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Even sets of nodes on sextic surfaces

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Abstract. We determine the possible even sets of nodes on sextic surfaces in $\mathbb{P}^3$, showing in particular that their cardinalities are exactly the numbers in the set $\{24, 32, 40, 56\}$. We also show that all the possible cases admit an explicit description. The methods that we use are an interplay of coding theory and projective geometry on one hand, and of homological and computer algebra on the other.

We give a detailed geometric construction for the new case of an even set of 56 nodes, but the ultimate verification of existence relies on computer calculations. Moreover, computer calculations have been used more than once in our research in order to get good guesses.

The construction gives a maximal family, unirational and of dimension 27, of nodal sextics with an even set of 56 nodes.

As in [Ca-Ca] (where other cases were described), each such nodal surface F is given as the determinant of a symmetric map $\varphi : \mathcal{E}^\vee \to \mathcal{E}$, for an appropriate vector bundle $\mathcal{E}$ depending on F. The first difficulty here is to show the existence of such vector bundles. This leads us to the investigation of a hitherto unknown moduli space of rank 6 vector bundles which we show elsewhere to be birational to a moduli space of plane representations of cubic surfaces in $\mathbb{P}^3$. The resulting picture shows a very rich and interesting geometry. The main difficulty is to show the existence of "good" maps φ, and the interesting phenomenon which shows up is the following: the "moduli space" of such pairs $(\mathcal{E}, \varphi)$ is (against our initial hope) reducible, and for a general choice of $\mathcal{E}$ the determinant of φ is the double of a cubic surface G. Only when the vector bundle $\mathcal{E}$ corresponds to a reducible cubic surface, we get an extra component of the space of such pairs $(\mathcal{E}, \varphi)$, and a general choice in this component yields one of our desired nodal sextic surfaces.

Introduction

Let F be a nodal surface in $\mathbb{P}^3$ of degree d, i.e., F has only μ nodes (ordinary double points) $P_1, \dots, P_\mu$ as singularities.

A natural and classical question is to ask for the maximum possible number of nodes $\mu(d)$ that such a surface F can have.

The theory of projectively dual surfaces shows easily that $\mu(d) < \frac{1}{2}d(d-1)^2$ for $d \geq 3$ and the slightly better inequality given by Bassett in 1907 (cf. [Bass]) was obtained using this method.

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The function $\mu(d)$ is only known for $d \leq 6$, and for $d \leq 5$ one has an explicit description of the nodal surfaces which attain the maximum $\mu(d)$: the Cayley cubic, the Kummer quartics, and the Togliatti quintics (cf. [Cay1], [Cay2], [Kum], [Tog1], [Tog2], [Bea] and also [Ca-Ce], [Ba1]).

An important tool to investigate the function $\mu(d)$ for small values of d ($d \leq 17$), and to characterize the maximizing surfaces, was introduced by Beauville in [Bea]: he attached a binary code to each nodal surface F and used coding theory in order to show that $\mu(5) = 31$.

The coding theory method was later used by Jaffe and Ruberman in order to show (see [Ja-Ru]) that $\mu(6) = 65$, but their proof is not so short as the one by Beauville, partly because at that time a complete knowledge of the cardinality of an even set of nodes[1] on a sextic was missing (the binary code consists of the even sets of nodes on F, introduced in [Cat1], where a complete classification of even sets for degree $d = 5$ was given).

Today we still do not know if the Barth sextics (see [Ba1]) are those which achieve the maximum $\mu(6) = 65$ and until now an explicit description of the possible even sets of nodes for sextic surfaces has been missing. A general structure theorem for even and $1/2$-even sets was given in [Ca-Ca], but the cases where the cardinality t of an even set would be > 40 were excluded as a consequence of a conceptual error which was pointed out to the authors by Duco van Straten. Thus the simple proof by J. Wahl (cf. [Wahl]) of $\mu(6) = 65$ also became invalid.

We rescue the situation here by showing the following

Main Theorem A. *Let F be a nodal surface of degree $d = 6$ in $\mathbb{P}^3$ with an even set of t nodes. Then $t \in \{24, 32, 40, 56\}$. These four possibilities occur and can be explicitly described.*

The situation is thus more complicated than for $d \leq 5$, and the list of possible cardinalities t is (cf. e.g. [Ca-Ca]):

$$\begin{aligned} d &= 3, \quad t = 4, \\ d &= 4, \quad t \in \{8, 16\}, \\ d &= 5, \quad t \in \{16, 20\}, \\ d &= 6, \quad t \in \{24, 32, 40, 56\}. \end{aligned}$$

We first show in Section 1 that an even set of 64 nodes cannot exist. The simple new idea is to study the so called extended code (cf. e.g. [Cat2]) and we then use a mixture of geometric and coding theory arguments, as was done in the papers cited above, for instance in [Ja-Ru], where the case of an even set of 48 nodes was excluded.

We then proceed, using the structure theorem of [Ca-Ca], to construct explicit cases of sextics with an even set of 56 nodes.

The bulk of the paper is devoted to this purpose, and we get the following result.

[1] We adopt here the terminology of [Ca-Ca] concerning the notion of even sets of nodes which was introduced in [Cat1]: namely, the strictly even sets of [Cat1] are called even sets, while the weakly even sets of [Cat1] are called half-even, or $1/2$-even sets of nodes.

Main Theorem B. *There is a family of nodal sextic surfaces with* 56 *nodes forming an even set, parametrized by a smooth irreducible rational variety* Φ_0 *of dimension* 33*, whose image* Ξ_0 *is a unirational subvariety of dimension* 27 *of the space of sextic surfaces. Moreover, the above family is versal, thus* Ξ_0 *yields an irreducible component of the subvariety of nodal sextic surfaces with* 56 *nodes.*

The fact that a maximal family of nodal sextics with 56 nodes forming an even set has dimension equal to 27 means that these nodes impose independent conditions on the space of sextic surfaces (cf. [Bu-Wa]). It is an interesting question to find the nodal surfaces of smallest degree which possess an even set of nodes failing to impose independent conditions.

As already mentioned, it follows from the more general result of [Ca-Ca] that every even nodal set on a sextic surface F occurs as the corank 2 degeneracy locus of a symmetric map $\varphi : \mathcal{E}^\vee \to \mathcal{E}$, for an appropriate vector bundle $\mathcal{E}$ depending on F (and the nodal set).

The method to construct $\mathcal{E}$ is based on Beilinson's theorem and on revisiting Horrocks' correspondence due to Charles Walter (cf. [Wal]), which was exploited in [Ca-Ca]. The bundle $\mathcal{E}$ is constructed starting from a submodule M of the intermediate cohomology module $H^1_*(\mathcal{F})$ of the quadratic sheaf $\mathcal{F}$ associated to the even set, and corresponding to the choice of a Lagrangian subspace U of $H^1(\mathcal{F}(1))$. The choice of M determines a unique vector bundle $\mathcal{E}$, if a certain generality assumption (which we call the *first assumption*) is satisfied.

The construction is quite explicit if we make another generality assumption, namely that the two nonzero degree components of the artinian graded module M, the previously mentioned U and another one denoted by W, both have dimension equal to 3. If we denote, as is customary, by V the vector space of linear forms on $\mathbb{P}^3$, then the module M is completely determined by the multiplication tensor $B \in U^\vee \otimes V^\vee \otimes W$ for M.

We then show that the tensor B determines explicitly the bundle $\mathcal{E}$ as the kernel of an exact sequence

$$0 \to \mathcal{E} \to U \otimes V \otimes \mathcal{O}(1) \xrightarrow{(B,\epsilon)} (W \otimes \mathcal{O}(1)) \oplus (U \otimes \mathcal{O}(2)) \to 0$$

where the first component is precisely B, and the second is the standard Euler map, here denoted by ϵ.

Section 3 then ends by showing that the family of pairs $(\mathcal{E}, \varphi)$ is parametrized (not uniquely) by the following family of pairs:

$$\begin{aligned}\mathfrak{M}_{AB} := \{(B, A) \mid {} & B \in U^\vee \otimes V^\vee \otimes W,\\ & A \in (U \otimes V) \otimes (U \otimes V) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(2)),\ A = {}^tA,\ (B, \epsilon) \cdot A = 0\}.\end{aligned}$$

$\mathfrak{M}_{AB}$ sits inside an affine space of dimension 816, and it is not possible to find the decomposition of $\mathfrak{M}_{AB}$ into irreducible components even by computer. It is clear that $\mathfrak{M}_{AB}$ dominates the space of the above tensors B, and, if $\mathfrak{M}_{AB}$ were irreducible, one would obtain the sextic surfaces immediately by a random choice.

However, for a long time all the random choices would always give the double of a cubic surface G as determinant of φ, and it looked like even sets with 56 nodes did not exist. We then tried to prove that this was indeed the case, and we had to find an explanation for the occurrence of the cubic surface G.

Now, it is classical that to a $3\times3\times4$ tensor B one can associate a cubic surface in $\mathbb{P}^3$ by taking the determinant of the corresponding 3×3 matrix of linear forms on $\mathbb{P}^3$. However, in our case we get a cubic surface G^* in the dual projective space $\mathbb{P}^{3\vee} = \mathrm{Proj}(V^\vee)$, together with two different realizations of G^* as a blow up of a projective plane $\mathrm{Proj}(U^\vee)$ (respectively, $\mathrm{Proj}(W)$) in a set of six points. These are the points where the Hilbert–Burch 3×4 matrix of linear forms on U drops rank by 1, and the rational map to $\mathbb{P}^{3\vee}$ is given by the system of cubics through the 6 points, a system which is generated by the determinants of the four 3×3 minors of the Hilbert–Burch matrix.

One passes from one realization to the other simply by transposing the tensor, and we will call this the trivial involution for $3\times3\times4$ tensors; but what we have discovered, through geometry, is the existence of another involution for $3\times3\times4$ tensors, which we call the *cross-product involution*.

This second involution associates to a general tensor $B \in U^\vee \otimes V^\vee \otimes W$ another tensor $\mathcal{B} \in W'^\vee \otimes V \otimes U'^\vee$, where $W' := \Lambda^2 W$ and $U'^\vee$ is defined as the kernel of the map $\Lambda^2 W^\vee \otimes V \to U^\vee \otimes W^\vee$ induced by contraction with B (cf. 4.19 for the proof that we have indeed a birational involution).

In fact, to $\mathcal{B}$ corresponds now a cubic surface $G \subset \mathbb{P}^3$, which is related to a general bundle $\mathcal{E}$ through the existence of an exact sequence

$$0 \to 6\mathcal{O} \to \mathcal{E} \to \tau \to 0,$$

where τ is an invertible sheaf on the cubic surface G. One can see more precisely that $\mathcal{B}$ determines a sheaf $\mathcal{G}$ on G such that $\tau = \mathcal{G}^{\otimes 2}(-1)$.

We found in this way a nice explanation of the phenomenon pointed out by computer calculations: we got as determinant the surface G counted twice, simply because, in view of the above exact sequence, for a smooth cubic surface (indeed, irreducible) all the symmetric endomorphisms $\phi \in H^0(S^2\mathcal{E})$ are induced by the inclusion $S^2(6\mathcal{O}) \to S^2\mathcal{E}$.

It was clear at this point that if $H^0(S^2\mathcal{E})$ always had dimension 21, then we could not get any nodal sextic surface of the desired type, but it was of course possible that the dimension could jump up for special surfaces G, and that our parameter space $\mathfrak{M}_{AB}$ would be reducible. As explained in Section 6, a small computational simplification and the reduction to finite fields allowed us to make many more random attempts, until the first sextic surface appeared. Since a determinantal approach predicts that the space of tensors B for which the dimension of $H^0(S^2\mathcal{E})$ jumps has codimension 7, it was only natural to guess that the case which works is the case of tensors $\mathcal{B}$ corresponding to reducible cubic surfaces. This guess turned out to be true.

The cross-product involution can also be phrased as a duality theorem for a certain moduli space of vector bundles on $\mathbb{P}^3$. Namely, we prove elsewhere the following

Theorem C. *Consider the moduli space* $\mathfrak{M}^s(6; 3, 6, 4)$ *of simple rank* 6 *vector bundles* $\mathcal{E}$ *on* $\mathbb{P}^3$ *with Chern polynomial* $1 + 3t + 6t^2 + 4t^3$ (cf. [Kob]), *and inside it the open set* $\mathfrak{A}$ *corresponding to the simple bundles with minimal cohomology, i.e., those with*

(1) $H^i(\mathcal{E}) = 0\ \forall i \geq 1$; (2) $H^i(\mathcal{E}(-1)) = 0\ \forall i \neq 1$;
(3) $H^i(\mathcal{E}(-2)) = 0\ \forall i \neq 1$; (4) $H^i(\mathcal{E}(-3)) = 0\ \forall i$;
(5) $H^i(\mathcal{E}(-4)) = 0\ \forall i$.

Then $\mathfrak{A}$ *is irreducible of dimension* 19 *and it is bimeromorphic to* $\mathfrak{A}^0$, *where* $\mathfrak{A}^0$ *is an open set of the G.I.T. quotient space of the projective space* $\mathfrak{B}$ *of tensors of type* $(3, 4, 3)$, $\mathfrak{B} := \{B \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W)\}$ *by the natural action of* $\mathrm{SL}(W) \times \mathrm{SL}(U)$.

Let moreover $[B] \in \mathfrak{A}^0$ *be a general point. Then to* $[B]$ *corresponds a vector bundle* $\mathcal{E}_B$ *on* $\mathbb{P}^3$, *and also a vector bundle* $\mathcal{E}_B^*$ *on* $\mathbb{P}^{3\vee}$, *obtained from the direct construction applied to the sheaf* $\mathcal{G}_B^*$ *defined by*

$$0 \to U \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \xrightarrow{B} W \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{G}_B^* \to 0.$$

$\mathcal{E}_B^*$ *is the vector bundle* $\mathcal{E}_{\mathcal{B}}$, *where* $[\mathcal{B}] \in \mathfrak{A}_*^0$ *is obtained from* B *via the cross-product involution.*

Section 6 is devoted to a brief account of the random approach which we already mentioned, while the two Macaulay scripts, which are needed for the ultimate verification of the existence of surfaces which have an even set of 56 distinct nodes as the only singularities, can be found on the web pages of the authors.

1. Excluding via coding theory

Throughout this section F will be a normal surface in $\mathbb{P}^3$ of degree d having at most rational double points as singularities, and possessing moreover μ nodes (ordinary double points) $P_1, \dots, P_\mu$ among its singularities.

We let $\pi : \tilde{F} \to F$ be the minimal resolution of the singularities of F. It is well known (cf. [Tju]) that $\tilde{F}$ is diffeomorphic to a smooth surface of degree d in $\mathbb{P}^3$; in particular $\tilde{F}$ is simply connected and for its second Betti number we have $b_2(\tilde{F}) = d(d^2 - 4d + 6) - 2$.

We let $A_1, \dots, A_\mu$ be the exceptional (-2)-curves ($\cong \mathbb{P}^1$) coming from the blow up of the nodes $P_1, \dots, P_\mu$, and we let H be the full transform of a plane section of F.

