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Group theory. — *Surface classification and local and global fundamental groups, I,* by FABRIZIO CATANESE, communicated on 10 February 2006.

Dedicated to Guido Zappa, the sweet (grand-?) father of Italian Algebra and Geometry, on the occasion of his 90-th birthday

ABSTRACT. — Given a smooth complex surface S, and a compact connected global normal crossing divisor $D = \bigcup_i D_i$, we consider the local fundamental group $\pi_1(T \setminus D)$, where T is a good tubular neighbourhood of D.

One has an exact sequence $1 \to \mathcal{K} \to \Gamma := \pi_1(T - D) \to \Pi := \pi_1(D) \to 1$, and the kernel $\mathcal{K}$ is normally generated by geometric loops γ_i around the curve D_i. Among the main results, which are strong generalizations of a well known theorem of Mumford, is the nontriviality of γ_i in $\Gamma = \pi_1(T - D)$, provided all the curves D_i of genus zero have self-intersection $D_i^2 \leq -2$ (in particular this holds if the canonical divisor K_S is nef on D), and under the technical assumption that the dual graph of D is a tree.

KEY WORDS: Fundamental groups; complex surfaces; 3-manifolds.

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1. INTRODUCTION

In his first mathematical paper [Mu61] David Mumford solved the conjecture of Abhyankar showing that, over the complex numbers $\mathbb{C}$, a normal singular point P of an algebraic surface X is indeed a smooth point if and only if it is topologically simple: more precisely, if and only if the local fundamental group $\pi_{1,\mathrm{loc}}(X, P)$ is trivial.

He derived from this result the interesting corollary that the local ring $\mathcal{O}_{X,P}$ of a normal singular point is factorial if and only if either P is a smooth point, or $\pi_{1,\mathrm{loc}}(X, P)$ is the binary icosahedral group, and the singularity is then analytically isomorphic to

$$\{(x, y, z) \in \mathbb{C}^3 \mid z^2 + x^3 + y^5 = 0\}$$

(a shorter independent proof of this corollary was later found by Shepherd-Barron, cf. [S-B99]; this proof is similar in spirit to the one by Lipman in [Lip69]).

Since the local fundamental group is the fundamental group of $U - \{P\}$ where U is a good neighbourhood of P in X, Mumford considered the minimal normal crossing resolution of the singularity, and derived the above theorem from the following.

Let $D = \bigcup_i D_i$ be a compact connected normal crossing divisor on a smooth algebraic surface S, such that the intersection matrix $(D_i \cdot D_j)$ is negative definite. Then the local fundamental group around D, i.e., the fundamental group $\Gamma := \pi_1(T - D)$ where T is a good tubular neighbourhood of D, is trivial if and only if D is an exceptional divisor of the first kind (i.e., D is obtained by successive blowing-ups starting from a smooth point of another algebraic surface).

Our purpose here is threefold:

1) First, we want to show that the theorem has more to do with a basic concept appearing in surface classification rather than with singularities; i.e., that the crucial hypothesis is not that the matrix $(D_i \cdot D_j)$ be negative definite, but that the canonical divisor K_S of S be nef on D (this happens for a minimal model of a nonruled algebraic surface). For a nonexpert: the condition that K_S be nef on D means that, if g_i is the genus of the smooth curve D_i, then $2g_i - 2 \geq D_i^2$ for each i.

As a matter of fact, this condition will only be needed for the curves D_i of genus zero, for which it reads $D_i^2 \leq -2$.

2) Second, since the structure of the group $\pi_1(D)$ is very well understood and there is an obvious surjection $\Gamma = \pi_1(T - D) \to \Pi := \pi_1(D)$, we want to study in general how big the kernel $\mathcal{K}$ of this surjection is. Then the result is that under the above nefness hypothesis each standard generator of $\mathcal{K}$, i.e., each simple loop γ_j around a component D_j, is nontrivial in $\pi_1(T - D)$.

More precisely, we would like to show that, apart from a well described family of exceptions, this generator γ_j has infinite order.

It is rather clear that, in order to have a very simple formulation, the hypothesis that K_S be nef on D is necessary.

In fact, if we let D be a line in $\mathbb{P}^2$, the local fundamental group around D is trivial and we have $K_{\mathbb{P}^2} D = -3$; similarly if we take a (-1)-curve (a smooth rational curve with self-intersection -1, hence with $K_S D = -1$).

A slightly more complicated example, obtained by blowing up the central point of a string of four (-2)-rational curves, shows that the local fundamental group may be nontrivial, yet some γ_i may be trivial, if we do not use the nefness assumption.

The simplest results we have in the direction explained above are the following Theorems A, B, C.

Among these, Theorem A is the simplest to state:

THEOREM 1 (Weak Plumbing Theorem A). *Let $D = \bigcup_i D_i$ be a connected compact (global) normal crossing divisor on a smooth complex surface S. Assume that the dual graph $\mathcal{G}$ of D is a tree. Let Σ be the boundary of a good tubular neighbourhood T of D, $T = \bigcup_i T_i$. The generator γ_i of the kernel $\cong \mathbb{Z}$ of $\pi_1(T_i - D_i) \to \pi_1(D_i)$ has a nontrivial image in $\pi_1(\Sigma) \cong \pi_1(T - D)$ under the assumption that the canonical divisor K_S of the surface S is nef on the components of D of genus 0, i.e., $K_S D_i \geq 0$ for each i such that D_i has genus zero.*

REMARK 1. Observe that we do not need S to be compact: this hypothesis would entail, by the index theorem, that the positivity index of the matrix $(D_i \cdot D_j)$ is ≤ 1. Therefore, our result concerns all the 3-manifolds Σ which are boundaries of complex surfaces obtained by plumbing smooth compact complex curves.

More generally, we have the more precise

THEOREM 2 (Strong Plumbing Theorem B). *Let $D = \bigcup_i D_i$ be a connected compact (global) normal crossing divisor on a smooth complex surface S. Assume that the dual* *graph $\mathcal{G}$ of D is a tree. Let Σ be the boundary of a good tubular neighbourhood T of D, $T = \bigcup_i T_i$. Then the generator γ_i of the kernel $\cong \mathbb{Z}$ of $\pi_1(T_i - D_i) \to \pi_1(D_i)$ has a nontrivial image in $\Gamma := \pi_1(\Sigma) \cong \pi_1(T - D)$ if*

(i) *D is minimal, i.e., it is not obtained by blowing up a (global) normal crossing divisor D', and moreover either*

(ii-1) *after successively blowing down all the rational (-1)-curves we get a divisor D' contained in a smooth complex surface S' and such that $K_{S'}$ is nef on the components D'_i corresponding to D_i of genus zero, or*

(ii-2) *if D_i has genus zero, then its self-intersection is negative.*

3) Our motivation for studying these questions came from the study of topological characterizations of the existence of fibrations on algebraic surfaces, especially in the noncompact case, where (cf. [Cat00]) one has to consider the fundamental group at infinity, which is a disjoint union of local fundamental groups $\pi_1(T - D)$.

The goal is to get new and simpler variants of the characterizations of the Zariski open sets which are complements of unions of fibres of a fibration containing all the singular fibres. These were given in [Cat00, Theorem 5.7] for constant moduli fibrations, and in [Cat03, Theorem 6.4] in the general case.

Indeed, in these theorems there is one condition pertaining to the fundamental group at infinity, namely that, given a certain group homomorphism, each γ_i maps to a certain element of infinite order.

So, a natural question is: when does each γ_i have infinite order in $\pi_1(T - D)$?

