

DEFORMATION TYPES OF REAL AND COMPLEX MANIFOLDS

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The present article grew directly out of the lecture delivered at the Chen-Chow Symposium, entitled "Algebraic surfaces : real structures, topological and differentiable types", and partly evolved through the transparencies I used later on to lecture at FSU Tallahassee, and at Conferences in Marburg and Napoli.

The main reason however to change the title was that later on, motivated by some open problems I mentioned in the Conference, I started to include in the article new results in higher dimensions, and then the theme of deformations in the large of real and complex manifolds emerged as a central one.

For instance, we could summarize the Leit-Faden of the article as a negative answer to the question whether, for a compact complex manifold which is a $K(\pi, 1)$ (we consider in some way these manifolds as being simple objects), the diffeomorphism type determines the deformation type.

The first counterexamples to this question go essentially back to some old papers of Blanchard and of Calabi (cf. [Bla1],[Bla2], [Cal] [Somm]), showing the existence of non Kähler complex structures on the product of a curve with a two dimensional complex torus (the curve has odd genus $g \geq 3$ for Calabi's examples which are, I believe, a special case of the ones obtained by Sommese via the Blanchard method). Our contribution here, in the more technical section 4, is to show that any deformation in the large of a product of a curve of genus $g \geq 2$ with a complex torus is again a variety of the same type. We show also that the Sommese-Blanchard complex structures give rise to infinitely many Kuranishi families whose dimension is unbounded, but we are yet unable to decide whether there are indeed infinitely many distinct deformation types.

As a preliminary result, we give a positive answer to a question raised by Kodaira and Spencer (cf. [K-S], Problem 8, section 22, page 907 of volume II of Kodaira's collected works), showing that any deformation in the large of a complex torus is again a complex torus. In the same section we also give a criterion for a complex manifold to be a complex torus, namely, to have the same integral cohomology algebra of a complex torus, and to possess n independent holomorphic and d-closed 1-forms.

Whether complex tori admit other complex structures with trivial canonical bundle is still an open question (the Sommese-Blanchard examples always give, on 3- manifolds diffeomorphic to tori, complex structures with effective anticanonical bundle).

The present research took place in the framework of the Schwerpunkt "Globale Methode in der komplexen Geometrie", and of EAGER. .

One might believe that the previous "pathologies" occur because we did not restrict ourselves to the class of Kähler manifolds. However, even the Kähler condition is not sufficient to ensure that deformation and diffeomorphism type coincide.

In fact, and it is striking that we have (non rigid) examples already in dimension 2, we illustrate in the last sections (cf.[Cat7],[Cat8] for full proofs), that there are Kähler surfaces which are $K(\pi, 1)$'s, and for which there are different deformation types, for a fixed differentiable structure.

These final examples tie up perfectly with the beginning of our journey, that is, the classification problem of real varieties.

The original point of view illustrated in the lecture was to zoom the focus , considering wider and wider classes of objects:

CLASSES OF OBJECTS

- Smooth real algebraic Varieties/ Deformation
- Smooth complex algebraic Varieties/ Deformation
- Complex Kähler Manifolds/ Deformation
- Symplectic Manifolds
- Differentiable Manifolds
- Topological Manifolds

In the article we could not illustrate all the possible aspects of the problem (we omitted for instance the topic of symplectic manifolds, see [Au] for a nice survey), we tried however to cover most of them.

The paper is organized as follows: in section 1 we recall the definition and the main properties and examples of real varieties, and we report on recent results ([C-F2]) on the Enriques classification of real surfaces.

Section 2 is devoted to some open questions, and illustrates the simplest classification problem for real varieties, namely the case of real curves of genus one.

In the next section we illustrate the role of hyperelliptic surfaces in the Enriques classification, and explain the main techniques used in the classification of the real hyperelliptic surfaces (done in [C-F2]).

Section 4 is mainly devoted, as already mentioned, to complex tori and to products $C \times T$, and there we barely mention the constructions of Sommese, Blanchard and Calabi. We end up with the classification and deformation theory of real tori (this is also a new result).

Section 5 is devoted to a detailed description of what we call the Blanchard-Calabi 3-folds (they are obtained starting from curves in the Grassmann variety $G(1, 3)$, via a generalization of the previous constructions). This approach allows to construct smooth submanifolds of the Kuranishi family of deformations of a Blanchard-Calabi 3-fold, corresponding to the deformations of X which preserve the fibration with fibres 2-dimensional tori.

We are moreover able to show that these submanifolds coincide with the Kuranishi family for the Sommese-Blanchard 3-folds, and more generally for the Blanchard-Calabi 3-folds whose associated ruled surface is non developable.

In this way we obtain infinitely many families whose dimension, for a fixed differentiable structure, tends to infinity. Finally, we sketch the Calabi construction of almost complex structures, without proving in detail that the Calabi examples are a special case of the Sommese-Blanchard 3-folds.

In section 6 we recall the topological classification of simply connected algebraic surfaces and briefly mention recent counterexamples to the Friedman-Morgan speculation that for surfaces of general type the deformation type should be determined by the differentiable type.

Sections 7 is devoted to some results and examples of triangle curves, whereas the last section explains some details of the construction of our counterexamples, obtained by suitable quotients of products of curves. For these, the moduli space for a fixed differentiable type is the same as the moduli space for a fixed topological type (and even the moduli space for a given fundamental group and Euler number, actually!), and it has two connected components, exchanged by complex conjugation.

1. WHAT IS A REAL VARIETY?

Let's then start with the first class, explaining what is a real variety, and what are the problems one is interested in (cf. [D-K2] for a broader survey).

In general, e.g. in real life, one wants to solve polynomial equations with real coefficients, and find real solutions. Some theory is needed for this.

First of all, given a system of polynomial equations, in order to have some continuity of the dependence of the solutions upon the choice of the coefficients, one has to reduce it to a system of homogeneous equations

$$\begin{aligned} f_1(z_0, z_1, \dots, z_n) &= 0 \\ &\dots\dots\dots \\ f_r(z_0, z_1, \dots, z_n) &= 0. \end{aligned}$$

The set X of non trivial complex solutions of this system is called an algebraic set of the projective space $\mathbb{P}_{\mathbb{C}}^n$, and the set $X(\mathbb{R})$ of real solutions will be the intersection $X \cap \mathbb{P}_{\mathbb{R}}^n$.

One notices that the set \bar{X} of complex conjugate points is also an algebraic set, corresponding to the system where we take the polynomials

$$\bar{f}_j(\bar{z}_0, \dots, \bar{z}_n) = 0$$

(i.e., where we conjugate the coefficients of the f'_j s).

X is said to be **real** if we may assume the f'_j s to have real coefficients, and in this case $X = \bar{X}$.

Assume that X is real: for reasons stemming from Lefschstz' s topological investigations, it is better to look at $X(\mathbb{R})$ as the subset of X of the points which are left fixed by the self mapping σ given by complex conjugation (curiously enough, Andre' Weil showed that one should use a similar idea to study equations over finite fields, in this case one lets σ be the map raising each variable z_i to its q -th power).

The simplest example of a real projective variety (a variety is an algebraic set which is not the union of two proper algebraic subsets) is a plane conic $C \subset \mathbb{P}_{\mathbb{C}}^2$. It is defined by a single quadratic equation, and if it is smooth, after a linear change of variables, its equation can be written (possibly multiplying it by -1) as

$$z_0^2 + z_1^2 + z_2^2 = 0$$

or as

$$z_0^2 - z_1^2 - z_2^2 = 0.$$

In the first case $X(\mathbb{R}) = \emptyset$, in the second case we have that $X(\mathbb{R})$ is a circle.

Indeed, the above cases exhaust the classification of smooth real curves of genus $= 0$.

We encounter genus $= 1$ when we proceed to an equation of degree 3.

For instance, if we consider the family of real cubic curves with affine equation

$$y^2 - x^2(x + 1) = t, \text{ we obtain}$$

- two ovals for $t < 0$
- one oval for $t > 0$.

However, every polynomial of odd degree has a real root, whence every real cubic curve has $X(\mathbb{R}) \neq \emptyset$.

On the other hand, there are curves of genus 1 and without real points, as we were taught by Felix Klein ([Kl]) and his famous **Klein's bottle**.

Simply look at a curve of genus 1 as a complex torus $X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. For instance, take $\tau = i$ and let σ be the antiholomorphic self map induced by $\sigma(z) = \bar{z} + 1/2$. By looking at the real part, we see that there are no fixpoints, thus $X(\mathbb{R}) = \emptyset$: indeed the well known Klein bottle is exactly the quotient X/σ , as an easy picture shows. [Remark: this real curve X is the locus of zeros of two real quadratic polynomials in $\mathbb{P}_{\mathbb{C}}^3$.]

The previous example shows once more the usefulness of the notion of an abstract manifold (or variety), which is indeed one of the keypoints in the classification theory.

Definition 1.1. *A smooth real variety is a pair (X, σ) consisting of a smooth complex variety X of complex dimension n and of an antiholomorphic involution $\sigma : X \rightarrow X$ (involution means: $\sigma^2 = Id$).*

I.e., let M be the differentiable manifold underlying X and let J be the complex structure of X (J is the linear map on real tangent vectors provided by multiplication by i): then the complex structure $-J$ determines a complex manifold which is called the conjugate of X and denoted by \bar{X} , and σ is said to be antiholomorphic if it provides a holomorphic map between the complex manifolds X and \bar{X} .

Main problems (let's assume X is compact)

- Describe the isomorphism classes of such pairs (X, σ) .

- Or, at least describe the possible topological or differentiable types of the pair (X, σ) .
- At least describe the possible topological types for the real part $X' := X(\mathbb{R}) = \text{Fix}(\sigma)$.

Remark 1.2. *One can generalize the last problem and consider real pairs $(Z \subset X, \sigma)$.*

Recall indeed that Hilbert's 16-th problem is a special case of the last question for the special case where $X = \mathbb{P}^2$, and Z is a smooth curve. In practice, the problem consists then in finding how many ovals $Z(\mathbb{R})$ can have, and what is their mutual disposition (one inside the other , or not).

For real algebraic curves one has the following nice

HARNACK'S INEQUALITY

Theorem 1.3. *Let (C, σ) be a real curve of genus g . Then the real part $C(\mathbb{R})$ consists of a disjoint union of t ovals = circles (topologically: S^1), where $0 \leq t \leq g + 1$.*

Remark 1.4. *Curves with $g + 1$ ovals are called $M(\text{aximal})$ -CURVES. An easy example of M -curves is provided by the hyperelliptic curves: take a real polynomial $P_{2g+2}(x)$ of degree $2g + 2$ and with all the roots real, and consider the hyperelliptic curves with affine equation*

$$z^2 = P_{2g+2}(x).$$

The study of real algebraic curves has been the object of many deep investigations, and although many questions remain still open, one has a good knowledge of them, for instance Seppälä and Silhol ([Se-Si]) and later Frediani ([Fre]) proved

Theorem 1.5. *Given an integer g , the subset of the moduli space of complex curves of genus g , given by the curves which admit a real structure, is connected.*

A clue point to understand the meaning of the above theorem is that a complex variety can have several real structures, or none. In fact the group $\text{Aut}(X)$ of biholomorphic automorphisms sits as a subgroup of index at most 2 in the group $\text{Dian}(X)$ consisting of $\text{Aut}(X)$ and of the antiholomorphic automorphisms. The real structures are precisely the elements of order 2 in $\text{Dian}(X) - \text{Aut}(X)$. Therefore, as we shall see, the map of a moduli space of real varieties to the real part of the moduli space of complex structures can have positive degree over some points, in particular, although the moduli space of real curves of genus g is not connected, yet its image in the moduli space of complex curves is connected!

In higher dimensions, there are many fascinating questions, and the next natural step is the investigation of the case of algebraic surfaces, which is deeply linked to the intriguing mystery of smooth 4- manifolds.

