THE ABSOLUTE GALOIS GROUP ACTS FAITHFULLY ON THE CONNECTED COMPONENTS OF THE MODULI SPACES OF SURFACES OF GENERAL TYPE

INGRID BAUER, FABRIZIO CATANESE AND FRITZ GRUNEWALD

Abstract. We show that the Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ operates faithfully on the set of connected components of the moduli spaces of surfaces of general type, and also that for each element $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ different from the identity and from complex conjugation, there is a surface of general type such that X and the Galois conjugate variety X^σ have nonisomorphic fundamental groups. The result was announced by the second author at the Alghero Conference 'Topology of algebraic varieties' in september 2006. Before the present paper was actually written, we received a very interesting preprint by Robert Easton and Ravi Vakil ([E-V07]), where it is proven, with a completely different type of examples, that the Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ operates faithfully on the set of irreducible components of the moduli spaces of surfaces of general type. We also give other simpler examples of surfaces with nonisomorphic fundamental groups which are Galois conjugate, hence have isomorphic algebraic fundamental groups.

Introduction

In the 60's J. P. Serre showed in [Ser64] that there exists a field automorphism $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, and a variety X defined over $\bar{\mathbb{Q}}$ such that X and the Galois conjugate variety X^σ have non isomorphic fundamental groups, in particular they are not homeomorphic.

In this note, using completely different methods, we show the following two results, which give a strong sharpening of the phenomenon discovered by Serre.

Theorem 0.1. *The absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of connected components of the (coarse) moduli spaces of minimal surfaces of general type.*

Theorem 0.2. *Assume that $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ is different from the identity and from complex conjugation. Then there is a minimal surface of general type X such that X and X^σ have non isomorphic fundamental groups.*

In order not to get confused by the above two statements, note that while the absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the set of connected components of the (coarse) moduli spaces of minimal surfaces of general type, it does not act on the set of isomorphism classes of fundamental groups of surfaces of general type. Observe that obviously complex conjugation does not change the isomorphism class of the fundamental group (X and $\bar{X}$ are diffeomorphic). Now, if we had an action on the set of isomorphism classes of fundamental groups, then all the normal closure of the $\mathbb{Z}/2$ generated by complex conjugation would

Date: January 8, 2014.

act trivially. This would contradict theorem 2 since complex conjugation does not lie in the centre of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$.

The surfaces we consider in this note are the socalled 'surfaces isogenous to a product' whose weak rigidity was proven in [Cat00], and which by definition are the quotient of a product of curves $(C_1 \times C_2)$ by the free action of a finite group G.

Therefore our method is strictly related to the socalled theory of 'dessins d' enfants' (see [Gro97]). Dessins d' enfants are, in view of Riemann's existence theorem (generalized by Grauert and Remmert in [GR58]), a combinatorial way to look at the monodromies of algebraic functions with only three branch points. It goes without saying that we make here an essential use of Belyi functions ([Belyi79]) and of their functoriality.

It would be interesting to obtain similar types of results with examples involving rigid varieties belonging to the class of varieties isogenous to a product. For instance we expect that similar results also hold if we restrict ourselves to consider only the socalled Beauville surfaces (see [Cat00] for the definition of Beauville surfaces and [Cat03],[BCG05], [BCG06] for further properties of these).

In the last section we use Beauville surfaces and polynomials with two critical values in order to produce simple examples of pairs of surfaces with nonisomorphic fundamental groups which are conjugate under the absolute Galois group (hence the two groups have isomorphic profinite completions).

1. Very special hyperelliptic curves

Fix a positive integer $g \in \mathbb{N}$, $g \geq 3$, and define, for any complex number $a \in \mathbb{C} \setminus \{-2g, 0, 1, \dots, 2g-1\}$, C_a as the hyperelliptic curve of genus g

$$w^2 = (z-a)(z+2g)\Pi_{i=0}^{2g-1}(z-i)$$

branched over $\{-2g, 0, 1, \dots, 2g-1, a\} \subset \mathbb{P}^1_{\mathbb{C}}$.

Lemma 1.1. *Consider two complex numbers a, b such that $a \in \mathbb{C} \setminus \mathbb{Q}$: then $C_a \cong C_b$ if and only if $a = b$.*

Proof.

One direction being obvious, assume that $C_a \cong C_b$. Then the two sets with $2g+2$ elements $B_a := \{-2g, 0, 1, \dots, 2g-1, a\}$ and $B_b := \{-2g, 0, 1, \dots, 2g-1, b\}$ are projectively equivalent over $\mathbb{C}$ (the latter set B_b has also cardinality $2g+2$ since $C_a \cong C_b$ and C_a smooth implies that also C_b is smooth).

In fact, this projectivity φ is defined over $\mathbb{Q}$, since there are three rational numbers which are carried into three rational numbers (since $g \geq 2$).

Since $a \notin \mathbb{Q}$ it follows that $b \notin \mathbb{Q}$ and φ maps $B := \{-2g, 0, 1, \dots 2g-1\}$ to B, and in particular φ has finite order. Since φ either leaves the cyclical order of $(-2g, 0, 1, \dots, 2g-1)$ invariant or reverses it, and $g \geq 3$ we see that there are 3 consecutive integers such that φ maps them to 3 consecutive integers. Therefore φ is either an integer translation, or an affine symmetry of the form $x \mapsto -x + 2n$. In the former case $\varphi = id$, since it has finite order, and in particular, $a = b$. In the latter case it must be $2g + 2n = \varphi(-2g) = 2g - 1$ and $2n = \varphi(0) = 2g - 2$, a contradiction.