Let X be the $\mathbb{Z}/2$-vector space freely generated by the A_i's and consider the map

$$\epsilon : X := \bigoplus_{i=1}^{\mu} (\mathbb{Z}/2) A_i \to H^2(\tilde{F}, \mathbb{Z}/2),$$

given by the reduction modulo two of the integral first Chern class of a divisor: $\epsilon(\sum_i a_i A_i) := c_1(\sum_i a_i A_i) \pmod 2$. Let U be the image of ϵ.

Since $A_i \cdot A_j = -2\delta_{ij}$, $A_i \cdot H = 0$, $H^2 = d$, it follows that U is an isotropic subspace of $H^2(\tilde{F}, \mathbb{Z}/2)$, and since the intersection product modulo 2 is non-degenerate its dimension does not exceed $b_2(\tilde{F})/2$.

In the case where the surface has even degree $d \equiv 0 \pmod 2$, we consider more generally $\tilde{X} := X \oplus (\mathbb{Z}/2)H$, $\tilde{\epsilon} : \tilde{X} \to H^2(\tilde{F}, \mathbb{Z}/2)$, and the corresponding isotropic subspace $\tilde{U} := \mathrm{Im}(\tilde{\epsilon})$.

Definition 1.1. 1) *The* strict code K *associated to the nodal set* $\{P_1, \dots, P_\mu\}$ *on the surface* F *is the binary code* $K := \ker(\epsilon)$.
2) *If* $d \equiv 0 \pmod 2$ *the* enlarged code $\tilde{K}$ *associated to the nodal set* $\{P_1, \dots, P_\mu\}$ *on the surface* F *is the binary code* $\tilde{K} := \ker(\tilde{\epsilon})$.

By the above inequality for $\dim U$, we get

$$\dim K \geq \mu - \frac{1}{2} b_2(\tilde{F}) = \mu - \frac{1}{2} d(d^2 - 4d + 6) + 1, \qquad \dim \tilde{K} \geq \mu - \frac{1}{2} d(d^2 - 4d + 6) + 2.$$

Remark 1.2. By Miyaoka's inequality (cf. [Miy]) $\mu \leq \frac{4}{9} d(d-1)^2$, therefore only for $d \leq 17$ we get for sure a nontrivial code K, because $\dim K \geq \mu - \frac{1}{2} d(d^2 - 4d + 6) + 1$.

Notice that the notion of an even, respectively half-even, set of nodes can be derived from the coding-theory framework.

Definition 1.3. *A vector* $v \in X$ *is completely determined by its* support $N_v := \{i \mid v_i = 1\}$. *The cardinality of the support is called the* weight *of* v *and denoted by* $w(v) := \# N_v$.

By the universal coefficients theorem and Lefschetz' $(1,1)$ *theorem the condition* $v \in K$ *is equivalent to the* 2*-divisibility of* $\sum_{i \in N_v} A_i$ *in* $\mathrm{Pic}(\tilde{F})$. *We denote by* L *a divisor on* $\tilde{F}$ *such that* $2L \equiv \sum_{i \in N_v} A_i$. *The class of* L *in* $\mathrm{Pic}(\tilde{F})$ *is uniquely determined, because* $\mathrm{Pic}(\tilde{F})$ *has no torsion. We then have a finite double cover* $\tilde{S}$ *of* $\tilde{F}$ *branched exactly on the nodal curves* A_i *such that* $i \in N_v$, *and moreover* $f_* \mathcal{O}_{\tilde{S}} = \mathcal{O}_{\tilde{F}} \oplus \mathcal{O}_{\tilde{F}}(-L)$. *Correspondingly, we have a double cover* $f : S \to F$, *with* $f_* \mathcal{O}_S = \mathcal{O}_F \oplus \mathcal{F}$, *ramified exactly in* $\Delta := \{P_i \in F \mid i \in N_v\}$ *(cf.* [Cat1] *and* [Ca-Ca] *for more details). These sets* Δ *are called* even sets of nodes *(cf.* [Cat1]*).*

Similarly, one defines a half-even set of nodes Δ *by the condition that its associated word* $\tilde{v} := (v_1, \dots, v_\mu, 1)$, *obtained by setting* $v_i = 1 \Leftrightarrow P_i \in \Delta$, *belongs to the enlarged code* $\tilde{K}$. *This condition is again equivalent to the existence of a divisor* L *in* $\mathrm{Pic}(\tilde{F})$ *with* $2L \equiv \sum_{i \in N_v} A_i + H$.

We define the weight *and* support *of* $\tilde{v}$ *as the weight and support of the word* $v := (v_1, \dots, v_\mu) \in X$ *(these notions are different from the corresponding ones in coding theory). Observe finally that* $K = \tilde{K} \cap \{\tilde{v} \mid \tilde{v}_{\mu+1} = 0\}$.

As shown in [Cat1, Prop. 2.11 and Prop. 2.13], the geometric interpretation of even sets of nodes in terms of double coverings allows one to give the following restrictions on the cardinality t of an even (resp. half-even) set of nodes:

Proposition 1.4. *Let* $t := w(u)$ *be the weight of a code word* u.

(1) *If* $u \in K$, *then* $t \equiv 0 \pmod 4$. *Moreover, if* d *is even, then* $t \equiv 0 \pmod 8$.
(2) *If (*d *is even and)* $u \in \tilde{K} \setminus K$, *then* $t \equiv d(2d-7)/2 \pmod 4$. *In particular, for* $d = 6$, $t \equiv -1 \pmod 4$.

Corollary 1.5. *Let* $d = 2(2k+1)$ *be twice an odd integer and assume that* $K, \tilde{K}$ *are the codes corresponding to an even set of nodes* Δ. *Then* $\tilde{K} = K$.

Proof. Our assumption is that the code $K \subset \tilde{K}$ contains the vector $\mathbb{I}$ whose coordinates are all equal to 1, except the last which equals 0. If we have a vector $w \in \tilde{K} \setminus K$ and let t be its weight, then the weight of $\mathbb{I} + w \in \tilde{K} \setminus K$ is congruent to $-t$ modulo 4. Since $t \equiv d(2d-7)/2 \equiv 2k+1 \pmod 4$, we have $-t \not\equiv d(2d-7)/2 \pmod 4$, contradicting (2) of the previous proposition. □

Let us examine by means of coding theory which even sets of nodes can occur on sextic nodal surfaces. The main result of this section is the following theorem.

Theorem 1.6. *On a sextic normal surface F with only rational double points as singularities there does not exist an even set of nodes of cardinality $t = 64$.*

In order to prove the theorem we first prove some preliminary results.

Lemma 1.7. *Suppose that there exists an even set Δ of nodes of cardinality $t = 64$ on a normal sextic surface F. Let $\gamma : F \dashrightarrow F^\vee \subset \mathbb{P}^3$ be the Gauss map of F, given by the partial derivatives $\partial F/\partial x_i$.*

(1) *γ corresponds to a linear subsystem $\mathcal{L}$ of $|5H - \sum_{i=1}^t A_i|$ on $\tilde{F}$ whose fixed part Φ is contained in the preimage of the singular points of F which are not the nodes of Δ.*
(2) *Let L be a divisor on $\tilde{F}$ with $2L \equiv \sum_{i=1}^t A_i$. Then $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(2H - L)) = 0$.*

Proof. The first assertion follows since the zero locus of the partial derivatives $\partial F/\partial x_i$ on F is exactly the singular locus of F, and at each node the derivatives $\partial F/\partial x_i$ define the maximal ideal. Thus γ is a morphism around each (-2)-curve A_i, which is indeed embedded as a plane conic.

Assertion (2) is proven by contradiction. Assume in fact that $C \in |2H - L|$. Then $C \cdot (5H - \sum_{i=1}^t A_i) = 60 - 64 = -4$.

However, $\Phi \cdot H = \Phi \cdot A_i = 0$ by our first assertion, whence $C \cdot (5H - \sum_{i=1}^t A_i)$ equals the intersection number of C with the movable part of the linear system $\mathcal{L}$, which is obviously non-negative.

We have obtained the desired contradiction. □

Proposition 1.8. *Suppose there exists an even set of nodes of cardinality 64 on a sextic normal surface with only rational double points as singularities, and let $f : S \to F$ be the corresponding finite double cover. Then $h^1(S, \mathcal{O}_S) = 5$.*

Proof. We have

$$h^1(S, \mathcal{O}_S) = h^1(F, \mathcal{F}) = h^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(-L)).$$

Moreover, since $H \cdot L = 0$ one easily sees that $h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(-L)) = 0$ and argues then that $h^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(-L)) = h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(2H + L)) = h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(2H - L)) = 0$ by the previous lemma, the second equality following from the fact that every divisor in $|2H + L|$ contains $\sum_{i=1}^t A_i$. Hence

$$\begin{aligned} -h^1(S, \mathcal{O}_S) &= -h^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(-L)) = \chi(\tilde{F}, \mathcal{O}_{\tilde{F}}(-L)) \\ &= \chi(\mathcal{O}_{\tilde{F}}) + \tfrac{1}{2}(-L) \cdot (-L - 2H) = 11 - 16 = -5. \end{aligned}$$ □

Lemma 1.9. *Suppose there exists an even set of nodes Δ of cardinality* 64 *on a sextic normal surface F with only rational double points as singularities. Let K, $\tilde{K}$ denote the corresponding binary codes. Then* $\dim \tilde{K} = 12 > \dim K = 11$.

Proof. By the previous proposition the surface S, the finite double cover of F ramified exactly along Δ, has invariants $p_g = 10$, $q = 5$, $K_S^2 = 48$, $\chi(S) = 6$. The corresponding (non-minimal) smooth surface $\tilde{S}$, the double cover of $\tilde{F}$, has the same invariants as S.

By [Bea, Lemma 2] or [Ja-Ru, Thm. 4.5] it follows that the code K has dimension $b_1(\tilde{S}) + 1 = 11$. We already remarked that $\dim \tilde{K} \geq 65 - 53 = 12$, and it is obvious that $K \subset \tilde{K}$ has codimension at most 1. □

Proof of Theorem 1.6. The conclusion of Lemma 1.9 contradicts Corollary 1.5. □

As an immediate consequence of Thm. 1.6 and [Ja-Ru, Sec. 7], we obtain the following.

Corollary 1.10. *Let F be a sextic normal surface in $\mathbb{P}^3$ with only rational double points as singularities with an even set of t nodes. Then* $t \in \{24, 32, 40, 56\}$.

Proof. Since $t \equiv 0 \pmod 8$, the inequality $t \leq 64$ follows from the classical inequalities of Bassett and of Miyaoka, and the case $t = 64$ has just been excluded. In [Ca-Ca] it is shown that $t \geq 24$, and that the cases $t = 24, 32, 40$ do exist.

The non-existence of even sets of 48 nodes on nodal sextics is proven in [Ja-Ru, Sec. 7]. □

2. Cohomology modules and bundle symmetric maps

In this section, after recalling the main result of [Ca-Ca], namely the correspondence between even sets of nodes and bundle symmetric maps, we shall give bounds for the cohomology groups $H^i(\mathcal{F}(j))$ of the quadratic sheaf $\mathcal{F}$ associated to an even set of nodes Δ.

We first recall the main result of [Ca-Ca], according to the following notation: $\delta \in \{0, 1\}$ and $\delta/2$-even stands for even if $\delta = 0$, and half-even if $\delta = 1$.

Theorem 2.1 ([Ca-Ca, Thm. 0.3]). *Let Δ be a $\delta/2$-even set of nodes on a normal surface F of degree d, let $f : S \to F$ denote a corresponding double cover of F, and let $\mathcal{F}$ be the anti-invariant part of $f_*\mathcal{O}_S$. Then there exists a locally free sheaf $\mathcal{E}$ on $\mathbb{P}^3$ and a symmetric map φ yielding an exact sequence*

$$0 \to \mathcal{E}^\vee(-d - \delta) \xrightarrow{\varphi} \mathcal{E} \to \mathcal{F} \to 0. \qquad (**)$$

In particular, $F = \{x \mid \det(\varphi(x)) = 0\}$, $\Delta = \{x \mid \operatorname{corank}(\varphi(x)) \geq 2\}$.

Conversely, assume that one is given an exact sequence as in $(**)$ with φ symmetric, such that $F = \{x \mid \det(\varphi(x)) = 0\}$ is a normal surface and $\Delta := \{x \mid \operatorname{corank}(\varphi(x)) \geq 2\}$ is a reduced set of t points. Then Δ is a $\delta/2$-even set of nodes on F.

The ideal of the reduced subscheme Δ is the second Fitting ideal of φ, i.e., on local trivializing affine sets for $\mathcal{E}$, it is given by the determinants of the $(\mathrm{rk}(\mathcal{E}) - 1)$-minors of φ.

We briefly explain how the sheaf $\mathcal{E}$ is explicitly constructed in [Ca-Ca] by means of C. Walter's interpretation of Horrocks' correspondence (cf. [Wal]).

Assume that the intermediate cohomology module $H^1_*(F, \mathcal{F}) := \bigoplus_{i\in\mathbb{Z}} H^1(F, \mathcal{F}(i))$ is known (it is an Artinian graded module over the polynomial ring of $\mathbb{P}^3$, $\mathcal{A} := \mathbb{C}[x_0, x_1, x_2, x_3]$).

One then considers the (Artinian) graded module

$$M := U \oplus \bigoplus_{i>(d-4)/2} H^1(F, \mathcal{F}(i)). \tag{2.1}$$

where, if d is even, U is a Lagrangian subspace in the Serre self-dual cohomology space $H^1(F, \mathcal{F}((d-4)/2))$, and $U := 0$ if d is odd.

Remark 2.2. Recall that the first syzygy bundle $\mathrm{Syz}^1(M)$ is obtained from a projective graded resolution of the module M by free $\mathcal{A}$-modules

$$0 \to P^4 \to \cdots \to P^1 \xrightarrow{\alpha_1} P^0 \xrightarrow{\alpha_0} M \to 0$$

as follows: the homomorphism $\alpha_1 : P^1 \to P^0$ induces a corresponding homomorphism $(\alpha_1)^\sim$ between the (Serre-) associated sheaves $(P^1)^\sim$ and $(P^0)^\sim$ and the first syzygy bundle of M is defined as $\mathrm{Syz}^1(M) := \ker(\alpha_1^\sim)$.

One has a natural homomorphism $\mathrm{Syz}^1(M) \to \mathcal{F}$ (cf. [Ca-Ca, pp. 240–241]) induced by truncation, whence one gets a homomorphism $H^0_*(\mathrm{Syz}^1(M)) \to H^0_*(\mathcal{F})$, which need not be surjective.

The bundle $\mathcal{E}$ is then defined as the direct sum of $\mathrm{Syz}^1(M)$ and a direct sum of line bundles, whose generators induce a minimal set of generators of the cokernel. One obtains in this way a surjection between $H^0_*(\mathbb{P}^3, \mathcal{E})$ and $H^0_*(F, \mathcal{F})$.

