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A characterization of varieties whose universal cover is the polydisk or a tube domain

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Abstract In this article we give necessary and sufficient conditions, in terms of certain tensors called semispecial tensors, respectively slope zero tensors, in order that the universal covering of a complex projective manifold be a symmetric domain of tube type. As an application, we give precisions of a result of Kazhdan showing that a Galois conjugate of such a manifold has the same universal covering.

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1 Introduction

The uniformization theorem states that any complex manifold C of dimension one which is not of special type (i.e., not $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{C}^*$, or an elliptic curve) has as universal

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covering the unit disk $\mathbb{B}_1 = \{z \in \mathbb{C} | |z| < 1\}$, which is biholomorphic to the upper half plane $\mathbb{H} = \{z \in \mathbb{C} | Im(z) > 0\}$.

A central problem in the theory of complex manifolds has been the one of determining the compact complex manifolds X whose universal covering $\tilde{X}$ is biholomorphic to a bounded domain $\Omega \subset \mathbb{C}^n$.

A first important restriction is given by theorems by Siegel and Kodaira, extending to several variables a result of Poincaré, and asserting that necessarily such a manifold X is projective and has ample canonical divisor K_X (see [17], [18], Theorem 8.4 page 144, where the Bergman metric is used, while the method of Poincaré series is used in [26], Theorem 3 page 117, see also [19], Chapter 5).

In particular X is a projective variety of general type embedded in projective space by a pluricanonical embedding associated to the sections of $\mathcal{O}_X(mK_X)$ for large m.

This is a restriction on X, whereas a restriction on Ω is given by another theorem of Siegel ([25], cf. also [14], Theorem 6.2[1]), asserting that Ω must be holomorphically convex.

The problems which naturally come up are then of two types:

Problem 1 Given a bounded domain $\Omega \subset \mathbb{C}^n$, when does there exist a properly discontinuous group $\Gamma \subset \mathrm{Aut}(\Omega)$ which acts freely on Ω and is cocompact (i.e., is such that $X =: \Omega/\Gamma$ is a compact complex manifold with universal cover $\cong \Omega$)?

The functions on Ω which yield then a pluricanonical embedding of X are classically called automorphic functions, and in [26, page 119] C.L. Siegel posed a second type of problem writing:

... we have no method of deciding whether a given algebraic variety of higher dimension can be uniformized by automorphic functions.

A more specific question than the one posed by Siegel is:

Problem 2 Given a bounded domain $\Omega \subset \mathbb{C}^n$, how can we tell when a projective manifold X with ample canonical divisor K_X has Ω as universal covering?

Obviously an answer to the second problem presupposes an answer to the first one.

For the first question it is natural to look at domains which have a big group of automorphisms, especially at **bounded homogeneous domains**, i.e., bounded domains such that the group $\mathrm{Aut}(\Omega)$ of biholomorphisms of Ω acts transitively. And especially at the **bounded symmetric domains**, the domains such that for each point $p \in \Omega$ there is a symmetry at p (an automorphism g with $g(p) = p$ and $(Dg)_p = -Identity$).

Bounded symmetric domains were classified by Elie Cartan in [6], and they are a finite number for each dimension n (see also [12], Theorem 7.1 page 383 and exercise D, pages 526–527, and [24] page 525 for a list of them). Among them are the so called **bounded symmetric domains of tube type**, which are biholomorphic to a **tube domain**, a generalized Siegel upper halfspace $T_{\mathcal{C}} = \mathbb{V} \oplus \sqrt{-1}\mathcal{C}$ where $\mathbb{V}$ is a real vector space and $\mathcal{C} \subset \mathbb{V}$ is a *symmetric cone*, i.e., a self dual homogeneous convex cone containing no full lines.

[1] We are indebted to Pascal Dingoyan for providing this reference.

Borel proved in [5] that for each bounded symmetric domain Ω Problem 1 has a positive answer; and such a compact free quotient $X = \Omega/\Gamma$ is called a compact Clifford-Klein form of the symmetric domain Ω.

Even if the bounded symmetric domains Ω are not the only ones for which Problem 1 has a positive answer (i.e., such a compact quotient X exists), Frankel proved in [9] that if Ω is a bounded convex domain, and Problem 1 has a positive answer, then Ω is a bounded symmetric domain.

Another theorem of Frankel ([10])[2] shows that K_X ample implies the splitting of a finite unramified covering of X as a product of a locally symmetric manifold and a *locally rigid* manifold, i.e., a manifold whose local group of isometries is discrete.