Indeed, for complex projective surfaces, we have the Enriques' classification of surfaces up to birational equivalence (equivalently, we have the classification

of minimal surfaces, i.e., of those surfaces S such that any holomorphic map $S \rightarrow S'$ of degree 1 is an isomorphism).

The Enriques (-Kodaira) classification of algebraic varieties should consist in subdividing the varieties X according to their so-called Kodaira dimension, which is a number $kod(X) \in \{-\infty, 0, 1, \dots, \dim(X)\}$, and then giving a detailed description of the varieties of **special type**, those for which $kod(X) < \dim(X)$ (the varieties for which $kod(X) = \dim(X)$ are called of **general type**).

As such, it has been achieved for curves:

Curve	Kod	g
$\mathbb{P}_{\mathbb{C}}^1$	$-\infty$	0
Elliptic curve: $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$	0	1
Curve of general type	1	≥ 2

and for surfaces, for instance the following is the Enriques' classification of complex projective surfaces (where C_g stands for a curve of genus g):

Surface	Kod	P_{12}	Structure
Ruled surface with $q = g$	$-\infty$	0	$\mathbb{P}_{\mathbb{C}}^1 \times C_g$
Complex torus	0	1	\mathbb{C}^2/Λ_4
K3 Surface	0	1	homeo to $X_4^2 \subset \mathbb{P}_{\mathbb{C}}^3$
Enriques surface	0	1	$K3/(\mathbb{Z}/2)$
Hyperelliptic surface	0	1	$(C_1 \times C'_1)/G$
Properly elliptic surface	1	≥ 2	$\dim \phi_{12}(S) = 1, \dots$
Surface of general type	2	≥ 2	$\dim \phi_{12}(S) = 2, ?$

Now, the Enriques classification of real algebraic surfaces has not yet been achieved in its strongest form, however it has been achieved for Kodaira dimension 0, thanks to Comessatti (around 1911, cf. [Com1],[Com2],[Com3]), Silhol, Nikulin, Kharlamov and Degtyarev (cf. references) and was finished in our joint work with Paola Frediani ([C-F2]) where we proved:

Theorem 1.6. *Let (S, σ) be a real hyperelliptic surface.*

- 1) *Then the differentiable type of the pair (S, σ) is completely determined by the orbifold fundamental group exact sequence.*
- 2) *Fix the topological type of (S, σ) corresponding to a real hyperelliptic surface. Then the moduli space of the real surfaces (S', σ') with the given topological type is irreducible (and connected).*
- 3) *Real hyperelliptic surfaces fall into 78 topological types. In particular, the real part $S(\mathbb{R})$ of a real hyperelliptic surface is either*

- *a disjoint union of c tori, where $0 \leq c \leq 4$*
- *a disjoint union of b Klein bottles, where $1 \leq b \leq 4$.*
- *the disjoint union of a torus and of a Klein bottle*
- *the disjoint union of a torus and of two Klein bottles.*

This result confirms a conjecture of Kharlamov that for real surfaces of Kodaira dimension ≤ 1 the deformation type of (S, σ) is determined by the topological type of (S, σ) . Our method consists in

1) Rerunning the classification theorem with special attention to the real involution.

2) Finding out the primary role of the **orbifold fundamental group**, which is so defined:

For X real smooth, we have a double covering $\pi : X \rightarrow Y = X / \langle \sigma \rangle$ (Y is called the **Klein variety** of (X, σ)), ramified on the real part of X , $X(\mathbb{R}) = \text{Fix}(\sigma)$.

In the case where $X(\mathbb{R}) = \emptyset$, $\pi_1^{\text{orb}}(Y)$ is just defined as the fundamental group $\pi_1(Y)$.

Otherwise, pick a point $x_0 \in X(\mathbb{R})$, thus $\sigma(x_0) = x_0$ and the action of σ on $\pi_1(X, x_0)$ allows to define $\pi_1^{\text{orb}}(Y)$ as a semidirect product of $\pi_1(X, x_0)$ with $\mathbb{Z}/2$.

One checks that the definition is independent of the choice of x_0 .

We have thus in all cases an exact sequence

$$1 \rightarrow \pi_1(X) \rightarrow \pi_1^{\text{orb}}(Y) \rightarrow \mathbb{Z}/2 \rightarrow 1.$$

This sequence is very important when X is a $K(\pi, 1)$, i.e., when the universal cover of X is contractible.

2. COMPLEX MANIFOLDS WHICH ARE $K(\pi, 1)$ 'S AND REAL CURVES OF GENUS 1.

Some interesting questions are, both for complex and real varieties

Question 1 : to what extent, if X is a $K(\pi, 1)$, does then $\pi_1(X)$ also determine the differentiable type of X , not only its homotopy type?

Question 2 : analogously, if (X, σ) is a $K(\pi, 1)$ and is real, how much from the differentiable viewpoint is (X, σ) determined by the orbifold fundamental group sequence?

There are for instance, beyond the case of hyperelliptic surfaces, other similar instances in the Kodaira classification of real surfaces (joint work in progress with Paola Frediani).

Question 3 : determine, for the real varieties whose differentiable type is determined by the orbifold fundamental group, those for which the moduli spaces are irreducible and connected.

Remark 2.1. *However, already for complex manifolds, the question whether (fixed the differentiable structure) there is a unique deformation type, has several negative answers, as we shall see in the later sections.*

To explain the seemingly superfluous statement "irreducible and connected", let us observe that : a hyperbola $\{(x, y) \in \mathbf{R}^2 | xy = 1\}$ is irreducible but not connected.

I want to show now an easy example, leading to the quotient of the above hyperbola by the involution $(x, y) \rightarrow (-x, -y)$ as a moduli space, and explaining the basic philosophy underlying the two above mentioned theorems concerning the orbifold fundamental group.

REAL CURVES OF GENUS ONE

Classically, the topological type of real curves of genus 1 is classified according to the number $\nu = 0, 1, \text{ or } 2$ of connected components (= ovals = homeomorphic to circles) of their real part. By abuse of language we shall also say: real elliptic curves, instead of curves of genus 1, although for many authors an elliptic curve comes provided with one point defined over the base field (viz. : the origin!).

The orbifold fundamental group sequence is in this case

$$1 \rightarrow H_1(C, \mathbb{Z}) \cong \mathbb{Z}^2 \rightarrow \pi_1^{\text{orb}}(C) \rightarrow \mathbb{Z}/2 \rightarrow 1.$$

and it splits iff $C(\mathbf{R}) \neq \emptyset$ (since $\pi_1^{\text{orb}}(C)$ has a representation as a group of affine transformations of the plane).

Step 1 : If there are no fixed points, the action s of $\mathbb{Z}/2$ on \mathbb{Z}^2 (given by conjugation) is diagonalizable.

PROOF:

let σ be represented by the affine transformation $(x, y) \rightarrow s(x, y) + (a, b)$. Now, s is not diagonalizable if and only if for a suitable basis choice, $s(x, y) = (y, x)$. In this case the square of σ is the transformation $(x, y) \rightarrow (x, y) + (a+b, a+b)$, thus $a+b$ is an integer, and therefore the points $(x, x-a)$ yield a fixed S^1 on the elliptic curve. Q.E.D.

Step 2 : If there are no fixed points, moreover, the translation vector of the affine transformation inducing σ can be chosen to be $1/2$ of the $+1$ -eigenvector e_1 of s . Thus we have exactly one normal form.

Step 3 : there are exactly 3 normal forms, and they are distinguished by the values $0, 1, 2$ for ν . Moreover, s is diagonalizable if and only if ν is even.

PROOF: if there are fixed points, then σ may be assumed to be linear, so there are exactly two normal forms, according to whether σ is diagonalizable or not. One sees immediately that ν takes then the respective values $2, 1$! Moreover, we have then verified that s is diagonalizable if and only if ν is even. Q.E.D.

We content ourselves now with

THE DESCRIPTION OF THE MODULI SPACE FOR THE CASE $\nu = 1$.

σ acts as follows: $(x, y) \rightarrow (y, x)$.

We look then for the complex structures J which make σ antiholomorphic, i.e., we seek for the matrices J with $J^2 = -1$ and with $Js = -sJ$.

The latter condition singles out the matrices $\begin{pmatrix} a & b \\ -b & -a \end{pmatrix}$

while the first condition is equivalent to requiring that the characteristic polynomial be equal to $\lambda^2 + 1$, whence, equivalently, $b^2 - a^2 = 1$.

We get a hyperbola with two branches which are exchanged under the involution $J \rightarrow -J$, but, as we already remarked, J and $-J$ yield isomorphic real elliptic curves, thus the moduli space is irreducible and connected. Q.E.D.

3. REAL HYPERELLIPTIC SURFACES ON THE SCENE

The treatment of real hyperelliptic surfaces is similar to the above sketched one of elliptic curves, and very much related to it, because the orbifold group has an affine representation. Recall:

Definition 3.1. *A complex surface S is said to be hyperelliptic if $S \cong (E \times F)/G$, where E and F are elliptic curves and G is a finite group of translations of E with a faithful action on F such that $F/G \cong \mathbf{P}^1$.*

HISTORICAL REMARK : the points of elliptic curves can be parametrized by meromorphic functions of $z \in \mathbb{C}$. Around 1880, thanks to the work of Appell, Humbert and Picard, there was much interest for the "hyperelliptic varieties" of higher dimension n , whose points can be parametrized by meromorphic functions of $z \in \mathbb{C}^n$, but not by rational functions of $z \in \mathbb{C}^n$. The classification of hyperelliptic surfaces was finished by Bagnera and De Franchis ([B-dF1],[B-dF2]) who were awarded the Bordin Prize in 1908 for this important achievement.

Theorem 3.2. *(Bagnera - de Franchis) Every hyperelliptic surface is one of the following, where E, F are elliptic curves and G is a group of translations of E acting on F as specified (ρ is a primitive third root of unity):*

1. $(E \times F)/G$, $G = \mathbb{Z}/2$ acts on F by $x \mapsto -x$.
2. $(E \times F)/G$, $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ acts on F by $x \mapsto -x$, $x \mapsto x + \epsilon$, where ϵ is a half period.
3. $(E \times F_i)/G$, $G = \mathbb{Z}/4$ acts on F_i by $x \mapsto ix$.
4. $(E \times F_i)/G$, $G = \mathbb{Z}/4 \oplus \mathbb{Z}/2$ acts on F_i by $x \mapsto ix$, $x \mapsto x + (1 + i)/2$.
5. $(E \times F_\rho)/G$, $G = \mathbb{Z}/3$ acts on F_ρ by $x \mapsto \rho x$.
6. $(E \times F_\rho)/G$, $G = \mathbb{Z}/3 \oplus \mathbb{Z}/3$ acts on F_ρ by $x \mapsto \rho x$, $x \mapsto x + (1 - \rho)/3$.
7. $(E \times F_\rho)/G$, $G = \mathbb{Z}/6$ acts on F_ρ by $x \mapsto -\rho x$.

In fact, the characterization of hyperelliptic surfaces is one of the two key steps of Enriques' classification of algebraic surfaces, we have namely:

Theorem 3.3. *The complex surfaces S with K nef, $K^2 = 0$, $p_g = 0$, and such that either S is algebraic with $q = 1$, or, more generally, such that $b_1 = 2$, are hyperelliptic surfaces if and only if $\text{kod}(S) = 0$ (equivalently, iff the Albanese fibres are smooth of genus 1).*

I want now to illustrate the main technical ideas used for the classification of real Hyperelliptic Surfaces.