Thus a first important step is to determine the intermediate (Artinian) cohomology module $H^1_*(F, \mathcal{F}) := \bigoplus_{i\in\mathbb{Z}} H^1(F, \mathcal{F}(i))$, in particular one has to determine the possible dimensions of its graded pieces, i.e., the numbers $h^1(\mathcal{F}(i))$. Later on, when we want to impose the surjectivity of $H^0_*(\mathbb{P}^3, \mathcal{E}) \to H^0_*(F, \mathcal{F})$, it will also be important to determine the dimensions $h^0(\mathcal{F}(i))$.

In short, the first necessary task is to determine the possible values for the cohomology table $h^j(\mathcal{F}(i))$ of $\mathcal{F}$ (a priori only $\chi(\mathcal{F}(i))$ is known, and it is determined by the degree d of F and the number t of nodes of Δ).

Besides geometrical estimates, an important tool used in [Ca-Ca] is the Beilinson complex (cf. [Bei]) constructed from the cohomology table $h^j(\mathcal{F}(i))$.

Remark 2.3. It is well known that for any coherent sheaf $\mathcal{G}$ on $\mathbb{P}^n$ the complex $\mathcal{K}^i := \bigoplus_j (H^{i-j}(\mathbb{P}^n, \mathcal{G}(j)) \otimes \Omega^{-j}(-j))$, called *Beilinson's monad*, has cohomology $H^i(\mathcal{K}^*)$ equal to $\mathcal{G}$ in degree $i = 0$ and 0 in all other degrees (cf. [Bei]).

In the case of even sets of nodes on sextics, [Ca-Ca] classifies the sets of cardinalities $t = 24, 32, 40$.

Let us therefore restrict ourselves to the case $d = 6$ and $t = 56$, and consider the module $M := U \oplus \bigoplus_{i>1} H^1(F, \mathcal{F}(i))$.

We shall use the analysis in [Ca-Ca, p. 254] for $\mathcal{F}|_H$, where H is a smooth plane section of F; it shows that $h^0(F, \mathcal{F}(1)) = h^2(F, \mathcal{F}(1)) = 0$ (it is shown there that this holds unless $\mathcal{F}|_H$ is of type $(2, 4)$, but if $\mathcal{F}|_H$ is of type $(2, 4)$ then so is F, in the sense of [Cat1, Thm. 2.2 and Thm. 2.16], and $t = 24$).

Hence we may assume that $h^0(F, \mathcal{F}(1)) = h^2(F, \mathcal{F}(1)) = 0$, so that by Riemann–Roch applied to $\tilde{F}$,

$$h^1(F, \mathcal{F}(1)) = -\chi(\mathcal{F}(1)) = -8 + \frac{t}{4} = 6.$$

According to the notation in [Ca-Ca, p. 254], set $2\tau := h^1(F, \mathcal{F}(1))$, $a := h^1(F, \mathcal{F}) = h^1(F, \mathcal{F}(2))$, $b := h^2(F, \mathcal{F}) = h^0(F, \mathcal{F}(2))$, the equalities following by Serre duality. Our previous calculation yields $\tau = 3$.

The exact sequence

$$\begin{aligned} &H^0(F, \mathcal{F}(1)) = 0 \to H^0(H, \mathcal{F}(1)|_H) \to H^1(F, \mathcal{F}) \to H^1(F, \mathcal{F}(1)) \\ &\quad \to H^1(H, \mathcal{F}(1)|_H) \to H^2(F, \mathcal{F}) \to H^2(F, \mathcal{F}(1)) \cong H^0(F, \mathcal{F}(1)) = 0 \end{aligned} \tag{2.2}$$

gives the relation $\chi(\mathcal{F}(1)|_H) - a + 2\tau + b = 0$. An application of Riemann–Roch on H yields $\chi(\mathcal{F}(1)|_H) = -3$ and the above relation becomes $0 \leq b = a - 3$.

Finally, notice that (see [Ca-Ca, formula (3.2), p. 248])

$$H^1(F, \mathcal{F}(-m)) \cong H^1(F, \mathcal{F}(m+2))^\vee = 0, \qquad m > 0,$$

and trivially also $H^0(F, \mathcal{F}(-m)) = 0,\ m > 0$.

Since the rank of $\mathcal{F}$ at the generic point of $\mathbb{P}^3$ is 0, a computation of the ranks of all the terms of Beilinson's monad of $\mathcal{F}(3)$ yields the relation $4b + 6\tau - 4a - c = 0$, i.e. $c = 12 - 2\tau = 6$.

Proposition 2.4. *Let F be a nodal surface of degree 6 with $\bar{t}$ nodes and with an even set Δ of $t = 56$ nodes, and $L \in \mathrm{Pic}(\tilde{F})$ the corresponding divisor such that $\sum_{i \in N_\Delta} A_i = 2L$. Then $b = h^0(2H - L) \leq 1$.*

Proof. Assume $b := h^0(F, \mathcal{F}(2)) = h^0(2H - L) \geq 2$ and write $|2H - L| = |M| + \Psi$, where Ψ is the fixed part of the linear system $|2H - L|$. Let $\mathcal{L}$ be the Gaussian linear subsystem of $5H - \sum_{i=1}^{\bar{t}} A_i$ and C any effective divisor in $|2H - L|$.

Since F is nodal it follows by our previous argument that $\mathcal{L}$ is free from base points, hence for any effective divisor $C' \leq C$ we have $C'\mathcal{L} \leq C\mathcal{L} = 60 - t = 4$.

Observe that by [Ja-Ru] the number $\bar{t}$ of nodes of F satisfies $\bar{t} \leq 65$; since $\mathcal{L}^2 \geq 150 - 2\bar{t} \geq 20$, the index theorem ensures that $(C')^2 \leq 0$. In particular, it follows that $M^2 = 0$.

Since the dual surface $F^\vee$ is birational to F and therefore it is still of general type, $\gamma(M)$ has degree at least 4.

Hence, by the previous calculation, $M\mathcal{L} = 4$ and $\Psi\mathcal{L} = 0$, the general curve in $|M|$ is irreducible (hence smooth) and maps $1:1$ to a quartic.

Since the arithmetic genus of M is at least 2, it follows that M maps $1:1$ to a plane quartic, and therefore its arithmetic genus is at most 3.

But then $4 \geq 2p_a(M) - 2 = M \cdot (M + K_F) = M \cdot K_F = 2M \cdot H$, i.e. $M \cdot H \leq 2$, a contradiction since F is of general type. $\square$

We can summarize the above discussion in the following statement.

Theorem 2.5. *Let F be a nodal surface of degree* 6 *with an even set of* 56 *nodes. Then M is an Artinian module of length* 2 *with Hilbert function* $(\tau, a) = (3, 3)$ *or* $(3, 4)$.

Proof. By hypothesis, $\tau = 3$. Proposition 2.4 yields $b = h^1(F, \mathcal{F}(2)) \leq 1$. Therefore only the following two cases are possible: $b = 0$, $a = 3$ or $b = 1$, $a = 4$. $\square$

In what follows we shall treat only the first case, for the second one we observe

Remark 2.6. Case $(3, 4)$ of Theorem 2.5 cannot be excluded by coding theory since there exists a 9-dimensional code $K \subset (\mathbb{Z}/2\mathbb{Z})^{56}$ with weights $(24, 32, 56)$.

Question. Does case $(3, 4)$ of Theorem 2.5 occur?

Proof. This code is constructed as follows: consider a code $U \subset (\mathbb{Z}/2\mathbb{Z})^{51}$ of dimension 8 and weights (24, 32), and let $e \in (\mathbb{Z}/2\mathbb{Z})^{56}$ be the vector with all coordinates equal to 1. It suffices to define K as the span of U and e.

The existence of U (cf. [McW-Sl, p. 229]) is easily established if we let $\mathbb{F}$ be the finite field with 2^8 elements, and ξ a generator of $\mathbb{F}^*$. Then ξ^5 is a primitive 51-st root of unity, it generates $\mathbb{F}$ as a field, thus $\mathbb{F} \cong (\mathbb{Z}/2\mathbb{Z})[\xi^5]/(P)$, where P is an irreducible polynomial of degree 8 dividing $x^{51} - 1$. By the Chinese remainder theorem $\mathbb{F} \cong (\mathbb{Z}/2\mathbb{Z})[\xi^5]/(P)$ is a direct summand of $(\mathbb{Z}/2\mathbb{Z})[x]/(x^{51} - 1) \cong (\mathbb{Z}/2\mathbb{Z})^{51}$, and it suffices to let U be the subspace of $(\mathbb{Z}/2\mathbb{Z})^{51}$ which corresponds to $\mathbb{F}$. $\square$

3. Hilbert function $(3, 3)$: general features

We shall assume, throughout the rest of the paper, that we have an even set Δ of 56 nodes, and that $b = 0$, i.e., $a = 3$ (cf. Theorem 2.5) . In other terms, the Artinian module M has dimension $\tau = 3$ in degree 1, dimension $a = 3$ in degree 2, and 0 in degree $\neq 1, 2$.

Therefore, the Beilinson table of $\mathcal{F}(3)$ is:

i

6	0	0	0	0
0	3	6	3	0
0	0	0	0	6

j

By Theorem 2.1, if we denote by $\mathcal{E}$ the previous $\mathcal{E}(3)$, we have a resolution of $\mathcal{F}(3)$ of the form

$$0 \to \mathcal{E}^\vee \xrightarrow{\varphi} \mathcal{E} \to \mathcal{F}(3) \to 0. \tag{3.1}$$

In this setting, the symmetric map φ appearing in the resolution of $\mathcal{F}(3)$ belongs to $H^0(\mathbb{P}^3, S^2(\mathcal{E})) \subset \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{E})$.

Definition 3.1. *Throughout the rest of the paper we denote by U a given Lagrangian 3-dimensional subspace of $H^1(\mathcal{F}(1))$, and we denote by W the 3-dimensional space $W := H^1(\mathcal{F}(2))$.*

Moreover, we shall denote by V the 4-dimensional vector space $V := H^0(\mathcal{O}(1))$. Later on, more generally, V shall denote a 4-dimensional vector space and we shall often continue to denote $\mathrm{Proj}(V)$ by $\mathbb{P}^3$.

Remark 3.2. Beilinson's theorem and the cohomology table for $\mathcal{F}(2)$ imply that $\mathcal{E}(-1)$ is obtained by adding a direct sum of line bundles to

$$\mathcal{E}'(-1) := \ker(U \otimes \Omega^1(1) \cong 3\Omega^1(1) \to W \otimes \mathcal{O} \cong 3\mathcal{O}),$$

and that (since Beilinson's complex has no cohomology in degree $\neq 0$) the above map is surjective; hence $\mathcal{E}'$ is a vector bundle with $\mathrm{rk}(\mathcal{E}') = 6$.

Consider now the Euler sequence

$$0 \to \Omega^1(1) \to V \otimes \mathcal{O} \cong 4\mathcal{O} \to \mathcal{O}(1) \to 0. \tag{3.2}$$

It implies that $h^0(\Omega^1(1)) = 0$ and $h^0(\Omega^1(2)) = 6$, thus $h^0(\mathcal{E}'(-1)) = 0$ and, since Beilinson's table for $\mathcal{F}(3)$ implies that $h^1(\mathcal{E}') = 0$, we infer that $h^0(\mathcal{E}') = 3 \times 6 - 4 \times 3 = 6$.

On the other hand, Beilinson's complex for $\mathcal{F}(3)$ yields an exact sequence

$$0 \to 3\mathcal{O}(-4) \to 6\Omega^2(2) \to 3\Omega^1(1) \oplus 6\mathcal{O} \to \mathcal{F}(3) \to 0,$$

and we make the following simplifying

First assumption. *$\mathcal{F}$ is generated in degree 3 and the linear map $H^0(\mathcal{E}') \to H^0(\mathcal{F}(3))$ is an isomorphism.*

Proposition 3.3. *According to the previous notation, the above first assumption implies that $\mathcal{E} = \mathcal{E}'$, equivalently, that $\mathrm{rk}(\mathcal{E}) = 6$. More precisely, it means that there exists a homomorphism $\beta : U \otimes \Omega^1(2) \cong 3\Omega^1(2) \to W \otimes \mathcal{O}(1) \cong 3\mathcal{O}(1)$ such that $\mathcal{E} = \ker \beta$ and that we have an exact sequence*

$$0 \to \mathcal{E} \to U \otimes \Omega^1(2) \cong 3\Omega^1(2) \xrightarrow{\beta} W \otimes \mathcal{O}(1) \cong 3\mathcal{O}(1) \to 0. \tag{3.3}$$

Conversely, if $\mathcal{E}$ is obtained in this way, it is a rank 6 bundle with an intermediate cohomology module M with the required Hilbert function of type $(3, 3)$. Moreover $H^0(\mathcal{E}^\vee) = 0$.

Proof. If $\mathcal{F}$ is generated in degree 3, there is an exact sequence

$$0 \to \tilde{G} \to 6\mathcal{O} \to \mathcal{F}(3) \to 0,$$

where $h^0(\tilde{G}) = h^1(\tilde{G}) = 0$ (cf. Beilinson's table).

Dualizing the sequence $0 \to \mathcal{E}' \to 3\Omega^1(2) \xrightarrow{\beta} 3\mathcal{O}(1) \to 0$ yields

$$0 \to 3\mathcal{O}(-1) \to 3T(-2) \to \mathcal{E}'^{\vee} \to 0.$$

Thus $h^0(\mathcal{E}'^{\vee}) = 0$.

Assume now that $H^0_*(\mathcal{E}') \to H^0_*(\mathcal{F}(3))$ is not surjective. Then, since by our assumption $H^0(\mathcal{E}') \to H^0(\mathcal{F}(3))$ is surjective, $\mathcal{E}$ will be obtained from $\mathcal{E}'$ by adding a direct sum of line bundles $\mathcal{O}(-m)$ where $m > 0$. This, however, leads to a contradiction, since then $\mathcal{O}(m)$ is a direct summand of $\mathcal{E}^{\vee}$ but it cannot embed in $\mathcal{E}$ since $H^0(\mathcal{E}'(-1)) = 0$ implies that $\mathrm{Hom}(\mathcal{O}(m), \mathcal{E}) = H^0(\mathcal{E}(-m)) = H^0(\mathcal{E}'(-m)) = 0$.

