' is obtained by deleting the first row of α, and therefore induces a unique ring structure on $\mathcal{S}$ as quotient of $\mathcal{B} = \mathcal{A}[w_0, w_1]$ by the three relations given by

$$\alpha \begin{pmatrix} 1 \\ w_0 \\ w_1 \end{pmatrix} = 0$$

and three more relations expressing $w_i w_j$ as linear combination of the other monomials whose coefficients are determined by the adjoint matrix of α.
3) The surfaces with $K^2 = 7$, $p_g = 4$ such that the canonical system is base point free form an irreducible unirational component of dimension 36 in the moduli space $\mathfrak{M}_{K^2=7, p_g=4}$ of surfaces with $K^2 = 7$, $p_g = 4$.

For a more precise formulation we refer to the original articles [Cat1] or [Cat3].

Remark 4.2 1) The previous theorem implies in particular that the canonical ring $\mathcal{S}$ is Gorenstein in codimension 3. Hence by the classical result of Buchsbaum and Eisenbud (cf. [B-E]) the ideal $\mathcal{I} \subset \mathcal{A}[w_0, w_1] =: \mathcal{B}$ defining $\mathcal{S}$ can be minimally generated by the $2k \times 2k$ - Pfaffians of a skewsymmetric $(2k+1) \times (2k+1)$ - matrix. Writing down explicitly the above six defining equations, we can see that $\mathcal{I}$ is generated by only five of them, hence in our case $k = 2$.

2) More precisely, we have a selfdual resolution of $\mathcal{S}$ as $\mathcal{B}$ - module as follows

$$0 \to \mathcal{B}(-9) \xrightarrow{f_2} \mathcal{B}(-5)^3 \oplus \mathcal{B}(-6)^2 \xrightarrow{f_1} \mathcal{B}(-4)^3 \oplus \mathcal{B}(-3)^2 \xrightarrow{f_0} \mathcal{B} \to \mathcal{S} \to 0,$$

where f_1 is alternating, f_0 is given by the Pfaffians of f_1 and $f_2 = {}^t f_0$.

3) Vice versa, assume we have a $\mathbb{C}$ - algebra $\mathcal{S}$, which admits a resolution as above: then under suitable open condition $\mathcal{S}$ is the canonical ring of a surface X with $p_g = 4$, $K^2 = 7$ and free canonical system.

Our aim is now to take the matrix P in antisymmetric and extrasymmetric format and try to find a deformation P_t of P with the following properties:

0) For $t \neq 0$ in at least one of the Pfaffians of degree 3 the generator u appears with a non zero coefficient;

1) for $t \neq 0$ there is a skewsymmetric 5×5 -matrix Q_t, such that the ideal $\mathcal{J}_t$ generated by the 4×4 - Pfaffians of Q_t coincides with $\mathcal{I}_t \cap \mathcal{B}$, where $\mathcal{I}_t$ is the ideal generated by the 4×4 - Pfaffians of P_t, for every $t \neq 0$;

2) the entries of Q_t have the right degrees, i.e. Q_t defines a map

$$\mathcal{B}(-5)^3 \oplus \mathcal{B}(-6)^2 \xrightarrow{Q_t} \mathcal{B}(-4)^3 \oplus \mathcal{B}(-3)^2;$$

3) $\mathcal{S}_t := \mathbb{C}[y_0, y_1, y_2, y_3, w_0, w_1, u]/\mathcal{I}_t$ is a flat family.

From the previous remark we conclude that $\mathcal{S}_t$ (for $t \neq 0$), constructed as above, is the canonical ring of a surface X with $p_g = 4$, $K^2 = 7$ and free canonical system. Hence once we have found a deformation P_t as above, we have explicitly deformed the surfaces with $p_g = 4$, $K^2 = 7$, such that $|K_S|$ has one base point and induces a birational map to the surfaces with $p_g = 4$, $K^2 = 7$ and free canonical system.