Definition 3.4. *The extended symmetry group \hat{G} is the group generated by G and a lift $\tilde{\sigma}$ of σ .*

Rerunning the classification theorem yields

Theorem 3.5. *Let (S, σ) , $(\hat{S}, \hat{\sigma})$ be isomorphic real hyperelliptic surfaces : then the respective extended symmetry groups \hat{G} are the same and given two Bagnera - De Franchis realizations $S = (E \times F)/G$, $\hat{S} = (\hat{E} \times \hat{F})/G$, there is an isomorphism $\Psi : E \times F \rightarrow \hat{E} \times \hat{F}$, of product type, commuting with the action of \hat{G} , and inducing the given isomorphism $\psi : S \xrightarrow{\cong} \hat{S}$. Moreover, let*

$\tilde{\sigma} : E \times F \rightarrow E \times F$ be a lift of σ . Then the antiholomorphic map $\tilde{\sigma}$ is of product type.

We need to give the list of all the possible groups \hat{G} .

Lemma 3.6. *Let us consider the extension*

$$(*) \quad 0 \rightarrow G \rightarrow \hat{G} \rightarrow \mathbb{Z}/2 = \langle \sigma \rangle \rightarrow 0.$$

We have the following possibilities for the action of σ on G : in what follows the subgroup T of G will be the subgroup acting by translations on both factors.

1. If $G = \mathbb{Z}/q$, $q = 2, 4, 3, 6$ then σ acts as $-Id$ on G and $\hat{G} = D_q$
2. If $G = \mathbb{Z}/2 \times \mathbb{Z}/2$, then either
 - (2.1) σ acts as the identity on G , $(*)$ splits, and $\hat{G} = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$
or
 - (2.2) σ acts as the identity on G , $(*)$ does not split, and $\hat{G} = \mathbb{Z}/4 \times \mathbb{Z}/2$ (and in this latter case the square of σ is the generator of T) or
 - (2.3) σ acts as $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $(*)$ splits, $\hat{G} = D_4$, the dihedral group, and again the square of a generator of $\mathbb{Z}/4\mathbb{Z}$ is the generator of T .
3. If $G = \mathbb{Z}/4 \times \mathbb{Z}/2$, then either $\hat{G} = T \times D_4 \cong \mathbb{Z}/2 \times D_4$, or \hat{G} is isomorphic to the group $G_1 := \langle \sigma, g, t, \mid \sigma^2 = 1, g^4 = 1, t^2 = 1, t\sigma = \sigma t, tg = gt, \sigma g = g^{-1}t\sigma \rangle$, and its action on the second elliptic curve F is generated by the following transformations: $\sigma(z) = \bar{z} + 1/2$, $g(z) = iz$, $t(z) = z + 1/2(1 + i)$. [The group G_1 is classically denoted by c_1 (cf. Atlas of Groups)].
In particular, in both cases $(*)$ splits.
4. If $G = \mathbb{Z}/3 \times \mathbb{Z}/3$, then we may choose a subgroup G' of G such that σ acts as $-Id \times Id$ on $G' \times T = G$ and we have $\hat{G} = D_3 \times \mathbb{Z}/3$.

MAIN IDEAS IN THE CLASSIFICATION OF REAL HYPERELLIPTIC SURFACES :

i) the first point is to show that the orbifold fundamental group has a unique faithful representation as a group of affine transformations with rational coefficients

ii) second, to distinguish the several cases, we use byproducts of the orbifold fundamental group, such as the structure of the extended Bagnera de Franchis group \hat{G} , parities of the (invariant ν of the) involutions in \hat{G} , their actions on the fixed point sets of transformations in G , the topology of the real part, and, in some special cases, some more refined invariants such as the translation parts of all possible lifts of a given element in \hat{G} to the orbifold fundamental group.

HOW TO CALCULATE THE TOPOLOGY OF THEIR REAL PART ?

We simply use the Albanese variety and what we said about the real parts of elliptic curves, that they consist of $\nu \leq 2$ circles. Since also the fibres are elliptic curves, we get up to 4 circle bundles over circles, in particular the connected components are either 2-Tori or Klein bottles.

For every connected component V of $Fix(\sigma) = S(\mathbb{R})$ its inverse image $\pi^{-1}(V)$ in $E \times F$ splits as the G -orbit of any of its connected components.

Let W be one such: then one can easily see that there is a lift $\tilde{\sigma}$ of σ such that $\tilde{\sigma}$ is an involution and W is in the fixed locus of $\tilde{\sigma}$: moreover $\tilde{\sigma}$ is unique.

Thus the connected components of $Fix(\sigma)$ correspond bijectively to the set \mathcal{C} obtained as follows: consider all the lifts $\tilde{\sigma}$ of σ which are involutions and pick one representative $\tilde{\sigma}_i$ for each conjugacy class.

Then we let \mathcal{C}' be the set of equivalence classes of connected components of $\cup Fix(\tilde{\sigma}_i)$, where two components A, A' of $Fix(\tilde{\sigma}_i)$ are equivalent if and only if there exists an element $g \in G$, such that $g(A) = A'$.

Let now $\tilde{\sigma}$ be an antiholomorphic involution which is a lift of σ and such that $Fix(\tilde{\sigma}) \neq \emptyset$.

Since $\tilde{\sigma}$ is of product type, $Fix(\tilde{\sigma})$ is a disjoint union of $2^{a_1+a_2}$ copies of $\mathbf{S}^1 \times \mathbf{S}^1$, where $a_i \in \{0, 1\}$.

In fact if an antiholomorphic involution $\hat{\sigma}$ on an elliptic curve C has fixed points, then $Fix(\hat{\sigma})$ is a disjoint union of 2^a copies of \mathbf{S}^1 , where $a = 1$ if the matrix of the action of $\hat{\sigma}$ on $H_1(C, \mathbb{Z})$ is diagonalizable, else $a = 0$.

What said insofar describes the number of such components V ; in order to determine their nature, observe that $V = W/H$, where W is as before and $H \subset G$ is the subgroup such that $HW = W$.

Since the action of G on the first curve E is by translations, the action on the first \mathbf{S}^1 is always orientation preserving. Whence, V is a Klein bottle if and only if H acts on the second \mathbf{S}^1 by some orientation reversing map, or, equivalently, iff H has some fixed point on the second \mathbf{S}^1 .

In fact, we have a restriction : let $h \in H$ be a transformation having a fixed point on the second \mathbf{S}^1 . Since the direction of this \mathbf{S}^1 is an eigenvector for the tangent action of h , it follows that the tangent action is given by multiplication by -1 .

FINALLY , EXPLICIT TABLES SHOW THAT REALLY EVERYTHING CAN BE NICELY WRITTEN DOWN AS IN THE BAGNERA DE FRANCHIS LIST (we refer to the original paper [C-F2] for details).

4. TORI AND CURVE TIMES A TORUS

In this section we begin analysing in detail the case of the simplest complex manifolds which are $K(\pi, 1)$'s, namely the complex tori.

It is well known that complex tori are parametrized by a connected family (with smooth base space), inducing all the small deformations (cf. [K-M]).

A first result we want to establish is the following

Theorem 4.1. *Every deformation of a complex torus of dimension n is a complex torus of dimension n .*

The proof starts with a sequence of more or less standard arguments. The first one is due to Kodaira ([Kol], Theorem 2, page 1392 of collected works, vol. III)

Lemma 4.2. *On a compact complex manifold X one has an injection*

$$H^0(d\mathcal{O}_X) \oplus H^0(d\bar{\mathcal{O}}_X) \rightarrow H_{DR}^1(X, \mathbb{C}).$$

PROOF. Suffices to show the injection $H^0(d\mathcal{O}_X) \rightarrow H_{DR}^1(X, \mathbb{R})$. by the map sending

$$\omega \rightarrow \omega + \bar{\omega}.$$

Else, there is a function f with $df = \omega + \bar{\omega}$, whence $\partial f = \omega$ and therefore $\bar{\partial}\partial f = d(\omega) = 0$. Thus f is pluriharmonic, hence constant by the maximum principle. Follows that $\omega = 0$. Q.E.D.

Lemma 4.3. *Assume that $\{X_t\}_{t \in \Delta}$ is a 1-parameter family of compact complex manifolds over the 1-dimensional disk, such that there is a sequence $t_\nu \rightarrow 0$ with X_{t_ν} Kähler.*

Then the weak 1-Kähler decomposition

$$H_{DR}^1(X_0, \mathbb{C}) = H^0(d\mathcal{O}_{X_0}) \oplus H^0(d\bar{\mathcal{O}}_{X_0}),$$

holds also on the central fibre X_0 .

PROOF. We have $f : \Xi \rightarrow \Delta$ which is proper and smooth, and $f_*(\Omega_{\Xi|\Delta}^1)$ is torsion free, whence (Δ is smooth of dimension 1) it is locally free of rank $q := (1/2) b_1(X_0)$.

In fact, there is (cf. [Mum], II 5) a complex of Vector Bundles on Δ ,

$$(*) \quad E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \dots E^n \text{ s.t.}$$

1) $R^i f_*(\Omega_{\Xi|\Delta}^1)$ is the i -th cohomology group of $(*)$, whereas

2) $H^i(X_t, \Omega_{X_t}^1)$ is the i -th cohomology group of $(*) \otimes \mathbb{C}_t$.

CLAIM.

Thus there are holomorphic 1-forms $\omega_1(t), \dots, \omega_q(t)$ defined in the inverse image $f^{-1}(U_0)$ of a neighbourhood (U_0) of 0, and linearly independent for $t \in U_0$.

PROOF OF THE CLAIM: assume that $\omega_1(t), \dots, \omega_q(t)$ generate the direct image sheaf $f_*(\Omega_{\Xi|\Delta}^1)$, but $\omega_1(0), \dots, \omega_q(0)$ be linearly dependent. Then, w.l.o.g. we may assume $\omega_1(0) \equiv 0$, i.e. there is a maximal m such that $\hat{\omega}_1(t) := \omega_1(t)/t^m$ is holomorphic. Then, since $\hat{\omega}_1(t)$ is a section of $f_*(\Omega_{\Xi|\Delta}^1)$, there are holomorphic functions α_i such that $\hat{\omega}_1(t) = \sum_{i=1, \dots, q} \alpha_i(t) \omega_i(t)$, whence it follows that $\hat{\omega}_1(t)(1 - t^m \alpha_1(t)) = \sum_{i=2, \dots, q} \alpha_i(t) \omega_i(t)$.

This however contradicts the fact that $f_*(\Omega_{\Xi|\Delta}^1)$ is locally free of rank q .

REMARK:

Let d_v be the vertical part of exterior differentiation, i.e., the composition of d with the projection $(\Omega_{\Xi}^2 \rightarrow \Omega_{\Xi|\Delta}^2)$. By our assumption, $\omega_i(t_\nu)$ is d_v -closed, whence, by continuity, also $\omega_i(0) \in H^0(d\mathcal{O}_{X_0})$.

END OF THE PROOF:

It follows from the previous lemma that, being $b_1 = 2q$, $H_{DR}^1(X_0, \mathbb{C}) = H^0(d\mathcal{O}_{X_0}) \oplus H^0(d\bar{\mathcal{O}}_{X_0})$.

Q.E.D.

Recall that for a compact complex manifold X , the Albanese Variety $Alb(X)$ is the quotient of the complex dual vector space of $H^0(d\mathcal{O}_X)$ by the minimal closed complex Lie subgroup containing the image of $H_1(X, \mathbb{Z})$.

The Albanese map $\alpha_X : X \rightarrow Alb(X)$ is given as usual by fixing a base point x_0 , and defining $\alpha_X(x)$ as the integration on any path connecting x_0 with x .

One says the the Albanese Variety is **good** if the image of $H_1(X, \mathbb{Z})$ is discrete in $H^0(d\mathcal{O}_X)$, and **very good** if it is a lattice. Moreover, the *Albanese dimension of X* is defined as the dimension of the image of the Albanese map.

With this terminology, we can state an important consequence of our assumptions.

