IRREDUCIBILITY OF THE SPACE OF CYCLIC COVERS OF ALGEBRAIC CURVES OF FIXED NUMERICAL TYPE AND THE IRREDUCIBLE COMPONENTS OF $Sing(\overline{\mathfrak{M}}_g)$

F. CATANESE

Dedicated to Shing Tung Yau with friendship and admiration on the occasion of his 60-th birthday.

INTRODUCTION

The first main purpose of this article is to prove irreducibility (equivalently, connectedness) results (Theorems 2.4, 4.10) for the space of cyclic covers of a fixed numerical type between complex projective curves, first in the case of smooth curves, and then in the case of stable curves.

In the smooth case the numerical type of a cyclic cover $C \to C'$ is given by the order d of the cyclic group G, by the genus g of the covering curve C, and by the branching datum, i.e., the equivalence class, for the natural action of $Aut(G) = (\mathbb{Z}/d)^*$, of the branching sequence $(k_1, \ldots, k_{d-1})$, where k_i is the number of branch points on C' such that their local monodromy is the element i of the group $G \cong \mathbb{Z}/d$.

This result, in the case where C is smooth and d is prime, was obtained long ago by Cornalba ([Cor87]); Barbara Fantechi [1] observed that Cornalba's proof does partially extend for C smooth, but, for arbitrary d, not for any numerical type.

Similar results were obtained by Biggers and Fried in [BF86], but concerning only the case where $C \to C'$ is unramified and cyclic.

Observe that, much more generally, given any finite group G, the space of Galois covers $C \to C'$ with group G, with C smooth, and with a fixed topological type is connected. This follows from Teichmüller theory, as shown in Proposition 4.13 of [Cat00] (see also sections 5 and 6 of [Cat08] for related topics).

This result reduces the question of determining the possible topological types to a question in group theory, namely finding the equivalence classes of monodromies $\mu : \pi_h := \pi_1(C') \to G$, for the action of $Aut(G)$ and for the action on the source of the Teichmüller group $\mathcal{T}_h = Out(\pi_h)$. In general the above numerical invariants of the cover

[1]Talk at MSRI, February 2009.

(the branching sequence is labeled here by the conjugacy classes in G) do not determine the topological type, but our first theorem says that they do determine it in the case where G is cyclic.

The second main purpose of this article is on the other hand (Theorem 4.14) the description of an irredundant irreducible decomposition of $Sing(\overline{\mathfrak{M}_g})$.

The determination of an irredundant irreducible decomposition for $Sing(\mathfrak{M}_g)$ was obtained by Cornalba in [Cor87] and [Cor08]; we give here a slightly shorter proof of Cornalba's description (3.4).

Our main novel contribution is then the complete explicit determination of the irreducible components which are contained in the boundary $\overline{\mathfrak{M}_g} \setminus \mathfrak{M}_g$.

Here a brief description of the contents of the paper.

In the first section we recall the general theory of cyclic coverings between normal varieties (and with factorial base), expecially important to construct families of cyclic covers: here a more general theory of abelian covers and their invariants was developed by Pardini in [Par91], extending the case of bidouble covers developed in [Cat84b].

In the second section we prove the irreducibility result for the space of cyclic covers of smooth curves with a fixed topological type (Theorem 2.4) and for arbitrary degree d.

In the third section we describe the relation of the above result with the determination of the irreducible components of the singular locus of the moduli space of curves $\mathfrak{M}_g$.

In fact, for $g \geq 3$, with exception of the locus of hyperelliptic curves of genus 3, all the loci that one obtains by letting d be a prime number are closed subvarieties of $\mathfrak{M}_g$ of codimension $c \geq 2$, hence they are irreducible sets whose union is $Sing(\mathfrak{M}_g)$. We give a shorter proof of Cornalba's theorem, using (in Proposition 3.2) numerical inequalities instead of degeneration arguments.

We restate, with some additions (which are then needed in section 4), Cornalba's main result in Theorem 3.4.

In the fourth section we consider the pair of a stable curve C and an automorphism γ, and we describe some numerical (and combinatorial) invariants of the pair (C, γ). A fundamental difference here, when trying to describe irreducible components of the 'space' of such coverings $C \to C/\gamma$, is that the topological type is not constant on connected components, but each such component is a finite union of locally closed irreducible sets (corresponding to a given topological type), only one of them being open and dense.

We call the pairs (C, γ) in this dense open set **maximal**, and we give a complete numerical-combinatorial description of such maximal coverings in the simpler case where the order d of the automorphism γ is a prime number (Theorem 4.10 deals only with the case where d is a prime number, we do not treat here the general case).

We also describe a 'simplification' algorithm showing how to obtain, from a fixed numerical-combinatorial type, the associated maximal numerical-combinatorial type.

Theorem 4.10 allows us to completely describe in the final section (Theorem 4.14) the irredundant irreducible decomposition of $Sing(\overline{\mathfrak{M}_g})$.

The irreducible components of $Sing(\overline{\mathfrak{M}_g})$ fall into two types: the ones of the first type are the closures of the irreducible components of $Sing(\mathfrak{M}_g)$, while the ones of the second type are the components which are completely contained in the boundary $\overline{\mathfrak{M}_g} \setminus \mathfrak{M}_g$.

The determination of these latter components is different than in the case of $Sing(\mathfrak{M}_g)$, and we have the following phenomenon.

The stable curves C with an elliptic tail E (i.e., E is a smooth elliptic curve intersecting $C'' := \overline{C \setminus E}$ in one point P) form a divisor, and admit an involution (an automorphism of order 2) which is the identity on C'', and on E is multiplication by -1 when one chooses P as the origin of the elliptic curve. If a curve C has several elliptic tails, the corresponding involutions are in the centre of $Aut(C)$ and do not contribute to $Sing(\overline{\mathfrak{M}_g})$.

It turns out that the irreducible components which are completely contained in the boundary correspond to cyclic automorphisms of prime order which are the identity on all components except one, are of maximal numerical-combinatorial type, and, if the order d equals 2, they act as the identity on the elliptic tails. For two exceptional cases, which we explicitly describe, we do not get irreducible components, but proper subsets.

In a brief final section we mention some problems and results related to the automorphism group $Aut(C)$ of a stable curve C of genus g (cf. [vO-V-07],[vO-V-10a],[vO-V-10b]) .

1. Cyclic covers of factorial varieties

Let X, Y be normal complex projective varieties, and let $\mathbb{C}(X)$, $\mathbb{C}(Y)$ be their respective function fields.

Assume that $\mathbb{C}(X)$ is a cyclic Galois extension of $\mathbb{C}(Y)$, and denote by

$$G \cong \mu_d := \{\zeta \in \mathbb{C}|\zeta^d = 1\}$$

its Galois group, by $\mathbb{Z}/d$ the group of characters

$$\mathbb{Z}/d \cong \{\chi|\exists m \in \mathbb{Z}/d, \chi(\zeta) = \zeta^m\}.$$

For each character χ of order d the extension is given by

$$\mathbb{C}(X) = \mathbb{C}(Y)(w), w^d = f(y) \in \mathbb{C}(Y),$$

where w is a χ-eigenvector.

Assume now that Y is factorial, so that f admits a unique prime factorization as a fraction of pairwise relatively prime sections of line

bundles, and we can write

$$w^d = \frac{\prod_i \sigma_i^{n_i}}{\prod_i \tau_j^{m_j}}.$$

Write now

$$n_i = N_i + dn'_i, \quad m_j = -M_j + dm'_j$$

with $0 \le N_i, M_j \le d-1$ and set

$$z := w \cdot \prod_i \sigma_i^{-n'_i} \prod_i \tau_j^{m'_j}.$$

Whence z is a rational section of a line bundle on Y and we have

$$z^d = \prod_i \sigma_i^{N_i} \prod_i \tau_j^{M_j}.$$

We put together the prime factors which appear with the same exponent and write:

$$z^d = \prod_{i=1}^{d-1} \delta_i^i.$$

Here each factor δ_j is reduced, but not irreducible, and corresponds to a Cartier divisor that we shall denote D_j. The local monodromy around D_j is easily calculated since, if we take a small loop

$$\delta_j = e^{2\pi\sqrt{-1}\,\theta}, \theta \in [0,1]$$

it lifts to the path

$$z = z_0 \cdot e^{\frac{2\pi j\sqrt{-1}\,\theta}{d}}$$

and its monodromy is

$$z_0 \mapsto e^{\frac{2\pi j\sqrt{-1}}{d}} z_0.$$

This shows that D_i is exactly the divisorial part of the branch locus $D := \sum_i D_i$ where the local monodromy is the i-th power of the standard generator $\gamma := e^{\frac{2\pi\sqrt{-1}}{d}}$ of $G \cong \mu_d$.

In the following we shall write characters additively, in the sense that we view them as $G^\star = Hom(G, \mathbb{Z}/d)$. We notice also that what said above applies to any character χ: to χ we associate the normal covering

$$Z_\chi := X/ker(\chi).$$

We have then a linear equivalence

$$(*) \ dL_\chi \equiv \sum_i \overline{\chi(i)} D_i$$

where $\overline{m}$, for $m \in \mathbb{Z}/d$, is the unique representative of the residue class lying in $\{0, 1, \ldots, d-1\}$.

We observe for further use the following formula:

$$(I) \ \overline{\chi(i)} + \overline{\chi'(i)} = \overline{(\chi + \chi')(i)} + \epsilon_{\chi,\chi'}^i d,$$

where $\epsilon^i_{\chi,\chi'} \in \{0,1\}$.

The following theorem is in part a special case of the structure theorem for Abelian coverings due to Pardini ([Par91], the existence result in terms of the basic linear equivalences was however already obtained by Comessatti in [Com30]): but here is explicitly stated the irreducibility criterion for the covering.