For the converse, we simply observe that if (3.3) holds, then $H^1(\mathcal{E}(-2)) \cong 3H^1(\Omega^1)$ and $H^1(\mathcal{E}(-1)) \cong 3H^0(\mathcal{O})$. Since $\mathcal{E}' \cong \mathcal{E}$ it follows right away that $H^0(\mathcal{E}^{\vee}) = 0$. □

Therefore we get the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \mathcal{E}^{\vee} \arrow[u] \arrow[r, "\varphi"] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F}(3) \arrow[r] & 0 \\
 & U^{\vee} \otimes T(-2) \arrow[u] \arrow[r, "\Phi"] & U \otimes \Omega^1(2) \arrow[d, "\beta"] & & \\
 & W^{\vee} \otimes \mathcal{O}(-1) \arrow[u, "{}^t\beta"] & W \otimes \mathcal{O}(1) \arrow[d] & & \\
 & 0 \arrow[u] & 0 & &
\end{tikzcd}$$

and the map φ yields, by composition, a homomorphism

$$\Phi \in \mathrm{Hom}(U^{\vee} \otimes T(-2), U \otimes \Omega^1(2))$$

which is symmetric since φ is symmetric. Conversely, such a homomorphism Φ determines φ if and only if $\beta\Phi = \Phi\, {}^t\beta = 0$; since, however, we choose Φ symmetric, the two conditions are equivalent to each other.

A more concrete way to set up the parameter space for such vector bundles is to replace $\mathrm{Hom}(T(-2), \Omega^1(2))$ by matrices of polynomials, as follows.

Recall $V := H^0(\mathbb{P}^3, \mathcal{O}(1))$ is the space of linear forms on $\mathbb{P}^3$. Applying $\mathrm{Hom}(-, \mathcal{O})$ to the Euler sequence and tensoring by $\mathcal{O}(1)$ yields, since $\mathrm{Hom}(\mathcal{O}(2), \mathcal{O}(1)) = 0$ and $\mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O}(1)) = 0$,

$$\mathrm{Hom}(\Omega^1(2), \mathcal{O}(1)) \cong \mathrm{Hom}(V \otimes \mathcal{O}(1), \mathcal{O}(1)). \tag{3.4}$$

Thus the map β factors through a map $B : U \otimes (V \otimes \mathcal{O}(1)) \to W \otimes \mathcal{O}(1)$ and the sheaf map B is surjective. This surjectivity is obviously equivalent to $H^0(B(-1)) : U \otimes V \to W$ being surjective. We shall often identify the sheaf map B with the corresponding tensor $H^0(B(-1)) \in U^\vee \otimes V^\vee \otimes W$.

Let ϵ be the tensor product of the identity map of the isotropic subspace U with the evaluation map $V \otimes \mathcal{O} \to \mathcal{O}(1)$. Then one sees easily that $\mathcal{E} = \ker(\beta) = \ker(B) \cap \ker(\epsilon)$, the short exact sequence (3.3) becomes

$$0 \to \mathcal{E} \to U \otimes V \otimes \mathcal{O}(1) \xrightarrow{(B,\epsilon)} (W \otimes \mathcal{O}(1)) \oplus (U \otimes \mathcal{O}(2)) \to 0, \tag{3.5}$$

and the previous diagram is replaced by

$$\begin{tikzcd}
 & 0 & 0 & \\
0 \arrow[r] & \mathcal{E}^\vee \arrow[u] \arrow[r, "\varphi"] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F}(3) \to 0 \\
 & U^\vee \otimes V^\vee \otimes \mathcal{O}(-1)) \arrow[u] \arrow[r, "A"] & U \otimes V \otimes \mathcal{O}(1) \arrow[d, "(B,\epsilon)"] & \\
 & \begin{matrix} W^\vee \otimes \mathcal{O}(-1) \\ \oplus \\ U^\vee \otimes \mathcal{O}(-2) \end{matrix} \arrow[u, "{}^t(B,\epsilon)"'] & \begin{matrix} W \otimes \mathcal{O}(1) \\ \oplus \\ U \otimes \mathcal{O}(2) \end{matrix} \arrow[d] & \\
 & 0 \arrow[u] & 0 &
\end{tikzcd}$$

where by a similar token to the one before, the map φ yields a symmetric matrix $A \in \mathrm{Mat}(12 \times 12, \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}(1)))$, and conversely such a matrix determines φ if and only if $(B, \epsilon) \cdot A = 0$.

Thus we obtain, as a parameter space for the symmetric resolutions of $\mathcal{F}(3)$ satisfying the open condition given by the first assumption, the variety of pairs

$$\begin{aligned} \mathfrak{M}_{AB} := \{(B, A) \mid {} & B \in \mathrm{Mat}(3 \times 12, \mathbb{C}), \\ & A \in \mathrm{Mat}(12 \times 12, H^0(\mathcal{O}_{\mathbb{P}^3}(2))),\ A = {}^tA,\ (B, \epsilon) \cdot A = 0\}. \end{aligned} \tag{3.6}$$

As a matter of fact, the equation of the surface F will then be given as the G.C.D. of the determinants of the 6×6 minors of the matrix A, whereas the even set of nodes Δ will be found to be given by the ideal of the determinants of the 5×5 minors of the matrix A.

A direct but complicated calculation shows that for a general choice of the parameter B (determining the bundle $\mathcal{E}$) the solution space for the A's (yielding the symmetric map φ) has positive dimension.

However, by computer algebra, one checks that a random choice of B and a random choice of A do not give a sextic surface with 56 nodes, but the square of a cubic surface. Notice that the condition that a given pair (B, A) yields a sextic with 56 nodes is an open condition, and therefore there can exist sextics with an even set of 56 nodes and satisfying the first assumption only if the above parameter space is reducible; we shall later show that this is indeed the case.

We finish this section by remarking that B is the multiplication matrix of the module $H^1_*(\mathcal{E})$ (i.e., the matrix of the only part of the multiplication map which is not a priori trivial).

Remark 3.4. The cohomology exact sequence associated to the following twist of (3.5):

$$0 \to \mathcal{E}(-2) \to U \otimes V \otimes \mathcal{O}(-1) \xrightarrow{(B,\epsilon)} (W \otimes \mathcal{O}(-1)) \oplus (U \otimes \mathcal{O}) \to 0$$

yields a canonical isomorphism $U \cong H^1(\mathcal{E}(-2))$.

Since there is a canonical isomorphism

$$H^0(\epsilon(-1)) : U \otimes V \to U \otimes H^0(\mathcal{O}(1)),$$

the projection $W \oplus (U \otimes V) \to W$ induces an isomorphism of the space $H^1(\mathcal{E}(-1)) =$ coker $H^0((B \oplus \epsilon)(-1))$ with W such that the map $B : U \otimes V \to W$ corresponds to the multiplication map of the cohomology module $H^1_*(\mathcal{E})$.

Remark 3.5. The condition that the linear map B has maximal rank 3 (which, as we observed, follows from the first assumption) is obviously equivalent to the condition that the module M is generated in degree -2. On the other hand, it also implies that there is an exact sequence

$$0 \to \mathcal{E} \to 9\mathcal{O}(1) \xrightarrow{b} 3\mathcal{O}(2) \to 0.$$

We proceed in the next section with the analysis of the vector bundles corresponding to a general choice of B, giving a geometrical explanation of the phenomenon of which the computer search made us aware.

4. General bundles and cubic surfaces

The main purpose of this section is to describe the beautiful geometry which relates the main component of the moduli space of our vector bundles to a given intermediate cohomology module M and the space of cubic surfaces viewed as blow ups of the projective plane in six points.

Let us first observe that, if the first assumption is satisfied, the vector bundle $\mathcal{E}$ is determined by the matrix B, hence we have an irreducible parameter space for our vector bundles, and each open condition, if satisfied at some point, is satisfied by the generic bundle $\mathcal{E}$.

Next, we have a surjection $H^0(\mathcal{E}) \to H^0(\mathcal{F}(3))$ and we have seen that both spaces are 6-dimensional, whence we get a homomorphism $\iota : 6\mathcal{O} \to \mathcal{E}$. We make the

Second assumption. 1) *$\iota : 6\mathcal{O} \to \mathcal{E}$ is injective, whence there is an exact sequence*

$$0 \to 6\mathcal{O} \to \mathcal{E} \to \tau \to 0, \tag{4.1}$$

2) *the torsion sheaf τ is $\mathcal{O}_G$-invertible, where G is the divisor of $\Lambda^6(\iota)$.*

Lemma 4.1. *Let a vector bundle $\mathcal{E}$ be given as in* 3.3 *or as in* 3.5. *Then its total Chern class is*

$$c(\mathcal{E})(t) = 1 + 3t + 6t^2 + 4t^3. \tag{4.2}$$

In particular, if the second assumption is satisfied, the divisor G is a cubic surface.

Proof. The sheaf τ has Chern polynomial

$$\begin{aligned} c(\tau) = c(\mathcal{E}) &= c(\Omega^1(2))^3 c(\mathcal{O}(1))^{-3} = (c(\mathcal{O}(1))^4 c(\mathcal{O}(2))^{-1})^3 c(\mathcal{O}(1))^{-3} \\ &= (1+t)^9(1+2t)^{-3} = (1+9t+36t^2+84t^3)(1-6t+24t^2-80t^3) \\ &= 1+3t+6t^2+4t^3. \end{aligned}$$

□

Remark 4.2. Observe that the space $H^0(S^2\mathcal{E})$ of symmetric morphisms from $\mathcal{E}^\vee$ to $\mathcal{E}$ contains $H^0(S^2(6\mathcal{O}))$ since to $\tilde{\alpha} \in H^0(S^2(6\mathcal{O}))$ corresponds $\alpha := \iota^\vee \tilde{\alpha} \iota$. For these morphisms one has $\det(\alpha) = \det(\tilde{\alpha})(\det(\iota))^2$, whence in this case $\mathrm{div}(\det(\alpha)) = 2G$, and not a sextic surface.

The next lemmas are meant to investigate the question: when does one have equality $h^0(S^2\mathcal{E}) = 21$, i.e., when $H^0(S^2\mathcal{E}) = H^0(S^2(6\mathcal{O}))$?

In order to answer this question, it is convenient first to analyze the geometry and the cohomology of the invertible sheaf τ on G.

Remark 4.3. Even without assuming τ to be $\mathcal{O}_G$-invertible, set $\tau' = \mathrm{Ext}^1(\tau, \mathcal{O})$. Then the dual of the previous exact sequence (4.1) gives

$$0 \to \mathcal{E}^\vee \to 6\mathcal{O} \to \tau' \to 0 \tag{4.3}$$

and:

(1) By (4.1) we clearly have $H^0(\tau) = H^1(\tau) = H^2(\tau) = 0$.
(2) From (4.3) and $h^i(\mathcal{E}^\vee) \cong h^{3-i}(\mathcal{E}(-4))$ we get $h^0(\tau') = 6$, $H^1(\tau') = H^2(\tau') = 0$.
(3) Since by definition $\tau' = \mathrm{Ext}^1(\tau, \mathcal{O})$, applying the functor $\mathrm{Hom}(\tau, -)$ to the exact sequence $0 \to \mathcal{O} \to \mathcal{O}(3) \to \mathcal{O}_G(3) \to 0$ we get $\tau' = \mathrm{Hom}(\tau, \mathcal{O}_G(3))$. Therefore, if $\tau = \mathcal{O}_G(D)$, then $\tau' = \mathcal{O}_G(3H - D)$.

Since $h^i(D) = 0$ for all i, $h^0(3H - D) = 6$, $h^i(3H - D) = 0$ for $i = 1, 2$, it follows by Riemann–Roch that $D^2 + DH = -2$ and $10 = 36 - 7DH + D^2$. Therefore $HD = 3$, $D^2 = -5$. By setting $\Delta := D + H$, it turns out that $\Delta H = 6$, $\Delta^2 + \Delta K_G = -2$.

We give elsewhere the proof of the following lemma.

Lemma 4.4. *Assume that G is a smooth cubic surface. Then there exists a realization of G as the image of the plane under the system $|3L - \sum_{i=1}^{6} E_i|$ of plane cubics through six points such that either $\Delta \equiv 2L$, i.e., Δ corresponds to the conics in the plane, or (up to permutations of the six points) $\Delta \equiv 3L - 2E_1 - E_2$.*

Remark 4.5. The complete linear system Δ has as image in $\mathbb{P}^5$ either the Veronese embedding of $\mathbb{P}^2$, or the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ through $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 2))$. In both cases we have a surface of minimal degree (= 4).

Thus we have concluded that either $D = 2L - H = -L + \sum E_i$, or $D = 3L - 2E_1 - E_2 - H = -E_1 + \sum_{i=3}^{6} E_i$. We shall see later that the latter case does not occur (Lemma 4.12).

Corollary 4.6. $H^i(\mathcal{O}_G(2D)) = 0$ *for* $i = 0, 2$, *and* $h^1(\mathcal{O}_G(2D)) = 6$.

Proof. The second part follows from the first by Riemann–Roch; for the first it suffices to intersect with L (using Serre duality in the case of $H^2(\mathcal{O}_G(2D))$). ◻

We are now ready to show that the smoothness assumption for the cubic surface G implies that all symmetric morphisms from $\mathcal{E}^\vee$ to $\mathcal{E}$ factor through and are induced by symmetric morphisms from $6\mathcal{O}$ to $6\mathcal{O}$, whence their determinant is a double cubic, instead of a nodal sextic (cf. Rem. 4.2).

Lemma 4.7. *Let*

$$0 \to \mathcal{F} \xrightarrow{i} \mathcal{E} \to \tau \to 0$$

be a locally free resolution of a coherent torsion sheaf τ, which is $\mathcal{O}_G$-invertible on a divisor G. Then we have an exact sequence

$$0 \to \Lambda^2\mathcal{F} \to \mathcal{F} \otimes \mathcal{E} \to S^2\mathcal{E} \to \tau \otimes \tau \to 0 \tag{4.4}$$

and a monad

$$0 \to S^2\mathcal{F} \to \mathcal{F} \otimes \mathcal{E} \to \Lambda^2\mathcal{E} \to 0 \tag{4.5}$$

whose cohomology in the middle is exactly $\mathrm{Tor}^1(\tau, \tau)$.

Proof. Recall that locally, by our assumption, we can write

$$\mathcal{E} = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \cdots \oplus \mathcal{O}e_r, \qquad \mathcal{F} = \mathcal{O}xe_1 \oplus \mathcal{O}e_2 \oplus \cdots \oplus \mathcal{O}e_r,$$

where x is a local equation for G.

Since $\mathrm{Tor}^1(\mathcal{B}, \mathcal{B}') = 0$ if $\mathcal{B}$ is locally free, we obtain the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 & 0 & 0 & \\
0 \arrow[r] & \mathrm{Tor}^1(\tau,\tau) \arrow[r] & \mathcal{F}\otimes\tau \arrow[r] \arrow[u] & \mathcal{E}\otimes\tau \arrow[r] \arrow[u] & \tau\otimes\tau \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[r] & \mathcal{F}\otimes\mathcal{E} \arrow[r] \arrow[u] & \mathcal{E}\otimes\mathcal{E} \arrow[r] \arrow[u] & \tau\otimes\mathcal{E} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[r] & \mathcal{F}\otimes\mathcal{F} \arrow[r] \arrow[u] & \mathcal{E}\otimes\mathcal{F} \arrow[r] \arrow[u] & \tau\otimes\mathcal{F} \arrow[r] \arrow[u] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] & \mathrm{Tor}^1(\tau,\tau) \arrow[u] & \\
 & & & & 0 \arrow[u] &
\end{tikzcd}$$

Hence the composite map $\mathcal{E}\otimes\mathcal{E} \to \tau\otimes\mathcal{E} \to \tau\otimes\tau$ is surjective and has kernel generated by $(\mathcal{F}\otimes\mathcal{E}) \oplus (\mathcal{E}\otimes\mathcal{F})$, as the diagram shows. Let K denote the kernel of the map $(\mathcal{F}\otimes\mathcal{E}) \oplus (\mathcal{E}\otimes\mathcal{F}) \to \mathcal{E}\otimes\mathcal{E}$. Then K contains the image of $\mathcal{F}\otimes\mathcal{F}$ under the injective map $(\mathrm{id}\otimes i, -\mathrm{id}\otimes i)$, where $i : \mathcal{F} \to \mathcal{E}$ is the inclusion.