Corollary 4.4. *Assume that $\{X_t\}_{t \in \Delta}$ is a 1-parameter family of compact complex manifolds over the 1-dimensional disk, such that there is a sequence $t_\nu \rightarrow 0$ with X_{t_ν} Kähler, and moreover such that X_{t_ν} has maximal Albanese dimension.*

Then the central fibre X_0 has a very good Albanese Variety, and has also maximal Albanese dimension.

PROOF. We use the fact (cf. [Cat4]) that, when the Albanese Variety is good, then the Albanese dimension of X is equal to $\max\{i | \Lambda^i H^0(d\mathcal{O}_X) \otimes \Lambda^i H^0(d\bar{\mathcal{O}}_X) \rightarrow H_{DR}^{2i}(X, \mathbb{C}) \text{ has non zero image}\}$.

If the weak 1-Kähler decomposition holds for X , then the Albanese dimension of X equals $(1/2) \max\{j | \Lambda^j H^1(X, \mathbb{C}) \text{ has non zero image in } H^j(X, \mathbb{C})\}$. But this number is clearly invariant by homeomorphisms.

Finally, the Albanese Variety for X_0 is very good since it is very good for X_{t_ν} and the weak 1-Kähler decomposition holds for X_0 .

Q.E.D.

Remark 4.5. *As observed in ([Cat6], 1.9), if a complex manifold X has a generically finite map to a Kähler manifold, then X is bimeromorphic to a Kähler manifold. This applies in particular to the Albanese map.*

Recall that (cf. [K-M]) a small deformation of a Kähler manifold is again Kähler. This is however (cf. [Hir]) false for deformations in the large, and shows that the following theorem (which pretty much follows the lines of corollary C of [Cat6]) is not entirely obvious (it answers indeed in the affirmative a problem raised by Kodaira and Spencer , page 464 of the paper [K-S], where the case $n=2$ was solved in Theorem 20.2)

Theorem 4.6. *A deformation in the large of complex tori is a complex torus.*

This follows form the following more precise statement: let X_0 be a compact complex manifold such that its Kuranishi family of deformations $\pi : \Xi \rightarrow \mathcal{B}$ enjoys the property that the set $\mathcal{B}(\text{torus}) := \{b | X_b \text{ is isomorphic to a complex torus}\}$ has 0 as a limit point.

Then X_0 is a complex torus.

We will use the following "folklore"

Lemma 4.7. *Let Y be a connected complex analytic space, and Z an open set of Y such that Z is closed for holomorphic 1-parameter limits (i.e., given any holomorphic map of the 1-disk $f : \Delta \rightarrow Y$, if there is a sequence $t_\nu \rightarrow 0$ with $f(t_\nu) \in Z$, then also $f(0) \in Z$). Then $Z = Y$.*

PROOF OF THE LEMMA.

By choosing an appropriate stratification of Y by smooth manifolds, it suffices to show that the statement holds for Y a connected manifold.

Since it suffices to show that Z is closed, let P be a point in the closure of Z , and let us take coordinates such that a neighbourhood of P corresponds to a compact polycylinder H in \mathbb{C}^n .

Given a point in Z , let H' be a maximal coordinate polycylinder contained in Z . We claim that H' must contain H , else, by the holomorphic 1-parameter limit property, the boundary of H' is contained in Z' , and since Z is open, by compactness we find a bigger polycylinder contained in Z , a contradiction which proves the claim.

Q.E.D. for the Lemma.

PROOF OF THE THEOREM.

It suffices to consider a 1-parameter family ($\mathcal{B} = \Delta$) whence we may assume w.l.o.g. that the weak 1-Lefschetz decomposition holds for each $t \in \Delta$.

By integration of the holomorphic 1-forms on the fibres (which are closed for t_ν and for 0), we get a family of Albanese maps $\alpha_t : X_t \rightarrow J_t$, which fit together in a relative map $a : \Xi \rightarrow J$ over Δ (J_t is the complex torus $(H^0(d\mathcal{O}_{X_t}))^\vee / H_1(X_t, \mathbb{Z})$).

Apply once more vertical exterior differentiation to the forms $\omega_i(t)$: $d_\nu(\omega_i(t))$ vanishes identically on X_0 and on X_{t_ν} , whence it vanishes identically in a neighbourhood of X_0 , and therefore these forms $\omega_i(t)$ are closed for each t .

Therefore our map $a : \Xi \rightarrow J$ is defined everywhere and it is an isomorphism for $t = t_\nu$.

Whence, for each t , α_t is surjective and has degree 1.

To show that α_t is an isomorphism for each t it suffices therefore to show that a is finite.

Assume the contrary: then there is a ramification divisor R of a , which is exceptional (i.e., if $B = a(R)$, then $\dim B < \dim R$).

By our hypothesis α_t is an isomorphism for $t = t_\nu$, thus R is contained in a union of fibres, and since it has the same dimension, it is a finite union of fibres. But if R is not empty, we reach a contradiction, since then there are some t 's such that α_t is not surjective.

Q.E.D.

Proposition 4.8. *Assume that X has the same integral cohomology algebra of a complex torus and that $H^0(d\mathcal{O}_X)$ has dimension equal to $n = \dim(X)$. Then X is a complex torus*

PROOF.

Since $b_1(X) = 2n$, it follows from 4.2 that the weak 1-Lefschetz decomposition holds for X and that the Albanese Variety of X is very good.

That is, we have the Albanese map $\alpha_X : X \rightarrow J$, where J is the complex torus $J = Alb(X)$. We want to show that the Albanese map is an isomorphism. It is a morphism of degree 1, since α_X induces an isomorphism between the respective fundamental classes of $H^{2n}(X, \mathbb{Z}) \cong H^{2n}(J, \mathbb{Z})$.

There remains to show that the bimeromorphic morphism α_X is finite.

To this purpose, let R be the ramification divisor of $\alpha_X : X \rightarrow J$, and B its branch locus, which has codimension at least 2. By means of blowing ups of J with non singular centres we can dominate X by a Kähler manifold $g : Z \rightarrow X$ (cf. [Cat6] 1.8, 1.9).

Let W be a fibre of α_X of positive dimension such that $g^{-1}W$ is isomorphic to W . Since Z is Kähler, $g^{-1}W$ is not homologically trivial, whence we find a differentiable submanifold Y of complementary dimension which has a positive intersection number with it. But then, by the projection formula, the image g_*Y has positive intersection with W , whence W is also not homologically trivial. However, the image of the class of W is 0 on J , contradicting that α_X induces an isomorphism of cohomology (whence also homology) groups.

Q.E.D.

Remark 4.9. *The second condition holds true as soon as the complex dimension n is at most 2. For $n = 1$ this is well known, for $n = 2$ this is also known, and due to Kodaira ([KoI]): in fact for $n = 2$ the holomorphic 1-forms are closed, and moreover $h^0(d\mathcal{O}_X)$ is at least $[(1/2)b_1(X)]$.*

For $n \geq 3$, the real dimension of X is greater than 5, whence, by the scobordism theorem ([Maz]), the assumption that X be homeomorphic to a complex torus is equivalent to the assumption that X be diffeomorphic to a complex torus.

André Blanchard ([Bla1]) constructed in the early 50's an example of a non Kähler complex structures on the product of a rational curve with a two dimensional complex torus. In particular his construction (cf. [Somm]), was rediscovered by Sommese, with a more clear and more general presentation, who pointed out that in this way one would produce exotic complex structures on complex tori.

The Sommese-Blanchard examples (cf. [Ue], where we learnt about them) are particularly relevant to show that the plurigenera of non Kähler manifolds are not invariant by deformation. In the following section 5 we will adopt the presentation of [Cat6]) : the possible psychological reason why we had forgotten about these 3-folds (and we are very thankful to Andrew Sommese for pointing out their relevance) is that these complex structures do not have a trivial canonical bundle. Thus remains open the following

QUESTION: let X be a compact complex manifold of dimension $n \geq 3$ and with trivial canonical bundle such that X is diffeomorphic to a complex torus : is then X a complex torus ?

The main problem is to show the existence of holomorphic 1-forms, so it may well happen that also this question has a negative answer.

On the other hand, as was already pointed out, the examples of Blanchard, Calabi and Sommese show that the answer to a similar question is negative also in the case of a product $C \times T$, where C is a curve of genus $g \geq 2$, and T is a complex torus.

For this class of varieties we still have the same result as for tori, concerning global deformations.

Theorem 4.10. *A deformation in the large of products $C \times T$ of a curve of genus $g \geq 2$ with a complex torus is again a product of this type.*

This clearly follows from the more precise statement: let $g \geq 2$ a fixed integer, and assume that X_0 be a compact complex manifold such that its Kuranishi family of deformations $\pi : \Xi \rightarrow \mathcal{B}$ enjoys the property that the set $\mathcal{B}'' := \{b | X_b \text{ is isomorphic to the product of a curve of genus } g \text{ with a complex torus}\}$ has 0 as a limit point.

Then X_0 is isomorphic to a product $C \times T$ of a curve of genus g with a complex torus T .

PROOF. Observe first of all that \mathcal{B}'' is open in the Kuranishi family \mathcal{B} , by the property that the Kuranishi family induces a versal family in each neighbouring point, and that every small deformation of a product $C \times T$ is of the same type.

Whence, by lemma 6.7 we can limit ourselves to consider the situation where \mathcal{B} is a 1-dimensional disk, and every fibre X_t is, for $t \neq 0$, a product of the desired form.

STEP I. There is a morphism $F : \Xi \rightarrow \mathcal{C}$, where $\mathcal{C} \rightarrow \mathcal{B}$ is a smooth family of curves of genus g .

Proof of step I.

We use for this purpose the isotropic subspace theorem of [Cat4], observing that the validity of this theorem does not require the full hypothesis that a variety X be Kähler, but that weaker hypotheses, e.g. the weak 1-Kähler and 2-Kähler decompositions do indeed suffice.

The first important property to this purpose is that the cohomology algebra $H^*(X_t, \mathbb{C})$ is generated by $H^1(X_t, \mathbb{C})$.

The second important property is that for a product $C \times T$ as above, the subspace $p_1^*(H^1(C, \mathbb{C}))$ is the unique maximal subspace V , of dimension $2g$, such that the image of $\Lambda^3(V) \rightarrow H^3(C \times T, \mathbb{C})$ is zero.

This is the algebraic counterpart of the geometrical fact that the first projection is the only surjective morphism with connected fibre of $C \times T$ onto a curve C' of genus ≥ 2 .

From the differentiable triviality of our family $\Xi \rightarrow \Delta$ follows that we have a uniquely determined such subspace V of the cohomology of X_0 , that we may freely identify to the one of each X_t .

Now, for $t \neq 0$, we have a decomposition $V = U_t \oplus \bar{U}_t$, where $U_t = p_1^*(H^0(\Omega_{C_t}^1))$.

By compactness of the Grassmann variety and by the weak 1-Kähler decomposition in the limit, the above decomposition also holds for X_0 , and U_0 is a maximal isotropic subspace in $H^0(d\mathcal{O}_{X_0})$. The Castelnuovo de Franchis theorem applies, and we get the desired morphism $F : \Xi \rightarrow \mathcal{C}$ to a family of curves.

STEP II. We produce now a morphism $G : \Xi \rightarrow \mathcal{T}$ where $\mathcal{T} \rightarrow \mathcal{B}$ is a family of tori.

Proof of step II.

Let $\alpha : \Xi \rightarrow \mathcal{J}$ be the family of Albanese maps. By construction, or by the universal property of the Albanese map, we have that F is obtained by taking projections of the Albanese maps, whence we have a factorization $\alpha : \Xi \rightarrow \mathcal{J} \rightarrow \mathcal{C} \rightarrow \mathcal{J}''$, where $\mathcal{J}'' \rightarrow \mathcal{B}$ is the family of Jacobians of \mathcal{C} .