Theorem 1.1. *i) Given a factorial variety Y, the datum of a pair (X, γ) where X is a normal variety and γ is an automorphism of order d such that, G being the subgroup generated by γ, one has $X/G \cong Y$, is equivalent to the datum of reduced effective divisors $D_1, \ldots D_{d-1}$ without common components, and of a divisor class L such that we have the following linear equivalence*

$$(*) \ dL \equiv \sum_i iD_i$$

and moreover, setting $m := G.C.D.\{i|D_i \neq 0\}$, either
(**) $m = 1$ or, setting $d = mn$, the divisor class

$$(***) \ L' := \frac{d}{m}L - \sum_i \frac{i}{m}D_i$$

has order precisely m in the Picard group.

ii) In fact, if $\mathbb{L}$ is the geometric line bundle whose sheaf of regular sections is $\mathcal{O}_Y(L)$, then X is the normalization of the singular covering

$$X' \subset \mathbb{L}, X' := \{(y,z)|z^d = \prod_{i=1}^{d-1} \delta_i^i(y)\}.$$

And γ acts by $z \mapsto e^{\frac{2\pi\sqrt{-1}}{d}}z$.

iii) The scheme structure of X is explicitly given as

$$X := Spec(\mathcal{O}_Y \oplus (\bigoplus_{\chi \in G^\star \backslash \{0\}} \mathcal{O}_Y(-L_\chi)))$$

where the divisor classes L_χ are recursively determined by $L_1 := L$, and by $L_{\chi+\xi} \equiv L_\chi + L_\xi + \sum_i \epsilon^i_{\chi,\xi}D_i$.

And where the ring structure is given by the multiplication maps

$$\mathcal{O}_Y(-L_\chi) \times \mathcal{O}_Y(-L_\xi) \to \mathcal{O}_Y(-L_{\chi+\xi})$$

determined by the section

$$\prod_i \delta_i^{\epsilon^i_{\chi,\xi}} \in H^0(\mathcal{O}_Y(-L_{\chi+\xi} + L_\chi + L_\xi)).$$

Remark 1.2. We can write more suggestively the ring structure as

$$z_\chi \cdot z_\xi = z_{\chi+\xi} \prod_i \delta_i^{\epsilon^i_{\chi,\xi}},$$

where z_χ is thought as a fibre variable on the geometric line bundle $\mathbb{L}_\chi$.

In other words, X is thus embedded in $\bigoplus_{\chi \in G^\star \backslash \{0\}} \mathbb{L}_\chi$.

Proof. We need only to show that X is irreducible iff either (**) or (***) holds.

But a covering is irreducible if and only if the covering monodromy is transitive, and this clearly holds if $m = 1$.

Else, we consider the quotient of X by the subgroup H of G generated by the inertia subgroups (i.e., $H = m\mathbb{Z}/d\mathbb{Z}$). Set $Z_m = X/H$, so that (**) holds, hence X is irreducible iff Z_m is irreducible.

More concretely, if $m > 1$, set $u := \dfrac{z^n}{\prod_i \delta_i^{\frac{i}{m}}}$, so that we have a factorization of the covering given by

$$ u^m = 1, \ z^n = u \prod_i \delta_i^{\frac{i}{m}}. $$

This means that we take $X \to Z_m \to Y$, and the last covering is étale with group $\cong \mathbb{Z}/m$.

The last covering is irreducible then if and only if the divisor class L' has order exactly m.

$\square$

Remark 1.3. The explicit description of cyclic covers (resp. : abelian covers) allows to construct families of varieties with a cyclic automorphism.

Let $\mathcal{Y} \to \mathcal{T}$ be a proper morphism with projective fibres, such that $\mathcal{Y}$ is factorial, and consider relative effective Cartier divisors $\mathcal{D}_i$ and a relative Cartier divisor $\mathcal{L}$ such that

$$ (*) \ d\mathcal{L} \equiv \sum_i i\mathcal{D}_i. $$

Then we have a finite Galois morphism $\mathcal{X} \to \mathcal{Y}$ over $\mathcal{Y} \to \mathcal{T}$ with an automorphism γ over $\mathcal{Y} \to \mathcal{T}$ such that γ generates the Galois group G and $\mathcal{X}/G = \mathcal{Y}$. In particular, for each fibre X_t, $X_t/G = Y_t$.

2. Cyclic covers of curves and their invariants

In this section C will be a projective complex curve of genus g, and γ an automorphism of C of order exactly d.

In this situation we can consider some obvious numerical invariants.

Definition 2.1. *Let $G \cong \mathbb{Z}/d$ be the subgroup generated by γ, and set $C' := C/G$, $h := genus(C')$.*

Denote by $k_i := deg(D_i)$ for $i = 1, \ldots, d-1$, and by $(k_1, \ldots k_{d-1})$ the branching sequence of γ.

Observe that if we change the chosen generator for G, then the covering does not change, only the identification of the Galois group with $\mathbb{Z}/d$ changes. Then we get another branching sequence, obtained by multiplying the indices of the elements of the given sequence with a

fixed element $r \in (\mathbb{Z}/d)^*$ (for example, for $r = -1$, we get the new sequence $(k_{d-1}, \ldots, k_1)$).

Definition 2.2. *We shall call a branching datum an equivalence class of branching sequences for the above action of $(\mathbb{Z}/d)^*$ by index multiplication. We shall denote it by $[(k_1, \ldots k_{d-1})]$.*

The next question is: which branching data do actually occur for g, d given?

A first restriction is given by the Hurwitz formula

$$2(g - 1) = d\{2(h - 1) + \sum_i k_i(1 - \frac{GCD(i,d)}{d})\},$$

which determines the genus of the quotient curve in terms of g, d and of the branching datum: it is necessary that h be a non negative integer number.

Moreover, by the previous theorem 1.1 a necessary and sufficient condition is, once the above formula yields $h \geq 0$, that

$$(*) \sum_i k_i i \equiv 0 \in \mathbb{Z}/d,$$

since in the Picard group of a curve a divisor is divisible by d iff its degree is divisible by d.

This motivates the following

Definition 2.3. *A branching datum corresponding to a sequence $[(k_1, \ldots k_{d-1})]$ is said to be admissible for d and g if (*) holds, and moreover*

$$h := 1 + \frac{2(g - 1)}{2d} - \frac{1}{2} \sum_i k_i(1 - \frac{GCD(i,d)}{d})$$

is a non negative integer.

Unless otherwise specified, we shall consider, given integers d, g, only admissible branching sequences.

One can view the same result from the point of view of Riemann's existence theorem: such pairs (C, γ) are determined by the following data: a curve C' of genus h, divisors D_i for all $i \in \mathbb{Z}/d$ and a surjective homomorphism $\psi : H_1(C' \backslash D, \mathbb{Z}) \to \mathbb{Z}/d$ such that the image of a small circle around a point $p \in D_i$ maps to the class of i in $\mathbb{Z}/d$.

We have the following exact sequence, where we write $D_1 = p_1 + \cdots + p_{k_1}$, $D_2 = p_{k_1+1} + \ldots p_{k_1+k_2}, \ldots,$

$$(**)0 \to A := (\oplus_j \mathbb{Z}p_j)/\mathbb{Z}(\sum_j p_j) \to H_1(C' \backslash D, \mathbb{Z}) \to H_1(C', \mathbb{Z}) \cong \mathbb{Z}^{2h} \to 0$$

which admits several splittings.

The condition $(*)$ pertains to the relation holding in the subgroup A, while we may choose a splitting such that $\mathbb{Z}^{2h}$ maps onto $\mathbb{Z}/d$. Topologically, this means that we choose a special symplectic basis

of $H_1(C', \mathbb{Z})$, such that the points p_j lie in the complement of the corresponding canonical dissection of the curve C', and then we take a disk Δ contained in this complement and containing the branch divisor D. Hence the ramified covering is just obtained glueing together a ramified covering of Δ with an unramified covering of $C' \setminus \Delta$.

For later use, we observe that, if d is a prime number p, then the Hurwitz formula is easier to write and we have

$$2(g - 1) = p\{2(h - 1) + k(1 - \frac{1}{p})\}, \ k := \sum_i k_i.$$

Theorem 2.4. *The pairs (C, G) where C is a complex projective curve of genus $g \geq 2$, and G is a finite cyclic group of order d acting faithfully on C with a given branching datum $[(k_1, \ldots k_{d-1})]$ are parametrized by a connected complex manifold $\mathcal{T}_{g;d,[(k_1,\ldots k_{d-1})]}$ of dimension $3(h-1)+k$, where $k := \sum_i k_i$.*

The image $\mathfrak{M}_{g;d,[(k_1,\ldots k_{d-1})]}$ of $\mathcal{T}_{g;d,[(k_1,\ldots k_{d-1})]}$ inside the moduli space $\mathfrak{M}_g$ is a closed subset of the same dimension $3(h-1)+k$.

Proof. Consider the Teichmüller space of curves C' of genus h with k marked points, and take a homomorphism ψ of the first homology group of C' onto $\mathbb{Z}/d$ sending the generator p_j of the subgroup $\mathbb{Z}p_j$, for $k_1 + \cdots + k_{i-1} + 1 \leq j \leq k_1 + \cdots + k_i$ to $i \in \mathbb{Z}/d$. Since the topological type of C' is fixed for the Teichmüller family, we choose a fixed splitting of $(**)$ and consider the surjection $H_1(C', \mathbb{Z}) \to \mathbb{Z}/d$ corresponding to a fixed primitive element $\Psi \in H^1(C', \mathbb{Z})$.