We have some partial result concerning this question, which we hope to be able to improve in the future:

THEOREM 3 (Plumbing Theorem C). *Let $D = \bigcup_i D_i$ be a connected compact (global) normal crossing divisor on a smooth complex surface S satisfying the assumptions of Theorem A (for instance, the dual graph $\mathcal{G}$ of D is still assumed to be a tree). Define D to be* **elementary infinite** *if either*

1) *$\mathcal{G}$ is a linear tree and there is a curve of positive genus, or*
2) *D is a* **comb** *(i.e., $\mathcal{G}$ contains only one vertex of valency 3) and there is a curve of positive genus, or all curves are of genus 0, but we are not in the exceptional cases* (Va) *and* (Vb) *of Theorem* 9.

Let Σ, Γ, γ_i be as in the previous theorems. Then each γ_i has infinite order in Γ if there is a sequence of moves, consisting in successively removing curves D_i which intersect two or more other curves, such that in the end one is left with a bunch of disjoint elementary infinite pieces.

Actually, since it can happen that the normal crossing configuration is not minimal, it would certainly be interesting to give general necessary and sufficient conditions also for the nontriviality of each γ_i (this might be very complicated, we fear).

For the applications mentioned above, however, we need to treat the general case and we cannot restrict ourselves to the situation where the dual graph is a tree, which is treated in this article.

As a matter of fact, at some point we thought we could easily reduce the case where the dual graph is not a tree to the difficult case where we have a tree: but about five years ago, when we were writing up a first version of the article, we realized that this reduction argument was not correct.

One reason why we have now written down the tree case is because this article owes much to Guido Zappa. When I started to think about these questions, I received a kind letter of Zappa, which was somehow related to my election as a corresponding member of the Accademia dei Lincei, and it was only natural to ask him some question in combinatorial group theory. Zappa not only answered, providing a result which is included in the article (cf. Proposition 4), but he was very kind to continue to read and answer my letters.

Thus this article is particularly appropriate for this special issue of the Rendiconti Lincei, dedicated to Guido Zappa. I am indebted to him, to his wife Giuseppina Casadio and also to Antonio Rosati for orienting my choice towards mathematics. Giuseppina Casadio ran some afternoon seminars in the Liceo Ginnasio 'Michelangelo' in the last year of my (classical studies) high-school. There I learnt such basic things as, for instance, congruences, and I was encouraged to take part in the Mathesis competitions first and the mathematical Olympics later. Rosati incited me over the summer to read parts of Courant and Robbins' book 'What is mathematics', and to apply for admission to the Scuola Normale Superiore di Pisa.

In Pisa the education was very analysis oriented, but later on in my life I discovered in myself something of an algebraist's soul which was longing to learn more.

For this part of my soul Zappa was a reference figure, and I was later quite happy to have finally a chance, during the Meetings of the Accademia, to discuss mathematical questions with him.

Another reason to write this article now is to take up the problem again, with the hope of finding soon the solution to the general case, and, even more, to propose further investigation of these three-manifold fundamental groups.

For instance, other general interesting questions are in our opinion:

1) How big is the kernel $\mathcal{K}$ of $\pi_1(T \setminus D) \to \pi_1(D)$?

2) What properties does $\mathcal{K}$ enjoy, for instance when is it not finitely generated (cf. [Cat03, Definition 3.1 and Lemma 3.4])?

2. A PRESENTATION OF THE LOCAL FUNDAMENTAL GROUP

Let us first of all set up the notation for our problem. We have a smooth complex surface S, and a compact connected global normal crossing divisor $D = \bigcup_i D_i$ contained in S; thus each D_i is a smooth curve of genus g_i and has a good tubular neighbourhood T_i which is a 2-disk bundle over D_i.

$T_i \setminus D_i$ is homotopy equivalent to its boundary Σ_i, which is an S^1-bundle over the compact Riemann surface D_i, and is completely classified by its Chern class, i.e., by the self-intersection number of D_i in S, as we are going to briefly recall.

Denote by m_i the opposite of the self-intersection number of D_i, so that $D_i^2 = -m_i$.

Let now q be a point of D_i. Then the bundle $\Sigma_i \to D_i$ is trivial over $D_i \setminus \{q\}$, and also over a neighbourhood V of q. The respective trivializations are clear if we identify topologically the associated line bundle as the line bundle corresponding to the divisor $-m_i q$.

Since $(D_i - q) \cap V$ is homotopy equivalent to S^1, and the glueing map on $S^1 \times S^1$ reads (we choose the trivialization over $D_i \setminus \{q\}$ in the source, and the one over V in the target)

$$(z, w) \mapsto (z, z^{-m_i} w),$$

from the first van Kampen theorem (cf. e.g. [dR69]) we derive a presentation for the fundamental group of Σ_i, which determines the central extension

$$1 \to \mathbb{Z}\gamma_i \to \pi_1(\Sigma_i) \to \pi_1(D_i) \to 1$$

provided by the homotopy exact sequence of the S^1-bundle.

In fact, in the inverse image of $D_i \setminus \{q\}$, homeomorphic to $(D_i \setminus \{q\}) \times S^1$, we take the lifts of some standard generators of the free group $\pi_1(D_i - q)$; we use for these lifts the usual notation $a_1(i), b_1(i), \ldots, a_{g_i}(i), b_{g_i}(i)$ (recall that g_i is the genus of D_i), and moreover we let γ_i be the generator of the fundamental group of the fibre S^1, with the standard complex counterclockwise orientation.

Since the fundamental group of a Cartesian product is a direct product, it follows, as already mentioned, that γ_i commutes with all other generators.

From the glueing map we get a single further relation:

$$\prod_{h=1}^{g_i} [a_h(i), b_h(i)] = \gamma_i^{-m_i}.$$

If we now take a good tubular neighbourhood T of D which is the union of the T_i's, we may assume moreover (by shrinking the T_i's, and by the implicit function theorem), that the intersection $T_i \cap T_j$ is biholomorphic to

$$\{(z_1, z_2) \mid |z_1 z_2| \leq 1,\ |z_i| \leq 2\},$$

where $z_1 = 0$, $z_2 = 0$ are the respective local equations of T_i, T_j at the point $p_{ij} := D_i \cap D_j$.

In each D_i let us consider a path L_i homeomorphic to a segment and going through all the points p_{ij} and let us mark a point $q_i \in L_i$ different from all the p_{ij}'s. We may assume that we thus get a linear tree L_i with the above points as vertices. Set $L = \bigcup_i L_i$, so L is naturally a graph.

It is important to notice that Σ has a natural projection onto D such that outside the points p_{ij} we have a fibre bundle with fibre S^1, whereas the fibre over p_{ij} is $\cong S^1 \times S^1$.

In fact, the local picture is given by

$$T_i \cap T_j = \{(z_1, z_2) \mid |z_1 z_2| \leq 1,\ |z_i| \leq 2\},$$

thus locally

$$\Sigma = \{(z_1, z_2) \mid |z_1 z_2| = 1,\ |z_i| \leq 2\} \cong S^1 \times S^1 \times [1/2, 2],$$

where the homeomorphism is given by the map sending (z_1, z_2) to $(z_1/|z_1|, z_2/|z_2|, |z_1|)$.

The projection sends $S^1 \times S^1 \times \{1\}$ to $(0, 0)$, whereas e.g. the observation that $S^1 \times S^1 \times [1/2, 1)$ is an S^1-bundle over $S^1 \times [1/2, 1) \cong$ punctured disk in the z_2 plane, allows us to define the projection for $|z_2| \geq 1$ as sending (z_1, z_2) to $(0, z_2(|z_2| - 1))$, and symmetrically for $|z_1| \geq 1$.

It is quite easy to see then that we can find a section of $\Sigma|_L \to L$, so we think of L as contained in $\Sigma|_L$.