The desired family of tori $\mathcal{T} \rightarrow \mathcal{B}$ is therefore the family of kernels of $\mathcal{J} \rightarrow \mathcal{J}''$.

To show the existence of the morphism G , observe that for $t \neq 0$, an isomorphism of X_t with $C_t \times T_t$ is given by a projection $G_t : X_t \rightarrow T_t$ which, in turn, is provided (through integration) via a complex subspace W_t of $H^0(\Omega_{X_t}^1)$ whose real span $H_t = W_t \oplus \bar{W}_t$ is \mathbb{R} -generated by a subgroup of $H^1(X_t, \mathbb{Z})$.

Although there are several choices for such a subgroup, we have at most a countable choice of those. Since for each t there is such a choice, by Baire's theorem there is a choice which holds for each $t \neq 0$. Let us make such a choice of $H = H_t \forall t$: then the corresponding subspace W_t has a limit in H for $t = 0$, and since the weak 1-Lefschetz decomposition holds for X_0 , and this limit is a direct summand for U_t , we easily see that this limit is unique, and the desired morphism G is therefore obtained.

STEP III. The fibre product $F \times_{\mathcal{B}} G$ yields an isomorphism onto $\mathcal{C} \times_{\mathcal{B}} \mathcal{T}$ for each fibre over $t \neq 0$, but since also the fibre over 0 is a torus fibration over C_0 , and the cohomology algebras are all the same, it follows that we have an isomorphism also for $t = 0$.

Q.E.D.

We end this section by giving an application of theorem 4.6.

Theorem 4.11. *Let (X, σ) be a real variety which is a deformation of real tori: then also (X, σ) is a real torus.*

The orbifold fundamental group sequence completely determines the differentiable type of the pair (X, σ) .

The real deformation type is also completely determined by the orbifold fundamental group of (X, σ) .

In turn, the orbifold fundamental group, if s is the linear integral transformation of $H_1(X, \mathbb{Z})$ obtained by conjugation with σ , is uniquely determined by the integer r which is the rank of the matrix $(s - Id)(\text{mod} 2)$ acting on $H_1(X, \mathbb{Z}/2)$ and, if $r \neq n = \dim X$, by the property whether the orbifold exact sequence does or does not split (if $r = n$ it always splits).

PROOF.

The first statement is a direct consequence of theorem 4.6.

Concerning the second statement, let $\tilde{\sigma}$ be a lifting of the antiholomorphic involution σ on the universal cover \mathbb{C}^n : then it follows as usual that the first derivatives are bounded, whence constant, and $\tilde{\sigma}$ is an affine map. We may moreover assume $\tilde{\sigma}$ to be linear in the case where σ admits a fixed point (in this case the orbifold exact sequence splits).

Let Λ be the lattice $H_1(X, \mathbb{Z})$: so we have first of all that $\tilde{\sigma}$ is represented by the affine map of $\Lambda \otimes \mathbb{R}$, $x \rightarrow s(x) + b$, where s is the isomorphism of Λ to itself, given by conjugation with σ .

Thus s is an integral matrix whose square is the identity, and it is well known (cf. e.g. [Cat 9], lemma 3.11) that we can thus split $\Lambda = U \oplus V \oplus W^+ \oplus W^-$, where s acts by $s(u, v, w^+, w^-) = (v, u, w^+, -w^-)$.

Since the number of $(+1)$ -eigenvalues is equal to the number of (-1) -eigenvalues, $\tilde{\sigma}$ being antiholomorphic, we get that W^+, W^- have the same dimension $n - r$ (whence $\dim U = \dim V = r$).

Since the square of σ is the identity, it follows that $\tilde{\sigma}^2$ is an integral translation. This condition boils down to $s b + b \in \Lambda$. If we write $b = (b_1, b_2, b^+, b^-)$, the previous condition means that $b_1 + b_2, 2b^+$ are integral vectors.

In order to obtain a simple normal form, we are allowed to take a different lift $\tilde{\sigma}$, i.e., to consider an affine map of the form $x \rightarrow s(x) + b + \lambda$, where $\lambda \in \Lambda$, and then to take another point $c \in \Lambda \otimes \mathbb{R}$ as the origin.

Then we get an affine transformation of the form $y \rightarrow sy + b + \lambda + sc - c$. The translation vector is thus $(b_1, b_2, b^+, b^-) + (\lambda_1, \lambda_2, \lambda^+, \lambda^-) + (c_1 - c_2, c_2 - c_1, 0, 2c^-)$.

We choose $c^- = -(1/2)b^-$ and $\lambda^- = 0$, $c_2 - c_1 = b_1 + \lambda_1$, $-\lambda_2 = b_1 + b_2 - \lambda_1$, and then we choose for λ^+ the opposite of the integral part of b^+ .

We obtain a new affine map $y \rightarrow sy + b'$ where $b' = (0, 0, b'^+, 0)$ and all the coordinates of b'^+ are either 0 or $1/2$. So, either $b' = 0$, or we can assume that $2b'^+$ is the first basis element of W^+ .

We obtain thus two cases :

- 1) There exists a lift $\tilde{\sigma}$ represented by the linear map s for a suitable choice of the origin.
- 2) There exists $\tilde{\sigma}$ represented, for a suitable choice of the origin, and for the choice of a suitable basis of Λ , by the affine map $y \rightarrow sy + 1/2e_1^+$ where e_1^+ is the first basis element of W^+ .

REMARK: Case 1) holds if and only if there is a fixed point of σ . Whereas, in case 2) we can never obtain that b'^+ be zero, therefore the square of any lift $\tilde{\sigma}$ is always given by a translation with third coordinate $2b'^+ \neq 0$, so the orbifold exact sequence does not split.

We conclude that 1) holds if and only if the orbifold exact sequence splits.

By the normal forms obtained above, it follows that the integer r and the splitting or non splitting property of the orbifold fundamental group sequence not only determine completely the orbifold fundamental group sequence, but also its affine representation.

To finish the proof of the theorem, assume that the normal form of $\tilde{\sigma}$ is preserved, whence also the affine representation of the orbifold fundamental group.

We are now looking for all the translation invariant complex structures which make the transformation $\tilde{\sigma}$ antiholomorphic. As before in the case of the elliptic curves, we look for the $2n \times 2n$ matrices J whose square $J^2 = -Id$, and such that $Js = -sJ$. The latter condition implies that J exchanges the eigenspaces of s in $\Lambda \otimes \mathbb{R}$.

We can calculate the matrices J in a suitable \mathbb{R} -basis of $\Lambda \otimes \mathbb{R}$ where s is diagonal, i.e., $s(y_1, y_2) = (y_1, -y_2)$. Then $J(y_1, y_2) = (Ay_2, By_1)$ and $Js = -sJ$ is then satisfied. The further condition $J^2 = -Id$ is equivalent to $AB = BA = -Id$, i.e., to $B = -A^{-1}$. Therefore, these complex structures are parametrized by $GL(n, \mathbb{R})$, which indeed has two connected components, distinguished by the sign of the determinant.

Since however we already observed that σ provides an isomorphism between (X, σ) and (\bar{X}, σ) , and \bar{X} corresponds to the complex structure $-J$, if J is the complex structure for X , we immediately obtain that A and $-A$ give isomorphic real varieties.

We are set for n odd, since $\det(-A) = (-1)^n \det A$.

In the case where n is even, observe that two matrices A and A' yield isomorphic real tori if and only if there is a matrix $D \in GL(\Lambda)$ such that D commutes with s , and such that D conjugates J to J' (then the diffeomorphism is orientation preserving for the orientations respectively induced by the complex structures associated to A , resp. A').

Thus D respects the eigenspaces of s , whence $D(y_1, y_2) = (D_1 y_1, D_2 y_2)$, and A is transformed to $A' = D_2 A (D_1)^{-1}$. Whence, we see that the sign of the determinant of A' equals the one of $(\det A)(\det D)$, whence we can change sign in any case by simply choosing D with $\det D = -1$.

Q.E.D.

5. THE BLANCHARD -CALABI THREEFOLDS

The Sommese-Blanchard examples ([Bla1], [Bla2], [Somm]) provide non Kähler complex structures X on manifolds diffeomorphic to a product $C \times T$, where C is a compact complex curve and T is a 2-dimensional complex torus.

In fact, in these examples, the projection $X \rightarrow C$ is holomorphic and all the fibres are 2-dimensional complex tori.

Also Calabi ([Cal]) showed that there are complex structures on a product $C \times T$ (that all these structures are non Kählerian follows also by the arguments of the previous theorem, else they would produce a complex product structure $C' \times T'$).

The result of Calabi is the following

Theorem 5.1. (CALABI) *Let C be a hyperelliptic curve of odd genus $g \geq 3$, and let T be a two dimensional complex torus. Then the differentiable manifold $C \times T$ admits a complex structure with trivial canonical bundle.*

We shall try to show that the construction of Calabi, although formulated in a different and very interesting general context, yields indeed a very special case of the construction of Sommese-Blanchard, which may instead be formulated and generalized as follows

Theorem 5.2. (*BLANCHARD-CALABI Jacobian 3-FOLDS*)

Let C be curve of genus $g \geq 0$, and let W be a rank 2 holomorphic vector bundle admitting four holomorphic sections $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ which are everywhere \mathbb{R} -linearly independent (for instance, if $W = L \oplus L$, where $H^0(L)$ has no base points, then four sections as above do exist).

Then the quotient X of W by the \mathbb{Z}^4 -action acting fibrewise by translations : $w \rightarrow w + \sum_{i=1, \dots, 4} n_i \sigma_i$ is a complex manifold diffeomorphic to the differentiable manifold $C \times T$, where T is a two dimensional complex torus, and will be called a **Jacobian Blanchard-Calabi 3-fold**.

The canonical divisor of X equals $K_X = \pi^*(K_C - \det W)$, where $\pi : X \rightarrow C$ is the canonical projection. Moreover, X is Kähler if and only if the bundle W is trivial (i.e., iff X is a holomorphic product $C \times T$).

Indeed one has $h^0(\Omega_X^1) = g$ unless W is trivial, while, if the vector bundle V is defined through the exact sequence

$$0 \rightarrow V \rightarrow (\mathcal{O}_C)^4 \rightarrow W \rightarrow 0,$$

then $h^1(\mathcal{O}_X) = g + h^0(V^\vee)$.

PROOF. The four holomorphic sections $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ make W a trivial \mathbb{R} -vector bundle, and with this trivialization we obtain that X is diffeomorphic to the product of C with a real four dimensional torus.

Let us now show that any vector bundle $W = L \oplus L$, where $H^0(L)$ has no base points, admits such sections $\sigma_1, \sigma_2, \sigma_3, \sigma_4$.

In fact, we have $s_1, s_2 \in H^0(L)$ without common zeros, so $s := {}^t (s_1, s_2)$ is a nowhere vanishing section of $W = L \oplus L$.

Use now the fact that $GL(2, \mathbb{C})$ operates on $L \oplus L$, and that, identifying \mathbb{C}^2 with the field H of Hamilton's quaternions, then $GL(2, \mathbb{C})$ contains the finite quaternion group $\mathcal{H} = \{+1, -1, +i, -i, +j, -j, +ij, -ij\}$, thus it suffices to define $\sigma_1 := s, \sigma_2 := is, \sigma_3 := js, \sigma_4 := ijs$.

In other more concrete words $\sigma_2 := {}^t (is_1, -is_2), \sigma_3 := {}^t (-s_2, s_1), \sigma_4 := {}^t (is_2, is_1)$.