□

Remark 1.2. *The previous lemma holds more generally under the assumption that $a, b \in \mathbb{C} \setminus \{-2g, 0, 1, \ldots, 2g-1\}$, provided $g \geq 6$.*

Proof.

The case where $a, b \in \mathbb{Q}$ is similar to the previous one: φ preserves the cyclical order of the two sets, and we are done if $\varphi(a) = b$ or there are 3 consecutive integers which are mapped by φ to 3 consecutive integers.

set B_b each consecutive triple of points is a triple of consecutive integers, unless one element in the triple is $-2g$ or b. This excludes six triples. Keep in mind that $a \in B_a$ and consider all the consecutive triples of integers in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$: at most two such triples are not a consecutive triple of points of B_a. We conclude that there is a triple of consecutive integers in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ mapping to a triple of consecutive integers under φ. Then either φ is an integer translation $x \mapsto x + n$, or it is a symmetry $x \mapsto -x + 2n$. In both case the intervals equal to the convex spans of the sets B_a, B_b are sent to each other by φ, in particular the length is preserved and the extremal point are permuted. If $a \in [-2g, 2g-1]$ also $b \in [-2g, 2g-1]$ and in the translation case $n = 0$, so that $\varphi(x) = x$ and $a = b$. We see right away that φ cannot be a symmetry, because only two points belong to the left half of the interval. If $a < -2g$ the interval has length $2g-1-a$, if $a > 2g-1$ the interval has length $2g+a$. We only need to exclude $a < -2g$, $b > 2g-1$, $2g-1-a = 2g+b$, i.e., $a = -b-1$. In this case $b = \varphi(a)$ leads to a contradiction since we should have $b = \varphi(a) = -a+2n = b+1+2n$. Else, $\varphi(x) = x + n$ and $\varphi(2g-1) = b$, $\varphi(2g-2) = 2g-1$, hence $n = 1$, and $\varphi(-2g) = -2g + 1$, a contradiction.

□

We shall assume from now on that $a, b \in \bar{\mathbb{Q}} \setminus \mathbb{Q}$ and that there is a field automorphism $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(a) = b$. (Obviously, for any $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ different from the identity, there are $a, b \in \bar{\mathbb{Q}} \setminus \mathbb{Q}$ with $\sigma(a) = b$ and $a \neq b$.)

Proposition 1.3. *Let P be the minimal polynomial of a and consider the field $L := \mathbb{Q}[x]/(P)$. Let C_x be the hyperelliptic curve over L*

$$w^2 = (z-a)(z+2g)\Pi_{i=0}^{2g-1}(z-i).$$

Then there is a rational function $F_x : C_x \to \mathbb{P}^1_L$ such that for each $a \in \mathbb{C}$ with $P(a) = 0$ it holds that the rational function F_a (obtained under the specialization $x \mapsto a$) is a Belyi function for C_a.

Proof. Let $f_x : C_x \to \mathbb{P}^1_L$ be the hyperelliptic involution, branched in $\{-2g, 0, 1, \ldots, 2g-1, x\}$. Then $P \circ f_x$ has as critical values:

- the images of the critical values of f_x under P, which are $\in \mathbb{Q}$,
- the critical values y of P, i.e. the zeroes of the discriminant $h_1(y)$ of $P(z) - y$ with respect to the variable z.

h_1 has degree $deg(P) - 1$, whence, inductively, we obtain $\tilde{f}_x := h \circ P \circ f$ whose critical values are all contained in $\mathbb{Q} \cup \{\infty\}$. If we take any root a of P, then obviously $\tilde{f}_a$ has the same critical values.

Let now $r_1, \ldots, r_n \in \mathbb{Q}$ be the (pairwise distinct) finite critical values of $\tilde{f}_x$. We set:

$$y_i := \frac{1}{\Pi_{j \neq i}(r_i - r_j)} .$$

Let $N \in \mathbb{N}$ be a positive integer such that $m_i := N y_i \in \mathbb{Z}$. Then we have that the rational function

$$g(t) := \Pi_i (t - r_i)^{m_i} \in \mathbb{Q}(t)$$

is ramified at most in ∞ and $r_1, \ldots r_n$. In fact, $g'(t)$ vanishes exactly when the logarithmic derivative $G(t) := \frac{g'(t)}{g(t)} = \sum_i m_i (\frac{1}{t - r_i})$ has a zero or a pole, but the choice made yields a zero of order n at ∞.

Therefore the critical values of $g \circ \tilde{f}_x$ are at most $0, \ \infty, \ g(\infty)$ (for details see [Wo01]).

We set $F_x := \Phi \circ g \circ \tilde{f}_x$ where Φ is the affine map $z \mapsto g(\infty)^{-1} z$, so that the critical valus of F_x are equal to $\{0, 1, \infty\}$. It is obvious by our construction that for any root a of P, F_a has the same critical values as F_x, in particular, F_a is a Belyi function for C_a.