Since the restriction of the fibration $\Sigma_i \to D_i$ to L_i is trivial, we see that, up to homotopy equivalence, $\Sigma|_L \to L$ is obtained from the manifolds $L_i^0 \times S^1$ (L_i^0 being a tubular neighbourhood of L_i in D_i) as follows.

We replace the product $B_{ij}^2 \times S^1$ (B_{ij}^2 being an open 2-dimensional ball around p_{ij} in D_i) by a product $A_{ij}^2 \times S^1$ (A_{ij}^2 being a 2-dimensional annulus around p_{ij} in D_i, $A_{ij}^2 \cong S^1 \times [1/2, 1)$). Then we glue together the pieces $A_{ij}^2 \times S^1$ and $A_{ji}^2 \times S^1$ identifying the (inner) boundaries $S^1 \times S^1$.

We now make another arbitrary choice for our presentation, namely, since the graph L is connected, we may take a connected subtree $L' \subset L$ containing all the points q_i.

We let one of them, say q_0, be the base point. For each q_i we get a canonical path in L' from q_0 to q_i, whence a canonical basis of $\pi_1(L)$ is given by the loops λ_{ij}, for $p_{ij} \notin L'$, obtained by going from q_0 to q_i along the canonical path, then going to p_{ij} inside L_i, then to q_j inside L_j, then back to q_0 again along the canonical path.

The above description makes it clear that, exchanging the roles of the two indices i, j, we get $\lambda_{ji} = \lambda_{ij}^{-1}$.

Let γ_i be the positively oriented generator of the infinite cyclic fundamental group of $(L_i^0 \times S^1) \cup L'$. Then we find immediately the following presentation for the fundamental group of Σ restricted to L^0 ($L^0 = \bigcup_i L_i^0$).

Generators:

- γ_i for each i,
- λ_{ij} for $p_{ij} \notin L'$.

In order to get the relations, set, for each $p_{ij} \in L$,

- $\gamma_{ij} = \gamma_j$ for $p_{ij} \in L'$,
- $\gamma_{ij} = \lambda_{ij}\gamma_j\lambda_{ij}^{-1}$ for $p_{ij} \notin L'$,

with the above convention that $\lambda_{ji} = \lambda_{ij}^{-1}$. Then we get the

Local commutation relations: $[\gamma_i, \gamma_{ij}] = 1$ (for each $p_{ij} \in L$).

To complete the presentation of $\pi_1(\Sigma)$, we use several times the first van Kampen theorem (cf. [dR69]), adding $\Sigma|_{D_i - L_i}$ to Σ restricted to L^0. Note that the S^1-bundle $\Sigma_i \to D_i$ is trivial on L_i^0, and also on $D_i - L_i$.

The corresponding fundamental group is obtained as the amalgamation by $\mathbb{Z}\gamma_i \times \mathbb{Z}\mu_i$ of the free product of the following two groups: the direct product $\mathbb{F}_{2g_i} \times \mathbb{Z}\gamma_i$ ($\mathbb{F}_{2g_i}$ = free group on $2g_i$ generators) and the cyclic group $\mathbb{Z}\gamma_i$.

Here, μ_i maps on the one side to the standard relation for the fundamental group Π_{g_i} of a compact curve of genus g_i, and on the other side to $\gamma_i^{m_i}$.

Now, μ_i is no longer trivial in $\pi_1(L^0)$, so we get the following extra

Generators: $a_1(i), b_1(i), \ldots, a_{g_i}(i), b_{g_i}(i)$ for each i,
Main relations:

$$\prod_{h=1}^{g_i}[a_h(i), b_h(i)] = \gamma_i^{-m_i} \prod_j \gamma_{ij}.$$

Moreover, since we have a direct product $\mathbb{F}_{2g_i} \times \mathbb{Z}\gamma_i$, we should not forget the obvious relations:

Global commutation relations: $[a_h(i), \gamma_i] = [\gamma_i, b_h(i)] = 1$.

3. PRESENTATION OF A SIMPLIFIED GROUP

Summarizing the result of the previous section, we have obtained the finitely presented group Γ with:

Generators:

- γ_i for each i,
- λ_{ij} for $p_{ij} \notin L'$,
- $a_1(i), b_1(i), \dots, a_{g_i}(i), b_{g_i}(i)$ for each i.

Relations:

- $[a_h(i), \gamma_i] = [\gamma_i, b_h(i)] = 1$ for each i, h,
- $\prod_{h=1}^{g_i} [a_h(i), b_h(i)] = \gamma_i^{-m_i} \prod_j \gamma_{ij}$ for each i,
- $[\gamma_i, \gamma_{ij}] = 1$ (for each $p_{ij} \in L$), where

  (I) $\gamma_{ij} = \gamma_j$ for $p_{ij} \in L'$,
  (II) $\gamma_{ij} = \lambda_{ij}\gamma_j\lambda_{ij}^{-1}$ for $p_{ij} \notin L'$,
  (III) $\lambda_{ji} = \lambda_{ij}^{-1}$.

REMARK 2. The projection $p : \Sigma \to D$ induces a surjection of fundamental groups $\Gamma \to \pi_1(D)$ with kernel $\mathcal{K}$ normally generated by the γ_i's. In fact, setting in the above presentation $\gamma_i = 1$ for all i, we get a free product of the fundamental groups $\pi_1(D_i)$ with the free group generated by the λ_{ij}'s (observe that $\lambda_{ji} = \lambda_{ij}^{-1}$, whence the rank of this free group is equal to the first Betti number of L).

DEFINITION 1. *The associated finitely presented* simplified group Γ' *is the group with:*

Generators:

- γ_i *for each* i,
- λ_{ij} *for* $p_{ij} \notin L'$,
- a_i, b_i *for each* i *such that* $g_i \geq 1$;

Relations:

(Global commutation relations) $[a_i, \gamma_i] = [\gamma_i, b_i] = 1$ *for each* i,
(Main relations) $[a_i, b_i] = \gamma_i^{-m_i} \prod_j \gamma_{ij}$ *for each* i,
(Local commutation relations) $[\gamma_i, \gamma_{ij}] = 1$ *(for each* $p_{ij} \in L$*) where, as above,* $\gamma_{ij} = \gamma_j$ *for* $p_{ij} \in L'$*, else (keeping in mind* $\lambda_{ji} = \lambda_{ij}^{-1}$*)* $\gamma_{ij} = \lambda_{ij}\gamma_j\lambda_{ij}^{-1}$.

REMARK 3. We can restrict ourselves to proving our results for the simplified groups Γ', which are also obtained from a plumbing procedure, replacing the (smooth) curves of genus ≥ 2 by genus 1 curves.

In fact, the simplified group Γ' is a homomorphic image of Γ, obtained by imposing the further relations

$$a_h(i) = b_h(i) = 1 \quad \text{for } h \geq 2.$$

Thus, if γ_j is nontrivial, respectively of infinite order, in the simplified group Γ' it is so a fortiori in the group Γ. Moreover, observe that our hypotheses only concern the nullity or positivity of the genus of D_j, and not its precise value.

For instance, the minimality of D in the category of normal crossing divisors amounts to the nonexistence of rational curves with self-intersection -1, and meeting at most two other curves each in at most one point. Thus, we see easily that hypothesis (i) of Theorem B is still satisfied for the simplified group, and likewise for the hypothesis of Theorem A.

However, the canonical divisor K' of the simplified surface may not be nef, since if there is a component D_i with genus ≥ 2, in the new configuration C we get a corresponding C_i with genus 1 and $K'C_i = -C_i^2 = -D_i^2 = -(2g(D_i) - 2) + KD_i$, which may become negative.