The natural attempt now is to deform the entries of P preserving the extrasymmetry (this should give automatically the flatness), so that one of the relations in degree 3 eliminates u. At first glance we see a natural way: putting the deformation parameter t in the only entry of degree 0 (replacing the 0 therein) and replacing the symmetrical "zero" to preserve the extrasymmetry. This actually works and we have the following

Theorem 4.3 *Let P be an antisymmetric and extrasymmetric matrix as in theorem 3.7. Consider the 1-parameter family of rings $\mathcal{S}_t = \mathbb{C}[y_0, y_1, y_2, y_3, w_0.w_1, u]/\mathcal{I}_t$ where the ideal $\mathcal{I}_t$ is given by the 4×4 pfaffians of the antysimmetric and extrasymmetric matrix*

$$
P_t = P + \begin{pmatrix}
0 & t & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 \\
 & & & & 0 & t\overline{Q}_1\overline{Q}_3 \\
-sym & & & & & 0
\end{pmatrix}.
$$

This is a flat family and describes a flat deformation of the surface corresponding to the matrix P to surfaces with $p_g = 4$, $K^2 = 7$ and with $|K|$ base point free. For $t \neq 0$, $\mathcal{S}_t$ is isomorphic to $\mathbb{C}[y_0, y_1, y_2, y_3, w_0.w_1]/J_t$ where J_t is the ideal generated by the 4×4 pfaffians of the matrix

$$
\begin{pmatrix}
0 & y_1 & -y_3 & -t^2 w_0 & -tQ \\
 & 0 & y_2 & t^3\overline{Q}_3 & tw_1 \\
 & & 0 & t^2 w_1 & t^2\overline{Q}_1 \\
 & & & 0 & -t^3 c - t^2 y_1 Q + t^2 y_3 w_0 \\
-sym & & & & 0
\end{pmatrix}.
$$

Proof.
We have to check that the properties 0)-3) described above are fullfilled by our pair P_t, Q_t.
0): We immediately see that the 4×4 - Pfaffian obtained eliminating the last two rows and columns eliminates the generator u;
1): once we eliminate u it is easy to write down explicitely $\mathcal{I}_t \cap \mathcal{B}$ for $t \neq 0$ and check that it coincides with the given ideal J_t (cf. Appendix 2);
2): obvious;
3): we expect that flatness holds in general once we preserve the extrasymmetry.

36

In our particular case, flatness follows auitomatically since:

a) $X_t :=$Proj$(\mathcal{B}/J_t)$ has dimension ≤ 2 by semicontinuity.

b) Then Q_t and its pfaffians give a resolution of J_t by [B-E].

c) Therefore the Hilbert polynomial of J_t is the same as the one of J_0, then the family is flat.

Finally, since the property that X_t has only R.D.P.'s as singularities is open, X_t is the canonical model of a surface of general type as required.

<div align="right">$Q.E.D.$</div>

Remark 4.4 As we have already said in the introduction, this particular degeneration was studied by Enriques in his book [Enr]. There Enriques states that such a degeneration should exist and suggest a way to construct such a family degenerating the canonical images. With the help of Macaulay 2 we have explicitly computed the degenerations of the canonical images corresponding to our family and we have found that the degeneration is not the one "predicted" by Enriques. In the following we recall briefly (see [Enr] for the details) Enriques' prediction and point out where is the difference.

The canonical image of a surface with $K^2 = 7$, $p_g = 4$ with base point free canonical system and birational canonical morphism is a surface in $\mathbb{P}^3$ of degree 7 with a singular curve of degree 7 and genus 4 having a triple point. Moreover the adjoint quadric is a quadric cone (with vertex in the singular point of the curve) whose intersection with the surface is given by the above curve of degree 7 counted twice.

As we have already seen, the canonical image of a surface with $K^2 = 7$, $p_g = 4$ with one simple case point for the canonical system and canonical map birational is a surface in $\mathbb{P}^3$ of degree 6 with a singular curve of degree 2 and a generalized tacnode. In this case there is an adjoint plane that is the plane through the conic ($\{x_3 = 0\}$ in our notation).

Enriques suggests to add to the sextic the adjoint plane. The intersection of this plane and the sextic is given by the singular conic and the line image of E both counted twice; this gives a reducible septic with a triple conic and a double line of "tacnodal type" (i.e. near a general point of the line the surface has two branches tangent on the line).