□

Since in the sequel we shall consider the normal closure (we prefer here, to avoid confusion, not to use the term 'Galois closure' for the geometric setting) $\psi_a : D_a \rightarrow \mathbb{P}^1_{\mathbb{C}}$ of each of the functions $F_a : C_a \rightarrow \mathbb{P}^1_{\mathbb{C}}$, we recall in the next section the 'scheme theoretic' construction of the normal closure.

2. Effective construction of normal closures

In this section we consider algebraic varieties over the complex numbers, endowed with their Hausdorff topology, and, more generally, 'good' covering spaces (i.e., between topological spaces which are arcwise connected and semilocally simply connected).

Lemma 2.1. *Let $\pi : X \rightarrow Y$ be a finite 'good' unramified covering space of degree d between connected spaces X and Y. Then the normal closure Z of $\pi : X \rightarrow Y$ is isomorphic to any connected component of*

$$W := W_\pi := (X \times_Y \ldots \times_Y X) \backslash \Delta \subset X^d \backslash \Delta,$$

where $\Delta := \{(x_1, \ldots, x_d) \in X \times_Y \ldots \times_Y X | \exists i \neq j \ , \ x_i = x_j\}$ is the big diagonal.

Proof. Choose base points $x_0 \in X$, $y_0 \in Y$ such that $\pi(x_0) = y_0$ and denote by F_0 the fibre over y_0, $F_0 := \pi^{-1}(\{y_0\})$.

We consider the monodromy $\mu : \pi_1(Y, y_0) \rightarrow \mathfrak{S}_d = \mathfrak{S}(F_0)$ of the unramified covering π. The monodromy of $\phi : W \rightarrow Y$ is induced by the diagonal product monodromy $\mu^d : \pi_1(Y, y_0) \rightarrow \mathfrak{S}(F_0^d)$, such that, for $(x_1, \ldots, x_d) \in F_0^d$, we have $\mu^d(\gamma)(x_1, \ldots, x_d) = (\mu(\gamma)(x_1), \ldots, \mu(\gamma)(x_d))$.

It follows that the monodromy of $\phi : W \rightarrow Y$, $\mu_W : \pi_1(Y, y_0) \rightarrow \mathfrak{S}(\mathfrak{S}_d)$ is given by left translation $\mu_W(\gamma)(\tau) = \mu(\gamma) \circ (\tau)$.

If we denote by $G := \mu(\pi_1(Y, y_0)) \subset \mathfrak{S}_d$ the monodromy group, it follows right away that the components of W correspond to the cosets $G\tau$ of G. Thus all the components yield isomorphic covering spaces.

□

The theorem of Grauert and Remmert ([GR58]) allows to extend the above construction to yield normal closures of morphisms between normal algebraic varieties.

Corollary 2.2. *Let $\pi : X \to Y$ be a finite map between normal projective varieties, let $B \subset Y$ be the branch locus of π and set $X^0 := X \setminus \pi^{-1}(B)$, $Y^0 := Y \setminus B$.*

If X is connected, then the normal closure Z of π is isomorphic to any connected component of the closure of $W^0 := (X^0 \times_{Y^0} \dots \times_{Y^0} X^0) \backslash \Delta$ in the normalization W^n of $W := \overline{(X \times_Y \dots \times_Y X) \setminus \Delta}$.

Proof.

The irreducible components of W correspond to the connected components of W^0, as well as to the connected components Z of W^n. So, our component Z is the closure of a connected component Z^0 of W^0. We know that the monodromy group G acts on Z^0 as a group of covering transformations and simply transitively on the fibre of Z^0 over y_0: by normality the action extends biholomorphically to Z, and clearly $Z/G \cong Y$.

□

3. Connected components of moduli spaces associated to very special hyperelliptic curves

Let a be an algebraic number, $g \geq 2$, and consider as in section 1 the hyperelliptic curve C_a of genus g defined by the equation

$$w^2 = (z - a)(z + 2g)\Pi_{i=0}^{2g-1}(z - i).$$

Let $F_a : C_a \to \mathbb{P}^1$ be the Belyi function constructed in proposition 1.3 and denote by $\psi_a : D_a \to \mathbb{P}^1$ the normal closure of C_a as in corollary 2.2.

Remark 3.1. *We denote by G_a the monodromy group of D_a and observe that there is a subgroup $H_a \subset G_a$ acting on D_a such that $D_a/H_a \cong C_a$.*

We choose a monodromy representation $\mu : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to G_a$ corresponding to the normal ramified covering $\psi_a : D_a \to \mathbb{P}^1$ and we denote the images of geometric loops around 0, 1, ∞ by $\tau_0,\ \tau_1,\ \tau_\infty$. Then we have that G_a is generated by $\tau_0,\ \tau_1,\ \tau_\infty$ and $\tau_0 \cdot \tau_1 \cdot \tau_\infty = 1$.