The proof of the main theorems follows by a reduction procedure which we examine in the next section.

4. Reduction to the case of a graph of rational curves

Recall that we are working in the simplified group.

In the case where we get a component of genus 1, we will be able to simultaneously remove the generators a_j, b_j, and replace the number m_j by an arbitrary integer n_j (in fact, one could say that we can have $n_j = \infty$, meaning that the corresponding main relation disappears).

If we can achieve this, certainly the nefness condition will continue to hold for the new configuration. To this end, fix the index j, write

$$a_j := a, \quad b_j := b, \quad \gamma := \gamma_j,$$

and consider the group G with generators

- γ_i for each i,
- a_i, b_i for the i's such that $g_i \geq 1$ and $i \neq j$,

and with relations

- $[a_i, \gamma_i] = [\gamma_i, b_i] = 1$ for each $i \neq j$,
- $[a_i, b_i] = \gamma_i^{-m_i} \prod_h \gamma_{ih}$ for each $i \neq j$,
- $[\gamma_i, \gamma_{ih}] = 1$ (for each $p_{ih} \in L$).

The group Γ is obtained from G by adding generators a, b, and relations

- $[a, \gamma] = [\gamma, b] = 1$ where $\gamma := \gamma_j$ is an element of G,
- $[a, b] = \gamma^{-m} \prod_h \gamma_{jh}$.

We may rewrite the last relation simply as

- $[a, b] = \gamma''$.

Note that, in the group G, $[\gamma, \gamma''] = 1$, since γ commutes with each γ_{jh}.

We now use:

PROPOSITION 4. *Given a group G and elements $\gamma, \gamma'' \in G$ such that $[\gamma, \gamma''] = 1$, let Γ be the quotient of the free product of G and a free group on two generators a, b, obtained by imposing the following relations:*

$$[a, \gamma] = [\gamma, b] = 1, \quad [a, b] = \gamma''.$$

Then the natural homomorphism of G into Γ is injective.

PROOF. We consider the quotient group Δ of Γ obtained by adding the commutation relations $[a, \gamma''] = [\gamma'', b] = 1$. An equivalent way to describe Δ is the following.

Let H be the Heisenberg group with generators a, b, c and relations $[a, b] = c$, $[a, c] = [b, c] = 1$. Then H is a two-step nilpotent group with infinite cyclic centre generated by c, and with abelianization free of rank 2. The elements in H can be uniquely written as words $a^m b^n c^k$, where k, m, n are integers.

Then we can define Δ as the quotient of the free product of H and G modulo the relations

$$\gamma'' = c, \quad [a, \gamma] = [\gamma, b] = 1.$$

At this point we are not able to give a unique representation for the elements of Δ, but we follow an idea of Guido Zappa.

Namely, we observe that every element of Δ can be written as a product

$$h = g_0 a^{m(1)} b^{n(1)} g_1 a^{m(2)} b^{n(2)} \cdots g_{r-1} a^{m(r)} b^{n(r)} g_r,$$

where each pair of exponents $(m(j), n(j))$ is $\neq (0, 0)$, $g_0, \dots, g_r$ are elements of G and we can assume that $g_1, \dots, g_{r-1}$ do not belong to the subgroup B generated by γ, γ'' in G (whereas g_0 and g_r could even be trivial).

It remains to see when two such products yield the same element h. Notice that the condition that $g_1, \dots, g_{r-1}$ do not belong to B follows from the property that r be minimal.

We claim that r is uniquely determined, and that the only allowed transformations of the minimal representation are obtained by letting factors γ, γ'' commute with a, resp. b.

More precisely, we claim that we get an equivalent minimal product iff:

- we replace each g_i ($i = 1, \dots, r-1$) by multiplying it by an element $g \in B$, and correspondingly:
- if g_i is replaced by $g_i g$, then g_{i+1} is replaced by $g^{-1} g_{i+1}$,
- if g_i is replaced by $g g_i$, then g_{i-1} is replaced by $g_{i-1} g^{-1}$.

This means that, for each i, the exponents $(m(j), n(j))$ are uniquely determined; moreover, the double coset $B g_i B$ is uniquely determined, and finally the product $g_0 \cdots g_r$ is uniquely determined. In particular, it follows that our element is in G iff $r = 0$, and in this case the representation is unique, which is precisely the assertion of the proposition.

To establish our claim, consider the equivalence classes of the products h described above. It suffices to show that we have an action of the generators of the group Δ which satisfies the defining relations for Δ. This is clear for the elements of the group G, and also for the generators a, b, and an easy verification shows that the relations are satisfied. □

REMARK 4. Notice that if we fix an integer n_j and in the group G we add the relation

$$1 = \gamma_j^{-n_j} \prod_h \gamma_{ih},$$

we obtain the fundamental group of the graph of curves where the elliptic curve C_j with self-intersection $(-m_j)$ has been replaced by a smooth curve $\cong \mathbb{P}^1$ with self-intersection $(-n_j)$. We can therefore by induction reduce to the case of a graph of rational curves.

5. THE CASE OF A TREE OF SMOOTH RATIONAL CURVES

Here we have a presentation with

Generators:

- γ_i for each i;

Relations:

- $1 = \gamma_i^{-m_i} \prod_j \gamma_{ij}$ for each i,
- $[\gamma_i, \gamma_j] = 1$ (for each $p_{ij} \in L$).

We first show the necessity of the nefness hypothesis in Theorem A.

EXAMPLE 1. Consider a diagram of type A_n, i.e., a linear tree with n vertices. Then our group, as we shall shortly see, is generated by $\gamma_1, \dots, \gamma_n$ with relations

$$\gamma_1^2 = \gamma_2, \quad \gamma_2^2 = \gamma_1\gamma_3, \quad \gamma_3^2 = \gamma_2\gamma_4, \ \dots, \ \gamma_{n-1}^2 = \gamma_{n-2}\gamma_n, \quad \gamma_n^2 = \gamma_{n-1}.$$

Therefore, the group is cyclic, generated by $\gamma := \gamma_1$ with $\gamma^{n+1} = 1$, and we have $\gamma_i = \gamma^i$.

Let $n = 4$, and let us now blow up the central point of intersection between C_2 and C_3. We then obtain a new generator γ' (the loop around the exceptional curve) and the relation $\gamma' = \gamma_2 \cdot \gamma_3$, but now $\gamma' = \gamma_2 \cdot \gamma_3 = 1$!

We recall that, for a tree of rational curves on a complex surface, the condition that the divisor K_S is nef reads

1) $D_i^2 \leq -2$.

If we are on an algebraic surface, the index theorem says that

2) the intersection matrix $(D_i \cdot D_j)$ has positivity index $b^+ \leq 1$.

An easy example where 1) holds but $b^+ = 1$ is provided by a tree of rational (-2)-curves, where all curves meet a central one (the dual graph is a star).

In fact, if D_0 is the central curve, we then have

$$(mD_0 + D_1 + \cdots + D_n)^2 = 2(-m^2 + mn - n),$$

which is positive for $1 < m < n - 1$.

Then the group is generated by $\gamma_1, \dots, \gamma_n, \delta$ with relations

$$\gamma_i^2 = \delta, \quad \delta^2 = \gamma_1\gamma_2 \cdots \gamma_n.$$

In this case the abelianization is the direct sum of cyclic groups of respective orders $2(n-4), 2, \dots, 2$, with generators induced by the respective residue classes of $\gamma_1, \gamma_1^{-1}\gamma_2, \dots, \gamma_1^{-1}\gamma_{n-1}$, whence our standard generators even have a nontrivial image in the maximal abelian quotient.