Enriques states that it is possible to construct a family of septics with a singular curve of degree 7 and genus 4 having a triple point that degenerates to the above configuration so that the singular septic degenerates to the union of the conic (counted three times) and the line.

We wrote the canonical images of the family described by the 4×4 pfaffians of the 5×5 skewsymmetric matrix in theorem 4.3, and, as anticipated, we did not get the situation predicted by Enriques. In fact the family of septics degenerates to the union of the sextic canonical image of the limit surface with a plane, but instead of the plane predicted by Enriques we have gotten the plane $\{x_2 = 0\}$ (the reduced tangent cone of the tacnode).

In fact, it is quite easy to compute also the degeneration of the adjoint quadric: consider the resolution of the canonical ring of a surface with base point free canonical system as $\mathcal{A}-$module as in theorem 4.1. From this resolution one can immediately see that the adjoint quadric must be the determinant of the right-bottom 2 by 2 minor of the resolution matrix in theorem 4.1 (i.e. $y_0 y_1 - y_2^2$ in the coordinates chosen there). This minor depends only on the two relations in degree 3 of the canonical ring.

It is now easy to compute it for our family: we have just to write down the two relations in degree 3, write the 2 by 2 matrix of the coefficients of w_0 and w_1 in this two equations, and then compute the determinant.

In the notation of theorem 4.3 the two relations in degree 3 (for $t \neq 0$) can be written as:

$$y_1 w_1 - w_0 y_2 + t y_3 \overline{Q}_3;$$
$$t y_1 \overline{Q}_1 + w_1 y_3 - Q y_2.$$

The equation of the quadric cone depends clearly on the coefficients of w_0 and w_1 in $\overline{Q}_1$ and $\overline{Q}_1$ (but notice that it is in every case independent of y_0, so it cannot be a smooth quadric, as expected), but the two equations degenerate respectively to $y_1 w_1 - w_0 y_2$ and $w_1 y_3 - Q y_2$; the 2 by 2 minor degenerates then to

$$\begin{pmatrix} -y_2 & y_1 \\ 0 & y_3 \end{pmatrix}$$

and the quadric cone degenerate to the union of the adjoint plane and the tacnodal plane.

Geometrically we could say that the septic degenerates to the union of a sextic and a plane, the adjoint quadric to the union of the same plane and a different plane (the tacnodal plane); the two "identical" planes "simplify" and we are left with the sextic and his adjoint plane.

References

[B-P-V]   W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces,* Springer-Verlag (1984).

[Ba]   Bauer, I. *Surfaces with $K^2 = 7$ and $p_g = 4$*, Mem. Am. Math. Soc. 721, 79 p. (2001)

[Bom]   E. Bombieri, *Canonical models of surfaces of general type.* Publ. Scient. I.H.E.S. **42** (1973), 447–495.

[B-E]   D. A. Buchsbaum, D. Eisenbud, *Algebra structures for finite free resolutions and some structure theorems for ideals of codimension 3.* Am. J. of Math. **99** (1977), 447–485.

[Cat1]   F. Catanese, *Commutative algebra methods and equations of regular surfaces.* "Alg. Geom. - Bucharest 1982", Springer LNM *1056* (1984), 68–111.

[Cat2]   F. Catanese, *On the moduli spaces of surfaces of general type.* J. Diff. Geom. **19** (1984), 483–515.

[Cat3]   F. Catanese, *Homological Algebra and Algebraic Surfaces.* Proceedings of Symposia in Pure Mathematics, vol. 62, **1** (1997), 3–56.

[Cil1]   C. Ciliberto,*Canonical surfaces with $p_g = p_a = 4$ and $K^2 = 5, \dots, 10$.* Duke Math. Journal **48**,1, 121–157 (1981).

[Cil2]   C. Ciliberto, *On the degree of the generators of the canonical ring of a surface of general type.*, Rend. Semin. Mat., Torino **41**, No.3 (1983), 83–111.

[Deb]   O. Debarre, *Inégalités numériques pour les surfaces de type général.* Bull. Soc. Math. France **110** (1982), 319–346.

[Dic1]   D. Dicks, *Surfaces with $p_g = 3$, $K^2 = 4$ and extension-deformation theory.* Warwick Ph. D. thesis (1988).