Fix now another integer $g' \geq 2$ and consider all the possible smooth complex curves C' of genus g'. Observe that the fundamental group of C' is isomorphic to the standard group

$$\Pi_{g'} := \langle \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'} | \Pi_{i=1}^{g'}[\alpha_i, \beta_i] = 1 \rangle.$$

Since $g' \geq 2$ there are epimorphisms (surjective homomorphisms) $\rho \colon \Pi_{g'} \to G_a$. For instance it suffices to consider the epimorphism $\theta : \Pi_{g'} \to \mathbb{F}_{g'}$ from $\Pi_{g'}$ to the free group $\mathbb{F}_{g'} :=< \lambda_1, \dots, \lambda_{g'} >$ in g' letters given by $\theta(\alpha_i) = \theta(\beta_i) = \lambda_i$,

$\forall\ 1 \leq i \leq g'$, and to compose θ with the surjection $\phi : \mathbb{F}_{g'} \to G_a$, given by $\phi(\lambda_1) = \tau_0$, $\phi(\lambda_2) = \tau_1$, and $\phi(\lambda_i) = 1$ for $3 \leq i \leq g'$.

Consider all the possible epimorphisms $\rho\colon \Pi_{g'} \to G_a$. Each such ρ gives a normal unramified covering $D' \to C'$ with monodromy group G_a.

Definition 3.2. *Let $\mathfrak{M}_a$ be the subset of the moduli space of surfaces of general type given by surfaces $S \cong (D_a \times D')/G_a$, where D_a, D' are as above and the group G_a acts by a diagonal action.*

From [Cat00] and especially Theorem 3.3 of [Cat03] it follows:

Proposition 3.3. *For each $a \in \bar{\mathbb{Q}}$, $\mathfrak{M}_a$ is a union of connected components of the moduli spaces of surfaces of general type.*

Moreover, for $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, $\sigma(\mathfrak{M}_a) = \mathfrak{M}_{\sigma(a)}$.

Proof. Since D_a is a triangle curve, the pair (D_a, G_a) is rigid, whereas, varying C' and ρ, we obtain the full union of the moduli spaces for the pairs (D', G_a), corresponding to the possible free topological actions of the group G_a on a curve D' of genus $|G_a|(g' - 1) + 1$.

Thus, the surfaces $S \cong (D_a \times D')/G_a$ give, according to the cited theorem 3.3 of [Cat03], a union of connected components of the moduli spaces of surfaces of general type.

Choose now a surface S as above (thus, $[S] \in \mathfrak{M}_a$) and apply the field automorphism $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ to a point of the Hilbert scheme corresponding to the 5-canonical image of S (which is isomorphic to S, since the canonical divisor of S is ample). We obtain a surface which we denote by S^σ.

By taking the fibre product of σ with $D_a \times D' \to S$ it follows that S^σ has an étale covering with group G_a which is the product $(D_a)^\sigma \times (D')^\sigma$.

Since $(C_a)^\sigma = C_{\sigma(a)}$ and since $\sigma(a)$ corresponds to another embedding of the field L into $\mathbb{C}$, it follows that $(F_a)^\sigma = F_{\sigma(a)}$, whence $(D_a)^\sigma = D_{\sigma(a)}$.

On the other hand, the quotient of $(D')^\sigma$ by the action of the group G_a has genus equal to the dimension of the space of invariants $dim(H^0(\Omega^1_{(D')^\sigma})^{G_a})$, but this dimension is the same $g' = dim(H^0(\Omega^1_{D'})^{G_a})$. Hence the action of G_a on $(D')^\sigma$ is also free (by Hurwitz' formula), and we have shown that S^σ is a surface whose moduli point is in $\mathfrak{M}_{\sigma(a)}$.

□

4. Proof of the main theorems

Prooof of theorem 1.

Given $a \in \bar{\mathbb{Q}}$, consider a connected component $\mathfrak{N}_a$ of $\mathfrak{M}_a$. Our main theorem 1 follows easily from the following

Main Claim: if $\mathfrak{N}_a = \sigma(\mathfrak{N}_a)$, then necessarily $a = \sigma(a)$.

Proof. Denote $\sigma(a)$ by b. We know that $C_a \cong D_a/H_a$, that $C_b \cong D_b/H_b$, and we have already shown that σ yields an isomorphism $G_a \cong G_b$. The assumption that $\mathfrak{N}_a = \sigma(\mathfrak{N}_a)$ implies, by theorem (3.3) of [Cat03], the condition that the pairs (D_a, G_a) and (D_b, G_b) are isomorphic as complex triangle curves.

It suffices therefore to show that under this isomorphism the subgroup H_a corresponds to the subgroup H_b, because then we conclude that the curves C_a, C_b are not only Galois conjugate, but also isomorphic.

And then by lemma 1 we conclude that $a = b$.

To show that H_a corresponds to the subgroup H_b, let K be the Galois closure of the field L (= splitting field of the field extension $\mathbb{Q} \subset L$), and view L as embedded in $\mathbb{C}$ under the isomorphism sending $x \mapsto a$.

Consider the curve $\hat{C}_x$ obtained from C_x by scalar extension $\hat{C}_x := C_x \otimes_L K$. Let also $\hat{F}_x := F_x \otimes_L K$ the corresponding Belyi function with values in $\mathbb{P}^1_K$.

Apply now the effective construction of the normal closure of section 2, and, taking a connected component of $(\hat{C}_x \times_{\mathbb{P}^1_K} \dots \times_{\mathbb{P}^1_K} \hat{C}_x) \setminus \Delta$ we obtain a curve D_x defined over K.