Therefore we get the complex

$$0 \to \mathcal{F}\otimes\mathcal{F} \to (\mathcal{F}\otimes\mathcal{E}) \oplus (\mathcal{E}\otimes\mathcal{F}) \to \mathcal{E}\otimes\mathcal{E} \to \tau\otimes\tau \to 0,$$

exact except possibly at $(\mathcal{F}\otimes\mathcal{E})\oplus(\mathcal{E}\otimes\mathcal{F})$, where the cohomology is equal to $K/(\mathcal{F}\otimes\mathcal{F})$.

Now let K_1 be the inverse image of $\mathrm{Tor}^1(\tau,\tau)$ in $\mathcal{F}\otimes\mathcal{E}$ via the short exact sequence $0 \to \mathcal{F}\otimes\mathcal{F} \to \mathcal{F}\otimes\mathcal{E} \to \mathcal{F}\otimes\tau \to 0$.

We claim that $K \cong K_1$. In fact, if $h_1 \oplus (-h_2) \in K$, the diagram shows that $h_1 \in K_1$ and moreover that h_2 is uniquely determined. Conversely, if $h_1 \in K_1$, then there is a (unique) element $h_2 \in \mathcal{E}\otimes\mathcal{F}$ with $h_1 = h_2 \in \mathcal{E}\otimes\mathcal{E}$.

This implies that $K/(\mathcal{F}\otimes\mathcal{F}) \cong \mathrm{Tor}^1(\tau,\tau)$.

Locally, K_1 is generated by $xe_1 \otimes e_1 \pmod{\mathcal{F}\otimes\mathcal{F}}$. Thus $K/\mathcal{F}\otimes\mathcal{F}$ is generated by $(xe_1\otimes e_1) \oplus (-e_1 \otimes xe_1) \in (\mathcal{F}\otimes\mathcal{E}) \oplus (\mathcal{E}\otimes\mathcal{F})$, which is an antisymmetric tensor, and we are done. $\square$

Corollary 4.8. *According to the previous notation, assume that G is a smooth cubic surface. Then $h^0(S^2\mathcal{E}) = 21$, $h^1(S^2\mathcal{E}) = 6$, $h^2(S^2\mathcal{E}) = h^3(S^2\mathcal{E}) = 0$. The same conclusion $h^0(S^2\mathcal{E}) = 21$ holds if more generally $\mathcal{E}$ satisfies the first and second assumption and $H^0(\tau^{\otimes 2}) = 0$.*

Proof. Split the long exact sequence (4.4) into

$$0 \to 15\mathcal{O} \to 6\mathcal{E} \to \mathcal{H} \to 0, \quad 0 \to \mathcal{H} \to S^2\mathcal{E} \to \tau^{\otimes 2} \to 0.$$

Recall from the construction of $\mathcal{E}$ that $h^0(\mathcal{E}) = 6$ and $h^i(\mathcal{E}) = 0$ for $i = 1, 2, 3$. The first corresponding long exact cohomology sequence yields $h^0(\mathcal{H}) = 36 - 15 = 21$ and $h^i(\mathcal{H}) = 0$ for $i = 1, 2, 3$.

Therefore it suffices to observe that if G is smooth, then by the previous corollary, one has $h^0(\tau^{\otimes 2}) = h^2(\tau^{\otimes 2}) = 0$, $h^1(\tau^{\otimes 2}) = 6$. The rest is straightforward. □

Recall now that the vector bundle $\mathcal{E}$, provided that the first assumption and the second assumption are satisfied, produces an invertible sheaf τ on a cubic surface G; conversely, given such a sheaf τ, one can construct $\mathcal{E}$ as an extension of $6\mathcal{O}$ and τ as in (4.1).

Setting as before $\tau' := \mathrm{Ext}^1(\tau, \mathcal{O})$, we see that such an extension is parametrized by $\mathrm{Ext}^1(\tau, 6\mathcal{O}) = H^0(6\,\mathrm{Ext}^1(\tau, \mathcal{O})) \cong \mathbb{C}^{36}$, if, as in Remark 4.3, we have $h^0(\tau') = 6$.

Lemma 4.9. *Let $\mathcal{E}$ be a vector bundle as in* (3.3) *with $h^0(\mathcal{E}) = 6$ and satisfying the second assumption. Then* $\mathrm{hom}(\mathcal{E}, \mathcal{E}) = 1$, *i.e., $\mathcal{E}$ is simple.*

Proof. We consider the exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E}, 6\mathcal{O}) \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}) \to \mathrm{Hom}(\mathcal{E}, \tau) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{O}).$$

We have $\mathrm{Ext}^1(\mathcal{E}, \mathcal{O}) \cong H^1(\mathcal{E}^\vee) \cong H^2(\mathcal{E}(-4))$ and from the exact sequence (3.3) we infer $H^2(\mathcal{E}(-4)) = 0$. Since $\mathrm{Hom}(\mathcal{E}, 6\mathcal{O}) = 0$ by Proposition 3.3, it follows that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \cong \mathrm{Hom}(\mathcal{E}, \tau)$.

We compute $\mathrm{hom}(\mathcal{E}, \tau)$ by considering the exact sequence

$$0 \to \mathrm{Hom}(\tau, \tau) \to \mathrm{Hom}(\mathcal{E}, \tau) \to \mathrm{Hom}(6\mathcal{O}, \tau).$$

Indeed, $\mathrm{hom}(\mathcal{O}, \tau) = h^0(\tau) = 0$ (since $h^0(\mathcal{E}) = 6$) and, since τ is $\mathcal{O}_G$-invertible, we have $\mathrm{hom}(\tau, \tau) = 1$. □

Lemma 4.10. *Assume that $h^0(\mathcal{E}^\vee) = 0$ (cf. the proof of Proposition* 3.3*), and that $\mathcal{E}$ is an extension as in* (4.1)*. Then the extension class in* $\mathrm{Ext}^1(\tau, 6\mathcal{O}) = H^0(6\,\mathrm{Ext}^1(\tau, \mathcal{O})) \cong \mathbb{C}^6 \otimes \mathbb{C}^6$ *is a rank* 6 *tensor (we shall refer to this statement by saying that* the extension does not partially split*). In particular, $\mathcal{E}$ is then uniquely determined up to isomorphism.*

Proof. The extensions which yield vector bundles form an open set.

We canonically view these extension classes as

$$\mathrm{Hom}(H^0(\tau'), H^0(6\mathcal{O})) = \mathrm{Hom}(H^0(\tau'), \mathbb{C}^6)$$

through the coboundary map of the corresponding exact sequence. We then have an action of $\mathrm{GL}(6, \mathbb{C})$ as a group of automorphisms of $6\mathcal{O}$, which induces an action on $\mathrm{Hom}(H^0(\tau'), H^0(6\mathcal{O})) = \mathrm{Hom}(H^0(\tau'), \mathbb{C}^6)$ which is immediately identified with the composition of the corresponding linear maps.

The extensions which yield vector bundles form an open set, which contains an open dense orbit, on which this action is free, namely, the tensors of rank 6.

If the rank of the tensor corresponding to an extension is $r < 6$, it follows that the extension is obtained from an extension $0 \to r\mathcal{O} \to \mathcal{E}'' \to \tau \to 0$ by taking a direct sum with $(6-r)\mathcal{O}$; but then $(6-r)\mathcal{O}$ is a direct summand of $\mathcal{E}^\vee$, a contradiction. $\square$

Corollary 4.11. *$\mathcal{E}$ as in Lemma* 4.10 *is a vector bundle if $H^0(\tau')$ has no base points.*

Proof. Our hypothesis shows that at each point of G the local extension class is non-zero, hence it yields a locally free sheaf. $\square$

Let us now show that the second case in Lemma 4.4 does not occur, since it produces a vector bundle $\mathcal{E}$ with a different intermediate cohomology from the one we require.

Lemma 4.12. *The second case in Lemma* 4.4 *does not occur, since otherwise the associated vector bundle $\mathcal{E}$ would have $h^2(\mathcal{E}(-3)) = 1 \neq b = 0$.*

Proof. Assume that $D = \sum_{i=3}^6 E_i - E_1$. Then the linear system $|2H - D| = |6L - E_1 - 2E_2 - 3(\sum_{i=3}^6 E_i)|$ has dimension greater than the expected dimension $27 - 28 = -1$, since it contains an effective divisor, $|2L - E_1 - (\sum_{i=3}^6 E_i)| + 2|2L - E_2 - (\sum_{i=3}^6 E_i)|$. This amounts to the non-vanishing of the cohomology group $H^2(-3H + D) = H^2(\tau(-3))$.

From the exact sequence (4.1) we infer that $h^2(\mathcal{E}(-3)) = 1$, whereas we assumed throughout that $h^2(\mathcal{E}(-3)) =: b = 0$, a contradiction. $\square$

We now assume that G is a smooth cubic surface, and that τ is an invertible sheaf on G, corresponding to the divisor class $-L + \sum_{i=1}^6 E_i$.

Consider the associated vector bundle $\mathcal{E}$; we want to verify that $\mathcal{E}$ has the required cohomology table, i.e., we want to calculate the dimensions $h^i(\mathcal{E}(-n))$ for $n = 0, 1, 2, 3$. This will allow us to verify that there are bundles $\mathcal{E}$ which satisfy the first and second assumptions.

Lemma 4.13. *Let G be a smooth cubic surface, and let τ be the invertible sheaf on G corresponding to the divisor class $-L + \sum_{i=1}^6 E_i$. Then the associated vector bundle $\mathcal{E}$ has the required cohomology table.*

Proof. Observe that:

- Clearly $h^0(\tau(-n)) = h^0(D - nH) = h^0(-(1+3n)L + (n+1)\sum_{i=1}^6 E_i) = 0$ for $n = 0, 1, 2, 3$.
- Also $h^2(\tau(-n)) = h^0((n-1)H - D) = h^0((3n-2)L - n\sum_{i=1}^6 E_i) = 0$ for $n = 0, 1, 2$, since a quartic with six double points is a union of four lines.
- $h^1(\tau(-3)) = h^1(-3H + D) = h^1(-(10L - 4\sum_{i=1}^6 E_i)) = 0$ by Ramanujam's vanishing trick for regular surfaces, since the linear system $|10L - 4\sum_{i=1}^6 E_i|$ contains a reduced and connected divisor, namely $Q_1 + \cdots + Q_5 + E_6$, where $Q_i \in |2L - \sum_{j=1}^6 E_j + E_i|$.
- Since $\chi(\tau(-n)) = 1 + \frac{1}{2}(D - nH)(D - (n-1)H) = \frac{3}{2}n(n-3)$, we also have $h^1(\tau) = 0$, $h^1(\tau(-1)) = h^1(\tau(-2)) = 3$, $h^2(\tau(-3)) = 0$. $\square$

We have seen how to a linear map $B : V \otimes U \to W$, where $V = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ denotes the space of linear forms, U is a fixed isotropic subspace in $H^1(\mathcal{F}(1))$, $W := H^1(\mathcal{F}(2))$, corresponds a homomorphism of vector bundles $B : V \otimes U \otimes \mathcal{O} \to W \otimes \mathcal{O}$, inducing $\beta : U \otimes \Omega^1(2) \to W \otimes \mathcal{O}(1)$, whence finally a vector bundle $\mathcal{E} := \ker(\beta)$ if β is surjective.

The second assumption yields a cubic surface $G \subset \mathrm{Proj}(V)$ and an invertible sheaf τ on G. If G is smooth, then the invertible sheaf $\tau(1)$ yields a birational morphism onto a Veronese surface, whence represents G as the blow up of a projective plane $\mathbb{P}^2$ in a subscheme ζ consisting of six points, and as the image of $\mathbb{P}^2$ through the linear system of cubic curves passing through ζ. The Hilbert–Burch theorem allows us to make an explicit construction which goes in the opposite direction.

Remark 4.14. Let U', W' be 3-dimensional vector spaces. Consider a $3 \times 3 \times 4$ tensor $\mathcal{B} \in (W'^{\vee} \otimes V \otimes U'^{\vee})$ and assume that the corresponding sheaf homomorphism $\hat{\mathcal{B}} : W' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^2}$ yields an exact sequence

$$0 \to W' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\hat{\mathcal{B}}} V \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(3) \to \mathcal{O}_{\zeta}(3) \to 0 \tag{4.6}$$

which is the Hilbert–Burch resolution of a codimension 2 subscheme ζ of length 6.

We obtain a canonical isomorphism $V \cong H^0(I_\zeta(3))$ and we let $G \subset \mathrm{Proj}(V)$ be the image of $\mathbb{P}^2$ under the rational map ψ associated to V. Under the above assumption on $\mathcal{B}$, and if moreover ζ is a local complete intersection, G is a normal cubic surface, and if we define $\mathcal{G} := \psi_* \mathcal{O}(1)$, then we have an exact sequence on $\mathrm{Proj}(V)$, corresponding again to $\mathcal{B}$:

$$0 \to W' \otimes \mathcal{O}(-1) \xrightarrow{\mathcal{B}} U'^{\vee} \otimes \mathcal{O} \to \mathcal{G} \to 0. \tag{4.7}$$

Under the more general assumption that $\mathcal{B}$ never drops rank by 2, $\mathcal{G}$ is an invertible sheaf on a cubic surface G, and $\mathcal{G} = \mathcal{O}_G(L)$ with $h^0(L) = 3$.

Conversely, given an exact sequence as in (4.7), the space $H^0(\mathcal{G})$, since $\mathcal{G}$ is generated by global sections, yields a morphism $\pi : G \to \mathrm{Proj}(H^0(\mathcal{G}))$. We calculate the Euler characteristic of $\mathcal{G}$:

$$\chi(\mathcal{G}(n)) = 3\left[\binom{n+3}{3} - \binom{n+2}{3}\right] = \frac{1}{2}(3n^2 + 9n + 6).$$

If the cubic surface G is normal, and we let G' be its minimal resolution, the Hilbert polynomial of $\mathcal{O}_{G'}(L)$ is equal to

$$\chi(L + nH) = 1 + \tfrac{1}{2}(L + nH)(L + (n+1)H) = 1 + \tfrac{1}{2}(L^2 + LH) + \tfrac{3}{2}n^2 + n(\tfrac{3}{2} + LH)$$

and therefore $LH = 3$, $L^2 = 1$. Since $L^2 = 1$, π is a birational morphism, and since L has genus 0 and degree three, G' is the birational image of $\mathbb{P}^2$ under a linear system of plane cubics $H^0(\mathcal{I}_\zeta(3))$, where ζ is a length six zero-dimensional subscheme.