A classical result of J. Hano (see [11] Theorem IV, page 886, and Lemma 6.2, page 317 of [21]) asserts that a bounded homogeneous domain that covers a compact complex manifold is symmetric. Henceforth we restrict our attention in this paper to Problem 2 for the case where Ω is a bounded symmetric domain.

In this respect the first breakthrough, giving an answer to C.L. Siegel's question in an important special case, was based on the theorems of Aubin and Yau (see [2,29]) showing the existence, on a projective manifold with ample canonical divisor K_X, of a Kähler–Einstein metric, i.e., a Kähler metric ω such that

$$Ric(\omega) = -\omega.$$

This theorem is indeed the right substitute for the uniformization theorem in dimension $n > 1$.

Yau showed in fact ([28]) that, for a projective manifold with ample canonical divisor K_X, the famous Yau inequality is valid

$$K_X^n \leq \frac{2(n+1)}{n} K_X^{n-2} c_2(X),$$

equality holding if and only if the universal cover $\widetilde{X}$ is the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$.

The uniformization theorems of Yau ([30,31]) for a manifold X with ample canonical bundle K_X go in the direction of providing further answers to Siegel's question, sketching sufficient (but not necessary) conditions in order that $\widetilde{X}$ be the product of a bounded symmetric domain with another manifold.

However Yau makes the unnecessary assumption that Ω^1_X splits as a direct sum

$$\Omega^1_X = V_1 \oplus V_2 \oplus \cdots \oplus V_k,$$

does not give an answer to the more specific Problem 2 and moreover, as we shall show here, his conditions for a summand V_j apply only for an irreducible factor of the universal cover which is a ball or a symmetric domain of tube type.

A very readable exposition of Yau's results, based on the concept of stability of the cotangent bundle Ω^1_X, is contained in the first section of [27]. Moreover Viehweg and Zuo, while still assuming the splitting of the cotangent bundle, characterize (statement

[2] We are indebted to Gang Tian for providing this reference.

(c.iii) in Theorem 1.4 of [27]) the addenda coming from bounded symmetric domains of rank ≥ 2 as those for which V_j is stable, but there is a symmetric power $S^m(V_j)$ which is not stable.[3]

In the special case where Ω^1_X splits as a sum of line bundles it follows from Yau's theorem that $\widetilde{X}$ is the polydisk $\mathbb{H}^n$, where $n = \dim(X)$.

The splitting of Ω^1_X as a sum of lines bundles is not a necessary condition, even if it does indeed hold on a finite unramified covering $X' \to X$.

The reason lies in the semidirect product (where $\mathcal{S}_n$ is the symmetric group):

$$1 \to \mathrm{Aut}(\mathbb{H})^n \to \mathrm{Aut}(\mathbb{H}^n) \to \mathcal{S}_n \to 1.$$

A necessary condition for a compact complex manifold of dimension n to be uniformized by a polydisk was found in [7], based on the consideration that the tensor (here $\odot$ denotes the symmetric product)

$$\tilde{\psi} =: \frac{dz_1 \odot \cdots \odot dz_n}{dz_1 \wedge \cdots \wedge dz_n}$$

is transformed by every automorphism g into $\sigma(g)\tilde{\psi}$, where $\sigma(g) = \pm 1$ is the signature of the permutation corresponding to g.

Namely, the tensor $\tilde{\psi}$ descends to a so called semi special tensor ψ on X, which is simply a non zero section of the sheaf $S^n(\Omega^1_X)(-K_X) \otimes \eta$, where η is an invertible sheaf such that $\eta^2 \cong \mathcal{O}_X$ (corresponding to the signature character).

The necessary condition about the existence of a semi special tensor was proven, in dimension $n \leq 3$, to be a sufficient condition for X to be uniformized by a polydisk ([7, Theorem 1.9.]).

Unfortunately, the above necessary condition is not sufficient for $n \geq 4$ (see [7, Theorem 1.10.]).

Our first result in this paper is the following necessary and sufficient condition for a compact complex manifold to be uniformized by a polydisk.

Theorem 1.1 *Let X be a compact complex manifold of dimension n. Then the following two conditions:*

(1) *K_X is ample*
(2) *X admits a semi special tensor $\psi \in H^0(S^n(\Omega^1_X)(-K_X) \otimes \eta)$ such that, given any point $p \in X$, the corresponding hypersurface $F_p =: \{\psi_p = 0\} \subset \mathbb{P}(TX_p)$ is reduced*

hold if and only if $X \cong (\mathbb{H}^n)/\Gamma$ (where Γ is a cocompact discrete subgroup of $\mathrm{Aut}(\mathbb{H}^n)$ *acting freely).*

Remark 1.1 The second condition is quite explicit, since it amounts to verifying that the polynomial ψ_p, obtained by evaluating ψ at the point p, is a squarefree polynomial: and to verify this it suffices to use the GCD of univariate polynomials.