Note that D_x is not geometrically irreducible, but once we tensor with $\mathbb{C}$ it splits into several components which are Galois conjugate and which are isomorphic to the conjugates of D_a.

Apply now the Galois automorphism σ to the triple $D_a \to C_a \to \mathbb{P}^1$. Since the triple is induced by the triple $D_x \to C_x \to \mathbb{P}^1_K$ by taking a tensor product $\otimes_K \mathbb{C}$ via the embedding sending $x \mapsto a$, the morphisms are induced by the composition of the inclusion $D_x \subset (C_x)^d$ with the coordinate projections, respectively buy the fibre product equation, it follows from proposition 1.3 that σ carries the triple $D_a \to C_a \to \mathbb{P}^1$ to the triple $D_b \to C_b \to \mathbb{P}^1$.

$\square$

If we want to interpret our argument in terms of Grothendieck's étale fundamental group, we define $C^0_x := F_x^{-1}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$, and accordingly $\hat{C}^0_x$ and D^0_x.

There are the following exact sequences for the Grothendieck étale fundamental group (compare Theorem 6.1 of [SGA1]):

$$1 \to \pi_1^{alg}(D_a) \to \pi_1^{alg}(D_x) \to Gal(\bar{\mathbb{Q}}/K) \to 1$$
$$1 \to \pi_1^{alg}(C_a) \to \pi_1^{alg}(C_x) \to Gal(\bar{\mathbb{Q}}/K) \to 1$$
$$1 \to \pi_1^{alg}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}) \to \pi_1^{alg}(\mathbb{P}^1_K \setminus \{0, 1, \infty\}) \to Gal(\bar{\mathbb{Q}}/K) \to 1$$

where H_a and G_a are the respective factor groups for the inclusions of the left hand sides, corresponding to the first and second sequence, and corresponding to the first and third sequence.

On the other hand, we also have the exact sequence

$$1 \to \pi_1^{alg}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}) \to \pi_1^{alg}(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}) \to Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to 1.$$

The finite quotient G_a of $\pi_1^{alg}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\})$ (defined over K) is sent by $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ to another quotient, corresponding to $D_{\sigma(a)}$, and the subgroup H_a, yielding the quotient C_a, is sent to the subgroup $H_{\sigma(a)}$.

Proof of theorem 2.

Assume that $\sigma(a) = b$, and that σ is neither complex conjugation nor the identity. Consider a surface X_a with $[X_a] \in \mathfrak{M}_a$ and its Galois conjugate $(X_a)^\sigma = X_{\sigma(a)} = X_b$.

Denote as before by $\mathfrak{N}_a$ the connected component of the moduli space of surfaces of general type containing $[X_a]$.

Assume now that the X_a and X_b have isomorphic fundamental groups. Since obviously the two surfaces have the same Euler number $e(X_a) = e((X_a)^\sigma)$ (since they have the same Hodge numbers $h^i(\Omega^j)$) we can apply again theorem 3.3. of [Cat03] and there are only two possibilities : either $[X_a]$ and $[X_b]$ belong to the same connected component $\mathfrak{N}_a$ of the moduli space, or $[X_b]$ belongs to the complex conjugate component $\mathfrak{N}_{\bar{a}}$ of the connected component $\mathfrak{N}_a$.

In the first case the main claim says that $\sigma(a) = a$, in the second case it says that $\sigma(a) = \bar{a}$.

Since however σ is neither complex conjugation nor the identity, we find an $a \in \bar{\mathbb{Q}}$ such that $b := \sigma(a) \neq a$ and $b := \sigma(a) \neq \bar{a}$.

Therefore the corresponding surfaces X_a and $(X_a)^\sigma$ have nonisomorphic fundamental groups.

□

Remark 4.1. *We observe that X_a and $(X_a)^\sigma$ have isomorphic Grothendieck étale fundamental groups. In particular, the profinite completion of $\pi_i(X_a)$ and $\pi_1((X_a)^\sigma)$ are isomorphic.*

It is not so easy to calculate explicitly the fundamental groups of the surfaces constructed above, since one has to explicitly calculate the monodromy of the Belyi function of the very special hyperelliptic curves.

Therefore we give in the next section an explicit example of two rigid surfaces with non isomorphic fundamental groups which are Galois conjugate.

5. An explicit example

In this section we provide, as we already mentioned, an explicit example of two surfaces with non isomorphic fundamental groups which are conjugate under the absolute Galois group, hence with isomorphic profinite completions of their respective fundamental groups. These surfaces are rigid.

We consider (see [BCG06] for an elementary treatment of what follows) polynomials with only two critical values: $\{0, 1\}$.

Let $P \in \mathbb{C}[z]$ be a polynomial with critical values $\{0, 1\}$.

In order not to have infinitely many polynomials with the same branching behaviour, one considers *normalized polynomials* $P(z) := z^n + a_{n-2}z^{n-2} + \ldots a_0$. The condition that P has only $\{0, 1\}$ as critical values, implies, as we shall briefly recall, that P has coefficients in $\bar{\mathbb{Q}}$. Denote by K the number field generated by the coefficients of P.