Recall that the symplectic group $Sp(2h, \mathbb{Z})$ acts transitively on the set of such primitive elements.

Applying remark 1.3 to our situation, we have a family of cyclic covers of the curves C', parametrized by Teichmüller space $\mathcal{T}$.

We want now to show that every pair (C, G) as in the statement occurs in our family. To this purpose, denote by C' the quotient curve, choose an isomorphism of G with $\mathbb{Z}/d$ and a marking of the branch points so that the branching sequence is $(k_1, \ldots, k_{d-1})$, and the divisor D_i consists of the points p_j with $k_1 + \cdots + k_{i-1} + 1 \leq j \leq k_1 + \cdots + k_i$.

We want to show that for a suitable diffeomorphism of C' which leaves the disk Δ pointwise fixed we can transform the resulting homomorphism Ψ into the standard one we have chosen.

To this purpose we take a product of Dehn twists over loops supported in $C' \setminus \Delta$, and we observe that these generate the mapping class group of C'. Since the mapping class group maps onto the symplectic group, our first statement is thus proven.

The group G has a linear representation on the tricanonical vector space $H^0(\mathcal{O}_C(3K_C))$, and on the family $\mathcal{T}_{g;d,[(k_1,\ldots k_{d-1})]}$ the dimension of the eigenspaces is semicontinuous, hence constant, since the sum is the fixed integer $5g - 5$.

We consider a vector space V of dimension $5g-5$ with a linear action of G having eigenspaces of the given dimensions. Inside $\mathbb{P}(V^\vee)$ we look at the locally closed subset $\mathcal{H}_g$ of the Hilbert scheme corresponding to tricanonically embedded smooth curves of genus g. Inside $\mathcal{H}_g$ we consider the closed subset $\mathcal{H}_g^G$ of subschemes which are G-invariant.

We claim that the image W of $\mathbb{P}GL(5g-5) \times \mathcal{T}_{g;d,[(k_1,...k_{d-1})]}$ is closed inside $\mathcal{H}_g^G$. If t_0 is in the closure of W, we can choose an analytic curve map $(T, t_0) \to \mathcal{H}_g^G$ such that T is biholomorphic to a 1-dimensional disk and there is a point t_1 such that t_1 maps to W.

Over T we have a family $\mathcal{C}$ of curves of genus g with an automorphism γ of order d, where γ generates G. Then $\mathcal{C}/G := \mathcal{C}'$ is a family of curves of genus h, and each point p in a fibre C'_{t_2} belongs to a section σ_p of $\mathcal{C}' \to T$, for which the stabilizer (i.e., the associated local monodromy subgroup) of $\sigma_p \cap C'_t$ is, $\forall t$, equal to the stabilizer of p (this follows since the action of G on $\mathcal{C}$ may locally be linearized, and the action on the base is the identity) and moreover the character of the tangent representation is also the same.

Hence the number of points k_i is constant for each $i = 1, \ldots, d-1$, and we have proven that W is closed, hence the image $\mathfrak{M}_{g;d,[(k_1,...k_{d-1})]}$ of $\mathcal{T}_{g;d,[(k_1,...k_{d-1})]}$ inside the moduli space $\mathfrak{M}_g$ is also closed.

Now the space $Kur(C)^G$ of G invariant deformations of our curves C correspond to the submanifold of the Kuranishi family of C obtained considering the subspace $H^1(\Theta_C)^G$ of $H^1(\Theta_C)$.

As shown in Pardini's article, there is an isomorphism between $H^1(\Theta_C)^G$ and $H^1(\Theta_{C'}(-D))$. This shows that the map between $\mathcal{T}_{g;d,[(k_1,...k_{d-1})]}$ and $Kur(C)^G$ is a local biholomorphism, hence our assertion on the dimension of $\mathfrak{M}_{g;d,[(k_1,...k_{d-1})]}$. $\qquad\square$

Remark 2.5. The general curve C inside $\mathfrak{M}_{g;d,[(k_1,...k_{d-1})]}$ has G as a maximal cyclic group of automorphisms unless possibly when we are in the cases

    (1) $h = 2$, $k = 0$;
    (2) $h = 1$, $k = 2$;
    (3) $h = 0$, $k = 3$ or 4.

We shall later see in theorem 3.4 that the only occurring case is the third.

Proof. If the group G is not a maximal cyclic subgroup of automorphisms, there would exist a nontrivial automorphism of C' leaving the divisor D invariant.

Since the divisor D can be chosen freely, this is a contradiction if C' has genus $h \geq 3$, and also in the case where $h = 2$ and D is non trivial.

By our assumption, in the case where C' has genus $h = 1$, D contains at least two points, and, if there are at least three points, we use that any automorphism of finite order on a general elliptic curve has order

2, hence ϕ leaves invariant a proper subset $D' \subset D$, with the property that the group H of automorphisms leaving D' invariant is finite. Then we derive a contradiction choosing the other points of D not to build a union of h-orbits for any element $h \in H$.

The same contradiction is derived in the case where $h = 0$ and $k \geq 5$.

If there is a nontrivial automorphism ϕ sending D to itself, and ϕ has only one orbit on D, then ϕ is cyclic of order k and the cross ratios of the k points satisfy an algebraic relation.

Otherwise there are at least two orbits, and a proper invariant subset $D' \subset D$, with at least 3 elements. Since the group H of automorphisms leaving D' invariant is finite, again we can choose the other points of D not to build a union of h-orbits for any element $h \in H$.

$\square$

3. IRREDUCIBLE COMPONENTS OF $Sing(\mathfrak{M}_g)$.

We begin this section with the obvious but very important observation that if a curve C of genus $g \geq 2$ has a nontrivial automorphism group, then (since $Aut(C)$ is finite) it has a non trivial automorphism γ of finite order d, and in particular it has an automorphism of prime order $p > 1$ (write $d = p \cdot m$).

By Theorem 2.4 and the ensueing remark the locus of curves with an automorphism of the same type as γ^m is strictly bigger than the locus of curves with an automorphism of the same type as γ, unless we are in 3 a priori possible special cases.

Therefore the locus of curves in $\mathfrak{M}_g$ with nontrivial automorphisms is the union of the irreducible closed subsets $\mathfrak{M}_{g;p,[(k_1,\ldots k_{p-1})]}$ (where p is a prime number and the sequence $(k_1, \ldots k_{p-1})$ is (g, p) admissible).

Remark 3.1. (Codimension of these loci)

$\mathfrak{M}_{g;p,[(k_1,\ldots k_{p-1})]}$ has dimension $3(h-1) + k$, while Hurwitz' formula reads out as

$$2(g-1) = 2p(h-1) + k(p-1) \Leftrightarrow 3(g-1) = 3p(h-1) + \frac{3}{2}k(p-1),$$

thus the codimension of this locus inside $\mathfrak{M}_g$ equals

$$c := 3(g-1) - 3(h-1) - k = 3(p-1)(h-1) + k\{\frac{3}{2}(p-1) - 1\} =$$

$$= 3(p-1)(h-1) + k(p-1) + k\{\frac{1}{2}(p-1) - 1\}.$$

If $h \geq 2$, then $c \geq 3$, if $h = 1$ we get $c \geq 4$ unless $p = 2$, in which case $k = 2(g-1)$, whence $c = (g-1)$, and $c \geq 2$ for $g \geq 3$. If $h = 0$, $c = (k-3)(p-1) + k\{\frac{1}{2}(p-1) - 1\}$, which is, for $p \geq 3$, ≥ 2 unless $k = 3, p = 3$: but then $g = 1$.

Finally, if $h = 0$, $p = 2$, $c = \frac{k}{2} - 3 \geq 2$ unless $k = 6$ or $k = 8$.

Hence the only exceptions to $c \geq 2$ are: all curves of genus 2 are hyperelliptic, hyperelliptic curves of genus 3 form a divisor inside $\mathfrak{M}_3$, and double covers of elliptic curves form a divisor inside $\mathfrak{M}_2$.

The next question is whether writing the locus of curves in $\mathfrak{M}_g$ with nontrivial automorphisms as the union of the irreducible closed subsets $\mathfrak{M}_{g;p,[(k_1,\dots k_{p-1})]}$ is an irredundant irreducible decomposition.

Proposition 3.2. *Assume that p, q are prime numbers, and that the component $\mathfrak{M}_{g;p,[(k_1,\dots k_{p-1})]}$ is contained in another (different) component $\mathfrak{M}_{g;q,[(k'_1,\dots k'_{q-1})]}$.*

For general $C \in \mathfrak{M}_{g;p,[(k_1,\dots k_{p-1})]}$, set $A := Aut(C)$, denote by $\gamma \in A$ the automorphism of order p with the first topological type, and by $\gamma' \in A$ the automorphism of order q with the second topological type. Set $C_1 := C/\gamma, C_2 := C/\gamma', C'' := C/A$, and let h_j be the genus of C_j, while we let h'' be the genus of C''; set finally $k := \sum_i k_i, k' := \sum_j k'_j$. Let G be the cyclic group generated by γ.

Then the normalizer A' of G in A is strictly bigger than G.

Proof. Denote by a the cardinality of A. Our assertion holds trivially if $a = 2p$, hence we shall assume that $a \geq 3p$.

If an element $\alpha \in A$ stabilizes a point which is fixed by γ, then α, γ generate a cyclic subgroup and, by our previous remark, $\alpha \in G$, with the possible exceptions $h_1 = 1, k = 2; h_1 = 0, k = 3$ or 4.

If in these exceptional cases $\alpha \notin G$, then obviously our assertion holds, hence we may reduce to consider the case where such an element α necessarily belongs to G.

In this case $C \to C_1$ is branched in k points, while the covering $C_1 \to C''$ is unramified over the images of these points.