We have the Koszul complex associated to (s_1, s_2) :

$$0 \rightarrow L^{-1} \rightarrow (\mathcal{O}_C)^2 \rightarrow L \rightarrow 0.$$

Now, an easy calculation shows that, defining V as the kernel subbundle of the linear map given by $\sigma := (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$

$$0 \rightarrow V \rightarrow (\mathcal{O}_C)^4 \rightarrow W \rightarrow 0,$$

then in the special case above we have $V = L^{-1} \oplus L^{-1}$.

Returning to the general situation, since now on a torus $T = W'/\Gamma$ there are canonical isomorphisms of $H^0(\Omega_T^1)$ with the space of linear forms on the vector space W' , and of $H^1(\mathcal{O}_T)$ with the quotient vector space $\text{Hom}(\Gamma, \mathbb{C})/H^0(\Omega_T^1)$, we obtain immediately

1) the exact sequence

$$0 \rightarrow \pi^*(\Omega_C^1) \rightarrow \Omega_X^1 \rightarrow (W^\vee) \rightarrow 0$$

2) an isomorphism $\mathcal{R}^1\pi_*(\mathcal{O}_X) \cong (V^\vee)$.

From the Leray spectral sequence follows immediately that $h^1(\mathcal{O}_X) = g + h^0(V^\vee)$.

If X is Kähler, then, since the first Betti number of X equals $4 + 2g$, then it must hold that $h^0(\Omega_X^1) = g + 2$, in particular $h^0(W^\vee) \geq 2$.

But (W^\vee) is a subbundle of a trivial bundle of rank 4, whence two linearly independent sections of (W^\vee) yield a composition $(\mathcal{O}_C)^2 \rightarrow (W^\vee) \rightarrow (\mathcal{O}_C)^4$ whose image is a trivial bundle of rank 1 or 2.

But in the former case the two sections would not be \mathbb{C} -linearly independent, whence the image must be a trivial bundle of rank 2 and taking determinants of the composition, we get that (W^\vee) is trivial if $h^0(W^\vee) \geq 2$.

Similarly, assume that $h^0(W^\vee) = 1$: then (W^\vee) has a trivial summand I of rank 1. Then we have a direct sum $(W^\vee) = I \oplus Q$ and dually W is a direct sum $\mathcal{O}_C \oplus Q^{-1}$. We use now the hypothesis that there are four holomorphic sections of W which are \mathbb{R} -independent at any point: it follows then that Q^{-1} admits two holomorphic sections which are everywhere \mathbb{R} -independent. Whence, it follows that also Q^{-1}, Q are trivial.

Q.E.D.

Definition 5.3. *Given a Jacobian Blanchard-Calabi 3-fold $\pi : X \rightarrow C$, any X -principal homogeneous space $\pi' : Y \rightarrow C$ will be called a Blanchard-Calabi 3-fold.*

Corollary 5.4. *The space of complex structures on the product of a curve C with a four dimensional real torus has unbounded dimension.*

PROOF.

Let G be the four dimensional Grassmann variety $G(1, 3)$: observe that the datum of an exact sequence

$$0 \rightarrow V \rightarrow (\mathcal{O}_C)^4 \rightarrow W \rightarrow 0$$

is equivalent to the datum of a holomorphic map $f : C \rightarrow G$, since for any such f we let V, W be the respective pull backs of the universal subbundle U and of the quotient bundle Q (of course, for the Blanchard-Calabi construction one needs the further open condition that the four induced sections of W be \mathbb{R} -linearly independent at each point).

We make throughout the assumption that f is not constant, i.e., W is not trivial. Then we have the following exact sequence:

$$0 \rightarrow \Theta_C \rightarrow (f)^*\Theta_G \rightarrow N_f \rightarrow 0,$$

where N_f , the normal sheaf of the morphism f , governs the deformation theory of the morphism f , in the sense that the tangent space to $\text{Def}(f)$ is the space $H^0(N_f)$, while the obstructions lie in $H^1(N_f)$.

By virtue of the fact that $\Theta_G = \text{Hom}(U, Q)$, and of the cohomology sequence associated to the above exact sequence, we get

$$0 \rightarrow H^0(\Theta_C) \rightarrow H^0(V^\vee \otimes W) \rightarrow H^0(N_f) \rightarrow H^1(\Theta_C) \rightarrow H^1(V^\vee \otimes W) \rightarrow H^1(N_f) \rightarrow 0,$$

and we conclude that the deformations of the map are unobstructed provided $H^1(V^\vee \otimes W) = 0$.

This holds, in the special case where $W = L \oplus L$, if the degree d of L satisfies $d \geq g$, since then $H^1(V^\vee \otimes W) = H^1((L \oplus L) \otimes (L \oplus L)) = 0$.

If $d \geq d$ the dimension of the space of deformations of the map f is given by $3g - 3 + 4h^0(2L) = 4d + 1 - g$, and this number clearly tends to infinity together with $d = \text{deg}(L)$.

On the other hand, we want to show that the deformations of the map f yield effective deformations of X as a Lie group principal fibration. This can be seen as follows.

Consider the exact sequence

$$0 \rightarrow \pi^*(W) \rightarrow \Theta_X \rightarrow \pi^*(\Theta_C) \rightarrow 0$$

and the derived direct image sequence

$$\begin{aligned} 0 \rightarrow (W) \rightarrow \pi_* \Theta_X \rightarrow (\Theta_C) \rightarrow \\ \rightarrow (V^\vee \otimes W) \rightarrow \mathcal{R}^1 \pi_* \Theta_X \rightarrow (\Theta_C) \otimes V^\vee \rightarrow \\ \rightarrow (\det(W) \otimes W) \rightarrow \mathcal{R}^2 \pi_* \Theta_X \rightarrow (\Theta_C) \otimes \det(W) \rightarrow 0, \end{aligned}$$

where we used that $\mathcal{R}^2 \pi_*(\mathcal{O}_X) \cong \Lambda^2(V^\vee) \cong \det(W)$.

Notice that $\pi_* \Theta_X$ is a vector bundle on the curve C , of rank either 2 or 3. In the latter case, since its image M in (Θ_C) is saturated (i.e., $(\Theta_C)/M$ is torsion free), then $M = (\Theta_C)$ and all the fibres of π are then biholomorphic.

In this case, for any tangent vector field on C , which we identify to the 0-section of W , we get a corresponding tangent vector field on W by using the fiberwise simply transitive action of \mathbb{R}^4 corresponding to the four chosen holomorphic sections. Since all the fibres are isomorphic, the corresponding tangent vector field on X is holomorphic, and the bundle W is then trivial.

Since we are assuming W not to be trivial, we get thus an exact sequence

$$0 \rightarrow (\Theta_C) \rightarrow (V^\vee \otimes W) \rightarrow \mathcal{R}^1 \pi_* \Theta_X$$

By the Leray spectral sequence follows that we have that $H^0(V^\vee \otimes W)/H^0((\Theta_C))$ injects into $H^0(\mathcal{R}^1 \pi_* \Theta_X)$ which is a direct summand of $H^1(\Theta_X) = H^0(\mathcal{R}^1 \pi_* \Theta_X) \oplus H^1(W)$.

Therefore the smooth space $\text{Def}(f)$ of deformations of the map f embeds into the Kuranishi space $\text{Def}(X)$ of deformations of X .

The space $H^1(W)$ is the classifying space for principal X homogeneous spaces (cf. [Ko2-3], [Shaf]) and therefore we see that this subspace of $H^1(\Theta_X)$ corresponds to actual deformations.

However for W of large degree we get $H^1(W) = 0$, so then the Blanchard-Calabi 3-folds coincide with the Jacobian Blanchard-Calabi 3-folds.

Q.E.D.

Remark 5.5. 1) In the Blanchard-Calabi examples one gets a trivial canonical bundle iff $\det(W) \equiv K_C$. This occurs in the special case where $W = L \oplus L$ and L is a theta-characteristic such that $H^0(L)$ is base point free. In particular, we have the Calabi examples where C is hyperelliptic of odd genus g and L is the $1/2(g-1)^{\text{th}}$ power of the hyperelliptic line bundle of C .

2) Start with a Sommese-Blanchard 3-fold with trivial canonical bundle and deform the curve C and the line bundle L in such a way that the canonical bundle becomes the pull back of a non torsion element of $\text{Pic}(C)$: then this is the famous example that the Kodaira dimension is not deformation equivalent for non Kähler manifolds (cf. [Ue]).

3) In the previous theorem we have followed the approach of [Cat6], correcting however a wrong formula for $\mathcal{R}^1\pi_*\mathcal{O}_X$ (which would give the dual vector bundle).

4) The approach of Sommese is also quite similar, and related to quaternion multiplication, which is viewed as the \mathbb{C} -linear map $\psi : H = \mathbb{C}^2 \rightarrow \text{Hom}_{\mathbb{R}}(H = \mathbb{R}^4, H = \mathbb{C}^2)$, defined by $\psi(q)(q') = qq'$.

Whence, quaternion multiplication provides four sections of $\mathcal{O}_{\mathbb{P}^1}(1)^2$ which are \mathbb{R} -linearly independent at each point.

Sommese explores this particular situation as an example of the natural fibration occurring in the more general context of the so called quaternionic manifolds (later on, the word "twistor fibration", for the other projection, has become more fashionable).

Sommese moreover observes that given a line bundle L such that $H^0(L)$ is base point free, the choice of two independent sections yields a holomorphic map to \mathbb{P}^1 and one can pull back $\mathcal{O}_{\mathbb{P}^1}(1)^2$ and the four sections: obtaining exactly the same situation we described above.

We learnt from Sommese that Blanchard knew these examples too (they are also described in [Ue]).

We will indeed prove in the sequel a much more precise statement concerning the deformations of Blanchard-Calabi 3-folds: we need for this purpose the following

Definition 5.6. A Blanchard-Calabi 3-fold is said to be developable if the corresponding ruled surface is developable, or, in other words, if and only if the derivative of the corresponding map $f : C \rightarrow G$ yields at each point p of C an element of $(V^\vee \otimes W)_p$ of rank ≤ 1 .

Remark 5.7. Observe in fact that (cf. [A-C-G-H] C-9, page 38) since f is non constant, and if there is no point of \mathbb{P}^3 contained in each line of the corresponding family of lines, f is the associated map to a holomorphic map $F : C \rightarrow \mathbb{P}^3$, so the union of the family of lines is the tangential developable of the image curve of the mapping F .

If such a point exists, it means that there is an effective divisor D on the curve C , and the bundle W is given as an extension

$$0 \rightarrow (\mathcal{O}_C)(-D) \rightarrow (\mathcal{O}_C)^3 \rightarrow W \rightarrow 0$$

and moreover a fourth section of W is given (which together with the previous three provides the desired trivialization of the underlying real bundle).

Remark 5.8. *By a small variation of a theorem of E. Horikawa concerning the deformations of holomorphic maps, namely Theorem 4.9 of [Cat4], we obtain in particular that, under the assumption $H^0((\Theta_C) \otimes V^\vee) = 0$, we have a smooth morphism $\text{Def}(\pi) \rightarrow \text{Def}(X)$. This assumption is however not satisfied in the case of Sommese-Blanchard 3-folds, at least in the case where the degree of L is large.*

Theorem 5.9. *Any small deformation of a non developable Blanchard-Calabi 3-fold such that $H^1(V^\vee \otimes W) = 0$ is again a Blanchard-Calabi 3-fold. In particular, a small deformation of a Sommese-Blanchard 3-fold with L of degree $d \geq g$ is a Blanchard-Calabi 3-fold.*

PROOF.

In this context recall once more (cf. [Ko2-3]) that the infinitesimal deformations corresponding to $H^1(W)$ correspond to the deformations of X as a principal homogenous space fibration over the principal Lie group fibration X , and these are clearly unobstructed. Indeed, for the Sommese-Blanchard examples, it will also hold $H^1(W) = 0$, if the degree d of L is large enough.