Fix the types the types $(m_1, \ldots, m_r)$ and $(n_1, \ldots, n_s)$ of the cycle decompositions of the respective local monodromies around 0 and 1: we can write our polynomial P in two ways, namely as:

$$P(z) = \prod_{i=1}^{r} (z - \beta_i)^{m_i},$$

and

$$P(z) = 1 + \prod_{k=1}^{s} (z - \gamma_k)^{n_k}.$$

We have the equations $F_1 = \sum m_i \beta_i = 0$ and $F_2 = \sum n_k \gamma_k = 0$ (since P is normalized). Moreover, $m_1 + \ldots + m_r = n_1 + \ldots + n_s = n = degP$ and therefore, since $\sum_j (m_j - 1) + \sum_i (n_i - 1) = n - 1$, we get $r + s = n + 1$.

Since we have $\prod_{i=1}^{r} (z - \beta_i)^{m_i} = 1 + \prod_{k=1}^{s} (z - \gamma_k)^{n_k}$, comparing coefficients we obtain further $n-1$ polynomial equations with integer coefficients in the variables β_i, γ_k, which we denote by $F_3 = 0, \ldots, F_{n+1} = 0$. Let $\mathbb{V}(n; (m_1, \ldots, m_n), (n_1, \ldots, n_s))$ be the algebraic set in affine $(n+1)$-space defined by the equations $F_1 = 0, \ldots, F_{n+1} = 0$. Mapping a point of this algebraic set to the vector $(a_0, \ldots, a_{n-2})$ of coefficients of the corresponding polynomial P we obtain a set

$$\mathbb{W}(n; (m_1, \ldots, m_n), (n_1, \ldots, n_s))$$

(by elimination of variables) in affine $(n-1)$ space. Both these are finite algebraic sets defined over $\mathbb{Q}$ since by Riemann's existence theorem they are either empty or have dimension 0.

Observe also that the equivalence classes of monodromies $\mu \colon \pi_1(\mathbb{P}^1 \setminus \{0.1.\infty\}) \to \mathfrak{S}_n$ correspond to the orbits of the group of n-th roots of 1 (we refer to [BCG06] for more details).

We recall also the following

Definition 5.1. *A smooth algebraic curve C is called a* triangle curve *iff there is a finite group G acting effectively on C with the property that $C/G \cong \mathbb{P}^1$ in such a way that $f \colon C \to \mathbb{P}^1$ is ramified only in $\{0, 1, \infty\}$.*

Example 5.2. *We calculate (here and in the following, either by a MAGMA routine, or, sometimes, more painfully by direct calculation) that $\mathbb{W} := \mathbb{W}(7; (2, 2, 1, 1, 1); (3, 2, 2))$ is irreducible over $\mathbb{Q}$. This implies that $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts transitively on $\mathbb{W}$. Looking at the possible monodromies, one sees that there are exactly two real non equivalent polynomials. Observe also that the equivalence classes of monodromies $\mu \colon \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to (S)_n$ correspond to the orbits of the group μ_n of n-th roots of 1 (we refer to* [BCG06] *for more details). In both cases, which will be explicitly described later on, the two permutations, of types $(2, 2)$ and $(3, 2, 2)$, are seen to generate $\mathfrak{A}_7$ and the respective normal closures of the two polynomial maps are easily seen to give (we use here the fact that the automorphism group of $\mathfrak{A}_7$ is $\mathfrak{S}_7$) nonequivalent triangle curves C_1, C_2.*

By Hurwitz's formula, we see that $g(C_i) = \frac{|\mathfrak{A}_7|}{2}(1 - \frac{1}{2} - \frac{1}{6} - \frac{1}{7}) + 1 = 241$.

We remark that $\mathfrak{A}_7$ admits generators a_1, a_2 of order 5 such that their product has order 5, hence we get a triangle curve C (of genus 505).

Consider the triangle curve C given by a **spherical system of generators of type** $(5, 5, 5)$ **of** $\mathfrak{A}_7$, i.e., generators a_1, a_2, a_3 of $\mathfrak{A}_7$ such that $a_1 \cdot a_2 \cdot a_3 = 1$.

Definition 5.3. *Let (a_1, a_2, a_3) and (b_1, b_2, b_3) be two spherical systems of generators of a finite group G of the same unordered type, i.e., $\{ord(a_1), ord(a_2), ord(a_3)\} = \{ord(b_1), ord(b_2), ord(b_3)\}$. Then (a_1, a_2, a_3) and (b_1, b_2, b_3) are called* **Hurwitz equivalent** *iff they are equivalent under the equivalence relation generated by*

$$(a_1, a_2, a_3) \equiv (a_2, a_2^{-1}a_1a_2, a_3),$$

$$(a_1, a_2, a_3) \equiv (a_1, a_3, a_3^{-1}a_2a_3).$$

It is well known that two such triangle curves are isomorphic, compatibly with the action of the group G, if and only if the two spherical systems of generators are *Hurwitz equivalent*.