We now proceed to analyse the different cases.

5A. *Case of a linear tree of rational curves*

LEMMA 5. *Assume that we have a linear tree of n smooth rational curves with self-intersection $(-m_i)$, where $m_i \geq 2$. Set inductively $a_1 := 1$, $a_2 := m_1$, $a_{i+1} := m_i a_i - a_{i-1}$. Then*

1) $a_{i+1} > a_i$;
2) *our group Γ is a cyclic group of order a_{n+1}, generated by γ_1;*
3) *the element γ_i equals $\gamma_1^{a_i}$, and is not trivial.*

PROOF. We can write our relations among $\gamma_1, \dots, \gamma_n$ as

$$\gamma_1^{m_1} = \gamma_2, \quad \gamma_2^{m_2} = \gamma_1\gamma_3, \ \dots, \ \gamma_i^{m_i} = \gamma_{i-1}\gamma_{i+1}, \ \dots, \ \gamma_{n-1}^{m_{n-1}} = \gamma_{n-2}\gamma_n, \quad \gamma_n^{m_n} = \gamma_{n-1}.$$

We then easily obtain

$$\gamma_{i+1} = \gamma_{i-1}^{-1}\gamma_i^{m_i} = \gamma_1^{-a_{i-1}}\gamma_i^{a_i m_i} = \gamma_1^{a_{i+1}},$$

which proves the first part of assertion 3); on the other hand, the last relation yields $\gamma_1^{a_{n+1}} = 1$, which proves assertion 2).

Notice that

$$a_{i+1} - a_i = m_i a_i - a_{i-1} - a_i = (m_i - 1)a_i - a_{i-1} > 0$$

since $m_i \geq 2$ and since by induction $a_i > a_{i-1}$.

Hence, assertion 1) is proved, and simultaneously we have shown that each γ_i is nontrivial. □

REMARK 5. The proof of the above lemma shows that in any case the local fundamental group of a tree of rational curves is cyclic, of order a_{n+1} if a_{n+1} is nonzero.

Assume now that all the numbers m_i are strictly positive. Then, if $m_i = 1$, we obtain $\gamma_i = \gamma_{i-1}\gamma_{i+1}$, and since the group is abelian, we may rewrite the relation $\gamma_{i-1}^{m_{i-1}} = \gamma_{i-2}\gamma_i$ as $\gamma_{i-1}^{m_{i-1}-1} = \gamma_{i-2}\gamma_{i+1}$, and similarly $\gamma_{i+1}^{m_{i+1}} = \gamma_i\gamma_{i+2}$ becomes $\gamma_{i+1}^{m_{i+1}-1} = \gamma_{i-1}\gamma_{i+2}$. This has the obvious geometrical meaning that we can blow down all the (-1)-curves, and then if at the end of the process K remains nef, our remaining elements γ_i are not trivial.

REMARK 6. Assume that we let $m_i \to \infty$. Then also $a_{i+1} \to \infty$, hence $a_{n+1} \to \infty$, whereas a_j remains constant for $j \leq i$. Hence, $\mathrm{ord}(\gamma_j) \to \infty$ for $j \leq i$. Changing the linear order of the linear tree to its inverse, we see that $\mathrm{ord}(\gamma_j) \to \infty$ also for $j \geq i$.

5B. Reduction to the case of a comb of rational curves

LEMMA 6. *Let G_1, G_2 be groups and let a_i be nontrivial elements in G_i for $i = 1, 2$ such that moreover a_2 has infinite order in G_2. If Γ is the quotient of the free product $G_1 * G_2$ by the relation $a_1 a_2 = 1$, then the natural homomorphism of G_1 in Γ is injective. Moreover, if a_1 does not generate G_1 and a_2 does not generate G_2, then Γ is always an infinite group.*

PROOF. The desired claim follows if we show that the elements in Γ are represented by elements of the set $\mathcal{W}$ of equivalence classes of "good" words

$$w = g_1(1) \cdot g_2(1) \cdot g_1(2) \cdots g_1(k) \cdot g_2(k) \cdot g_1(k+1),$$

where $g_2(i)$ does not belong to the subgroup generated by a_2 for $1 \le i \le k$, and $g_1(j)$ does not belong to the subgroup generated by a_1 for $2 \le j \le k$, and w is equivalent to w' if and only if the following conditions hold:

1) $k = k'$,
2) there exist integers ("r" for right, "λ" for left) $r_1, \lambda_2, r_2, \lambda_3, \dots, r_k, \lambda_{k+1}$ such that the word w' equals

$$(g_1(1)a_1^{r_1}) \cdot (a_2^{r_1} g_2(1) a_2^{\lambda_2}) \cdot (a_1^{\lambda_2} g_1(2) a_1^{r_2}) \\ \cdots (a_1^{\lambda_k} g_1(k) a_1^{r_k}) \cdot (a_2^{r_k} g_2(k) a_2^{\lambda_{k+1}}) \cdot (a_1^{\lambda_{k+1}} g_1(k+1)).$$

We let the elements of Γ operate by left multiplication as follows:

- for $\gamma_1 \in G_1$ we let $\gamma_1 w := (\gamma_1 g_1(1)) \cdot g_2(1) \cdot g_1(2) \cdots g_1(k) \cdot g_2(k) \cdot g_1(k+1)$,
- for $\gamma_2 \in G_2$ not in the subgroup generated by a_2 we let

$$\gamma_2 w := e_1 \cdot \gamma_2 \cdot g_1(1) \cdot g_2(1) \cdot g_1(2) \cdots g_1(k) \cdot g_2(k) \cdot g_1(k+1),$$

  (e_i being the identity element of G_i), while we set
- $a_2^r w := a_1^{-r} w$.

We obtain a homomorphism of each G_i into the group $\mathcal{S}(\mathcal{W})$ of permutations of $\mathcal{W}$, and moreover the transformation associated to $a_1 a_2$ is by definition the identity, whence we get a homomorphism of Γ into $\mathcal{S}(\mathcal{W})$.

Moreover, Γ acts transitively on $\mathcal{W}$. Representing each element of Γ by a good word w, we see that if w is the identity this implies that $k = 0$, and $g_1(1) = e_1$.

Thus the action on e_1 establishes a bijection between Γ and $\mathcal{W}$; in particular, since the words with $k = 0$ correspond to the elements of G_1, G_1 injects into $\mathcal{W}$, whence into Γ. Notice finally that if a_2 generates G_2 then G_1 is isomorphic to Γ, and similarly if a_1 generates G_1.

On the other hand, if a_i does not generate G_i for $i = 1, 2$, then k can be arbitrarily large, whence Γ is surely infinite. □

COROLLARY 7. *Let $G_1, \dots, G_r$ be groups and let a_i for $i = 1, \dots, r$ be a nontrivial element in G_i. If Γ is the quotient of the free product $G_1 * \cdots * G_r$ by the relation $a_1 \cdots a_r = 1$, then for $r \ge 3$ the natural homomorphism of G_1 in Γ is injective. Moreover, if $r \ge 4$, then the group Γ is infinite.*

PROOF. Apply Lemma 6, considering that $a_2 \cdots a_r$ is an element of infinite order in $G_2 * \cdots * G_r$. In the case $r \geq 4$, apply the lemma to $G_1 * G_2$ and $G_3 * \cdots * G_r$, taking into consideration that both are infinite and not cyclic. □

With the aid of the foregoing corollary we are able to reduce the proof of our main results to a very special case.