Assume then that the normalizer of G in A is equal to G.

Then the k branch points of $C \to C_1$ have different images in C'', hence, if k'' denotes the number of branch points of $C \to C''$, $k'' \geq k$.

Hurwitz' formula yields then:

$$2(g-1) = p[2(h_1-1)+k\frac{(p-1)}{p}] \geq a[2(h''-1)+k\frac{(p-1)}{p}+\frac{1}{2}(k''-k)],$$

whence

$$(***) \quad 2(h_1-1)+k\frac{(p-1)}{p} \geq \frac{a}{p}[2(h''-1)+k\frac{(p-1)}{p}+\frac{1}{2}(k''-k)].$$

By counting the number of moduli for the two families, we get that

$$3(h_1-1)+k \leq 3(h''-1)+k'' \Leftrightarrow 2(h_1-1) \leq 2(h''-1)+\frac{2}{3}(k''-k).$$

The previous inequality $(***)$ is contradicted if $h'' \geq 1$, since $\frac{a}{p} \geq 3$.

If $h'' = 0$, then by the above inequality $k'' - k \geq 3h_1$, hence

$$2(h_1-1) \geq \frac{a}{p}[-2+\frac{3}{2}h_1]+(\frac{a}{p}-1)k\frac{(p-1)}{p} =$$

$$\frac{3a}{2p}[h_1 - 4/3] + (\frac{a}{p} - 1)k\frac{(p-1)}{p}.$$

If $h_1 \geq 2$ we get, since $\frac{a}{p} \geq 3$, $h_1 \leq \frac{8}{5}$, a contradiction.

If $h_1 = 1$ we get

$$\frac{1}{2}\frac{a}{p} \geq (\frac{a}{p} - 1)k\frac{(p-1)}{p},$$

and, since $k \geq 2$, we obtain $\frac{1}{2}\frac{a}{p} \geq (\frac{a}{p} - 1)$, contradicting $\frac{a}{p} \geq 3$.

If instead $h_1 = 0$ we get

$$2(\frac{a}{p} - 1) \geq (\frac{a}{p} - 1)k\frac{(p-1)}{p} \Leftrightarrow 0 \geq -2 + k\frac{(p-1)}{p},$$

hence $g \leq 1$, a contradiction.

□

In the following theorem, which is the main result of Cornalba in [Cor87] and [Cor08], we shall use the same notation introduced in Proposition 3.2.

Remark 3.3. In order to understand some statement in the following theorem, observe that, for $g \geq 3$, the singular locus of $\mathfrak{M}_g$ consists of the union of all the closed irreducible sets $\mathfrak{M}_{g;p,[(k_1,...k_{p-1})]}$, with exception of the divisor $\mathfrak{M}_{3;2,[(6)]} \subset \mathfrak{M}_3$, corresponding to the locus of hyperelliptic curves.

This holds since $\mathfrak{M}_g$ is locally the quotient of the Kuranishi family of a curve C by the action of $Aut(C)$. And, by Chevalley's theorem ([Chev55]), this quotient is smooth if and only if the action of $Aut(C)$ is generated by pseudoreflections. A pseudoreflection has a fixed locus which is a divisor, hence the only pseudoreflection which occurs is the hyperelliptic involution in genus $g = 3$.

An irreducible closed set $\mathfrak{M}_{g;p,[(k_1,...k_{p-1})]}$ is then an irreducible component of $Sing(\mathfrak{M}_g)$ if it is not properly contained in another irreducible set $\mathfrak{M}_{g;q,[(k'_1,...k'_{q-1})]}$, which is different from $\mathfrak{M}_{3;2,[(6)]}$.

Theorem 3.4. *Assume that p, q are prime numbers, and that the component $\mathfrak{M}_{g;p,[(k_1,...k_{p-1})]}$ is contained in a different component $\mathfrak{M}_{g;q,[(k'_1,...k'_{q-1})]}$.*

Let $A' \neq G$ be the normalizer of G in A, and set $G' := A'/G$.

We have exactly the following cases:

(1) *$h_1 = 2, k = 0$ (hence $g = p + 1$) and $G' \cong \mathbb{Z}/2$ is generated by the hyperelliptic involution; A' is a dihedral group D_p.*

   For $p = 2$ we have $g = 3$ and $A' \cong (\mathbb{Z}/2)^2$.

   In this case there are two possibilities for the component $\mathfrak{M}_{g;q,[(k'_1,...k'_{q-1})]}$, one being the divisor of hyperelliptic curves of genus $g = 3$, and the other being the 4-dimensional locus of double covers of elliptic curves.

   For p odd, any element of order 2 in $A' \setminus G$ has exactly 6 fixed points, and the quotient curve C_2 has genus $h_2 = 1 + \frac{p-3}{2}$;

whence we land in the component $\mathfrak{M}_{p+1;2,6}$ which has dimension $3\frac{p-3}{2} + 6 \geq 6 > 3$, and we have a strict inclusion

$$\mathfrak{M}_{p+1;p,[(0,\ldots 0)]} \subset_{\neq} \mathfrak{M}_{p+1;2,6}.$$

(2) *$h_1 = 1, k = 2$ (hence $g = p$) and $G' \cong \mathbb{Z}/2$ is generated by a transformation $z \mapsto -z + a$; we have that $[(k_1, \ldots k_{p-1})]$ is the class of $k_1 = 1, k_{p-1} = 1, k_i = 0$ for $i \neq 1, p-1$; A' is a dihedral group D_p and the genus of C equals p.*

For $p = 2$ we have $A' \cong (\mathbb{Z}/2)^2$ and there is only one possibility for the component $\mathfrak{M}_{g;q,[(k'_1,\ldots k'_{q-1})]}$, namely the divisor of hyperelliptic curves of genus $g = 3$.

This case yields a component of the singular locus of $\mathfrak{M}_3$.

For p odd, any element of order 2 in $A' \setminus G$ has exactly 4 fixed points, and the quotient curve C_2 has genus $h_2 = 1 + \frac{p-3}{2}$; whence we land in the component $\mathfrak{M}_{p;2,4}$ which has dimension $\frac{3}{2}(p-3) + 4 \geq 4 > 3$, and we have again a proper inclusion.

(3) *$h_1 = 0, k = 4$, (hence $g = p - 1$) and $\mathbb{Z}/2 \subset G'$ occurs for arbitrary p; if $\tau \in G'$ is a nontrivial element, then it is a double transposition of the four branch points.*

If τ permutes P_1 with P_2, and P_3 with P_4, then the local monodromies of the four points are either $m_1 = m_2 = 1, m_3 = m_4 = p - 1$ or $m_1 = 1, m_2 = p - 1, m_3 = n, m_4 = p - n$.

In the first case (where $m_1 = m_2 = 1, m_3 = m_4 = p - 1$), then τ lifts and τ and G generate a group $\cong (\mathbb{Z}/2p)$ for p odd, in the second case they generate a dihedral group D_p.

Thus in the case where the local monodromies are as in the first case (for instance for $p = 3$, since then the two cases coincide), $A' \cong D_p \times (\mathbb{Z}/2)$.

In the other case where the local monodromies are (equivalent to) $m_1 = 1, m_2 = p - 1, m_3 = n, m_4 = p - n, \ n \neq 1, n \neq p - 1$ then $A' \cong D_p$.

(4) *$h_1 = 0, k = 3$ (hence $g = \frac{p-1}{2}$ and $p \geq 5$) and $\mathbb{Z}/2 \subset G'$ occurs for arbitrary p, while $\mathbb{Z}/3 \subset G'$ occurs for $p \equiv 1(3)$.*

$G' \cong \mathfrak{S}_3$ cannot occur.

(5) *In the case $h_1 = 0, k = 3$, $\mathbb{Z}/2 \cong G'$ we have that A' is cyclic of order $2p$, and contained in the isotropy subgroup of a unique point of C.*

This is the only case where A' is contained in the isotropy subgroup of a point of C.

Remark 3.5. 1) In many cases one can directly show, once one has an inclusion $\mathfrak{M}_{g;p,[(k_1,\ldots k_{p-1})]} \subset \mathfrak{M}_{g;q,[(k'_1,\ldots k'_{q-1})]}$, that this inclusion is strict: for instance it suffices that $\mathfrak{M}_{g;q,[(k'_1,\ldots k'_{q-1})]}$ does not appear in the list given in Theorem 3.4.

2) If the group A' is $(\mathbb{Z}/2p)$ with p odd the component $\mathfrak{M}_{g;q,[(k'_1,\dots k'_{q-1})]}$ clearly corresponds to the only element of order 2 in A'. If the group A' is a dihedral group D_p, then we know that all elements of order 2 are conjugated, hence the corresponding components are of the same topological type.

Proof.

The quotient curve C_1 has a nontrivial group of automorphisms $G' := A'/G$, which preserves the set of k branch points.

Our main observation is however that the curve C_1 and the k branch points are general, therefore we conclude first of all that either $h_1 = 2, k = 0$ and $G' \cong \mathbb{Z}/2$ is generated by the hyperelliptic involution, or $k \geq 1$.

Since these branch points can be chosen arbitrarily, it follows that the genus $h_1 \leq 1$. And we can use the result of remark 2.5.

If $h_1 = 1$, then, since $k \geq 2$, we obtain by the generality assumption on the branch points that $k = 2$, and G' has order 2. In fact, given two general points x, y in a general elliptic curve, there is no translation leaving the set $\{x, y\}$ invariant, while there is a unique transformation $\tau_a : z \mapsto -z + a$ exchanging x, y: the one with $a = x + y$.