Remark 5.4. *An easy MAGMA routine shows that there is exactly one Hurwitz equivalence class of triangle curves given by a* spherical system of generators of type $(5, 5, 5)$ of $\mathfrak{A}_7$. *In other words, if D_1 and D_2 are two triangle curves given by spherical systems of generators of type $(5, 5, 5)$ of $\mathfrak{A}_7$, then D_1 and D_2 are not only isomorphic as algebraic curves, but they have the same action of G.*

Let C be as above the triangle curve given by a(ny) spherical systems of generators of type $(5, 5, 5)$ of $\mathfrak{A}_7$, and consider the two triangle curves C_1 and C_2 as in example 5.2. Clearly $\mathfrak{A}_7$ acts freely on $C_1 \times C$ as well as on $C_2 \times C$ and we obtain two non isomorphic so-called *Beauville surfaces* $S_1 := (C_1 \times C)/G$, $S_2 := (C_2 \times C)/G$.

Obviously, these two surfaces have the same topological Euler characteristic. If they had isomorphic fundamental groups, by theorem 3.3 of [Cat03], S_2 would be the complex conjugate surface of S_1. In particular, C_1 would be the complex conjugate triangle curve of C_2: but this is absurd since we shall show that both C_1 and C_2 are real triangle curves.

Proposition 5.5. *There is a field automorphism $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $S_2 = (S_1)^\sigma$.*

Proof. We know that $(S_1)^\sigma = ((C_1)^\sigma \times (C)^\sigma)/G$. Since there is only one Hurwitz class of triangle curves given by a spherical system of generators of type $(5, 5, 5)$ of $\mathfrak{A}_7$, we have $(C)^\sigma \cong C$ (with the same action of G).

□

We determine now explicitly the respective fundamental groups of S_1 and S_2.

In general, let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_m)$ be two sets of spherical generators of a finite group G of respective order types $r := (r_1, \ldots, r_n)$, $s := (s_1, \ldots, s_m)$. We denote the corresponding 'polygonal' curves by D_1, resp. D_2.

Assume now that the diagonal action of G on $D_1 \times D_2$ is free. We get then the smooth surface $S := (D_1 \times D_2)/G$, isogenous to a product.

Denote by $T_r := T(r_1, \ldots, r_n)$ the *polygonal group*

$$\langle x_1, \ldots, x_{n-1} | x_1^{r_1} = \ldots = x_{n-1}^{r_{n-1}} = (x_1x_2 \ldots x_{n-1})^{r_n} = 1\rangle.$$

We have the exact sequence (cf. [Cat00] cor. 4.7)

$$1 \to \pi_1 \times \pi_2 \to T_r \times T_s \to G \times G \to 1,$$

where $\pi_i := \pi_1(D_i)$.

Let Δ_G be the diagonal in $G \times G$ and let H be the inverse image of Δ_G under $\Phi : T_r \times T_s \to G \times G$. We get the exact sequence

$$1 \to \pi_1 \times \pi_2 \to H \to G \cong \Delta_G \to 1.$$

Remark 5.6. *$\pi_1(S) \cong H$ (cf.* [Cat00] *cor. 4.7).*

We choose now an arbitrary spherical system of generators of type $(5,5,5)$ of $\mathfrak{A}_7$, for instance $((1,7,6,5,4),(1,3,2,6,7),(2,3,4,5,6))$. Note that we use here MAGMA's notation, where permutations act on the right (i.e., ab sends x to $(xa)b$).

A MAGMA routine shows that

$$((1,2)(3,4),(1,5,7)(2,3)(4,6),(1,7,5,2,4,6,3)) \tag{1}$$

and

$$((1,2)(3,4),(1,7,4)(2,5)(3,6),(1,3,6,4,7,2,5)) \tag{2}$$

are two representatives of spherical generators of type $(2,6,7)$ yielding two non isomorphic triangle curves C_1 and C_2, each of which is isomorphic to its complex conjugate. In fact, an alternative direct argument is as follows: first of all C_i is isomorphic to its complex conjugate triangle curve since, for an appropriate choice of the real base point, complex conjugation sends $a \mapsto a^{-1}, b \mapsto b^{-1}$ and then one sees that the two corresponding monodromies are permutation equivalent (see Figure 1 and Figure 2).

Moreover, since $Aut(\mathfrak{A}_7) = \mathfrak{S}_7$, if the two triangle curves were isomorphic, then the two monodromies were conjugate in $\mathfrak{S}_7$. That this is not the case is seen again by the following pictures.

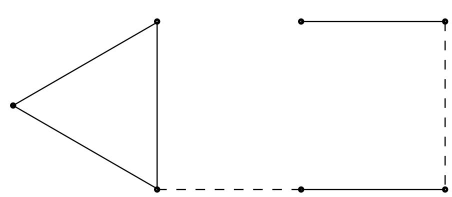

FIGURE 1. Monodromy corresponding to (1)

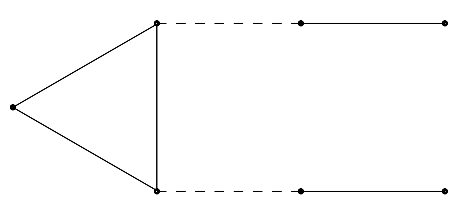

FIGURE 2. Monodromy corresponding to (2)