PROPOSITION 8. *Let γ_i be one of our generators of the group Γ, in the case where the hypotheses of Theorem B are satisfied. Then γ_i is nontrivial except possibly if the tree is nonlinear and the curve D_i is the only one which intersects at least three other irreducible components of D (we shall then say that the tree is a* **comb**, *and that D_i is the rim of the comb).*

PROOF. The case where the tree is linear was already dealt with. So, assume that there exists a curve D_j with $i \neq j$ such that D_j intersects at least three other irreducible components of D. Consider the group G obtained as the quotient of Γ by the relation $\gamma_j = 1$.

If $D - D_j$ (the difference of divisors, and not of sets) has r connected components $D(1), \dots, D(r)$, we see immediately that G is the quotient of the free product $G_1 * \cdots * G_r$ by the relation $a_1 \cdots a_r = 1$, where G_h is the fundamental group of the boundary of a good tubular neighbourhood of $D(h)$, and a_h is the loop around the unique irreducible component of $D(h)$ meeting D_j. By our corollary, and since by induction we may assume that each a_i, $i = 1, \dots, r$, is nontrivial, we conclude that each G_h injects into G, and a fortiori into Γ.

Hence, all elements γ_i with $i \neq j$ are nontrivial. □

5C. The rim of a comb of rational curves

Assume that we have a unique curve D_j such that $D - D_j$ has $r \geq 3$ connected components $D(1), \dots, D(r)$, each being a chain of smooth rational curves. Set for convenience $\gamma := \gamma_j$. We shall then say as before that we have a **comb** with **rim** D_j and with **strings** $D(1), \dots, D(r)$.

Then, for each chain $D(h)$, we can order the generators in such a way that we obtain the relations

$$\gamma_1^{m_1} = \gamma_2, \quad \gamma_2^{m_2} = \gamma_1\gamma_3, \ \dots, \ \gamma_i^{m_i} = \gamma_{i-1}\gamma_{i+1}, \quad \gamma_{n-1}^{m_{n-1}} = \gamma_{n-2}\gamma_n.$$

Proceeding as in Section 5A, we infer that $\gamma = \gamma_1^{a_n}$, where $a_n > 0$ is defined inductively as there.

Finally, letting $(-m)$ be the self-intersection of D_j, we obtain a relation

$$\gamma^m = \beta_1^{d_1} \cdots \beta_r^{d_r},$$

where the β_h's are loops, for each chain $D(h)$, around the end opposite to D_j.

We are left with the following

THEOREM 9. *Let $\Gamma(m, b_1, \dots, b_r; d_1, \dots, d_r)$, for integers $m \geq 2$ and $b_i > d_i \geq 1$, be the group with*

(i) *generators $\gamma, \beta_1, \dots, \beta_r$ and relations*
(ii) $\gamma = \beta_1^{b_1} = \cdots = \beta_r^{b_r}$ *(recall that the integers b_h are ≥ 2), and*
(iii) $\gamma^m = \beta_1^{d_1} \cdots \beta_r^{d_r}$.

Then the (central) element γ is nontrivial inside Γ and indeed of infinite order unless we are in the following exceptional cases with $r = 3$, and where $c = 1, 2$ and $1 \leq t \leq n - 1$:

(Va) $(b_1, b_2, b_3) = (2, 2, n)$, $n \geq 2$, $(d_1, d_2, d_3) = (1, 1, t)$,
(Vb) $(b_1, b_2, b_3) = (2, 3, n)$, $3 \leq n \leq 5$, $(d_1, d_2, d_3) = (1, c, t)$.

PROOF. STEP I. We may assume that G.C.D.$(b_i, d_i) = 1$ for each i. This is a consequence of the following Logical Principle Lemma of combinatorial group thery.

LEMMA 10 (Logical Principle Lemma). *Let G be a finitely presented group*

$$G = \langle \beta_1, \dots, \beta_r \mid R_1(\beta) = \cdots = R_s(\beta) = 1 \rangle.$$

Then, setting $\beta_1 = \beta^k$, i.e., taking the new group $G'' := G * \mathbb{Z}/\langle\langle \beta_1 \beta^{-k} \rangle\rangle$, we get $\mathrm{ord}_{G''}(\beta) = k \cdot \mathrm{ord}_G(\beta_1)$, *while, for $j \geq 2$,* $\mathrm{ord}_{G''}(\beta_j) = \mathrm{ord}_G(\beta_j)$.

PROOF. The situation is a particular case of Lemma 6 with $a_1 = \beta_1$ and $a_2 = \beta^{-k}$. The injectivity of the map $G \to G''$ implies the desired assertion. □

Clearly, if $c_i =$ G.C.D.(b_i, d_i) and Δ is the group $\Gamma(m, b_1/c_1, \dots, b_r/c_r; d_1/c_1, \dots, d_r/c_r)$, an iterated application of the Logical Principle shows that the order of γ is the same in Γ and in Δ.

STEP II. Let $T := T(m, b_1, \dots, b_r; d_1, \dots, d_r)$ be the quotient of the group $\Gamma(m, b_1, \dots, b_r; d_1, \dots, d_r)$ by the central cyclic subgroup $C(\gamma)$ generated by γ. Then by Step I, T is isomorphic to the polygonal group $T(b_1, \dots, b_r)$ with generators $\delta_1, \dots, \delta_r$ and relations $\delta_1^{b_1} = \cdots = \delta_r^{b_r} = \delta_1 \cdots \delta_r = 1$.

In fact $T(m, b_1, \dots, b_r; d_1, \dots, d_r)$ is the quotient of the free product of cyclic groups of respective orders b_i by the relation $\beta_1^{d_1} \cdots \beta_r^{d_r} = 1$. But, since G.C.D.$(b_i, d_i) = 1$, each $\beta_i^{d_i} := \delta_i$ is a generator of the respective cyclic group.

STEPS III–V. We thus have a central extension

$$1 \to C(\gamma) \to \Gamma(m, b_1, \dots, b_r; d_1, \dots d_r) \to T(b_1, \dots, b_r) \to 1,$$

where $C(\gamma)$ is the cyclic central subgroup generated by γ, and the quotient $T := T(b_1, \dots, b_r)$ is the polygonal group defined above.

Our strategy will consist in proving that either

(III) the image of γ is nontrivial in $\mathbb{Q}$-homology (i.e., in the abelianization of Γ tensored with $\mathbb{Q}$), whence a fortiori γ has infinite order in Γ, or

(IV) $H^1(\Gamma, \mathbb{Q}) = 0$; however, then, in the nonexceptional cases, Γ differs from T because it has cohomological dimension 3 instead of 2, and thus in any case γ has infinite order in Γ.

(V) then treats the exceptional cases using integral homology and matrix representations.

STEP III. The above odd looking alternative is a consequence of the following

PROPOSITION 11. *Let Γ be the above group $\Gamma(m, b_1, \dots, b_r; d_1, \dots, d_r)$. Then the image of γ in $H_1(\Gamma, \mathbb{Q})$ is a generator, and it is nonzero if and only if $m \neq \sum_i d_i/b_i$.*

PROOF. Let $[\gamma], [\beta_i]$ be the respective images of γ, β_i in $H_1(\Gamma, \mathbb{Q})$. Then they generate it and the only relations are

$$[\beta_i] = (1/b_i)[\gamma], \qquad \Big(m - \sum_i d_i/b_i\Big)[\gamma] = 0.$$

Hence, $[\gamma]$ generates $H_1(\Gamma, \mathbb{Q})$, and $H_1(\Gamma, \mathbb{Q}) \neq 0$ if and only if $m = \sum_i d_i/b_i$. □

STEP IV. Assume now that $H_1(\Gamma, \mathbb{Q}) = 0$, and observe that, because of our plumbing construction, Γ is the fundamental group of an orientable 3-manifold $M := \Sigma$. In particular, $H_1(M, \mathbb{Q}) = H_1(\Gamma, \mathbb{Q}) = 0$, and by Poincaré duality and ordinary duality $H^1(M, \mathbb{Q}) = H^2(M, \mathbb{Q}) = 0$, while $H^3(M, \mathbb{Q}) \cong \mathbb{Q}$. Let N be the universal covering of M. Then we have a spectral sequence $H^p(\Gamma, H^q(N, Q))$ converging to the graded module associated to a suitable filtration of $H^{p+q}(M, Q)$, for each ring Q ($Q = \mathbb{Z}$ or $\mathbb{Q}$ in our application).