Proposition 4.15. *More generally, assume that $\mathcal{G}$ is an invertible sheaf on a (not necessarily irreducible) cubic surface G, given by Lemma* 4.7*. Then the Cartier divisor L has degree* 3*, and the morphism π yields a birational morphism of a component of the surface G onto the plane. If G is irreducible and $\tau := \mathcal{G}^{\otimes 2}(-1)$, then $H^0(\tau^{\otimes 2}) = 0$.*

Proof. It suffices to consider two general divisors L_1, L_2 in the linear system $|L|$, and to consider the Hilbert polynomials of the sheaves appearing in the two exact sequences

$$0 \to \mathcal{O}_G(nH) \to \mathcal{O}_G(nH + L_1) \to \mathcal{O}_{L_1}(nH + L_1) \to 0,$$
$$0 \to \mathcal{O}_{L_2}(nH) \to \mathcal{O}_{L_2}(nH + L_1) \to \mathcal{O}_{L_1 \cap L_2}(nH + L_1) \to 0.$$

The conclusion is that $H^0(\mathcal{O}_{L_1 \cap L_2}(nH + L_1)) = 1$ for all $n \gg 0$, thus L_1 and L_2 meet transversally in a smooth point.

Let us now assume that G is irreducible, and consider the inverse of the birational morphism π. We can factor it as a sequence of blow ups $\sigma : Y \to \mathbb{P}^2$ followed by a projection $p : Y \to G$, to which corresponds a sublinear system of a complete linear system on Y, which reads on the plane as $|H| = |bL - \sum_i a_i E_i|$. Here, $b = H \cdot L$, and if $b \geq 3$, clearly $L \cdot (4L - 2H) < 0$, hence $|4L - 2H| = \emptyset$ and our desired vanishing is proven.

If instead $b \leq 2$, since $\dim |H| \geq 3$, it follows that $H = 2L - E$ and G is a linear projection of the cubic scroll $Y \subset \mathbb{P}^4$ with centre a point in $\mathbb{P}^4 \setminus Y$. We claim, however, that this case does not occur, essentially because otherwise $\mathcal{G} = p_*(\mathcal{O}_Y(L))$ would not be invertible.

As an alternative argument, observe that the factorization $\sigma = \pi \circ p$ is not possible, since σ is an isomorphism on the complement of the line $E \subset Y$, while the inverse image of the double curve of G is a conic (possibly reducible) contained in Y. □

Definition 4.16. *We now define the* direct construction *of the bundle $\mathcal{E}$ relying on our results above.*

Consider a sheaf $\mathcal{G}$ defined by an exact sequence as in (4.7)*, and which is invertible on a cubic surface G (i.e.,* $\mathrm{rk}(\mathcal{G} \otimes \mathbb{C}_y) \leq 1$ *at each* $y \in \mathbb{P}^3$*).*

Define $\tau := \mathcal{G}^{\otimes 2}(-1)$ and let $\mathcal{E}$ be a vector bundle which is an extension of $6\mathcal{O}$ *by τ as in* (4.1) *(here and elsewhere, $\mathcal{O} := \mathcal{O}_{\mathrm{Proj}(V)}$).*

Proposition 4.17. *$\mathcal{E}$ as above is unique up to isomorphism in the following cases:*

(1) *if G is a smooth cubic surface;*
(2) *if G is reducible to the union of a plane T and a smooth quadric Q intersecting transversally.*

Proof. As before, it suffices to show that $\dim \mathrm{Ext}^1(\tau, \mathcal{O}) = 6$.

Now, $\mathrm{Ext}^1(\tau, \mathcal{O}) = H^0(\mathcal{E}xt^1(\tau, \mathcal{O}))$ and the exact sequence

$$0 \to \mathcal{O}(-3) \to \mathcal{O} \to \mathcal{O}_G \to 0$$

yields

$$0 \to \mathcal{H}om(\tau, \mathcal{O}_G) \to \mathcal{E}xt^1(\tau, \mathcal{O}(-3)) \to \mathcal{E}xt^1(\tau, \mathcal{O})$$

where the last map is 0 on G. Hence, $\mathrm{Ext}^1(\tau, \mathcal{O}) \cong \mathrm{Hom}(\tau, \mathcal{O}_G(3)) = H^0(\tau^\vee(3))$.

Using the previous notation for the Cartier divisors corresponding to $\mathcal{G}$ and $\mathcal{O}_G(1)$, we want to show that $h^0(\mathcal{O}_G(4H-2L)) = 6$.

Assume first that G is a smooth cubic surface: then by Riemann–Roch it suffices to show the vanishing of the first cohomology group $h^1(\mathcal{O}_G(4H-2L)) = 0$.

We argue as before using Ramanujam's vanishing theorem, since

$$h^1(\mathcal{O}_G(4H-2L)) = h^1(\mathcal{O}_G(-5H+2L)) = h^1\Big(\mathcal{O}_G\Big(-13L+5\sum_i E_i\Big)\Big),$$

and $|13L-5\sum_i E_i| \supset |10L-4\sum_i E_i| + |H|$ contains a reduced and connected divisor.

In the second case, observe that there is no birational morphism of a smooth quadric Q onto the plane, thus $\mathcal{G}$ defines π which is an isomorphism on the plane, and has degree zero on Q.

Since we know that π is an embedding of $Q\cap T$, $\pi|_Q$ is the projection of $Q \cong \mathbb{P}^1\times\mathbb{P}^1$ on the second factor, followed by the isomorphism of $\mathbb{P}^1$ onto $Q\cap T$.

Since $\tau^\vee(3)|_T = \mathcal{O}_T(2)$, $\tau^\vee(3)|_Q = \mathcal{O}_Q(4,0)$, and $H^0(\tau^\vee(3)|_T = \mathcal{O}_T(2)) \to H^0(\tau^\vee(3)|_Q = \mathcal{O}_Q(4,0))$ is surjective, we obtain

$$\dim \operatorname{Ext}^1(\tau, \mathcal{O}) = h^0(\tau^\vee(3)) = 6. \qquad \square$$

Remark 4.18. Indeed, the above proof shows that if $G = T \cup Q$ with Q a smooth quadric, and $\mathcal{G}$ is invertible, then necessarily T and Q intersect transversally.

We now observe that Lemma 4.7 provides a resolution of $\tau := \mathcal{G}^{\otimes 2}(-1)$ starting from (4.7). We take the second symmetric power of the sequence (4.7) to obtain a resolution

$$\begin{aligned}0 \to (\Lambda^2 W')\otimes\mathcal{O}(-2) &\xrightarrow{\neg\mathcal{B}(-1)} (U'^\vee\otimes W')\otimes\mathcal{O}(-1)\\ &\xrightarrow{\tilde{\mathcal{B}}} (S^2U'^\vee)\otimes\mathcal{O} \to \tau(1) \to 0, \end{aligned} \tag{4.8}$$

where $\neg\mathcal{B}$ is the contraction given by the composition of the natural inclusion from $(\Lambda^2 W')\otimes\mathcal{O}(-1)$ to $(W'\otimes W')\otimes\mathcal{O}(-1)$ with the map $\mathcal{B}\otimes \mathrm{id}_{W'}(-1)$, while $\tilde{\mathcal{B}}$ is the composition of $\mathrm{id}_{U'^\vee}\otimes\mathcal{B}$ with the surjection of $(U'^\vee\otimes U'^\vee\otimes\mathcal{O}$ onto $(S^2U'^\vee)\otimes\mathcal{O}$.

Consider now the exact sequence defining $\mathcal{E}$,

$$0 \to 6\mathcal{O} \to \mathcal{E} \to \tau \to 0,$$

and the above projective resolution of τ; by the mapping cone construction (cf. e.g. [Eis, pp. 650–651]) we obtain a projective resolution of $\mathcal{E}$:

$$\begin{aligned}0 \to \Lambda^2 W'\otimes\mathcal{O}(-3) &\xrightarrow{\neg\mathcal{B}(-2)} U'^\vee\otimes W'\otimes\mathcal{O}(-2)\\ &\xrightarrow{(\tilde{\mathcal{B}}(-1),\lambda)} 6\mathcal{O}\oplus(S^2U'^\vee\otimes\mathcal{O}(-1)) \to \mathcal{E} \to 0. \end{aligned} \tag{4.9}$$

We now want to find a relation between the multiplication map $B : U\otimes V \to W$, where U (resp. W) denotes as usual $H^1(\mathcal{E}(-2))$ (resp. $H^1(\mathcal{E}(-1))$), and the above map $\mathcal{B} : W'\otimes V^\vee \to U'^\vee$ (cf. (4.7)). Let $\mathcal{E}$ be the unique sheaf given by B (cf. (3.5)).

We split the above resolution (4.9) of $\mathcal{E}$ into two short exact sequences, denoting by $\mathcal{K}$ the image of $(\tilde{\mathcal{B}}(-1), \lambda)$. This gives $H^1_*(\mathcal{E}) \cong H^2_*(\mathcal{K}) = \ker[H^3_*(\Lambda^2 W' \otimes \mathcal{O}(-3)) \to H^3_*(U'^\vee \otimes W' \otimes \mathcal{O}(-2))]$. Fixing these isomorphisms, we can make the following identifications: $U \cong \ker[\neg\mathcal{B} : \Lambda^2 W' \otimes V^\vee \to U'^\vee \otimes W']$, $W = \Lambda^2 W'$, and the multiplication map B is given as the composition in the following diagram:

$$\begin{tikzcd} U \otimes V \arrow[r, "B"] \arrow[d, hook] & W = \Lambda^2 W' \\ \Lambda^2 W' \otimes V^\vee \otimes V \arrow[ur, "\neg"'] & \end{tikzcd}$$

of the natural inclusion with the natural contraction $\neg : V^\vee \otimes V \to \mathbb{C}$.

One can also obtain the above factorization in the following alternative way: Beilinson's complex yields the short exact sequence

$$0 \to U \otimes \Omega^2(2) \to W \otimes \Omega^1(1) \oplus 6\mathcal{O} \to \mathcal{E} \to 0,$$

where U (resp. W) denotes as usual $H^1(\mathcal{E}(-2))$ (resp. $H^1(\mathcal{E}(-1))$).

From the above we get

$$\begin{aligned} 0 \to (U \otimes \mathcal{O}(-2)) \oplus (W \otimes \mathcal{O}(-3)) \to (U \otimes \Lambda^3 V \otimes \mathcal{O}(-1)) \\ \oplus (W \otimes \Lambda^3 V \otimes \mathcal{O}(-2)) \to 6\mathcal{O} \oplus (W \otimes \Lambda^2 V(-1)) \to \mathcal{E} \to 0. \end{aligned} \tag{4.10}$$

Comparing (4.9) and (4.10), we obtain the following identifications:

(1) $W' \cong \Lambda^2 W$, $U \cong \ker[\neg\mathcal{B} : \Lambda^2 W' \otimes V^\vee \to U'^\vee \otimes W']$,
(2) $U'^\vee \otimes W' \cong (W \otimes \Lambda^3 V)/U$, $S^2 U'^\vee \cong (W \otimes \Lambda^2 V)/(U \otimes V)$.

Based on the above considerations we give the following

Definition-Proposition 4.19. *The* cross-product involution on tensors of type $3 \times 3 \times 4$ *is given as follows: to a* 5*-uple* $(U', W', V, \delta', \mathcal{B})$*, where* U', W' *are* 3*-dimensional vector spaces,* V *is a* 4*-dimensional vector space,* $\mathcal{B} \in \mathrm{Hom}(W' \otimes V^\vee, U'^\vee) = W'^\vee \otimes V \otimes U'^\vee$, $\delta' : \Lambda^3 W' \cong \mathbb{C}$ *an isomorphism, we associate the* 5*-uple* $(U, W, V^\vee, \delta, B)$*, where:*

(1) $W := \Lambda^2 W'$ *and, since* W *is then canonically isomorphic to* $W'^\vee$*, by the duality* $W' \otimes \Lambda^2 W' \to \mathbb{C}$ *induced by* δ'*, we let* $\delta := \delta'^\vee$*,*
(2) $U := \ker[\neg\mathcal{B} : \Lambda^2 W' \otimes V^\vee \to U'^\vee \otimes W']$*, where* $\neg\mathcal{B}$ *is the contraction with the tensor* $\mathcal{B}$ *described above;*
(3) $B \in \mathrm{Hom}(U \otimes V, W) = W \otimes V^\vee \otimes U^\vee$ *is the composition of the inclusion* $U \otimes V \hookrightarrow \Lambda^2 W' \otimes V^\vee \otimes V$ *with the natural contraction* $\neg$.

The dimension of U *is equal to three if we make*

Main assumption. $\neg\mathcal{B}$ *is surjective (this in turn obviously implies the injectivity of the map* $\mathcal{B} : U' \to V \otimes W$*).*

The cross-product involution is then defined through the associated 5*-uple on the open set of tensors satisfying the main assumption, and it is involutive whenever the composition is defined.*

Proof. Given a 5-uple $(U', W', V, \delta, \mathcal{B})$, let $(U, W, V^\vee, \delta, B)$ be its corresponding 5-uple, to which corresponds a third 5-uple $(U'', W'', V, \delta, B'')$. We have $\mathcal{B} \in W'^\vee \otimes V \otimes U'^\vee$, $B \in W \otimes V^\vee \otimes U^\vee$, $B'' \in W \otimes V \otimes (U'')^\vee$.

We claim that there exists a canonical isomorphism $U'^\vee = (U'')^\vee$, equivalently, a canonical isomorphism $U'' = U'$.

To show this, we shall first observe that both spaces can be canonically regarded as subspaces of $W \otimes V$, and then, since they have the same dimension (for U'', this is a consequence of the hypothesis that $(U, W, V^\vee, \delta, B)$ also satisfies the main assumption), it will suffice to show that $U' \subset U''$.

U'' is the kernel of $\neg B : \Lambda^2 W' \otimes V \to U^\vee \otimes W'$. Upon identifying $\Lambda^2 W'$ with W, the previous map becomes $\neg B : W \otimes V \to U^\vee \otimes W^\vee$. We now consider $\mathcal{B}$ as the map $\mathcal{B} : U' \to W \otimes V$. It suffices to show $(\neg B) \circ \mathcal{B}(U') = 0$, i.e., by dualizing, that $\mathcal{B}^\vee \circ (\neg B)^\vee (U \otimes W) = 0$. This is a consequence of the commutativity of the following diagram:

$$\begin{tikzcd}
U \otimes W \arrow[r, "(\neg B)^\vee"] \arrow[d, hook] & V^\vee \otimes W^\vee \arrow[r, "\mathcal{B}^\vee"] & U'^\vee \\
(\Lambda^2 W' \otimes V^\vee) \otimes W \arrow[r, "\neg\mathcal{B}\otimes \mathrm{id}_W"'] \arrow[ur, "\neg"'] & (U'^\vee \otimes W') \otimes W \arrow[ur, "\neg"'] &
\end{tikzcd}$$ □

5. The explicit unirational family

Up to now we have studied extensively the vector bundles $\mathcal{E}$ such that an even set of 56 nodes on a sextic surface F should come from a symmetric homomorphism associated to a section of $S^2\mathcal{E}$.