[Dic2]   D. Dicks, *Surfaces with $p_g = 3$, $K^2 = 4$.* Warwick preprint (1988).

[Enr]   F. Enriques, *Le Superficie Algebriche.* Zanichelli, Bologna (1949).

[Fran]   A. Franchetta, *Su alcuni esempi di superficie canoniche.* Rend. Sem. Mat. Roma 3, (1939). 23–28.

[Fra]   P. Francia, *On the base points of the bicanonical system.* "Problems in the theory of surfaces and their classification" (Cortona, 1988), 141–150, Sympos. Math., XXXII, Academic Press, London, 1991.

[Gri]   E. Griffin, *Families of quintic surfaces and curves*, Compositio Math. **55** (1986), 33–62.

[HorQ]   E. Horikawa, *On deformations of quintic surfaces.* Inv. Math. **31** (1975), 43–85.

[HorI-V]   E. Horikawa, *Algebraic surfaces of general type with small c_1^2.* I, Ann. of Math. (2) **104** (1976), 357–387; II, Inv. Math. **37** (1976), 121–155; III, Inv. Math. **47** (1978), 209–248; IV, Inv. Math. **50** (1979), 103–128; V, J. Fac. Sci. Univ. Tokyo, Sect. A. Math. *283* (1981),745–755.

[Kod]   K. Kodaira, *On characteristic systems of families of surfaces with ordinary singularities in a projective space.* Amer. J. of Math. **87**, (1965), 227–256.

[Kur]   M. Kuranishi, *Deformations of compact complex manifolds.* Presses de l'Universite de Montreal (1969).

[dJ-vS]   de Jong, T.; van Straten, D. Deformations of the normalization of hypersurfaces. (English) Math. Ann. 288, No.3, 527-547 (1990).

[Lau]   H. B. Laufer, *On minimally elliptic singularities.* Am. J. of Math. **99** (1977), 1257–1295.

[Max]   E. A. Maxwell, *Regular canonical surfaces of genus three and four.* Proceedings of the Cambridge Philosophical Society **23** (1937), 306-310.

[Pal]   V. P. Palamodov, *Deformations of complex spaces.* Russ. Math. Surveys **31**, No. 1 (1976), 129–197.

[Rei0]   M. Reid, *Problem 12, Part 2* Problemes ouverts, Séminaire de Mathématiques Supérieures, Univ. Montréal (1980).

[Rei1]   M. Reid, *Infinitesimal view of extending a hyperplane section - deformation theory and computer algebra.* Alg. Geometry, L'Aquila 1988, LNM *1417*, Springer Verlag (1990), 214–286.

[Rei2]   M. Reid, *Surfaces with $p_g = 3$, $K^2 = 4$ according to E. Horikawa and D. Dicks.* Proc. of Alg. Geometry mini Symposium, Tokyo Univ. Dec 1989, (1989), 1–22.

[Rei3]   M. Reid, *Graded rings and birational geometry.* Proc. of algebraic geometry symposium (Kinosaki, Oct 2000), K. Ohno (Ed.), 1–72.

[Zuc]   F. Zucconi, *Surfaces with canonical map of degree three and $K^2 = 3p_g − 5$.*, Osaka J. Math. 34, No.2, 411–428

Address of the authors until August 31, 2001:
Mathematisches Institut der Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen

Address of the authors after September 1, 2001:
Mathematisches Institut der Universität Bayreuth
Lehrstuhl Mathematik VIII
Universitätstr. 30
95447 Bayreuth