Assume that the vanishing $H^1(V^\vee \otimes W) = 0$ holds: then the deformations of the map f are unobstructed, and it is then clear that we obtain a larger family of deformations of X , parametrized by an open set in $H^0(N_f) \oplus H^1(W)$, and that this family yields deformations of X as a Blanchard-Calabi 3-fold (one can show that these are the deformations of X which preserve the fibration π).

All that remains is thus to show that the Kodaira Spencer map of this family is surjective.

Since we already remarked that we have an isomorphism $W \cong \pi_*(\Theta_X)$, then $H^1(W) \cong H^1\pi_*(\Theta_X)$, and we only have to control $H^0(\mathcal{R}^1\pi_*\Theta_X)$.

To this purpose we split the long exact sequence of derived direct images into the following exact sequences

$$\begin{aligned} 0 &\rightarrow (\Theta_C) \rightarrow V^\vee \otimes W \rightarrow \mathcal{R} \rightarrow 0 \\ 0 &\rightarrow \mathcal{R} \rightarrow \mathcal{R}^1\pi_*\Theta_X \rightarrow \mathcal{K} \rightarrow 0 \\ 0 &\rightarrow \mathcal{K} \rightarrow ((\Theta_C) \otimes V^\vee) \rightarrow W \otimes \Lambda^2(V^\vee) \end{aligned}$$

and observe that all we need to prove is the vanishing $H^0(\mathcal{K}) = 0$.

In fact, then $H^0(N_f) = H^0(\mathcal{R}) = H^0(\mathcal{R}^1\pi_*\Theta_X)$.

By the last exact sequence, it would suffice to show the injectivity of the linear map $H^0((\Theta_C) \otimes V^\vee) \rightarrow H^0(W \otimes \Lambda^2(V^\vee))$.

But indeed we will show that the sheaf \mathcal{K} is the zero sheaf.

Consider again in fact the beginning of the exact sequence of derived direct images

$$\begin{aligned} 0 &\rightarrow (W) \rightarrow \pi_*\Theta_X \rightarrow (\Theta_C) \rightarrow \\ &\rightarrow (V^\vee \otimes W) : \end{aligned}$$

and observe then that the homomorphism $(\Theta_C) \rightarrow (V^\vee \otimes W) = \mathcal{R}^1\pi_*(\pi^*(W))$ is indeed the derivative of the map $f : C \rightarrow G$.

Since the homomorphism of sheaves we are considering is $(\Theta_C) \otimes V^\vee = \mathcal{R}^1\pi_*(\pi^*(\Theta_C)) \rightarrow \mathcal{R}^2\pi_*(\pi^*(W)) = W \otimes \Lambda^2(V^\vee)$ the sheaf map $(\Theta_C) \otimes V^\vee \rightarrow W \otimes \Lambda^2(V^\vee)$ is induced by wedge product of $(\Theta_C) \rightarrow (V^\vee \otimes W)$ with the identity of V^\vee , therefore we will actually have that the sheaf \mathcal{K} is zero if the subsheaf of $(V^\vee \otimes W)$ given by the image of $(\Theta_C) \rightarrow (V^\vee \otimes W)$ consists of tensors of generical rank 2.

But this holds by virtue of the hypothesis that X be non developable.

Finally, let X be a Sommese-Blanchard 3-fold with L of degree $d \geq g$: then we have the desired vanishing $H^1(V^\vee \otimes W) = 0$.

The condition that the associated ruled surface is non developable follows from a direct calculation which shows that, if $f(t)$ is given as the subspace generated by the columns of a 4×2 matrix $B(t)$, then the 4×4 matrix $B(t)B'(t)$ has determinant equal to the square (up to sign) of the Wronskian determinant of the section s .

But s is non constant, for a Blanchard-Calabi 3-fold, therefore the Wronskian determinant is not identically zero and we are done.

Q.E.D.

Remark 5.10. *How does the developable case occur and which are its small deformations? The condition that the four sections should in every point of C provide a real basis of the fibre of W simply means that there is no real point in the developable surface associated to $f : C \rightarrow G$.*

A more detailed analysis of the developable Blanchard-Calabi 3-folds and of their deformations could allow a positive answer to the following

QUESTION: Do the above Blanchard-Calabi 3-folds provide infinitely many deformation types on the same differentiable manifold $C \times T$? (In other words, do these just give infinitely many irreducible components of the "moduli space", or also "connected" components?)

Calabi's construction is also quite beautiful, so that we cannot refrain from indicating its main ideas.

The first crucial observation is that, interpreting \mathbb{R}^7 as the space C^i of imaginary Cayley numbers, for any oriented hypersurface $M \subset \mathbb{R}^7$, the Cayley product produces an almost complex structure as follows:

$$J(v) = (v \times n)^i,$$

where v is a tangent vector at the point $x \in M$, n is the normal vector at $x \in M$, and w^i stands for the imaginary part of a Cayley number w .

For instance , this definition provides the well known non integrable almost complex structure on the 6-sphere S^6 .

Moreover, Calabi shows that the almost complex structure is special hermitian, i.e., that its canonical bundle is trivial.

Second, Calabi analyses when is the given complex structure integrable, proving in particular the following. If we write $\mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^4$, according to the decomposition $C = H \oplus He_5$ (here H is the space of Hamilton's quaternions), and M splits accordingly as a product $S \times \mathbb{R}^4$, then the given complex structure is integrable if and only if S is a minimal surface in \mathbb{R}^3 .

The third ingredient is now a classical method used by Schwarz in order to construct minimal surfaces. Namely, let C be a hyperelliptic curve, so that the canonical map is a double cover of a rational normal curve of degree $g - 1$. If $g = 3$ we have then a basis of holomorphic differentials $\omega_1, \omega_2, \omega_3$ such that the sum of their squares equals zero. Similarly, for every odd genus, via an appropriate projection to \mathbb{P}^2 , we obtain three linearly independent holomorphic differentials, without common zeros, and satisfying also the relation $\omega_1^2 + \omega_2^2 + \omega_3^2 = 0$.

The integral of the real parts of the ω_i 's provides a multivalued map of C to \mathbb{R}^3 which is a local embedding. Since another local determination differs by translation, the tangent space to a point in C is then naturally a subspace of \mathbb{R}^3 , whence Cayley multiplication provides a well defined complex structure on $C \times \mathbb{R}^4$. Since moreover this complex structure is also invariant by translations on \mathbb{R}^4 , we can descend a complex structure on $C \times T$.

From the constructions of Blanchard and Calabi we deduce a negative answer to the problem mentioned in remark 2.1 and in the introduction.

Corollary 5.11. *Products of a curve of genus $g \geq 1$ with a 2-dimensional complex torus provide examples of complex manifolds which are a $K(\pi, 1)$, and for which there are different deformation types.*

We shall see in the next section that the answer continues to be negative even if we restrict to Kähler, and indeed projective manifolds, even in dimension = 2.

6. MODULI SPACES OF SURFACES OF GENERAL TYPE

In this section we begin to describe a recent result ([Cat8]), showing the existence of complex surfaces which are $K(\pi, 1)$'s and for which there are different deformation types.

Before we get into the details of the construction, it seems appropriate to give a more general view of the status of the art concerning deformation, differentiable and topological types of algebraic surfaces of general type.

Let S be a minimal surface of general type: then to S we attach two positive integers $x = \chi(\mathcal{O}_S)$, $y = K_S^2$ which are invariants of the oriented topological type of S .

The moduli space of the surfaces with invariants (x, y) is a quasi-projective variety (cf. [Gie]) defined over the integers, in particular it is a real variety.

Fixed (x, y) we have several possible topological types, but indeed only two if moreover the surface S is simply connected. These two cases are distinguished as follows:

- S is EVEN, i.e., its intersection form is even : then S is a connected sum of copies of a $K3$ surface and of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$.
- S is ODD : then S is a connected sum of copies of $\mathbb{P}_{\mathbb{C}}^2$ and $\mathbb{P}_{\mathbb{C}}^{2\,opp}$.

Remark 6.1. $\mathbb{P}_{\mathbb{C}}^{2\,opp}$ stands for the same manifold as $\mathbb{P}_{\mathbb{C}}^2$, but with reversed orientation.

It is rather confusing, especially if one has to do with real structures, that some authors use the symbol $\mathbb{P}_{\mathbb{C}}^2$ for $\mathbb{P}_{\mathbb{C}}^{2\,opp}$.

In general, the fundamental group is a powerful topological invariant. Invariants of the differentiable structure have been found by Donaldson and Seiberg-Witten, and one can easily show that on a connected component of the moduli space the differentiable structure remains fixed.

Up to recently, the converse question $DEF = DIFF ?$ was open, but recently counterexamples have been given, by Manetti ([Man3]) for simply connected surfaces, by Kharlamov and Kulikov ([K-K]) for rigid surfaces, while we have found rather simple examples ([Cat8]):

Theorem 6.2. *Let S be a surface isogenous to a product, i.e., a quotient $S = (C_1 \times C_2)/G$ of a product of curves by the free action of a finite group G . Then any surface with the same fundamental group as S and the same Euler number of S is diffeomorphic to S . The corresponding moduli space M_S^{top} is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation. There are infinitely many examples of the latter case.*

Corollary 6.3. 1) $DEF \neq DIFF$.

2) There are moduli spaces without real points

3) There are complex surfaces whose fundamental group cannot be the fundamental group of a real surface.

For the construction of these examples we imitate the hyperelliptic surfaces, in the sense that we take $S = (C_1 \times C_2)/G$ where G acts freely on C_1 , whereas the quotient C_2/G is $\mathbb{P}_{\mathbb{C}}^1$.

Moreover, we assume that the projection $\phi : C_2 \rightarrow \mathbb{P}_{\mathbb{C}}^1$ is branched in only three points, namely, we have a so called TRIANGLE CURVE.

What happens is that if two surfaces of such sort are antiholomorphic, then there would be an antiholomorphism of the second triangle curve (which is rigid).

Now, giving such a branched cover ϕ amounts to viewing the group G as a quotient of the free group on two elements. Let a, c be the images of the two generators, and set $abc = 1$.

We find such a G with the properties that the respective orders of a, b, c are distinct, whence an antiholomorphism of the triangle curve would be a lift of the standard complex conjugation if the 3 branch points are chosen to be real, e.g. $-1, 0, +1$.

Such a lifting exists if and only if the group G admits an automorphism τ such that $\tau(a) = a^{-1}, \tau(c) = c^{-1}$.

An appropriate semidirect product will be the desired group for which such a lifting does not exist.

For this reason, in the next section we briefly recall the notion and the simplest examples of the so called triangle curves.

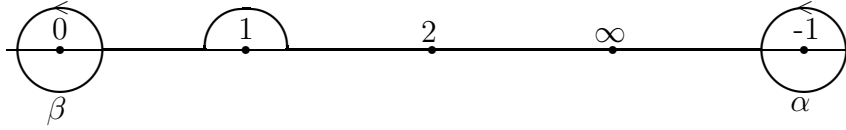
7. SOME NON REAL TRIANGLE CURVES

Consider the set $B \subset \mathbb{P}_{\mathbb{C}}^1$ consisting of three real points $B := \{-1, 0, 1\}$.

We choose 2 as a base point in $\mathbb{P}_{\mathbb{C}}^1 - B$, and we take the following generators α, β, γ of $\pi_1(\mathbb{P}_{\mathbb{C}}^1 - B, 2)$ such that

$$\alpha\beta\gamma = 1$$

and α, γ are free generators of $\pi_1(\mathbb{P}_{\mathbb{C}}^1 - B, 2)$ with α, β as indicated in the following picture



With this choice of basis, we have provided an isomorphism of $\pi_1(\mathbb{P}_{\mathbb{C}}^1 - B, 2)$ with the group

$$T_{\infty} := \langle \alpha, \beta, \gamma \mid \alpha\beta\gamma = 1 \rangle.$$

For each finite group G generated by two elements a, b , passing from Greek to latin letters we obtain a tautological surjection

$$\pi : T_{\infty} \rightarrow G.$$

I.e., we set $\pi(\alpha) = a, \pi(\beta) = b$ and we define $\pi(\gamma) := c$. (then $abc = 1$). To π we associate the Galois covering $f : C \rightarrow \mathbb{P}_{\mathbb{C}}^1$, branched on B and with group G .