Clearly, $H^1(N, \mathbb{Q}) = 0$, hence $H^2(M, \mathbb{Q}) = 0$ implies $H^2(\Gamma, \mathbb{Q}) = 0$.

We can moreover apply (cf. [Wei94, 6.8.2]) the Lyndon–Hochschild–Serre spectral sequence associated to the exact sequence

$$1 \to C(\gamma) \to \Gamma := \Gamma(m, b_1, \dots, b_r; d_1, \dots, d_r) \to T \to 1,$$

whose E_2 term is $H^p(T, H^q(C(\gamma), \mathbb{Q}))$ and which converges to a graded quotient of $H^{p+q}(\Gamma, \mathbb{Q})$.

Now, if γ had finite order, then $H^i(C(\gamma), \mathbb{Q}) = 0$ for each $i \geq 1$, whence $H^i(\Gamma, \mathbb{Q}) = H^i(T, \mathbb{Q})$ for each $i \geq 0$.

We get therefore an obvious contradiction in the case where $H^2(T, \mathbb{Q}) \neq 0$.

Observe that the polygonal group T is a quotient of the group Π with generators $\beta_1, \dots, \beta_r$ and with relation $\beta_1 \cdots \beta_r = 1$. Π is the fundamental group of $\mathbb{P}^1_{\mathbb{C}}$ minus r points, and T is the orbifold fundamental group of the maximal Galois cover C of $\mathbb{P}^1$ branched at these points with respective ramification multiplicities exactly equal to $b_1 - 1, \dots, b_r - 1$.

If T is infinite, then C is not compact, otherwise $C \cong \mathbb{P}^1$, by the Riemann mapping theorem. Hence if T is infinite, then $H^2(\mathbb{P}^1, \mathbb{Q}) \cong \mathbb{Q} \cong H^2(T, \mathbb{Q})$ and we have found the required contradiction.

Otherwise, T is finite, and $C \to \mathbb{P}^1$ has a finite degree d. As is well known, by the formula of Hurwitz, then $2 - 2/d = \sum_i (1 - 1/b_i)$, which implies that $r \leq 3$, and since $r \geq 3$ we get $r = 3$ and $\sum_i (1 - 1/b_i) > 1$, an inequality which leads us to the exceptional cases for (b_1, b_2, b_3), corresponding to the Platonic solids and to the Klein groups

(Va) $(2, 2, n)$, $n \geq 2$ $(d = 2n)$, $(d_1, d_2, d_3) = (1, 1, t)$,
(Vb) $(2, 3, n)$, $3 \leq n \leq 5$ $(d = 12, 24, 60)$, $(d_1, d_2, d_3) = (1, c, t)$

(here $c = 1, 2$ and $1 \leq t \leq n - 1$).

STEP VA. Assume we are in the exceptional case (Va). We shall explicitly prove that the group Γ is finite, find a faithful matrix representation, and find that the period of γ equals exactly $2p$, where $p := (m - 1)n - t$. Thus, the order of γ is always ≥ 2.

In fact, we can change the presentation of the group, eliminating $\gamma = \beta_3^{b_3} = \beta_3^n$ and obtaining the relation $\beta_3^{mn-t} = \beta_1\beta_2$. Then $\beta_1\beta_2 = \beta_3^{mn-t} = \beta_1^2\beta_3^p$, whence $\beta_2 = \beta_1\beta_3^p$.

Setting for simplicity $a := \beta_1$, $b := \beta_3$, we get the presentation

$$\Gamma = \langle a, b \mid a^2 = b^n = ab^p ab^p \rangle.$$

Since $a^2 = ab^p ab^p$, we get $b^{-p} = ab^p a^{-1}$, whence $b^{-pn} = ab^{pn}a^{-1}$ and since a commutes with $b^n = a^2$, finally $b^{-pn} = b^{pn}$, i.e., $b^{2pn} = 1 = a^{4p}$.

It follows that the order of the group Γ is at most $4pn$, and that equality holds if the period of a is exactly equal to $4p$.

In order to show that the period of a is exactly equal to $4p$ we use the representation $\rho : \Gamma \to \mathrm{GL}(2, \mathbb{C})$ such that

$$\rho(a) = \begin{pmatrix} 0 & \zeta_{4p} \\ \zeta_{4p} & 0 \end{pmatrix}, \qquad \rho(b) = \begin{pmatrix} \zeta_{2np} & 0 \\ 0 & u\zeta_{2np}^{-1} \end{pmatrix}$$

where $\zeta_h := \exp(2\pi i/h)$, and u is a p-th root of 1 such that $u^n = \zeta_p$ (recall that, since we assumed G.C.D.$(n, t) = 1$, also G.C.D.$(p, n) = 1$).

One can indeed verify that $\rho(a^2) = \rho(b^n) = \rho((ab^p)^2) = \zeta_{2p} \cdot \mathrm{Id}$, as claimed. □

STEP VB. Assume that we are in the exceptional case (Vb).

In this case, we shall first try to show that the image of γ in the abelianization G of Γ is nontrivial.

Eliminating γ we get $\beta_1 = \beta_3^{mn-t}\beta_2^{-c}$, thus Γ is generated by $a := \beta_2$ and $b := \beta_3$ with relations

$$a^3 = b^n = b^{p+n}a^{-c}b^{p+n}a^{-c},$$

where $p := n(m - 1) - t$, as above.

Letting A, B be the respective images of a, b in the abelianization of Γ, we obtain

$$3A - nB = 0, \qquad 2cA = (2p + n)B.$$

Since $3 - 2c = \pm 1$ (according to the respective cases $c = 1$, $c = 2$), we get the relation $\pm A + 2pB = 0$, thus G is cyclic with generator B.

Moreover, the relation $nB = 3A = -(\pm 6pB)$ shows that B has period $f := n \pm 6p$.

Now, if $m \geq 2$, then $p > 0$, thus if $c = 1$ then $f > n$, whence $nB \neq 0$, as we wanted to show.

If instead $m \geq 2$ and $c = 2$, then the absolute value of the period equals $6p - n = n[6(m - 1) - 1] = 6t$, which is clearly $> n$ as soon as $m \geq 3$.

If instead $m = 2$, the absolute value of the period is $> n$ iff $4n > 6t$, which holds unless $\frac{2}{3}n \leq t \leq n-1$, i.e., unless $t = n-1$. But in this case one has $f = 5n - 6(n-1) = 6-n$, thus $nB = 0$ since $6 - n$ divides n.

Similarly, if $m = 1$ and $p = -t$, we have $f = \pm 6t - n$, and $nB \neq 0$ if $c = 2$, whereas if $c = 1$ we can reach this conclusion unless n is a multiple of $6t - n$, that is, unless $t = 1$ and $n = 3, 4, 5$.