[3] They however took for granted Yau's wrong assertion, that if $S^m(V_j)$ is not stable, then it should have a direct factor of rank one having the same slope.

Our second and third results show that semispecial tensors, and a generalization of them, the slope zero tensors (see [4] for the related concepts of slope and stability) work out in a more general setting, and give a necessary and sufficient condition for a complex compact manifold X to be uniformized by a bounded symmetric domain of tube type.

Here a slope zero tensor is a non zero section $\psi \in H^0(S^{nm}(\Omega^1_X)(-mK_X))$, where m is a positive integer.

Theorem 1.2 *Let X be a compact complex manifold of dimension n. Then the following two conditions:*

(1) *K_X is ample*
(2) *X admits a semi special tensor ψ;*

hold if and only if $X \cong \Omega/\Gamma$, where Ω is a bounded symmetric domain of tube type with the special property

(*) Ω is a product of irreducible bounded symmetric domains D_j of tube type whose rank r_j divides the dimension n_j of D_j, and Γ is a cocompact discrete subgroup of $\mathrm{Aut}(\Omega)$ *acting freely.*

Moreover, the degrees and the multiplicities of the irreducible factors of the polynomial ψ_p determine uniquely the universal covering $\widetilde{X} = \Omega$.

Theorem 1.3 *Let X be a compact complex manifold of dimension n. Then the following two conditions:*

(1) *K_X is ample*
(2) *X admits a slope zero tensor $\psi \in H^0(S^{mn}(\Omega^1_X)(-mK_X))$, (here m is a positive integer);*

hold if and only if $X \cong \Omega/\Gamma$, where Ω is a bounded symmetric domain of tube type and Γ is a cocompact discrete subgroup of $\mathrm{Aut}(\Omega)$ *acting freely.*

Moreover, the degrees and the multiplicities of the irreducible factors of the polynomial ψ_p determine uniquely the universal covering $\widetilde{X} = \Omega$.

In particular, for $m = 2$, we get that the universal covering $\widetilde{X}$ is a polydisk if and only if ψ_p is the square of a squarefree polynomial.

We obtain as a corollary a simple proof of a variant of Kazhdan's Theorem [13] about the Galois conjugates of an arithmetic projective manifold X. Namely, we have the following application.

Corollary 1.4 *Assume that X is a projective manifold with K_X ample, and that the universal covering $\tilde{X}$ is a bounded symmetric domain of tube type.*

Let $\sigma \in$ $\mathrm{Aut}(\mathbb{C})$ *be an automorphism of $\mathbb{C}$.*

Then the conjugate variety X^σ has universal covering $\tilde{X^\sigma} \cong \tilde{X}$.

Our paper leaves two questions open:

(1) Is it possible (as in [7]) to remove the assumption that K_X is ample, replacing it by the condition that X be of general type?
(2) Study necessary and sufficient conditions for the case where there are irreducible factors which are bounded symmetric domains not of tube type: these should probably involve subbundles of higher rank of the bundles $S^k(\Omega^1_X)(-mK_X)$.

The paper is organized as follows: in Sect. 2 we recall a result by Korányi and Vági which plays a central role for our theorems, since it determines the holonomy invariant hypersurfaces in the tangent space to an irreducible symmetric bounded domain.

After this, in Sects. 3 and 4, we provide the proofs of our two main Theorems 1.1 and 1.2, using the existence of the Kähler–Einstein metric, the classical theorems of De Rham and Berger and the Bochner principle, in order to show the sufficiency of the condition of the existence of a semispecial tensor.

In Sect. 4 we show that this condition is also necessary for every bounded symmetric domain of tube type satisfying (*), thereby partly generalizing the result of Korányi and Vági (we prove semi-invariance of our tensor for the full group).

We conclude with the Kazhdan type Corollary 1.4, a couple of examples, and the proof of Theorem 1.3.

2 Preliminaries

2.1 Symmetric bounded domains and its invariant polynomials

Let $D \subset \mathbb{C}^n$ be a bounded symmetric domain in its circle realization around the origin $0 \in \mathbb{C}^n$. Let $\mathrm{Aut}(D)$ be the full group of automorphisms of D. It is well-known that $\mathrm{Aut}(D)$ is the full group of isometries of D endowed with its Bergman metric.