If instead $h_1 = 0$, it must be $k = 4$ (there exists always permutations which preserve the cross ratio, given by double transpositions) or $k = 3$.

Let's examine now more closely the several possibilities.

Case 1.

For $h_1 = 2, k = 0$ we observe that the hyperelliptic involution acts on the first homology group as -1, and -1 is an automorphism of G, whence the hyperelliptic involution lifts to an automorphism of C.

If $p \neq 2$ we get that A' is a dihedral group, while for $p = 2$ we have $g = 3$ and $A' \cong (\mathbb{Z}/2)^2$.

In the latter case we have a bidouble cover of $\mathbb{P}^1$ with branch divisors of degrees $0, 2, 4$, hence C has two more involutions, with respective quotients of genus 1, or 0. In particular C is hyperelliptic and the larger component consists either of the hyperelliptic curves of genus 3 (this component has dimension 5), or of the double covers of elliptic curves (this component has dimension 4).

In the case where p is an odd prime, all elements in $A' \setminus G$ are conjugate, and we may take any $\gamma \in A' \setminus G$.

Consider the standard presentation

$$D_p = \langle x, y; y^2 = x^p = xyxy = 1 \rangle,$$

and observe that all the local monodromies are conjugate of y, and denote then by H for each branch point the corresponding subgroup. So that the fibre over this branch point is in bijection with the set of cosets wH.

It is then easy to see that y has exactly one fixed point lying over each of the 6 branch points in $\mathbb{P}^1$: since, if H is as above, $H = z\{1, y\}z^{-1}$,

$$ywH = wH \Leftrightarrow w^{-1}yw \in H \Leftrightarrow w^{-1}yw = zyz^{-1} \Leftrightarrow zw \in \{1, y\}.$$

Case 2.

For $h_1 = 1, k = 2$, we may assume that the local monodromies of the two points x, y are equal to $1, p - 1$, since their sum is 0. In this case the automorphism τ_a lifts to the cyclic covering and we have again a dihedral group.

For $p = 2$ we have again a bidouble cover of $\mathbb{P}^1$ with branch divisors of degrees $0, 2, 4$ hence C has two more involutions, with respective quotients of genus 2, or 0.

If p is odd, the covering $C \to C' = \mathbb{P}^1$ is branched in 4 points, and the normal form of the monodromies is then (y, y, y, y, x, x^{-1}) using the standard presentation of D_p (we do not really need this assertion for the forthcoming argument). For each involution there is exactly, as before, one fixed point lying above each branch point of $C_1 \to C' = \mathbb{P}^1$.

Case 4.

For $h_1 = 0, k = 3$, let the local monodromies be $m_1, m_2, m_3 \in \{1, \ldots, p-1\}$. Without loss of generality we may assume $m_1 = 1$, and recall the obvious inequality $2p - 1 \geq 1 + m_2 + m_3 \equiv 0(p)$.

Hence $1 + m_2 + m_3 = p$ and we may write $m_1 = 1, m_2 = m, m_3 = p - 1 - m$, with $1 \leq m \leq p - 2$.

Let $\tau \in G'$: if τ has a fixed point, then we assume this point to be the point P_1. Then τ transposes P_2 and P_3, whence it lifts if and only if $m = p - 1 - m$, i.e., $m = \frac{p-1}{2}$ (observe that here $p \geq 3$).

If instead τ cyclically permutes the branch points $P_1 \mapsto P_2 \mapsto P_3 \mapsto P_1$, then τ lifts if and only if $m \cdot (p - 1 - m) \equiv 1(p)$.

In the field $\mathbb{Z}/p$ (recall $p \geq 3$) this means that $m^2 + m + 1 \equiv 0$ has a solution, and this occurs if and only if there exists m which is is a nontrivial third root of 1 (since $(m^2 + m + 1)(m - 1) = m^3 - 1$).

Thus this holds if and only if $p \equiv 1(3)$.

Observe that the two cases for the existence of such a τ are mutually exclusive: if the local monodromies are $1, \frac{p-1}{2}, \frac{p-1}{2}$, then they cannot be all equal, since $p = 3$ leads to a curve C of genus $g = 1$. Therefore it cannot occur that $G' \cong \mathfrak{S}_3$.

Case 3.

For $h_1 = 0, k = 4$, let the local monodromies be $m_1, m_2, m_3, m_4 \in \{1, \ldots, p-1\}$. Without loss of generality we may assume $m_1 = 1$, and recall that $3p - 2 \geq 1 + m_2 + m_3 + m_4 \equiv 0(p)$.

Hence $1 + m_2 + m_3 + m_4 = p$ or $1 + m_2 + m_3 + m_4 = 2p$ and we may write $m_1 = 1, m_2 = m, m_3 = n, m_4 = p - 1 - m - n$, or $m_1 = 1, m_2 = m, m_3 = n, m_4 = 2p - 1 - m - n$.

Since the four points have a general cross ratio, $\tau \in G'$ is a double transposition and without loss of generality we may assume that τ permutes P_1 with P_2, and P_3 with P_4.

Whence τ lifts if and only if $m^2 \equiv 1(p)$, $m_4 \equiv mn(p)$. Hence $m \equiv \pm 1$, and $1 + m + n + mn \equiv 0(p)$, i.e.,

$$(1 + m)(1 + n) \equiv 0(p) \Leftrightarrow m \equiv -1(p) \text{ or } n \equiv -1(p).$$

Hence the solutions are with $m \equiv -1$, n arbitrary, $m_4 \equiv -n$, or $m_1 = m_2 = 1, m_3 = m_4 = p - 1$.

The other assertions follow then easily as before, since if τ exchanges two points with the same monodromy, then its lift commutes with γ; else, if it exchanges two points with opposite monodromies, then its lift conjugates γ with its inverse.

Observe finally that in the second case if it were possible to exchange P_1 with P_3 it should hold: $n^2 \equiv 1(p)$, which is excluded by the assumption $n \neq 1, n \neq p - 1$.

Assertion 5.

It is easy to describe explicitly the case where $h_1 = 0, k = 3$ and τ has order 2. Up to an automorphism of the cyclic group, we can assume $m_1 = p - 2$, $m_2 = m_3 = 1$.

In other words, the function field of C is $\mathbb{C}(x, z)$, where $z^p = (x^2 - 1)$. The involution is $\tau(x) = -x$, which lifts to C, and the quotient of C by τ is the curve with function field $\mathbb{C}(z)$.

Setting $y := x^2$, $xz := u$, we have $z^p = (y - 1)$, $u^{2p} = y^p(y - 1)^2$, and $\mathbb{C}(x, z) = \mathbb{C}(y, u)$ expresses C as a cyclic covering of the $\mathbb{P}^1$ with function field $\mathbb{C}(y)$.

The only fixed point for the cyclic group G' is the one lying over ∞, while the automorphism γ has three fixed points, lying over $x = \infty, x = 1, x = -1$.

τ exchanges the second and third point of these three, and leaves fixed other p points, the ones lying over $x = 0$.

Assume now that A' is contained in the isotropy subgroup of a point $P_0 \in C$, hence in particular A' is cyclic. From remark 2.5 and the above description of the possible cases, we see that we necessarily are in the case where $h_1 = 0, k = 3$, and τ leaves one point fixed. Thus we are in case 4, with τ of order 2.

$\square$

4. Spaces of cyclic covers of stable curves

In this section we consider a stable curve C of genus $g \geq 2$ and $\gamma \in Aut(C)$ an automorphism, of order d. At a later point we shall make the simplifying assumption that d is a prime number.

Consider the decomposition of C into irreducible components

$$C = \cup_{i \in I} C_i.$$

We partition the set of irreducible components into three subsets:

$$I_0 := \{i \in I | \gamma|_{C_i} = Id_{C_i}\}; \ I_1 := \{i \in I\backslash I_0 | \gamma(C_i) = C_i\}; I_2 := I\backslash\{I_0 \cup I_1\}.$$

Since each irreducible component of the space of such pairs (C, γ) will be the union of certain strata, we try to look immediately for such strata which are open. For this purpose we use deformation theory.

Recall that the Kuranishi space of such a curve C is smooth and locally biholomorphic to $Ext^1(\Omega_C^1, \mathcal{O}_C)$, and the subspace of local deformations of the pair (C, γ) will be locally biholomorphic to $Ext^1(\Omega_C^1, \mathcal{O}_C)^G$, where G is the cyclic group generated by γ.

The local to global spectral sequence yields an exact sequence

$$0 \to \oplus_i H^1(\Theta_{C_i}(-\sum_{j \neq i}(C_i \cap C_j))) \to \mathrm{Ext}^1(\Omega_C^1, \mathcal{O}_C) \to (\oplus_{p \in Sing(C)}\mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)_p) \to 0.$$

It is an exact sequence of G-vector spaces, hence the sequence of G-invariants is also exact.

In particular we have a surjection:

$$\mathrm{Ext}^1(\Omega_C^1, \mathcal{O}_C)^G \to (\oplus_{p \in Sing(C)}\mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)_p)^G.$$

The first consequence is that we can smooth all the G-fixed nodes p such that

$$\mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)_p^G = \mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)_p \cong \mathbb{C}.$$

By the well known Cartan's lemma ([Car57]) the action of G can be linearized around p, and, if we set $\zeta := exp(2\pi\, i/d)$, there are local holomorphic coordinates (x, y) such that $C = \{xy = 0\}$, and either

$$\gamma(x, y) = (\zeta^m x, \zeta^n y)$$

or

$$\gamma(x, y) = (y, \zeta^{2m}x).$$

$\mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C) \cong \mathbb{C}$ is identified with the space of local deformations of the singularity, i.e., $\mathbb{C} = \{t \in \mathbb{C}\}$ is the parameter space for the family of curves

$$\{(x, y, t) | xy = t\}.$$

In the first case G acts on the family by

$$\gamma(x, y, t) = (\zeta^m x, \zeta^n y, \zeta^{m+n}t).$$

Hence $\mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)_p^G = \mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)_p \cong \mathbb{C}$ if and only if $m + n \equiv 0(d)$, i.e., exactly when we have a local family of curves with a G-action.