E-mail addresses:
Ingrid.Bauer@uni-bayreuth.de
Fabrizio.Catanese@uni-bayreuth.de
Roberto.Pignatelli@uni-bayreuth.de

```
-- APPENDIX 1
-- This script checks that the relations we found in Theorem 2.13 are all the
-- relations of the canonical ring till degree 6, so they are all the
-- relations by theorem 1.8.
-- First we write the ring: with all the variables and parameters we need
R=QQ[u,w_1,w_0,y_0,y_1,y_2,y_3,
     a0,a1,a2,b0,d0,d1,t,Q,Q1,Q3,c,q00,q01,q02,q11,q12,q22,
     q100,q101,q103,q111,q113,q3,q300,q301,q303,q311,q313,q333,
     c000,c001,c002,c003,c011,c012,c013,c022,c023,c033,
     c111,c112,c113,c122,c123,c133,c222,c223,c333,
     MonomialOrder=>Lex
       ]
-- Now we write the matrix in theorem 3.7
M=matrix{
{0,0,w_0,Q,w_1,u},
{0,0,y_1,y_3,y_2,w_1},
{-w_0,-y_1,0,-u+(c+a0*w_1*y_0-a1*w_1*y_1+a2*w_1*y_2+b0*w_0*y_0),y_3*Q3,Q*Q3},
{-Q,-y_3,u-(c+a0*w_1*y_0-a1*w_1*y_1+a2*w_1*y_2+b0*w_0*y_0),0,
               (Q1+d0*w_0+d1*w_1)*y_1,w_0*(Q1+d0*w_0+d1*w_1)},
{-w_1,-y_2,-y_3*Q3,-(Q1+d0*w_0+d1*w_1)*y_1,0,0},
{-u,-w_1,-Q*Q3,-w_0*(Q1+d0*w_0+d1*w_1),0,0}
};
pfaff=pfaffians(4,M);
-- Here we restrict to 9 pfaffians and check that they are enough to
-- generate the whole pfaffian ideal (the second line gives ''true''
-- as output)
pfaff9=submatrix(gens(pfaff),,{0,1,5,2,6,9,3,4,7});
gens(pfaff) % ideal(pfaff9)==0
-- Then we write explicitly all the polynomials in the matrix
ourideal:=substitute(pfaff9,{
         Q=>y_0*y_0+q01*y_0*y_1+q02*y_0*y_2+q11*y_1*y_1+q12*y_1*y_2+q22*y_2*y_2,
         Q1=>q100*y_0*y_0+q101*y_0*y_1+q103*y_0*y_3+q111*y_1*y_1+q113*y_1*y_3,
         Q3=>q3*w_0+q300*y_0*y_0+q301*y_0*y_1+q303*y_0*y_3+q311*y_1*y_1+
             q313*y_1*y_3+q333*y_3*y_3,
         c=>c000*y_0*y_0*y_0+c001*y_0*y_0*y_1+c002*y_0*y_0*y_2+c003*y_0*y_0*y_3+
             c011*y_0*y_1*y_1+c012*y_0*y_1*y_2+c013*y_0*y_1*y_3+c022*y_0*y_2*y_2+
             c023*y_0*y_2*y_3+c033*y_0*y_3*y_3+c111*y_1*y_1*y_1+c112*y_1*y_1*y_2+
             c113*y_1*y_1*y_3+c122*y_1*y_2*y_2+c123*y_1*y_2*y_3+c133*y_1*y_3*y_3
```

```
                  +c222*y_2*y_2*y_2+c223*y_2*y_2*y_3+c333*y_3*y_3*y_3
            })
-- Here we define the ideal of the monomials in all the degrees till 6
linear:=ideal(y_0,y_1,y_2,y_3);
quadrics:=ideal(w_0,w_1)+linear^2;
cubics:=ideal(mingens(ideal(u)+linear*quadrics));
quartics:=ideal(mingens(quadrics^2+linear*cubics));
quintics:=ideal(mingens(quadrics*cubics+linear*quartics));
sextics:=ideal(mingens(cubics*cubics+quadrics*quartics+linear*quintics));
-- finally we compute a system of generators, degree by degree, of the
-- resulting quotient. All of them turn out to be composed exactly by
-- P_n elements (resp. 4,12,26,47,75), that concludes the argument in the
-- proof of theorem 2.13
K=mingens ideal((gens linear) % gb ourideal);
twoK=mingens ideal((gens quadrics) % gb ourideal);
threeK=mingens ideal((gens cubics) % gb ourideal);
fourK=mingens ideal((gens quartics) % gb ourideal);
fiveK=mingens ideal((gens quintics) % gb ourideal);
sixK=mingens ideal((gens sextics) % gb ourideal);

restart

-- APPENDIX 2

R=QQ[t,u,y_1..y_3,Q,Q1,Q3,w_0..w_1,c,
      Degrees=>{1,3,1,1,1,2,2,2,2,2,3}]