Let $K \subset \mathrm{Aut}(D)$ be the isotropy group of D at the origin $0 \in \mathbb{C}^n$, so that we have $D = \mathrm{Aut}(D)/K$.

A polynomial $f \in \mathbb{C}[X_1, \ldots, X_n]$ is said to be K-semi-invariant if there is a character $\chi : K \to \mathbb{C}^*$ such that, for all $g \in K$, $f(gX) = \chi(g) f(X)$. Since K is compact we have: $|\chi(g)| = 1$.

An *inner function* on D is a bounded holomorphic function on D such that $|f^*(z)| = 1$ for almost every $z \in S$ where S is the Shilov boundary of D and f^* is the radial limit of f, see [20, page 185] for details.

Let $D = D' \times D''$ be the decomposition of D as a product of two domains where D' is of tube type and D'' has no irreducible factor of tube type.

The following theorem collects several results in [20].

Theorem 2.1 *[20, Korányi-Vági] Let $D = D' \times D''$ be the above decomposition and let moreover*

$$D' = D'_1 \times D'_2 \times \cdots \times D'_p$$

be the decomposition of D' as a product of irreducible tube type domains D'_j, $(j = 1, \ldots, p)$.

Then there exists, for each $j = 1, \ldots, p$, a unique polynomial $N_j(z_j)$, which is an inner function on D'_j such that: for all inner function f on D there exist a constant $c \in \mathbb{C}$ and exponents k_j with

(1)

$$f = c \prod_{j=1}^{p} N_j^{k_j},$$

hence in particular

(2)

$$f(z', z'') = f(z')$$

where z' denotes a vector in the domain D' and $z'' \in D''$.

The second part of the above theorem follows from part (iii) in Theorem 3.3. of [20, page 187]. The first part is contained in Lemma 2.5. and Lemma 2.3 of [20, pages 184,182].

It is well-known that the isotropy group K acts transitively on the Shilov boundary S of D. So a K-semi-invariant polynomial f is, up to a multiple, an inner function. Notice that, by the uniqueness in the above result, the polynomials N_j are K-semi-invariant. Thus, we have the following corollary.

Corollary 2.2 *The above theorem holds replacing inner function by K-semi-invariant polynomial.*

The same result was rediscovered by Mok in [22].

It is very important to observe that the polynomials N_j have degree equal to the $rank(D'_j)$ of the irreducible domain D'_j. Here $rank(D'_j)$ denotes the dimension r of the maximal totally geodesic embedded polydisc $\mathbb{H}^r \subset D'_j$. Therefore $rank(D'_j) \leq dim(D'_j)$ and equality holds if and only if $D'_j = \mathbb{H}$.

Remark 2.1 For the explicit form of the polynomials N_j see [20, page 183]. In Sects. 4.2 and 4.3 we will see that the polynomials N_j are used to construct an $\mathrm{Aut}(D)^0$-invariant tensor $\tilde{\psi}$.

2.2 Irreducible symmetric domains of tube-type whose dimension is divisible by its rank

Recall the notation for the classical domains:

- $I_{n,p}$ is the omain $D = \{Z \in M_{n,p}(\mathbb{C}) : \mathrm{I}_p -^t Z \cdot \overline{Z} > 0\}$.
- II_n is the intersection of the domain $I_{n,n}$ with the subspace of skew symmetric matrices.
- III_n is instead the intersection of the domain $I_{n,n}$ with the subspace of symmetric matrices.
- IV_n, the so called Lie Balls, are described in Sect. 4.3.

Theorem 2.3 *Let D be an irreducible symmetric domain of tube-type. Let $d = dim(D)$ be the complex dimension of D and r its rank.*

If d is multiple of r then one of the following holds:

(i) *D is of type $I_{n,n}$, $n \geq 1$. In this case $r = n$ and $d = n^2$,*
(ii) *D is of type II_{2k}, $k \geq 1$. In this case $r = k$ and $d = k(2k-1)$,*
(iii) *D is of type III_{2k+1}, $k \geq 0$. In this case $r = 2k+1$ and $d = (2k+1)(k+1)$,*

(iv) *D is of type IV_{2k}, $k \geq 2$. In this case $r = 2$ and $d = 2k$,*
(v) *D is the exceptional domain of dimension $d = 27$ and rank $r = 3$.*

Proof The proof follows from the classification of irreducible bounded symmetric domains, see e.g., [24, page 525]. □

3 Manifolds uniformized by a polydisk

Here we prove Theorem 1.1.