In the second case G acts on the family by

$$\gamma(x, y, t) = (y, \zeta^{2m}x, \zeta^{2m}t)$$

hence $\mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)_p^G = \mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)_p \cong \mathbb{C}$ if and only if $2m \equiv 0(d)$, i.e., exactly when we have a local family of curves with a G-action.

Definition 4.1. *We shall say that a pair (C, γ) is **simplifiable** if it admits a small deformation to a pair with a smaller number of nodes, whereas we shall say that (C, γ) is **maximal** if it is not simplifiable.*

Remark 4.2. (1) Assume that $i, j \in I_0$ and that $p \in C_i \cap C_j$. Then the node p can be smoothed.

Hence, if (C, γ) is maximal, $\forall i, j \in I_0$ we have $C_i \cap C_j = \emptyset$.

(2) if $i_0 \in I_0$ and $\exists p \in C_j \cap C_{i_0}$, then $\gamma(C_j) = C_j$.

(3) if p is a node such that $\gamma(p) = p$, and $p \in C_i \cap C_j$, $(i \neq j)$, then either

(3i) $\gamma(C_i) = C_i$, $\gamma(C_j) = C_j$ or

(3ii) $\gamma(C_i) = C_j$.

In the first subcase, if we set as above $\zeta := exp(2\pi\ i/d)$, there are local holomorphic coordinates (x, y) such that $C = \{xy = 0\}$, $C_i = \{y = 0\}$, $C_j = \{x = 0\}$, and

$$\gamma(x, y) = (\zeta^{m_i} x, \zeta^{m_j} y).$$

The node is smoothable if and only if $m_i + m_j = d$ (we take $m_i \in \{0, 1, \ldots, d-1\}$).

In the second case we have:

$$\gamma(x, y) = (y, \zeta^{2m} x),$$

and the node is smoothable if and only if $\zeta^{2m} = 1$.

Lemma 4.3. *If d is prime, then each node $p \in C$ not fixed by γ can be smoothed. Hence, if (C, γ) is maximal and d is prime, then every node $p \in C$ is fixed by γ.*

Proof. Since d is prime we see then that the orbit of p has d elements, and there is a bijection between G and the orbit $G(p)$.

Look however at the summand of $(\oplus_{p' \in Sing(C)} \mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)_{p'})^G$ corresponding to $(\oplus_{p' \in G(p)} \mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)_{p'})^G \subset (\oplus_{p' \in G(p)} \mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)_{p'})$.

It is not empty since $\oplus_{p' \in G(p)} \mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)_{p'}$ corresponds to the representation of G on the orbit $G(p)$: this yields a small deformation smoothing all nodes in the orbit $G(p)$.

$\square$

Proposition 4.4. *Let $\hat{\gamma}$ be the permutation of I induced by γ.*

If (C, γ) is maximal and d is prime, then $\hat{\gamma} = $ Identity (i.e., $I_2 = \emptyset$).

Proof. Observe preliminarly that if $\hat{\gamma}(i) = j \neq i$ and $C_i \cap C_j \neq \emptyset$, then d is divisible by 2. Since then, by the previous lemma, if we take $p \in C_i \cap C_j$, then $\gamma(p) = p$, so that $\hat{\gamma}$ transposes i and j.

Since d is prime, $d = 2$ and by the previous remark each node $p \in C_i \cap C_j$ is smoothable.

Hence, since we assume (C, γ) to be maximal, $C_i \cap C_j = \emptyset$.

Let now $i \in I_2$. There exists, by the connectedness of C, a j such that $C_i \cap C_j \neq \emptyset$.

Let $p \in C_i \cap C_j$. Since $\gamma(p) = p$, $\hat{\gamma}$ leaves the set $\{i, j\}$ invariant.

But $\hat{\gamma}(i) = i$ contradicts $i \in I_2$, while $\hat{\gamma}(i) = j$ contradicts our previous observation. Hence $I_2 = \emptyset$.

$\square$

Proposition 4.5. *If (C, γ) is maximal and d is prime then, for each $i \in I_0$, C_i is smooth.*

If $i \in I_1$ and p is a node of C_i, then γ does not exchange the two branches of p.

Proof.
If $i \in I_0$, then each node p of C_i is smoothable.

If $i \in I_1$, and p is a node of C_i, then we know that $\gamma(p) = p$.

If γ exchanges the two branches, then $d = 2$, and as we saw before the node is smoothable.

If γ does not exchange the two branches, the local action is $\gamma(x, y) = (\zeta^m x, \zeta^n y)$, and the node is smoothable iff $m + n = d$.

$\square$

We are ready to define the numerical type (it is indeed a combinatorial type, but we call it a numerical type to keep the analogy with the smooth case) of maximal pairs (C, γ) in the case where d is a prime number.

Definition 4.6. *Let (C, γ) be a maximal pair for d prime.*

We attach to (C, γ) a graph whose vertices correspond to the set I, and whose edges correspond to the nodes p. Each vertex i has a multilabeling, first of all a labeling by the genus g_i of C_i, and then a colouring 0, or 1, according to $i \in I_0$, or $i \in I_1$.

For $i \in I_1$, we associate to i a branching sequence $(k'_1, \ldots, k'_{d-1})$ corresponding to the fixed points of $\gamma|_{C_i}$ which are not nodes.

For each edge p connecting i and j, $i \in I_1$, $i \neq j$, we give labels $m(p, i) \in \{1, \ldots, d-1\}$, $m(p, j) \in \{0, 1, \ldots, d-1\}$ according to the local action at the fixed point $p \in C_i$, respectively $p \in C_j$.

If instead $i = j$, we look at the action on the two branches and obtain an unordered pair $n_1(p, i), n_2(p, i)$.

In order to understand the notion of admissibility which will be given next, observe that for each $j \in I_1$ we shall consider a curve C''_j which is the normalization of C_j, hence the genus C''_j equals g_j plus the number of nodes of C_j. Then G acts on C''_j and we denote by $C'_j := C''_j/G$ the quotient curve, by g'_j the genus of C'_j, and by r_j the number of branch points of $C''_j \to C'_j$.

Remark 4.7. C_j and C_j/G can be easily reconstructed by the marking of certain pairs of branch points on C'_j.

Definition 4.8. *Let d be a prime number.*

Then an admissible automorphism graph is a connected graph with the following properties.

It has set of vertices $I = I_0 \cup I_1$, and each vertex $i \in I_0$ is labelled by an integer g_i, while each vertex $i \in I_1$ is labelled by an integer g_i, and by a branching sequence $(k'_1(i), \ldots, k'_{d-1}(i))$.

No edge can connect two vertices in I_0, but an edge can connect a vertex $i \in I_1$ with itself (i.e., the graph has loops).

Each edge p connecting i and j, $i \neq j$, is labelled by $m(p, i) \in \{0, 1, \ldots, d-1\}$, and $m(p, j) \in \{0, 1, \ldots, d-1\}$ in such a way that $m(p, i) = 0$ if and only if $i \in I_0$, and moreover $m(p, i) + m(p, j) \neq d$.

If instead $i = j$, we label the loop by an unordered pair of integers in $\{1, \ldots, d-1\}$

$$n_1(p, i), n_2(p, i) \neq 0$$

such that $n_1(p, i) + n_2(p, i) \neq d$.

Define, for each $i \in I_1$, the branching integer $k_m(i)$ as the sum of $k'_m(i)$ with the number $k''_m(i)$ of occurrences of the integer m among the integers $m(p, i)$, for p an edge connecting i with $j \neq i$, or among the pairs of integers $n_1(p, i), n_2(p, i)$, for p a loop based at i.

*Then the sequence $(k_1(i), \ldots, k_{d-1}(i))$ must be **admissible** in the sense that there exists a non negative integer g'_i such that, setting $g''_i := g_i + \nu_i$, ν_i being the number of loops based at the vertex i, and setting $k(i) := \sum_m k_m(i)$, we have*

$$2(g''_i - 1) = d\{2(g'_i - 1) + k(i)(1 - \frac{1}{d})\}.$$

The genus g of the graph is as usual defined as

$$g := \sum_{i \in I} g_i + b^1,$$

where b^1 is the first Betti number of the connected graph.

Remark 4.9. There is an obvious action of $(\mathbb{Z}/d)^*$ on the branching sequences and on the edge labelings, and the equivalence classes of the admissible graphs for this action are denoted the **numerical types** of the maximal pairs (C, G) where G is a cyclic group of automorphisms of prime order d.

Theorem 4.10. *The pairs (C, G) where C is a stable projective curve of genus $g \geq 2$, and G is a finite cyclic group of prime order d acting faithfully on C with a given numerical type associated to an admissible automorphism graph $\mathcal{G}$ are parametrized by a non empty connected complex manifold $\mathcal{T}_{g;d,[\mathcal{G}]}$.*

The image $\overline{\mathfrak{M}}_{g;d,[\mathcal{G}]}$ of $\mathcal{T}_{g;d,[\mathcal{G}]}$ inside the compactified moduli space $\overline{\mathfrak{M}}_g$ is a locally closed subset of the same dimension whose closure consists of the coverings whose numerical type can be simplified to the numerical type of $\mathcal{T}_{g;d,[\mathcal{G}]}$.