We have almost shown, however, in Corollary 4.8 and Proposition 4.15, that all such sections have as determinant the square of a cubic surface G, if the cubic G appearing in the direct construction is irreducible.

It seems therefore only natural to try to see what happens for a reducible cubic, hoping that then $h^0(S^2\mathcal{E}) > 21$.

We assume henceforth that G is the union of a smooth quadric Q with a plane T intersecting transversally. We have already observed in the proof of Proposition 4.17 that in this case there is a unique choice for $\mathcal{G}$, likewise for $\mathcal{E}$.

Lemma 5.1. *If G is the union of a smooth quadric Q and of a plane T which intersect transversally, then $h^0(\tau^{\otimes 2}) = 1$ and $h^0(S^2\mathcal{E}) = 22$.*

Proof. In this case the sheaf $\mathcal{G}$ corresponds to the sheaf $\mathcal{O}_Q(0, 2)$ on Q and to $\mathcal{O}_T(1)$ on T. Therefore $\tau := \mathcal{G}^{\otimes 2}(-1)$ corresponds to $\mathcal{O}_Q(-1, 3)$ on Q and to $\mathcal{O}_P(1)$ on P. Thus the sections of $H^0(\tau^{\otimes 2})$ vanish identically on Q and correspond to sections of $\mathcal{O}_T(2)$ vanishing on $Q \cap T$.

The second statement then follows from the proof of Corollary 4.8. □

We also remark that, since we assume $Q \cap T$ is smooth, G is unique up to projective equivalence.

We shall now give an explicit tensor B_0 whose associated sheaf is the unique $\mathcal{G}$ on a reducible cubic of the form $T \cup Q$, where Q and $Q \cap T$ are smooth, and compute explicitly that the tensor corresponding to the unique $\mathcal{E}$ obtained from the direct construction is again B_0; this will allow us to calculate explicitly the determinant of a generic symmetric map $\mathcal{E}^\vee \to \mathcal{E}$, and to show that it is a nodal sextic.

Lemma 5.2. *Consider the following* $3 \times 3 \times 4$ *tensor* B_0*:*

$$B_0 = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} + x_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.1}$$

The sheaf $\mathcal{G}_0$ *associated to* B_0 *is an invertible sheaf on the reducible cubic* $G_0 := \{x_0(\sum_i x_i^2) = 0\}$. *Its class in* $\mathfrak{A}^0$ *is invariant under the cross-product involution. More precisely, the vector bundle* $\mathcal{E}_0$ *obtained from* B_0 *via the direct construction has the required cohomology table and its multiplication matrix is again* B_0 *for the intermediate cohomology module* $M_0 := H^1_*(\mathcal{E}_0)$. *Moreover,* $h^0(S^2\mathcal{E}_0) = 22$.

Proof. The determinant of B_0 equals G_0. On the plane $x_0 = 0$ the Pfaffians are $x_1 = x_2 = x_3 = 0$ and μ has rank 2. Elsewhere G_0 is smooth, whence $\mathcal{G}$ is everywhere invertible.

It is a routine calculation to verify that its class is invariant under the cross-product involution. The direct construction then gives a unique vector bundle $\mathcal{E}$ (cf. Proposition 4.17), and we claim that its cohomology table is as required. As in the proof of Lemma 4.13 it suffices to calculate the dimensions $h^i(\tau(-n))$ for $0 \le n \le 3$, and to show they vanish for $i = 0, i = 2$.

We use the exact sequence

$$0 \to \tau \to \mathcal{O}_Q(-1,3) \oplus \mathcal{O}_T(1) \to \mathcal{O}_{Q\cap T}(1) \cong \mathcal{O}_{\mathbb{P}^1}(2) \to 0,$$

which is easily seen to be exact on global sections, hence the case $i = 0$ follows right away.

For the case $i = 2$ we observe that in the exact cohomology sequence

$$0 \to H^1(\tau(-n)) \to H^1(\mathcal{O}_Q(-1-n, 3-n)) \xrightarrow{\psi} H^1(\mathcal{O}_{Q\cap T}(1-n)) \to H^2(\tau(-n)) \to 0$$

ψ is surjective, since its cokernel is isomorphic to $H^2(\mathcal{O}_Q(-2-n, 2-n))$ whose dimension equals $h^0(\mathcal{O}_Q(n, n-4)) = 0$ (since $n < 4$).

The last claim follows from Lemma 5.1. $\square$

Remark 5.3. Consider the invertible sheaf given by

$$0 \to 3\mathcal{O}(-1) \xrightarrow{B_0} 3\mathcal{O} \to \mathcal{G}_0 \to 0. \tag{5.2}$$

We have $h^0(\mathcal{G}_0) = 3$ and the associated morphism π from G_0 to $\mathbb{P}^2$ is determined by the rational map given by the entries of any column of $\mathrm{Ad}(B_0)$:

$$\mathrm{Ad}(B_0) = \begin{pmatrix} x_1^2 + x_0^2 & x_1x_2 - x_0x_3 & x_1x_3 + x_0x_2 \\ x_1x_2 + x_0x_3 & x_2^2 + x_0^2 & x_2x_3 - x_0x_1 \\ x_1x_3 - x_0x_2 & x_2x_3 + x_0x_1 & x_3^2 + x_0^2 \end{pmatrix}.$$

We see right away that π is the identity on the plane T, and the projection along one ruling from the quadric Q to $Q \cap T$.

Proposition 5.4. *For a section $\phi \in H^0(S^2\mathcal{E}_0)$, denote by $\varphi \in \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{E})$ its associated symmetric morphism. Then, for ϕ general, $F := \{x \mid \det(\varphi) = 0\}$ is a nodal sextic surface with, as singularities, exactly an even set of* 56 *nodes* $\Delta = \{x \mid \mathrm{corank}(\varphi) = 2\}$.

Proof. The required computations were performed and can be verified by using the computer-algebra system [Gr-St] over a finite field, or over $\mathbb{Q}$.

The first step is to compute explicitly the fibre over B_0 inside the variety of pairs $\mathfrak{M}_{AB}$ (cf. (3.6)), i.e., the vector space of symmetric matrices $A \in \mathrm{Mat}(12 \times 12, H^0(\mathcal{O}_{\mathbb{P}^3}(2)))$ satisfying the equation $(B_0, \epsilon) \cdot A = 0$.

Step two: for a random A in such a fibre one computes the G.C.D. of two (different) 6×6 minors of A: if the G.C.D. has degree 6, then it is the equation of the sextic F.

Step three: one verifies with the jacobian criterion that the singular locus of F consists exactly of a 0-dimensional subscheme of length 56.

Step four: one verifies that the ideal sheaf of the singular locus is a radical ideal. Then the singularities are just a set of nodes.

A further (but not absolutely necessary) check consists in verifying that the scheme Δ coincides with the subscheme formed by those 56 reduced points; this can be performed by computing a set of 5×5 minors of A sufficient to generate the ideal of the 56 points. $\square$

Remark 5.5. Since the space of reducible cubic surfaces has dimension 12 (9 + 3), we obtain an explicit family parametrized by a rational variety Φ_0 of dimension $33 = 21+12$.

Proof. We simply construct a parameter space by choosing a 12-dimensional subgroup $H \subset \mathrm{PGL}(V)$ such that the orbit of G_0 dominates the space of reducible cubics, and then we take as parameter space $H \times \mathbb{P}(H^0(S^2\mathcal{E}))$.

Then to the pair (g, ϕ_0) corresponds the vector bundle $g^*(\mathcal{E}) := \mathcal{E}_{gG_0}$ and the section $g^*(\phi_0)$, and, correspondingly, the sextic surface $g^*(\det(\varphi) = 0)$. $\square$

Lemma 5.6. *The morphism $\Phi_0 \to \mathbb{P}(S^6V)$ associating to (g, φ) the corresponding nodal sextic $F = \det(\varphi)$ has fibres of dimension* 6*.*

Proof. Recall first that we have already shown that the surface $g(G_0)$ uniquely determines a vector bundle $\mathcal{E}$ and conversely.

Second, observe that if F has exactly 56 nodes then F determines the quadratic sheaf $\mathcal{F}$ uniquely (observe moreover that $\mathcal{F}$ has only constant automorphisms).

Suppose that there are two different vector bundles $\mathcal{E}, \mathcal{E}'$ and respective morphisms φ, φ' forming exact sequences as in Theorem 2.1 which define isomorphic cokernels $\mathcal{F}, \mathcal{F}'$.

By abuse of notation we identify $\mathcal{F}'$ with $\mathcal{F}$ and assume that we have $\gamma : \mathcal{E} \to \mathcal{F}$, $\gamma' : \mathcal{E}' \to \mathcal{F}$ inducing such isomorphisms of the respective cokernels with $\mathcal{F}$.

A first question is whether there exist lifts $\alpha : \mathcal{E}' \to \mathcal{E}$ and $\beta : \mathcal{E}'^\vee \to \mathcal{E}^\vee$ of the identity $\mathrm{id}_\mathcal{F}$ on $\mathcal{F}$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}^\vee \arrow[r, "\varphi"] & \mathcal{E} \arrow[r, "\gamma"] & \mathcal{F} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{E}'^\vee \arrow[r, "\varphi'"] \arrow[u, "\beta"'] & \mathcal{E}' \arrow[r, "\gamma'"] \arrow[u, "\alpha"'] & \mathcal{F} \arrow[r] \arrow[u, "\mathrm{id}_\mathcal{F}"'] & 0 \end{tikzcd} \tag{5.3}$$

If such an α exists, then necessarily the submodule $M := \mathrm{Im}(H^1_*(\gamma))$ of $H^1_*(\mathcal{F})$ equals the submodule $M' := \mathrm{Im}(H^1_*(\gamma'))$.

Assume now that $M' = M$; then since any automorphism of M lifts to an isomorphism of two minimal resolutions of M, this automorphism induces an isomorphism α of the respective first syzygy bundles, here $\mathcal{E}$, resp. $\mathcal{E}'$ (cf. Section 2). We can henceforth assume that if $M = M'$, then $\mathcal{E} = \mathcal{E}'$, and then, using $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = \mathbb{C}$ (cf. Lemma 4.9) we conclude that α is multiplication by a constant, necessarily $\neq 0$.

From the exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{E}^\vee) \to \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{E}) \xrightarrow{\gamma\circ} \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{F}),$$

it follows that there exists a unique homomorphism β making the diagram commute. Again, using $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = \mathbb{C}$, we infer that β is multiplication by a non-zero constant, and we have thus shown that if $M = M'$ then the sections ϕ and ϕ' are proportional.

On the other hand, the choice of M is completely determined by the choice of a Lagrangian subspace U of the 6-dimensional space $H^1(\mathcal{F}(1))$, and we saw that for each choice of U there is a bundle $\mathcal{E}$ and a ϕ yielding an exact sequence as in Theorem 2.1, with M equal to the image of $H^1_*(\mathcal{E})$.

We are done, since the dimension of the Lagrangian Grassmannian $\mathrm{LGr}(3, 6)$ equals 6. □

We want to show that the explicit unirational family that we constructed is locally maximal. To this purpose, observe that to a pair (g, ϕ_0) corresponds a vector bundle $g^*(\mathcal{E}_0)$ and a section $g^*(\phi_0)$, but more precisely a tensor $g^*(B)$ and a matrix of quadratic forms A_{g,ϕ_0} representing $g^*(\phi_0)$ as in (3.5).

Thus Φ_0 maps in a generically finite way to the variety of pairs $\mathfrak{M}_{AB}$ (cf. (3.6)) and we can consider the $\mathrm{GL}(U) \times \mathrm{GL}(W)$-orbit of its image.

Observe that we then obtain an irreducible algebraic set Ψ_0 of dimension $33 + 1 + 9 + 9 - 1 = 51$.

The following lemma shows that Ψ_0 is indeed a component of $\mathfrak{M}_{AB}$.

Lemma 5.7. *Let $(B_0, A_0) \in \mathfrak{M}_{AB}$ be a general point of the fibre over B_0. Then the tangent space to $\mathfrak{M}_{AB}$ at the point (B_0, A_0) has dimension* 51.

Proof. Fix the pair $(B_0, A_0) \in \mathfrak{M}_{AB}$. For a generic pair $(B, A) \in \mathrm{Mat}(3, 12, \mathbb{C}) \times \mathrm{Mat}_{\mathrm{Sym}}(12, 12, H^0(\mathcal{O}_{\mathbb{P}^3}(2)))$ we search for the solutions of the equations

$$(B_0 + tB, \epsilon)(A_0 + tA) \equiv 0 \pmod{t^2}.$$

The above equation is equivalent to the two equations $BA_0 + B_0A = 0$, $\epsilon A = 0$, and we have to compute the space of solutions.

Again this is done by means of a computer-algebra system over a finite field; it suffices to choose a point A_0 at random for which the tangent space has dimension 51. The computation works out successfully. □

We can now summarize the result of the construction of the above explicit family:

Main Theorem B. *There is a family of nodal sextic surfaces with* 56 *nodes forming an even set, parametrized by a smooth irreducible rational variety* Φ_0 *of dimension* 33*, whose image* Ξ_0 *is a unirational subvariety of dimension* 27 *of the space of sextic surfaces. Moreover, the above family is versal, thus* Ξ_0 *yields an irreducible component of the subvariety of nodal sextic surfaces with* 56 *nodes.*

Proof. The first assertions were proven between Lemma 5.1 and Lemma 5.6.

Let Ξ be the subvariety of nodal sextic surfaces with 56 nodes. Since the property that the set of nodes is even is a topological property (cf. for instance [Cat1], [Cat2]), it follows that there is an open and closed set $\Xi' \subset \Xi$ such that for $F \in \Xi'$ the set of 56 nodes is even. We only need to prove that $\Xi_0 \subset \Xi'$ is open.

But Ξ' contains the open set Ξ'' such that, for $F \in \Xi''$, $H^1(\mathcal{F}(2))$ has dimension 3 and the first assumption is satisfied.

We can form a variety Ψ' consisting of quadruples (F, U, B, ϕ) where:

(i) $F \in \Xi'$ is a sextic surface,
(ii) $U \subset H^1(\mathcal{F}(1))$ is a Lagrangian subspace,
(iii) B is the multiplication tensor for the intermediate cohomology submodule M of $H^1_*(\mathcal{F})$ determined as in (2.1) by the choice of U,
(iv) if $\mathcal{E}$ is the unique vector bundle determined by B as in (3.5), then ϕ is a section of the vector bundle $S^2\mathcal{E}$ such that $\det(\varphi) = F$.