By a *semi special tensor ψ with reduced divisor* we mean, as in [7, Definition 1.3], a semi special tensor

$$\psi \in H^0(S^n(\Omega^1_X)(-K_X) \otimes \eta)$$

such that the homogeneous polynomial ψ_p, obtained by evaluating the tensor on the fibre over the point $p \in X$ (ψ_p is a polynomial of degree n on the tangent space TX_p), is not divisible by a square.

Proposition 1.4. and its proof in [7, page 162] shows that, as explained in the introduction, (1) and (2) are necessary if $X \cong (\mathbb{H}^n)/\Gamma$ (where Γ is a cocompact discrete subgroup of $\mathrm{Aut}(\mathbb{H}^n)$ acting freely).

Assume now that (1) and (2) hold and let $\widetilde{X}$ be the universal cover of X.

Proceeding as in [7, page 160] the semispecial tensor ψ pulls back to a special tensor $\tilde{\psi} =: \Psi$ on $\tilde{X}$ which is parallel with respect to the Levi-Civita connection associated to the Kähler-Einstein metric (this follows from the Bochner principle, see [15,30], page 272 and [31], page 479).

Fix a point $x \in \widetilde{X}$ and let $H_x \subset \mathrm{U}(T_x\widetilde{X})$ be the restricted holonomy group with respect to the Levi-Civita connection associated to the Kähler–Einstein metric.

Since Ψ is parallel there exists a degree n polynomial $f := \psi_x$ on $T_x\widetilde{X}$ such that

$$F_x = \{v_x \in T_x\widetilde{X} : \psi_x(v_x) =: \Psi(x, v_x) = 0\}$$

is H_x-invariant.

This implies that $f = \psi_x$ is H_x-semi-invariant.

Notice that $f = \psi_x$ is not divisible by a square since ψ_x has a reduced divisor F_x.

Since $\widetilde{X}$ has no flat De Rham factor (otherwise X is flat and the canonical divisor K_X cannot be ample) we use the second author's Proposition A.1 (appendix to [7], page 178) implying that there is a decomposition of the vector space $T_x\widetilde{X}$ as $T_x\widetilde{X} = V_1 \oplus V_2$ and where $f(v_1, v_2) = f(v_1)$ depends only on the variable v_1.

Moreover V_1 is the tangent space at the origin of a bounded symmetric domain $D \subset \mathbb{C}^m$ such that the action of H_x on V_1 is equal to the action of the connected component K^0 of the isotropy group K at the origin $0 \in \mathbb{C}^m$.

Let us use now Theorem 2.1 and the notation therein.

We obtain that f splits as

$$f = c \prod_{j=1}^{p} N_j^{\epsilon_j}$$

where $\epsilon_j \in \{0, 1\}$. Then we get

$$n = deg(f) = \sum_{j=1}^{p} \epsilon_j deg(N_j) = \sum_{j=1}^{p} \epsilon_j r_j.$$

We also have that

$$\sum_{j=1}^{p} \epsilon_j r_j = n \geq m = dim(D) = \sum_{j=1}^{p} dim(D'_j) + dim(D'') \geq \sum_{j=1}^{p} r_j + dim(D'')$$

since $r_j \leq dim(D'_j)$.

We conclude that $\epsilon_j = 1 \, \forall j, n = m, p = n, dim(D'') = 0$, and moreover $dim(D'_j) = r_j = 1$ for $j = 1, \dots, n$.

This shows that $H_x = K^0$ splits as $K^0 = \mathrm{U}(1)^n$ and completes the proof that $\widetilde{X}$ is a polydisk $\mathbb{H}^n$.

4 Manifolds uniformized by a tube domain

Here we give the proof of Theorem 1.2.

4.1 Sufficient conditions

We want here to show that if K_X is ample, and X admits a semispecial tensor ψ, then the universal covering $\tilde{X}$ is a product of irreducible symmetric domains of tube type whose rank divides the dimension.

We proceed as for the proof of Theorem 1.1.

Namely, we write the universal cover $\tilde{X}$, according to the theorems of De Rham and Berger (see [3] and also [23]), as the product $\tilde{X} = D_1 \times D_2 = D'_1 \times D''_1 \times D_2$ where (since there are no flat factors, as already observed):

- D_2 is the product of the irreducible factors of dimension ≥ 2 for which the holonomy group is transitive (actually, it is the unitary group)
- D_1 is a bounded symmetric domain
- D'_1 is the product of all the irreducible bounded symmetric domains of tube type
- D''_1 is the product of all the irreducible bounded symmetric domains which are neither a ball nor are of tube type.

Consider now the pull back tensor $\Psi = \tilde{\psi}$, and consider coordinates (u, w, z) according to the product decomposition $\tilde{X} = D'_1 \times D''_1 \times D_2$.