If $\mathcal{T}_{g;d,[\mathcal{G}]}$ contains only stable singular curves, then $\overline{\mathfrak{M}}_{g;d,[\mathcal{G}]}$ is not a divisor in the moduli space $\overline{\mathfrak{M}}_g$, unless we are in the following two cases:

    (1) *$d = 2$, $C = C_1 \cup C_2$, where $1 \in I_0$, $2 \in I_1$, and $g_2 = 1$ (elliptic tail).*

(2) $d = 2$, $C = C_1 \cup C_2$, where $1, 2 \in I_1$, and $g_i = 1$ ($g = 2$ case).

Proof.

We consider $\mathcal{T}_{g;d,[\mathcal{G}]}$ as a product of two products of Teichmüller spaces: firstly

$$\Pi_{i \in I_0} \mathcal{T}_{g_i, r_i},$$

where r_i is the number of edges which touch the vertex $i \in I_0$, and secondly

$$\Pi_{j \in I_1} \mathcal{T}_{g'_j, r_j}.$$

In the second case, for each $j \in I_1$, and for each point in $\mathcal{T}_{g'_j, r_j}$ we construct a curve C''_j which has an automorphism γ of order d with quotient a curve C'_j of genus g'_j, and such that $C''_j \to C'_j$ is branched on r_j points, and with the branching indices determined by $\mathcal{G}$.

Then we construct the family of curves C_j from the family of curves C''_j glueing certain pairs of ramification points according to the pattern determined by $\mathcal{G}$.

Observe that the family of such coverings $C''_j \to C'_j$ is an irreducible family in view of Theorem 2.4, since the numerical type determines the branching datum, and the local monodromy at the points of C'_j corresponding to the nodes of C_j is also determined by the admissible automorphism graph.

Hence the same holds for the family of such coverings C_j (for $j \in I_1$), and we finally obtain in this way a family with connected base of curves C with an automorphism γ of topological type determined by the graph $\mathcal{G}$.

Since by our assumption for each curve C in the family we have

$$(\oplus_{p \in Sing(C)} \mathcal{E}xt^1(\Omega^1_C, \mathcal{O}_C)_p)^G = 0,$$

it is easy to see that our family equals the Kuranishi family $\mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C)^G$ of pairs (C, γ) at each point.

Hence the image $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ of $\mathcal{T}_{g;d,[\mathcal{G}]}$ is a locally closed subset of the same dimension as $\mathcal{T}_{g;d,[\mathcal{G}]}$. Its closure consists of pairs (C, G) corresponding to pairs (C, γ) which admit a G-invariant local deformation containing a maximal pair (C, γ).

If $\mathcal{T}_{g;d,[\mathcal{G}]}$ contains only stable singular curves, then the locus $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ is not a divisor if the general curve has at least two nodes.

If C is stable and has only one node, then if C is irreducible its normalization has no automorphisms provided C is general. If instead $C = C_1 \cup C_2$, we should have that all smooth curves C_1 of genus g_1 and all smooth curves C_2 of genus g_2 occur.

Assume that $1 \in I_0$ and $2 \in I_1$: then it must be $g_2 = 2$ or $g_2 = 1$. Since however the node must be, for $g_2 = 2$, a fixed point for the hyperelliptic involution, we do not have a divisor for $g_2 = 2$ and this case must be excluded.

If instead $1, 2 \in I_1$, both curves would have genus equal to 2 or 1. If however $g_1 = 2$, then the node must be a fixed point for the hyperelliptic involution, and we no longer have a divisor.

$\square$

We need the following result for the forthcoming theorem.

Lemma 4.11. *Let C be a stable curve with elliptic tails $E_1, \ldots E_r$. Then the automorphism group of C is a direct sum $(\oplus_{i=1,\ldots r} \mathbb{Z}/m_i)^r \oplus \Gamma$ where the first addendum is generated by the multiplication by an m_i-th root of unity around the node p_i of each elliptic tail E_i (hence $m_i \in \{2, 4, 6\}$). The quotient of the Kuranishi family of C by $Aut(C)$ is then singular unless Γ is trivial and $m_i = 2 \ \forall i$.*

Proof. It suffices to define Γ as the subgroup which acts as the identity on each elliptic tail E_i.

Recall that, by the cited theorem of Chevalley ([Chev55]), the quotient of a smooth manifold of dimension N by a finite group Γ acting with a fixed point P is smooth at the image point of P if and only if the group is generated by pseudoreflections (these are transformations which are biholomorphic to linear maps with exactly $N - 1$ eigenvalues equal to 1). In particular, if the quotient is smooth the locus of fixed points is a union of divisors.

The only pseudoreflections correspond to elliptic tails and to the case where the automorphism of E_i is of order 2. The assertion follows now immediately.

$\square$

Definition 4.12. (Enlargement).

Let γ be an automorphism of prime order $d > 1$, with (C, γ) maximal (hence $I = I_0 \cup I_1$).

Enlargement of type 1).

Assume there is a component C_j, where $j \in I_1$, such that C_j does not intersect any component C_i with $i \in I_0$. Then we can consider an automorphism γ'' such that $\gamma|_{C_j} =$ Identity, and $\gamma'' = \gamma$ on the other components.

If C_j has nodes, we obtain a non maximal pair (C, γ''), where γ'' has the same order of γ, but if we smooth the nodes of C_j, we obtain a maximal pair (C''', γ''). Denote by $\mathcal{G}''$ the associated graph.

Then the closed subvariety $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ is contained in the closed subvariety $\overline{\mathfrak{M}_{g;d,[\mathcal{G}'']}}$, and properly contained unless C_j is an elliptic tail and $d = 2$ (if C_j is smooth of genus 0, observe that it intersects the other components in at least 3 points, which are fixed by γ, contradicting $j \in I_1$).

Enlargement of type 2).

Assume that there is a component C_j with $j \in I_1$, such that C_j intersects some components C_i with $i \in I_0$. Assume further that $I_1 \neq \{j\}$.

Then we can consider an automorphism γ'' such that $\gamma|_{C_j} = $ Identity, and $\gamma'' = \gamma$ on the other components. We obtain a non maximal pair (C, γ''), where γ'' has the same order of γ. If we first smooth the nodes of C_j, we obtain a non maximal pair (C'', γ''); however, if we smooth the union of C_j'' with the components C_i with $i \in I_0$ and which intersect C_j'', we obtain a maximal pair (C''', γ'''). Denote by $\mathcal{G}'''$ the associated graph.

Then the closed subvariety $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ is contained in the closed subvariety $\overline{\mathfrak{M}_{g;d,[\mathcal{G}''']}}$, and properly contained unless C_j is an elliptic tail and $d = 2$.

Maximal enlargement.

Take a component C_j with $j \in I_1$, assume further that $I_1 \neq \{j\}$.

Then we can consider an automorphism γ'' such that $\gamma|_{C_i} = $ Identity for $i \neq j$, and $\gamma'' = \gamma$ on C_j.

We obtain a non maximal pair (C, γ''), where γ'' has the same order of γ. If we smooth the nodes of $C \setminus C_j$, we obtain a maximal pair (C'', γ'').

Denote by $\mathcal{G}'''$ the associated graph.

Then the closed subvariety $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ is contained in the closed subvariety $\overline{\mathfrak{M}_{g;d,[\mathcal{G}''']}}$, and properly contained unless $d = 2$ and, $\forall h \in I_1 \setminus \{j\}$, C_h is an elliptic tail.

Clearly a maximal enlargement can be obtained as a sequence of enlargements of type 1 and 2.

Before stating the main theorem of this section, let's discuss two cases to which we shall refer as to ' the exceptional cases'.

Definition 4.13. *Consider a curve C as in 5 of theorem 3.4, i.e., with a cyclic group A' of automorphisms of order $2p$, where $p \geq 3$ is a prime number.*

Denote by γ' an automorphism in A' of order p, and by γ the automorphism of order 2.

Denote by P_0 the only fixed point of A' on C, and let P_1, P_2 be the other two fixed points of γ', which are exchanged by γ.

II-a).

Define C_0' to be the nodal curve obtained by C identifying P_1, P_2, and let C_3' be another smooth curve of genus at least 1, and $P_3 \in C_3'$ an arbitrary point.

Let C' be the stable curve obtained as $C' = C_0' \cup C_3'$, identifying $P \in C_0'$ with $P_3 \in C_3'$, and let γ' be the automorphism induced by γ' on C_0', extended as the identity on C_3'.

Let γ be the automorphism induced by γ on C_0', extended as the identity on C_3'.

Then we set (C'', γ) to be a smoothing of C' at the node of C_0'.

Here, C_0'' is smooth hyperelliptic of genus $\frac{p+1}{2}$.

II-b).

Define C'_0 to be the curve C, let C'_3 be another smooth curve of genus at least 1, and $P_3 \in C'_3$ an arbitrary point, and consider moreover two isomorphic 1-pointed smooth curves of genus at least 1 $(C'_1, P'_1) \cong (C'_2, P'_2)$.

Let C' be the stable curve obtained as $C' = C'_0 \cup C'_1 \cup C'_2 \cup C'_3$, obtained identifying $P \in C'_0$ with $P_3 \in C'_3$, and $P_h \in C'_0$ with $P'_h \in C'_h$, for $h = 1, 2$.

Let γ' be the automorphism induced by γ' on C'_0, extended as the identity on the other components C'_i.

Let γ be the automorphism induced by γ on C'_0, extended as the identity on C'_3, and exchanging C'_1 with C'_2 according to the given isomorphism and its inverse.

Then we set (C'', γ) to be a smoothing of C' at the nodes corresponding to P_2, P_3.