Then we see that the map $\Psi' \to \mathfrak{M}_{A,B}$ is an embedding. Now Lemma 5.7 shows that $\Psi_0 \subset \Psi'$ is open, and we are done. □

It is a natural question to ask if the above is the unique irreducible component of the subvariety of nodal sextic surfaces with 56 nodes forming an even set. For this purpose one should first settle the case of Hilbert function $(3, 4)$ for M.

6. The random approach

Let $\mathbb{M}$ be a variety defined over a finite field of order q and let $\mathbb{M}_0 \subset \mathbb{M}$ be a subvariety of codimension k. The random approach consists in finding a point in $\mathbb{M}_0$ by choosing points in $\mathbb{M}$ at random. Since the probability of hitting a point of $\mathbb{M}_0$ is q^{-k}, it is evident that this method is only successful if the computation time to decide whether a point of $\mathbb{M}$ actually belongs to $\mathbb{M}_0$ is small enough (cf. [Sch], [Sch-To]).

In this section we show how this method was applied to find the first examples of sextic surfaces with an even set of 56 nodes.

Let $\mathcal{A}$ denote the coordinate ring of $\mathbb{P}^3$ and let B be the multiplication matrix of the intermediate cohomology module M. If B is general, since $\mathcal{E}$ is a syzygy bundle (cf. Section 3), it follows (cf. 3.5 and 4.9) that M has a resolution of the form

$$0 \leftarrow M \leftarrow 3\mathcal{A}[2] \leftarrow 9\mathcal{A}[1] \leftarrow 6\mathcal{A} \oplus 6\mathcal{A}[-1] \leftarrow 9\mathcal{A}[-2] \leftarrow 3\mathcal{A}[-3] \leftarrow 0. \qquad (6.1)$$

In an analogous way to the one followed after the exact sequence (3.5) we find that the symmetric morphisms $\varphi : \mathcal{E}^\vee \to \mathcal{E}$ are exactly induced by the symmetric morphisms $a : 9\mathcal{O}(-1) \to 9\mathcal{O}(1)$ such that $b \circ a = 0$, according to the following diagram:

$$\begin{tikzcd}
0 & 3\mathcal{O}(2) \arrow[l] & 9\mathcal{O}(1) \arrow[l, "b"'] & \mathcal{E} \arrow[l] \\
0 \arrow[r] & 3\mathcal{O}(-2) \arrow[r, "{}^t b"] \arrow[u, "0"'] & 9\mathcal{O}(-1) \arrow[r] \arrow[u, "a"'] & \mathcal{E}^\vee \arrow[u, "\varphi"']
\end{tikzcd}$$

It is clear that the replacement of (A, B) with (a, b) reduces the memory and the time required for computations.

Repeated random choices of b allow one to find an $\mathcal{E}$ with

$$h^0(S^2\mathcal{E}) = \dim\{a : 9\mathcal{O}(-1) \to 9\mathcal{O}(1) \mid a = {}^t a,\ b \circ a = 0\} \geq 22.$$

This property leads to the definition of $\mathbb{M}$ and $\mathbb{M}_0$.

Definition 6.1. *Let* $\mathbb{M}$ *be the Zariski open set*

$$\mathbb{M} := \{b : 9\mathcal{O}(1) \to 3\mathcal{O}(2) \mid M := \operatorname{coker}(b) \textit{ has a resolution as in } (6.1) \textit{ and } \iota : 6\mathcal{O} \to \mathcal{E} := \operatorname{Syz}_1(M) \textit{ is injective}\}$$

and let

$$\mathbb{M}_0 := \{b \in \mathbb{M} \mid h^0(S^2\mathcal{E}) \geq 22\}.$$

We already remarked that $\mathbb{M}$ is non-empty. A resolution for $S^2\mathcal{E}$ is provided by the following lemma.

Lemma 6.2. *If* $0 \to A \to B \to C \to \mathcal{E} \to 0$ *is an exact sequence of locally free sheaves, then the following sequence is also exact:*

$$0 \to S^2 A \to A \otimes B \to \Lambda^2 B \oplus (A \otimes C) \to B \otimes C \to S^2 C \to S^2\mathcal{E} \to 0.$$

Proof. By hypothesis we have $0 \to B/A \to C \to \mathcal{E} \to 0$. Therefore we get $0 \to \Lambda^2(B/A) \to (B/A)\otimes C \to S^2 C \to S^2\mathcal{E} \to 0$. Resolutions for $\Lambda^2(B/A)$ and $(B/A)\otimes C$ are standard, respectively $0 \to S^2 A \to A \otimes B \to \Lambda^2 B \to \Lambda^2(B/A) \to 0$ and $0 \to A \otimes C \to B \otimes C \to (B/A) \otimes C \to 0$. The resolution for $S^2\mathcal{E}$ stated in the lemma is the mapping cone of the previous resolutions. □

Hence it was guessed that the "good" locus has codimension 7:

Proposition 6.3. *The condition* $h^0(S^2\mathcal{E}) \geq 22$ *is expected to hold on a codimension* 7 *algebraic subset of* $\mathbb{M}$.

Proof. By applying the previous lemma to a minimal free resolution of $\mathcal{E}$ we get a (not necessarily minimal) free resolution of $S^2\mathcal{E}$:

$$0 \to 6\mathcal{O}(-6) \to 27\mathcal{O}(-5) \xrightarrow{r_3} \begin{matrix} 18\mathcal{O}(-4) \\ \oplus \\ 18\mathcal{O}(-3) \\ \oplus \\ 36\mathcal{O}(-4) \end{matrix} \xrightarrow{r_2} \begin{matrix} 54\mathcal{O}(-3) \\ \oplus \\ 54\mathcal{O}(-2) \end{matrix} \xrightarrow{r_1} \begin{matrix} 21\mathcal{O}(-2) \\ \oplus \\ 36\mathcal{O}(-1) \\ \oplus \\ 21\mathcal{O} \end{matrix} \xrightarrow{r_0} S^2\mathcal{E} \to 0$$

Denote by $\mathcal{K}_i$ the image of the map r_i, split the above exact sequence into short exact sequences and look at the associated long exact cohomology sequences. From

$$0 \to H^0(\mathcal{K}_1) \to H^0(21\mathcal{O}) \xrightarrow{S^2\iota^0} H^0(S^2\mathcal{E}) \to H^1(\mathcal{K}_1) \to 0$$

and since $S^2\iota^0$ is injective ($\iota : 6\mathcal{O} \to \mathcal{E}$ being injective), we get, using also the other cohomology sequences, $0 = H^0(\mathcal{K}_1) \cong H^1(\mathcal{K}_2) \cong H^2(\mathcal{K}_3)$ and $H^0(S^2\mathcal{E})/H^0(21\mathcal{O}) \cong H^1\mathcal{K}_1 \cong H^2\mathcal{K}_2$.

We also have the short exact sequence

$$H^2(\mathcal{K}_3) = 0 \to H^3(6\mathcal{O}(-6)) \to H^3(27\mathcal{O}(-5)) \to H^3(\mathcal{K}_3) \to 0,$$

hence $H^3(\mathcal{K}_3)$ has dimension 48.

Finally, the exact sequence

$$0 \to H^2(\mathcal{K}_2) \to H^3(\mathcal{K}_3) \xrightarrow{\alpha} H^3(54\mathcal{O}(-4)) \to H^3(\mathcal{K}_2) \to 0,$$

since $H^3(\mathcal{K}_2) \cong H^2(\mathcal{K}_1) \cong H^1(S^2\mathcal{E})$, yields

$$0 \to \frac{H^0(S^2\mathcal{E})}{H^0(21\mathcal{O})} \to \mathbb{C}^{48} \xrightarrow{\alpha} \mathbb{C}^{54} \to H^1(S^2\mathcal{E}) \to 0. \tag{6.2}$$

Therefore the condition that $H^0(S^2\mathcal{E}) \cong H^0(21\mathcal{O})$ is equivalent to the linear map α having maximal rank, and since we know that this happens in general, the condition $h^0(S^2\mathcal{E}) \geq 22$ holds in a determinantal subscheme of $\mathbb{M}$ of expected codimension $54 - 48 + 1 = 7$. $\square$

Let $\mathfrak{M}_{ab}$ be the variety, analogous to $\mathfrak{M}_{AB}$ (cf. (3.6)), of pairs (b, a) such that $a = {}^t a$, $ab = 0$. Computations similar to the ones in Lemma 5.7 verified over a finite field that at a random point (b_0, a_0) the variety of pairs $\mathfrak{M}_{ab}$ is smooth of dimension 123. A standard argument then ensures the existence of a lift of the pair (b_0, a_0) from a finite field to a number field (cf. [Sch]).

This random approach, and the remark that the space of reducible cubic surfaces is a codimension 7 subvariety of the projective space of cubic surfaces, led then to the explicit family constructed in the previous section.

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References

[Ba1] Barth, W.: Two projective surfaces with many nodes, admitting the symmetries of the icosahedron. J. Algebraic Geom. **5**, 173–186 (1996) Zbl 0860.14032 MR 1358040

[Ba2] Barth, W.: Counting singularities of quadratic forms on vector bundles. In: Vector Bundles and Differential Equations (Nice, 1979), Progr. Math. 7, Birkhäuser, Boston, MA, 1–19 (1980) Zbl 0442.14021 MR 0589218

[Bass] Basset, A. B.: The maximum number of double points on a surface. Nature **73**, 246 (1905) JFM 37.0646.03

[Bea] Beauville, A.: Sur le nombre maximum de points doubles d'une surface dans $\mathbb{P}^3$ ($\mu(5) = 31$). In: Algebraic Geometry (Angers, 1979), A. Beauville (ed.), Sijthoff & Noordhoff, 207–215 (1980) Zbl 0445.14016 MR 0605342

[Bei] Beilinson, A.: Coherent sheaves on $\mathbb{P}^N$ and problems of linear algebra. Functional Anal. Appl. **12**, 214–216 (1978) Zbl 0424.14003 MR 0509388

[Bu-Wa] Burns, D. M. Jr., Wahl, J. M.: Local contributions to global deformations of surfaces. Invent. Math. **26**, 67–88 (1974) Zbl 0288.14010 MR 0349675

[Ca-Ca] Casnati, G., Catanese, F.: Even sets of nodes are bundle symmetric. J. Differential Geom. **47**, 237–256 (1997). Erratum, J. Differential Geom. **50**, 415 (1998) Zbl 0896.14017 MR 1601608

[Cat1] Catanese, F.: Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications. Invent. Math. **63**, 433–465 (1981) Zbl 0472.14024 MR 0620679

[Cat2] Catanese, F.: Generalized Kummer surfaces and differentiable invariants of Noether–Horikawa surfaces. I. In: Manifolds and Geometry (Pisa, 1993), Sympos. Math. 36, Cambridge Univ. Press, Cambridge, 132–177 (1996) Zbl 0872.14031 MR 1410071

[Ca-Ce] Catanese, F., Ceresa, G.: Constructing sextic surfaces with a given number d of nodes. J. Pure Appl. Algebra **23**, 1–12 (1982) Zbl 0484.14011 MR 0638117

[Cay1] Cayley, A.: A memoir on cubic surfaces. Trans. London Math. Soc. **159**, 231–326 (1869) JFM 02.0576.01

[Cay2] Cayley, A.: A third memoir on quartic surfaces. Proc. London Math. Soc. **3**, 234–266 (1871) JFM 03.0391.02

[Eis] Eisenbud, D.: Commutative Algebra. With a View Toward Algebraic Geometry. Grad. Texts in Math. 150, Springer, New York (1995) Zbl 0819.13001 MR 1322960

[End] Endraß, S.: Minimal even sets of nodes. J. Reine Angew. Math. **503**, 87–108 (1998) Zbl 0911.14017 MR 1650323

[Gr-St] Grayson, D., Stillman, M.: Macaulay 2—a software system for algebraic geometry and commutative algebra. http://www.math.uiuc.edu/Macaulay2 (1999)

[Ja-Ru] Jaffe, D. B., Ruberman, D.: A sextic surface cannot have 66 nodes. J. Algebraic Geom. **6**, 151–168 (1997) Zbl 0884.14015 MR 1486992

[Kob] Kobayashi, S.: On moduli of vector bundles. In: Complex Geometry and Analysis (Pisa, 1988), Lecture Notes in Math. 1422, Springer, Berlin, 45–57 (1990) Zbl 0718.32020 MR 1055842

[Kum] Kummer, E.: Ueber diejenigen Flächen, welche mit ihren reciprok polaren Flächen von gleicher Ordnung sind und dieselben Singularitäten besitzen. Berl. Monatsber. **1878**, 25–36 JFM 10.0510.03

[McW-Sl] MacWilliams, F. J., Sloane, N. J. A.: The Theory of Error-Correcting Codes I, II. North-Holland Math. Library 16, North-Holland, Amsterdam (1977) Zbl 0369.94008 MR 0465509

[Miy] Miyaoka, Y.: The maximal number of quotient singularities on surfaces with given numerical invariants. Math. Ann. **268**, 159–171 (1984) Zbl 0521.14013 MR 0744605

[Sch] Schreyer, F.-O.: Small fields in constructive algebraic geometry. In: Moduli of Vector Bundles (Sanda and Kyoto, 1994), Lecture Notes in Pure Appl. Math. 179, Dekker, 221–228 (1996) Zbl 0876.14040 MR 1397991

[Sch-To] Schreyer, F.-O., Tonoli, F.: Needles in a haystack: special varieties via small fields. In: Computations in Algebraic Geometry with Macaulay 2, D. Eisenbud et al. (eds.), Springer, 251–279 (2002). Zbl 0994.14037 MR 1949554

[Tju] Tjurina, G. N.: Resolution of singularities of plane (= flat) deformations of double rational points. Funktsional. Anal. i Prilozhen. **4**, no. 1, 77–83 (1970) (in Russian) Zbl 0221.32008 MR 0267129

[Tog1] Togliatti, E. G.: Sulle forme cubiche dello spazio a cinque dimensioni aventi il massimo numero finito di punti doppi. In: Scritti Mat. off. a Luigi Berzolari, 577–593 (1936) Zbl 0016.22102; Ancora sulle forme cubiche dello spazio a 5 dimensioni aventi il massimo numero finito di punti doppi. In: Atti 1. Congr. Un. Mat. Ital. (Firenze 1937), 254–258 (1938) JFM 64.0697.04

[Tog2] Togliatti, E. G.: Una notevole superficie di 5^o ordine con soli punti doppi isolati. Vierteljschr. Naturforsch. Ges. Zürich **85**, 127–132 (1940) JFM 66.0796.02 MR 0004492

[Wahl] Wahl, J.: Nodes on sextic hypersurfaces in P^3. J. Differential Geom. **48**, 439–444 (1998) Zbl 0931.14025 MR 1638049

[Wal] Walter, C.: Pfaffian subschemes. J. Algebraic Geom. **5** (1996), 671–704. Zbl 0864.14032 MR 1486985