We are then left with two cases to consider, the first where $c = 2$, the second where $c = 1$. For the latter case, we use directly a result which goes back essentially to Felix Klein ([Klein]), and is clearly stated by Milnor in [Mil75]:

Given a triangle group $T := T(1, b_1, b_2, b_3; 1, 1, 1)$ which is elliptic, i.e., such that $\sum_i 1/b_i > 1$, its inverse image $\hat{T}$ in $\mathrm{SU}(2, \mathbb{C})$ has the presentation

$$\hat{T} = \langle \gamma, \beta_1, \beta_2, \beta_3 \mid \gamma = \beta_1^{b_1} = \beta_2^{b_2} = \beta_3^{b_3} = \beta_1\beta_2\beta_3 \rangle.$$

It follows that $\hat{T}$ is isomorphic to our group Γ, thus we have a nontrivial central extension of T by the central element γ of order two.

In the former case, we have the presentation

$$\Gamma = \langle \gamma, \delta_1, \delta_2, \delta_3 \mid \gamma = \delta_1^2 = \delta_2^3 = \delta_3^n,\ \gamma^2 = \delta_1\delta_2\delta_3^{n-1} \rangle.$$

Again here we use the extended triangle group $\hat{T}$, setting

$$\delta_1 := \beta_1, \qquad \delta_2 := \beta_2, \qquad \delta_3 := \beta_3^{-1}.$$

Then we see that we get a homomorphic image of Γ, where γ maps onto an element of order 2 (that we still denote by γ).

We are finished with (Vb). $\square$

6. PROOFS OF THE MAIN THEOREMS

PROOF OF THEOREM A. By Remark 3 we may replace Γ by its homomorphic image given by the simplified group, i.e., we may assume $g_i = 1$ or 0.

If $g_i \geq 1$, by Remark 4, we may again take a homomorphic image of Γ corresponding to changing g_i to 0 and making m_i arbitrarily large (i.e., making the self-intersection extremely negative).

Thus we may assume that we have a tree of rational curves, where $-m_i \leq -2$ for all i.

If the tree is linear, the statement follows by Lemma 5.

If we have a comb of rational curves, and γ_i corresponds to the rim of the comb, then the nontriviality of γ_i follows by Theorem 9 and by the subsequent Steps III, IV, V; else, it follows by Proposition 8.

The remaining cases are taken care of again by Proposition 8. $\square$

PROOF OF THEOREM B. Observe that if (ii-1) holds, and $g_i = 0$, then if D_i' is a curve we have $K_{S'} \cdot D_i' \geq 0$, hence also $K_S \cdot D_i \geq 0$.

Thus we see that all the curves D_i with $g_i = 0$ have self-intersection $D_i^2 = -m_i \leq -1$, therefore (ii-1) implies (ii-2) and we proceed with assumption (ii-2), without forgetting the other assumption of minimality in the GNC category. This implies that if $g_i = 0$ and $D_i^2 = -1$, then D_i meets at least three other components.

We can then use exactly the same strategy used for Theorem A, since the case of a linear tree follows automatically, and curves with self-intersection -1 occur only as rims, and in this case the possibility $m = 1$ is handled in Theorem 9 and in the subsequent Steps III, IV, V. □

PROOF OF THEOREM C. We again follow the strategy of proof of Theorem A.

If we have a linear tree, and there is a curve of positive genus, then we may conclude that each γ_i has infinite order by Remark 6.

If we have a comb, then we know by Theorem 9 that the generator γ corresponding to the rim has infinite order apart from the exceptional cases (Va), (Vb). Let moreover γ_i belong, say, to the string $D(1)$. Then we have shown in 5A (cf. Lemma 5) that $\gamma = \gamma_1^{a_n}$ and $\gamma_i = \gamma_1^{a_i}$, where $1 \leq a_i \leq a_n$. Hence, also γ_1 and γ_i have infinite order in the nonexceptional cases.

Similarly we are done if we have a comb and there is a curve D_i of positive genus, since we may then reduce to the case where all the genera are 0, but m_i is arbitrary, hence we are not in the exceptional cases.

So our statement is proven for elementary infinite pieces, and the rest follows easily by induction, since we may apply Lemma 6 and Corollary 7. □

NOTE. When I presented these results at the AMS Meeting in NY, November 3–5, 2000, Walter Neumann mentioned that our presentation of the local fundamental group of neighbourhoods of divisors in complex surfaces is similar to the method of [Neu81] of solid tori decompositions for 3-manifolds, in turn based on the methods earlier introduced by Waldhausen ([Wald67], [Wald68]), who studied the problem whether such manifolds are determined by their fundamental groups.

We would also like to mention that Wagreich ([Wag71]) and Karras ([Kar75]) determined the cases where D comes from a singularity and the group Γ is solvable.

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REFERENCES

[BPV84] W. BARTH - C. PETERS - A. VAN DE VEN, *Compact Complex Surfaces*. Springer, 1984.

[Cat84] F. CATANESE, *On the moduli spaces of surfaces of general type*. J. Differential Geom. 19 (1984), 483–515.

[Cat91] F. CATANESE, *Moduli and classification of irregular Kähler manifolds (and algebraic varieties) with Albanese general type fibrations*, with appendix by Arnaud Beauville. Invent. Math. 104 (1991), 263–289.

[Cat00] F. CATANESE, *Fibred surfaces, varieties isogenous to a product and related moduli spaces.* Amer. J. Math. 122 (2000), 1–44.

[Cat03] F. CATANESE, *Fibred Kähler and quasi-projective groups*. Adv. Geom. 2003, suppl., S13–S27.

[Kar75] U. KARRAS, *Klassifikation 2-dimensionaler Singularitäten mit auflösbaren lokalen Fundamentalgruppen*. Math. Ann. 213 (1975), 231–255.

[Klein] F. KLEIN, *Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree*. Translated into English by George Gavin Morrice. Dover, New York, 1956.

[Kot99] D. KOTSCHICK, *On regularly fibered complex surfaces*. In: Proc. Kirbyfest (Berkeley, CA, 1998), Geom. Topol. Monogr. 2, Geom. Topol. Publ., Coventry, 1999, 291–298.

[Lip69] J. LIPMAN, *Rational singularities, with applications to algebraic surfaces and unique factorization*. Inst. Hautes Études Sci. Publ. Math. 36 (1969), 195–279.

[Mil75] J. MILNOR, *On the 3-dimensional Brieskorn manifolds* $M(p,q,r)$. In: Knots, Groups, and 3-Manifolds, Ann. of Math. Stud. 84, Princeton Univ. Press, 1975, 175–225.

[Mu61] D. MUMFORD, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*. Inst. Hautes Études Sci. Publ. Math. 9 (1961), 5–22.

[Neu81] W. D. NEUMANN, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*. Trans. Amer. Math. Soc. 268 (1981), 299–344.

[dR69] G. DE RHAM, *Lectures on Introduction to Algebraic Topology*. Notes by V. J. Lal. Tata Inst. Fund. Res. Lectures on Math. 44, Tata Inst. Fund. Res., Bombay, 1969.

[S-B99] N. SHEPHERD-BARRON, *Mumford's theorem on surface singularities*. E-mail to the author, April 9, 1999.

[Wag71] P. WAGREICH, *Singularities of complex surfaces with solvable local fundamental group*. Topology 11 (1971), 51–72.

[Wald67] F. WALDHAUSEN, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. I, II.* Invent. Math. 3 (1967), 308–333; ibid. 4 (1967), 87–117.

[Wald68] F. WALDHAUSEN, *On irreducible 3-manifolds which are sufficiently large*. Ann. of Math. (2) 87 (1968), 56–88.

[Wei94] C. A. WEIBEL, *An Introduction to Homological Algebra*. Cambridge Stud. Adv. Math. 38, Cambridge Univ. Press, Cambridge, 1994.

[Zap98-9] G. ZAPPA, Letters to the author, June 24, 1998, May 22 and June 4, 1999.

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