Theorem 4.14. *Assume that $g \geq 2$, and consider the closed subvarieties $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ inside the compactified moduli space $\overline{\mathfrak{M}_g}$, such that*

(1) *d is a prime number*
(2) *the cyclic group G either has order $d \neq 2$ or it acts trivially on the elliptic tails.*
(3) *the subset I_1 contains exactly one element*
(4) *I_0 is not empty (hence $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ contains only singular stable curves).*
(5) *$\mathfrak{M}_{g;d,[\mathcal{G}]}$ is not one of the two exceptional cases II-a, II-b for (C', γ').*

The above components $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ are then all distinct, for different d and different topological types, and provide the irreducible components of $Sing(\overline{\mathfrak{M}_g})$ which do not intersect $\mathfrak{M}_g$.

Proof. $\overline{\mathfrak{M}_g}$ is locally the quotient of the Kuranishi family of a stable curve C by the group $Aut(C)$.

Hence $\overline{\mathfrak{M}_g}$ is smooth unless C has an automorphism of prime order.

Moreover, as we already recalled, by Chevalley's theorem, the quotient is smooth if and only if the action of the group $Aut(C)$ is generated by pseudoreflections.

By our previous remark the only pseudoreflections correspond to reflections on an elliptic tail. Hence, by lemma 4.11, it suffices to take care of the cases where $d \neq 2$, or, if $d = 2$, we can assume that γ acts trivially on the elliptic tails.

By the definition of maximal enlargement given in 4.12 we may restrict ourselves to consider only irreducible components satisfying the assumption that the subset I_1 contains exactly one element.

We want first to see when two irreducible components $\overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$ and $\overline{\mathfrak{M}_{g;d',[\mathcal{G}']}}$, satisfying our assumptions, can be contained into each other.

We assume $\overline{\mathfrak{M}_{g;d',[\mathcal{G}']}} \subset \overline{\mathfrak{M}_{g;d,[\mathcal{G}]}}$.

Let C' be a general curve in $\mathfrak{M}_{g;d',[\mathcal{G}']}$, and C a general curve in $\mathfrak{M}_{g;d,[\mathcal{G}]}$. Since C is a smoothing of C', we have an automorphism γ of C' such that the pair (C',γ) deforms to the pair (C,γ).

Observe that there is a unique component C'_j with $j \in I'_1$. Since C' is general, and (C',γ') is maximal, C'_j intersects all other components C'_i, while each of these other components C'_i is smooth, and intersects only the component C'_j.

We infer easily from the above observation that there are only two possible cases.

Case a) : $\gamma(C'_j) = C'_j$

Case b) : C' has exactly two components, C'_j and C'_i, and γ exchanges them.

Case b) leads to a contradiction, since then $d = 2$ and C would be smooth (while $I_0 \neq \emptyset$).

In case a), if γ were the identity on C'_j, we would derive a contradiction. In fact, for each component C'_i with $i \in I_0$ such that γ is not the identity on C'_i, we have that C'_i is smooth, and all the points of intersection with C'_j can be chosen freely. This implies that either C'_i is an elliptic tail, and γ the elliptic involution, or $C'_i \cong \mathbb{P}^1$, and γ has at least three fixed points on C'_i . The second alternative implies that γ is the identity of C'_i, a contradiction. Hence, for all such $i \in I_0$, the first alternative holds, contradicting our assumption 2.

We conclude that both automorphisms γ, γ' leave then C'_j invariant and are different from the identity.

We have that C contains an irreducible component C_i on which γ is the identity. It follows that C_i comes from deforming a component C'_i with $i \in I'_0$.

Then both γ, γ' leave C'_i pointwise fixed, in particular a node $P \in C'_i \cap C'_j$.

Since γ, γ' belong to the isotropy group of P, a cyclic subgroup, either

I) $\gamma = \gamma'$ on C'_j, so that in particular $d = d'$, or

II) we are in the exceptional situation $h_1 = 0, k = 3$ of Theorem 3.4, and γ has order $d = 2$.

If I) holds, then it follows that, for each component C'_i with $i \in I'_0$, C'_i is left invariant by γ, and there are two possibilities:

A) γ is the identity on C'_i and no nodes $P \in C'_i \cap C'_j$ are smoothed;

B) γ is not the identity on C'_i and some nodes $P \in C'_i \cap C'_j$ are smoothed;

Case B) leads however to a contradiction, by the generality of C', because C'_i admits no nontrivial automorphism unless it is an elliptic tail and $d' = 2$. But this possibility is excluded by our assumptions.

The conclusion is that $C = C'$ and $\gamma = \gamma'$, as we wanted to show.

Let's consider now the second possibility II).

Here, as we saw in the proof of theorem 3.4, γ' has exactly 3 fixed points on the normalization of C_j', one being our P, fixed by γ also, and the other two, P_1, P_2 being exchanged by γ.

The first conclusion is that there is only one component C_i' sent to itself by γ.

Since moreover, for each node of C_j', γ' does not exchange the two branches, the only possibilities are :

II-a) C_j' has a node, and there is only another component C_i', meeting C_j' precisely in one point P.

II-b) C_j' is smooth, and there are just three other components: C_i' meeting C_j' precisely in one point P, and then C_{i_1}', C_{i_2}', exchanged by γ and such that $C_j' \cap C_{i_h}'$ consists precisely of the point P_h.

The above possibilities correspond exactly to the two exceptional cases for (C', γ'), hence our proof is finished.

$\square$

5. Some open questions.

The first natural question that would be worth to investigate is: what is the description of the numerical type of an automorphism of a stable curve C of non prime order d?

We have seen in section 4 that the assumption that d be a prime number is repeatedly used, so the combinatorial description is likely to be rather more complicated.

Morally, however, one should expect that again a similar result to theorem 4.10 holds true in the non prime case.

More generally, an interesting problem is the investigation of the group of automorphisms of a stable singular curve.

Remark 5.1. Consider a stable curve consisting of a rational smooth component C_0, intersecting g elliptic tails $C_1, \ldots C_g$ in nodes $p_1, \ldots p_g$.

Then, if $C_1, \ldots C_g$ and $p_1, \ldots p_g \in C_0$ are general, $Aut(C)$ has cardinality at least 2^g. If instead the elliptic curves are equianharmonic, and the points $p_1, \ldots p_g$ are roots of unity in the complex line $\mathbb{C}$, then $Aut(C)$ has cardinality $(2g) \cdot 6^g$.

This number is by far larger than the Hurwitz bound $84(g-1)$ for the cardinality of $Aut(C)$ for a smooth curve of genus g.

Concerning the questions about determining the Hurwitz bound for stable curves, and the geometrical description of the stable curves C of genus g such that $Aut(C)$ attains the maximal allowed cardinality, which we had posed in the first version of this paper, we have been informed by Gavril Farkas that these issues have been thoroughly investigated and completely solved by van Opstall and Veliche (see [vO-V-07], [vO-V-10a], [vO-V-10b]).

References

[BF86] Biggers, R.; Fried, M. *Irreducibility of moduli spaces of cyclic un-ramified covers of genus g curves.* Trans. Am. Math. Soc. 295, 59-70 (1986).

[Car57] Cartan, H. *Quotient d'un espace analytique par un groupe d'automorphismes.* A symposium in honor of S. Lefschetz, Algebraic geometry and topology. pp. 90–102. Princeton University Press, Princeton, N. J. 1957.

[Cat84b] Catanese, F. *On the moduli spaces of surfaces of general type.* J. Differential Geom. 19 (1984), no. 2, 483–515.

[Cat88] Catanese, F. *Moduli of algebraic surfaces.* Theory of moduli (Montecatini Terme, 1985), 1–83, Lecture Notes in Math., 1337, Springer, Berlin, (1988).

[Cat00] Catanese, F., *Fibred Surfaces, varieties isogeneous to a product and related moduli spaces.* Amer. J. Math. 122 (2000), no. 1, 1–44.

[Cat08] Catanese, F. *Differentiable and deformation type of algebraic surfaces, real and symplectic structures.* Symplectic 4-manifolds and algebraic surfaces, 55–167, Lecture Notes in Math., 1938, Springer, Berlin, (2008).

[Chev55] Chevalley, C. *Invariants of finite groups generated by reflections.* Amer. J. Math. 77 (1955), 778–782.

[Com30] Comessatti, A. *Sulle superficie multiple cicliche.* Rendiconti Seminario Padova 1, 1-45 (1930)

[Cor87] Cornalba, M. *On the locus of curves with automorphisms.* Ann. Mat. Pura Appl., IV. Ser. 149, 135-151 (1987).

[Cor08] Cornalba, M. *Erratum: On the locus of curves with automorphisms.* Ann. Mat. Pura Appl. (4) 187, No. 1, 185-186 (2008).

[LW10] Liu, Wenfei, *Stable Degenerations of Surfaces Isogenous to a Product II.* arXiv:0911.5177, to appear on Trans. A.M.S.

[Par91] Pardini, R. *Abelian covers of algebraic varieties.* J. Reine Angew. Math. 417 (1991), 191–213.

[vO-V-07] van Opstall, M.A., Veliche, R. *Maximally symmetric stable curves.* Michigan Math. Jour. 55, n.3, (2007), 513-534.

[vO-V-10a] van Opstall, M.A., Veliche, R. *Cubic graphs with most automorphisms.* J. Graph Theory 64 (2010), n. 2, 99115.

[vO-V-10b] van Opstall, M.A., Veliche, R. *Maximally symmetric stable curves II.* arXiv:math/0608799v1, version of August 28 2010.

Author' s Address:
F. Catanese,
Lehrstuhl Mathematik VIII,
Mathematisches Institut der Universität Bayreuth
NW II, Universitätsstr. 30
95447 Bayreuth
email: fabrizio.catanese@uni-bayreuth.de