Notice that the Fermat curve $C := \{(x_0, x_1, x_2) \in \mathbb{P}_{\mathbb{C}}^2 \mid x_0^n + x_1^n + x_2^n = 0\}$ is in two ways a triangle curve, since we can take the quotient of C by the group $G := (\mathbb{Z}/n)^2$ of diagonal projectivities with entries n -th roots of unity, but also by the full group $A = \text{Aut}(C)$ of automorphisms, which is a semidirect product of the normal subgroup G by the symmetric group permuting the three coordinates. For G the three branching multiplicities are all equal to n , whereas for A they are equal to $(2, 3, 2n)$.

Another interesting example is provided by the **Accola curve** (cf. [Acc1], [Acc2]), the curve Y_g birational to the affine curve of equation

$$y^2 = x^{2g+2} - 1.$$

If we take the group $G \cong \mathbb{Z}/2 \times \mathbb{Z}/(2g+2)$ which acts multiplying y by -1 , respectively x by a primitive $2g+2$ -root of 1, we realize Y_g as a triangle curve with branching multiplicities $(2, 2g+2, 2g+2)$. G is not however the

full automorphism group, in fact if we add the involution sending x to $1/x$ and y to iy/x^{g+1} , then we get the direct product $\mathbb{Z}/2 \times D_{2g+2}$ (which is indeed the full group of automorphisms of Y_g as it is well known and as also follows from the next lemma), a group which represents Y_g as a triangle curve with branching multiplicities $(2, 4, 2g + 2)$.

One can get many more examples by taking (a la Macbeath. [Mcb]) unramified coverings of the above curves (associated to characteristic subgroups of the fundamental group). The following natural question arises then: which are the curves which admit more than one realization as triangle curves? It is funny to observe:

Lemma 7.1. *Let $f : C \rightarrow \mathbb{P}_{\mathbb{C}}^1 = C/G$ be a triangle covering where the branching multiplicities m, n, p are all distinct (thus we assume $m < n < p$). Then the group G equals the full group A of automorphisms of C .*

Idea:

I. By Hurwitz's formula the cardinality of G is in general given by the formula

$$|G| = 2(g - 1)(1 - 1/m - 1/n - 1/p)^{-1}.$$

II. Assume that $A \neq G$ and let $F : \mathbb{P}_{\mathbb{C}}^1 = C/G \rightarrow \mathbb{P}_{\mathbb{C}}^1 = C/A$ be the induced map. Then $f' : C \rightarrow \mathbb{P}_{\mathbb{C}}^1 = C/A$ is again a triangle covering, otherwise the number of branch points would be ≥ 4 and we would have a non trivial family of such Galois covers with group A .

III. We claim now that the three branch points of f cannot have distinct images through F : otherwise the branching multiplicities $m' \leq n' \leq p'$ for f' would be not less than the respective multiplicities for f , and by the analogous of formula I for $|A|$ we would obtain $|A| \leq |G|$, a contradiction. The rest of the proof is complicated. End.

We come now to our particular triangle curves. Let r, m be positive integers $r \geq 3, m \geq 4$ and set

$$p := r^m - 1, \quad n := (r - 1)m .$$

Notice that the three integers $m < n < p$ are distinct. Let G be the following semidirect product of \mathbb{Z}/p by \mathbb{Z}/m :

$$G := \langle a, c \mid a^m = 1, c^p = 1, aca^{-1} = c^r \rangle$$

One sees easily that the period of b is exactly n .

Proposition 7.2. *The triangle curve C associated to π is not antiholomorphically equivalent to itself (i.e., it is not isomorphic to its conjugate).*

Idea:

We derive a contradiction assuming the existence of an antiholomorphic automorphism σ of C .

STEP I : $G = A$, where A is the group of holomorphic automorphisms of C , $A := \text{Bihol}(C, C)$.

Proof. This follows from the previous lemma 2.3.

STEP II : if σ exists, it must be a lift of complex conjugation.

Proof. In fact σ normalizes $\text{Aut}(C)$, whence it must induce an antiholomorphism of $\mathbb{P}_{\mathbb{C}}^1$ which is the identity on B , and therefore must be complex conjugation.

STEP III : complex conjugation does not lift.

Proof. This is purely an argument about covering spaces: complex conjugation acts on $\pi_1(\mathbb{P}_{\mathbb{C}}^1 - B, 2) \cong T_{\infty}$, as it is immediate to see with our choice of basis, by the automorphism τ sending α, γ to their respective inverses.

Thus, complex conjugation lifts if and only if τ preserves the normal subgroup $K := \ker \pi$. In turn, this means that there is an automorphism $\rho : G \rightarrow G$ with

$$\rho(a) = a^{-1}, \quad \rho(c) = c^{-1}.$$

Recall now the relation $aca^{-1} = c^r$: applying ρ , we would get $a^{-1}c^{-1}a = c^{-r}$, or, equivalently,

$$a^{-1}ca = c^r.$$

But then we would get $aca^{-1} = c^r = a^{-1}ca = a^{m-1}ca^{m-1} = c^{r^{m-1}}$, which holds only if

$$r \equiv r^{m-1} \pmod{p}.$$

Since $p = r^m - 1$ we obtain, after multiplication by r , that we should have $r^2 \equiv 1 \pmod{p}$ but this is the desired contradiction, because $r^2 - 1 < r^m - 1 = p$. \square

8. THE EXAMPLES OF SURFACES ISOGENOUS TO A PRODUCT

Definition 8.1. *A projective surface S is said to be isogenous to a (higher) product if it admits a finite unramified covering by a product of curves of genus ≥ 2 .*

Remark 8.2. *In this case, (cf. [Cat7], props. 3.11 and 3.13) there exist Galois realizations $S = (C_1 \times C_2)/G$, and each such Galois realization dominates a uniquely determined minimal one. S is said to be of nonmixed type if G acts via a product action of two respective actions on C_1, C_2 . Else, G contains a subgroup G^0 of index 2 such that $(C_1 \times C_2)/G^0$ is of nonmixed type.*

Proposition 8.3. *Let S, S' be surfaces isogenous to a higher product, and let $\sigma : S \rightarrow S'$ be an antiholomorphic isomorphism. Let moreover $S = (C_1 \times C_2)/G$, $S' = (C'_1 \times C'_2)/G'$ be the respective minimal Galois realizations. Then, up to possibly exchanging C'_1 with C'_2 , there exist antiholomorphic isomorphisms $\tilde{\sigma}_i$, $i = 1, 2$ such that $\tilde{\sigma} := \tilde{\sigma}_1 \times \tilde{\sigma}_2$ normalizes the action of G , in particular $\tilde{\sigma}_i$ normalizes the action of G^0 on C_i .*

Proof. Let us view σ as yielding a complex isomorphism $\sigma : S \rightarrow \bar{S}'$. Consider the exact sequence corresponding to the minimal Galois realization $S = (C_1 \times C_2)/G$,

$$1 \rightarrow H := \Pi_{g_1} \times \Pi_{g_2} \rightarrow \pi_1(S) \rightarrow G \rightarrow 1.$$

Applying σ_* to it, we infer by theorem 3.4 of ([Cat7]) that we obtain an exact sequence associated to a Galois realization of \bar{S}' . Since σ is an isomorphism, we get a minimal one, which is however unique. Whence, we get an isomorphism $\tilde{\sigma} : (C_1 \times C_2) \rightarrow (\bar{C}'_1 \times \bar{C}'_2)$, which is of product type by the rigidity lemma (e.g., lemma 3.8 of [Cat7]). Moreover this isomorphism must normalize the action of $G \cong G'$, which is exactly what we claim. □

In ([Cat7], cf. correction in [Cat8]) we have proven:

Theorem 8.4. *Let S be a surface isogenous to a product, i.e., a quotient $S = (C_1 \times C_2)/G$ of a product of curves by the free action of a finite group G . Then any surface S' with the same fundamental group as S and the same Euler number of S is diffeomorphic to S . The corresponding moduli space $M_S^{\text{top}} = M_S^{\text{diff}}$ is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.*

We are now going to explain the construction of our examples :

Let G be the semidirect product group we constructed in section 2, and let C_2 be the corresponding triangle curve.

Let moreover g'_1 be any number greater or equal to 2, and consider the canonical epimorphism ψ of $\Pi_{g'_1}$ onto a free group of rank g'_1 , such that in terms of the standard bases $a_1, b_1, \dots, a_{g'_1}, b_{g'_1}$, respectively $\gamma_1, \dots, \gamma_{g'_1}$, we have $\psi(a_i) = \psi(b_i) = \gamma_i$.

Compose then ψ with any epimorphism of the free group onto G , e.g. it suffices to compose with any μ such that $\mu(\gamma_1) = a$, $\mu(\gamma_2) = b$ (and $\mu(\gamma_j)$ can be chosen whatever we want for $j \geq 3$).

For any point C'_1 in the Teichmüller space we obtain a canonical covering associated to the kernel of the epimorphism $\mu \circ \psi : \Pi_{g'_1} \rightarrow G$, call it C_1 .

Definition 8.5. *Let S be the surface $S := (C_1 \times C_2)/G$ (S is smooth because G acts freely on the first factor).*

Theorem 8.6. *For any two choices $C'_1(I), C'_1(II)$ of C'_1 in the Teichmüller space we get surfaces $S(I), S(II)$ such that $S(I)$ is never isomorphic to $\bar{S}(II)$. Varying C'_1 we get a connected component of the moduli space, which has only one other connected component, given by the conjugate of the previous one.*

The last result that we have obtained as an application of these ideas is the following puzzling:

Theorem 8.7. *Let S be a surface in one of the families constructed above. Assume moreover that X is another complex surface such that $\pi_1(X) \cong \pi_1(S)$. Then X does not admit any real structure.*

Proof. (idea)

Observe that since S is a classifying space for the fundamental group of $\pi_1(S)$, then by the isotropic subspace theorem of ([Cat4]) the Albanese mapping of X maps onto a curve $C'(I)_2$ of the same genus as C'_2 .

Consider now the unramified covering \tilde{X} associated to the kernel of the epimorphism $\pi_1(X) \cong \pi_1(S) \rightarrow G$.

Again by the isotropic subspace theorem, there exists a holomorphic map $\tilde{X} \rightarrow C(I)_1 \times C(I)_2$, where moreover the action of G on \tilde{X} induces actions of G on both factors which either have the same topological types as the actions of G on C_1 , resp. C_2 , or have both the topological types of the actions on the respective conjugate curves.

By the rigidity of the triangle curve C_2 , in the former case $C(I)_2 \cong C_2$, in the latter $C(I)_2 \cong \bar{C}_2$.

Assume now that X has a real structure σ : then the same argument as in [C-F2] section 2 shows that σ induces a product antiholomorphic map $\tilde{\sigma} : C(I)_1 \times C(I)_2 \rightarrow C(I)_1 \times C(I)_2$. In particular, we get a non constant antiholomorphic map of C_2 to itself, contradicting proposition 2.3 . \square

In [Cat7], [Cat8] we gave the following

Definition 8.8. *A Beauville surface is a rigid surface isogenous to a product.*

Beauville gave examples of these surfaces ([Bea]), as quotients of the product $C \times C$ where C is the Fermat quintic curve. This example is real.

It would be interesting to

- Classify all the Beauville surfaces, at least those of non mixed type.
- Classify all the non real Beauville surfaces, especially those which are not isomorphic to their conjugate surface.

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