-- the 4 x 4 - Pfaffians of the matrix M define the relations
-- of the canonical ring of a surface with K^2 = 7, p_g = 4 such that the
-- canonical map has one base point and is birational.


M=matrix{
{0,0,w_0,Q,w_1,u},
{0,0,y_1,y_3,y_2,w_1},
{-w_0,-y_1,0,-u+c,y_3*Q3,Q*Q3},
{-Q,-y_3,u-c,0,Q1*y_1,w_0*Q1},
{-w_1,-y_2,-y_3*Q3,-Q1*y_1,0,0},
{-u,-w_1,-Q*Q3,-w_0*Q1,0,0}};
M=map(R^{-1,-2,3:0,1},R^{-2,-1,3:-3,-4},M)
```

```
-- we calculate the 4 x 4 - Pfaffians of M and extract the 9
-- ``important'' as above;
pfaff15=pfaffians(4,M);
pfaff9=submatrix(gens(pfaff15),,{0,1,5,2,6,9,3,4,7});
gens(pfaff15) % ideal(pfaff9)==0

-- we write the sixteen syzygies of them
syzs=syz pfaff9

-- we define the matrix $Mt = M + tM1$, which is the deformation of $M$ whose
-- pfaffians we want to understand;

M1=matrix(R,{
          {0,1,0,0,0,0},
          {-1,0,0,0,0,0},
          {0,0,0,0,0,0},
          {0,0,0,0,0,0},
          {0,0,0,0,0,Q3*Q1},
          {0,0,0,0,-Q3*Q1,0}
          });
M1=map(R^{-1,-2,3:0,1},R^{-2,-1,3:-3,-4},M1)
Mt=M+t*M1;
-- we calculate the 15 Pfaffians of Mt and verify that the same nine
-- Pfaffians of Pt again generate the whole ideal (the output of the third
-- line below is ``true'';
defpfaff=pfaffians(4,Mt);
defpfaff9=submatrix(gens(defpfaff),,{0,1,5,2,6,9,3,4,7});
gens(defpfaff) % ideal(defpfaff9)==0

-- If t in different from 0, one can eliminate the variable u using the first
-- equation in defpfaff9.

elimu=defpfaff9_(0,0)

-- we will use the following trick to eliminate u:
-- u appears only in degrees smaller than 2 in defpfaff9; we
-- multiply defpfaff9 by t^2, and reduce by elimu;
-- what we get is the same ideal defpfaff as before for every t
-- different from zero!
```

```
-- Finally we divide by t, wherever it is possible;
-- here it is crucial that we choose a monomial order such that tu
-- is the leading term of elimu: this forces the result to be independent of u;

defpfaff9withoutu:= divideByVariable((t^2*defpfaff9) % elimu,t);

-- the following 5 generators are enough; we choose a strange order in order
-- to get a nicer result;
fiveequationsfortnotzero=submatrix(defpfaff9withoutu,,{5,4,2,3,1});
defpfaff9withoutu % ideal(fiveequationsfortnotzero)==0

-- now we look for the 5x5 matrix inducing these equations as pfaffians:

lookforQ=syz fiveequationsfortnotzero;


-- among the 36 syzygies (Macaulay found a lot of them because he is
-- considering also the case t=0) one can easily find something that looks
-- interesting
almostQ=submatrix(lookforQ,,{1,3,2,5,9})
-- this matrix is not (yet) antisymmetric; we change coordinates in the source
-- and in the target in order to make it antisymmetric;
one:=matrix(R,{{1}})
diag1=one++one++one++(t*one)++(-1*one)
diag2=one++one++(-1*one)++(-t^2*one)++(-t*one)
Qt=diag1*almostQ*diag2

-- finally we check, whether for t different from 0, the Pfaffians of Qt and
-- the Pfaffians of Mt after having eliminated u generate the same ideal;
pfaffQt=pfaffians(4,Qt);
pfaffQred=divideByVariable(gens pfaffQt,t);
fiveequationsfortnotzero %  pfaffQred==0
pfaffQred % fiveequationsfortnotzero==0
```