The two corresponding homomorphisms $\Phi_1 : T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7$ and $\Phi_2 : T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7$ give two exact sequences

$$1 \to \pi_1(C_1) \times \pi_1(C) \to T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7 \to 1,$$

and

$$1 \to \pi_1(C_2) \times \pi_1(C) \to T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7 \to 1,$$

yielding two non isomorphic fundamental groups $\pi_1(S_1) = \Phi_1^{-1}(\Delta_{\mathfrak{A}_7})$ and $\pi_1(S_2) = \Phi_2^{-1}(\Delta_{\mathfrak{A}_7})$ fitting both in an exact sequence of type

$$1 \to \Pi_{241} \times \Pi_{505} \to \pi_1(S_j) \to \Delta_{\mathfrak{A}_7} \cong \mathfrak{A}_7 \to 1,$$

where $\Pi_{241} \cong \pi_1(C_1) \cong \pi_1(C_2)$, $\Pi_{505} = \pi_1(C)$.

Remark 5.7. *1) Using the same trick that we used for our main theorems, namely, using a surjection of a group $\Pi_g \to \mathfrak{A}_7$ we get infinitely many examples of pairs of fundamental groups which are nonisomorphic, but which have isomorphic profinite completions. Each pair fits into an exact sequence*

$$1 \to \Pi_{241} \times \Pi_{g'} \to \pi_1(S_j) \to \mathfrak{A}_7 \to 1.$$

2) Many more explicit examples as the one above (but with cokernel group different from $\mathfrak{A}_7$) can be obtained using polynomials with two critical values.

A construction of polynomials with two critical values having a very large Galois orbit was proposed to us by D. van Straten.

Remark 5.8. *In the sequel to this work we plan to show that the absolute Galois group in fact acts faithfully also on the set of Beauville surfaces.*

Acknowledgements. The research of the authors was performed in the realm of the Forschergruppe 'Classification of algebraic surfaces and compact complex manifolds' of the D.F.G.. We would like to thank Ravi Vakil for his interest in our work and for pointing out a minor error in a first version of this note.

References

[Bea] A. Beauville, *Surfaces algébriques complexes*, **Asterisque 54, Soc. Math. France** (1978).

[BCG05] I. Bauer, F. Catanese, F. Grunewald, *Beauville surfaces without real structures.* In: Geometric methods in algebra and number theory, **Progr. Math., 235**, Birkhäuser (2005), 1–42.

[BCG06] I. Bauer, F. Catanese, F. Grunewald, *Chebycheff and Belyi polynomials, dessins d'enfants, Beauville surfaces and group theory.* **Mediterranean J. Math. 3, no.2**, (2006) 119–143.

[Belyi79] G.V. Belyĭ, *On Galois extensions of a maximal cyclotomic field,* **Izv. Akad. Nauk SSSR Ser. Mat. 43:2** (1979), 269–276. Translation in Math. USSR- Izv. **14** (1980), 247–256.

[Bom] E. Bombieri, *Canonical models of surfaces of general type*, **Publ. Math. I.H.E.S., 42** (1973), 173–219.

[Cat00] F. Catanese, *Fibred surfaces, varieties isogenous to a product and related moduli spaces*, **Amer. J. Math. 122, no.1** (2000), 1–44.

[Cat03] F. Catanese, *Moduli Spaces of Surfaces and Real Structures*, **Ann. Math. 158** (2003), n.2, 577-592.

[E-V07] R. W. Easton, R. Vakil, *Absolute Galois acts faithfully on the components of the moduli space of surfaces: a Belyi-type theorem in higher dimension*, preprint april 12 2007.

[Groth60] A. Grothendieck, *Techniques de construction et théoremes d'existence en géométrie algébrique. IV, Les schemas de Hilbert*, Sem. Bourbaki, Vol. 13, (1960-61), 1–28.

[GR58] Grauert, H., Remmert, *Komplexe Räume,* **Math. Ann. 136** (1958), 245–318.

[SGA1] A. Grothendieck (dirigé par), *Revêtements étales et groupe fondamental,* Séminaire de géométrie algébrique du Bois Marie 1960-61 **Springer Lecture Notes in Math. 224**, (1971). Reedited by **Société Mathématique de France** in the series 'Documents mathématiques (2003).

[Gro97] A. Grothendieck, *Esquisse d'un programme.* In: 'Geometric Galois actions, 1', **London Math. Soc. Lecture Note Ser., 242. Cambridge Univ. Press, Cambridge** (1997), 5–48.

[Ser64] J..P. Serre, *Exemples de variétés projectives conjuguées non homéomorphes.* **C. R. Acad. Sci. Paris 258**, (1964) 4194–4196.

[Wo01] J. Wolfart, "*Kinderzeichnungen und Uniformisierungstheorie*", Manuskript, Frankfurt (2001), Homepage.

Authors' addresses:

Ingrid Bauer, Fabrizio Catanese
Lehrstuhl Mathematik VIII, Universität Bayreuth
D-95440 Bayreuth, Germany
(email: Ingrid.Bauer@uni-bayreuth.de, Fabrizio.Catanese@uni-bayreuth.de)

Fritz Grunewald
Mathematisches Institut der Heinrich-Heine Universität Düsseldorf
Universitätsstrasse
Düsseldorf
(email: grunewald@math.uni-duesseldorf.de)