Let $a = dim(D'_1), b = dim(D''_1), r = dim(D_2)$.

Then the tensor ψ_x in a point x can be written as

$$\psi_x = f(u, w, z)(du_1 \wedge \cdots \wedge du_a)^{-1} \wedge (dw_1 \wedge \cdots \wedge dw_b)^{-1} \wedge (dz_1 \wedge \cdots \wedge dz_r)^{-1}$$

and it is holonomy invariant.

By the same argument as in the previous section (Proposition A.1 of the appendix to [7], page 178, and the theorem of Korányi-Vági) we have:

$$f(u, w, z) = f(u).$$

Write the (restricted) holonomy group as $K_1' \times K_1'' \times K_2$ and observe that none of the subgroups K_2, K_1', K_1'' is contained in the special unitary group, otherwise the Kähler-Einstein metric is Ricci flat in a certain direction, contradicting the ampleness of K_X.

Hence for instance K_2 acts nontrivially on $(dz_1 \wedge \cdots \wedge dz_r)^{-1}$, while it acts trivially on f.

This would contradict the holonomy invariance of the tensor unless there is no factor D_2. The same identical argument implies that there is no factor D_1'', hence D is a product of irreducible bounded symmetric domains of tube type.

We write now accordingly D as a product of such irreducible bounded symmetric domains of tube type

$$D = \prod_{j=1}^{h} \Omega_j$$

and we take variables $(z_1, \ldots, z_h)$ with $z_j \in \Omega_j$, and write, if $z_j = (z_{j,1}, \ldots, z_{j,n_j})$, and $n_j = dim(\Omega_j)$,

$$dz_j^{top} =: dz_{j,1} \wedge \cdots \wedge dz_{j,n_j}.$$

By the theorem of Korányi-Vági, up to a constant we can write

$$\psi_x = N_1^{m_1}(z_1) \cdots N_h^{m_h}(z_h)(dz_1^{top} \wedge \cdots \wedge dz_h^{top})^{-1}.$$

We impose invariance for each holonomy subgroup K_j.

We know that K_j acts on $N_j(z_j)$ by a character $\chi_j(g)$, and similarly K_j acts on (dz_1^{top}) by a character χ_j'.

Recall that, Ω_j being a circular domain, K_j contains the diagonal subgroup $S_j = \{e^{i\theta} I_{n_j}\}$.

Restricting to S_j we see that, if ϕ_j is the tautological character, then $\chi_j|_{S_j} = \phi_j^{r_j}$, $\chi_j'|_{S_j} = \phi_j^{n_j}$, hence, by S_j invariance, we conclude that

$$m_j r_j = n_j, \ \forall j = 1, \ldots, h.$$

We are done since we observe that the classification Theorem 2.3 shows that the pair of integers (r_j, n_j), under the condition $r_j | n_j$, completely determines the irreducible bounded symmetric domain of tube type Ω_j.

4.2 Necessary conditions

As we observed in the introduction the ampleness of the canonical line bundle K_X is a result of Kodaira, i.e., condition (1) is necessary.

We shall give two proofs that condition (2), i.e., the existence of a semi special tensor, is necessary.

Our first proof relies on the foundations of the theory of bounded symmetric domains of tube type by means of their associated cones $\mathcal{C}$ and their Jordan algebras, developed for instance in [8].

The second proof is a case by case computation which works just for the classical domains but provides an explicit expression for the semi special tensor.

Both proofs are based on the fact that, if Ω is a bounded symmetric domain, and

$$\Omega = \Pi_{j=1}^{h} \Omega_j$$

is its decomposition as a product of irreducible bounded symmetric domains, then we have a semidirect product

$$1 \to \Pi_{j=1}^{h} \mathrm{Aut}(\Omega_j) \to \mathrm{Aut}(\Omega) \to \mathcal{S} \to 1$$

where $\mathcal{S} \subset \mathcal{S}_h$.

This follows from the fact that the De Rham decomposition of the universal cover of a complete Riemannian manifold is unique up to the ordering of the factors (see [16], Theorem 6.2 of Chapter IV).

As in the proof of Proposition 1.4 in [7] and by the above exact sequence it is enough to construct, for each irreducible bounded symmetric domain of tube type D, a special tensor Ψ semi-invariant by the group of holomorphic automorphisms $\mathrm{Aut}(D)$.

Then such a tensor Ψ necessarily descends to a semi special tensor ψ on any quotient X of Ω.

Let D be an irreducible bounded symmetric domain of tube type. Following [8, Chapter X] D is biholomorphic, via the Cayley map, to a tube domain $T_{\mathcal{C}} = \mathbb{V} + i\mathcal{C}$ where $\mathbb{V}$ is a real finite dimensional vector space and $\mathcal{C} \subset \mathbb{V}$ is a so called symmetric cone.

Both D and $T_{\mathcal{C}}$ are open subsets of the Hermitian Jordan algebra $\mathbb{V}_{\mathbb{C}} := \mathbb{C} \otimes \mathbb{V}$ which is the complexification of a simple Euclidean Jordan algebra whose real vector space is $\mathbb{V}$.

Let $\tilde{\psi}$ be the tensor defined as follows

$$\tilde{\psi} := \frac{\det(dz)^{\frac{n}{r}}}{\mathrm{K}} \tag{1}$$

where $n = dim(D)$, r is the rank of D, $\det(\cdot)$ is defined in [8, page 29] and K is a generator of $\wedge^n(\mathbb{V}_{\mathbb{C}})$, viewed as a non vanishing holomorphic top-degree form.

Notice that $\det(\cdot)$ is also denoted by $\Delta(\cdot)$ and called the Koecher norm in [20]. It is the same polynomial N_j we encountered before.

Let $G(T_{\mathcal{C}})$ be the group of biholomorphic maps of the tube $T_{\mathcal{C}}$ and let $G(T_{\mathcal{C}})^0$ its identity component.

Lemma 4.1 *$\tilde{\psi}$ is invariant by $G(T_{\mathcal{C}})^0$.*

Proof According to Theorem X.5.6 in [8, page 207] the group $G(T_{\mathcal{C}})$ is generated by the involution $j(z) := -z^{-1}$ and the subgroups $G(\mathcal{C})$ and N^+. So it is enough to show that $\tilde{\psi}$ is invariant by $j(z) := -z^{-1}$ and by the subgroups $G(\mathcal{C})$ and N^+. □

That $\tilde{\psi}$ is invariant by the translations of N^+ is obvious.

The invariance by $G(\mathcal{C})^0$ follows from Proposition III.4.3 in [8, page 53].

To show that $j^*\tilde{\psi} = \tilde{\psi}$ we will use the results in [8, Chapter II] about the so called quadratic representation $P(\cdot)$, and also Lemma 1.1 and Proposition 1.2 of [1] stating the crucial properties:

- $P(x)(x^{-1}) = x$
- $P(x)^{-1} = P(x^{-1})$
- $Dj(x) = P(x)^{-1}$
- $Det(P(x)) = det(x)^{\frac{2n}{r}}$
- $det(P(y)\cdot x) = (det y)^2 \cdot det\, x.$

We have then:

$$
\begin{aligned}
j^*\tilde{\psi} &= \frac{\det(\mathrm{d}j(z))^{\frac{n}{r}}}{j^*\mathrm{K}} = \frac{\det(P(z)^{-1}\cdot \mathrm{d}z)^{\frac{n}{r}}}{j^*\mathrm{K}}\\
&= \frac{\det(P(z^{-1})\cdot \mathrm{d}z)^{\frac{n}{r}}}{j^*\mathrm{K}} = \frac{((\det z^{-1})^2\cdot \det(\mathrm{d}z))^{\frac{n}{r}}}{j^*\mathrm{K}} = \frac{(\det z)^{\frac{-2n}{r}}\cdot \det(\mathrm{d}z)^{\frac{n}{r}}}{j^*\mathrm{K}}\\
&= \frac{(\det z)^{\frac{-2n}{r}}.\det(\mathrm{d}z)^{\frac{n}{r}}}{\mathrm{Det}(P(z)^{-1})\mathrm{K}} = \frac{(\det z)^{\frac{-2n}{r}}.\det(\mathrm{d}z)^{\frac{n}{r}}}{(\det z)^{\frac{-2n}{r}}\mathrm{K}} = \tilde{\psi}.
\end{aligned}
$$

This completes the proof of the claim.

It is known that either $G(T_{\mathcal{C}})$ is connected or has just two connected components. Indeed, $\mathrm{Aut}(D)$ is connected unless D is a classical domain $D = I_{n,n}$ $(n \geq 2)$ or $D = IV_{2k}$ where $\mathrm{Aut}(D)$ has two connected components [6, page 152].

So we get the following corollary.

Lemma 4.2 *$\tilde{\psi}$ is semi-invariant by $G(T_{\mathcal{C}})$. That is to say, there exists a character $\chi : G(T_{\mathcal{C}}) \to \{1, -1\}$ such that for $g \in G(T_{\mathcal{C}})$:*

$$
g.\tilde{\psi} = \chi(g)\tilde{\psi}
$$

A simple example where the above character χ is not trivial is given by the domain $I_{2,2}$. Indeed, in this case

$$
\tilde{\psi} = \frac{(\mathrm{d}z_1 \otimes \mathrm{d}z_4 - \mathrm{d}z_2 \otimes \mathrm{d}z_3)^2}{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4}
$$

where $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in I_{2,2}$. The transposition $\tau(Z) = Z^t$ belongs to $\mathrm{Aut}(I_{2,2})$ and $\tau^*\tilde{\psi} = -\tilde{\psi}$ since

$$\tau^*(\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4) = \mathrm{d}z_1 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_4 = -\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4.$$

4.3 Necessary conditions found classically

Here we construct explicitly $\tilde{\psi}$ for the classical bounded symmetric domains of tube type D whose rank r divides the dimension $n = dim(D)$. We give a simpler direct proof of the semi-invariance of our tensor $\tilde{\psi}$ by $\mathrm{Aut}(D)$ the full automorphism group of the domain D.

We will follow the standard Elie Cartan's notation.

Domains of type $I_{n,n}$

The Cartan–Harish-Chandra realization of $I_{n,n} := SU(n,n)/S(U(n) \times U(n))$ is the domain $\Omega = \{Z \in M_{n,n}(\mathbb{C}) : \mathrm{I}_n - Z^t \cdot \overline{Z} > 0\}$.

To an element $\gamma \in SU(n,n)$ corresponds the transformation

$$\gamma(Z) = (AZ + B) \cdot (CZ + D)^{-1}.$$

As in [7, p. 174] the function $\gamma \mapsto \chi(\gamma) \in \mathbb{C}^*$ defined by the equation:

$$det(\mathrm{d}\gamma(Z)) = \chi(\gamma) \cdot det(CZ + D)^{-2} \cdot det(\mathrm{d}Z)$$

is a character of $SU(n,n)$.

Indeed, if $\gamma' \in SU(n,n)$ is another isometry, say $\gamma'(Z) = (A'Z + B') \cdot (C'Z + D')^{-1}$, then

$$det(\mathrm{d}(\gamma \cdot \gamma')(Z)) = \chi(\gamma.\gamma') \cdot det((CA' + DC')Z + CB' + DD')^{-2} \cdot det(\mathrm{d}Z)$$

and by direct computation we have

$$\begin{aligned} det(\mathrm{d}(\gamma \cdot \gamma')(Z)) &= \chi(\gamma) \cdot det(C\gamma'(Z) + D)^{-2} \cdot det(\mathrm{d}\gamma'(z)) \\ &= \chi(\gamma)\chi(\gamma') \cdot det(C\gamma'(Z)+D)^{-2} \cdot det(C'Z+D')^{-2} \cdot det(\mathrm{d}Z) \end{aligned}$$

so that we only have to show that

$$det((CA' + DC')Z + CB' + DD')^{-2} = det(C\gamma'(Z) + D)^{-2} \cdot det(C'Z + D')^{-2}$$

which is equivalent to

$$det((CA' + DC')Z + CB' + DD') = det(C\gamma'(Z) + D) \cdot det(C'Z + D')$$

but indeed

$$\begin{aligned} det(C\gamma'(Z)+D)\cdot det(C'Z+D') &= det(C(A'Z+B')\cdot(C'Z+D')^{-1}+D)\\ &\quad \cdot det(C'Z+D'),\\ &= det(C(A'Z+B')+D(C'Z+D')),\\ &= det((CA'+DC')Z+CB'+DD'). \end{aligned}$$

This shows that $\chi(\gamma)$ is a character of $SU(n,n)$. Actually, any character of $SU(n,n)$ is trivial since $SU(n,n)$ is a connected semisimple Lie group. Hence the

Claim 4.3 $\chi(\gamma)\equiv 1$, *i.e., the character* χ *is trivial.*

Thus, for $\gamma\in SU(n,n)$, we get the formula

$$det(\mathrm{d}\gamma(Z)) = det(CZ+D)^{-2}\cdot det(\mathrm{d}Z).$$

The Jacobian determinant of γ is $det(CZ+D)^{-2n}$, i.e., $\gamma^*\mathrm{K} = det(CZ+D)^{-2n}\mathrm{K}$, where K is the holomorphic volume form of $I_{n,n}$. For $I_{n,n}$ our tensor $\tilde{\psi}$ is given by

$$\tilde{\psi} = \frac{det(\mathrm{d}Z)^n}{\mathrm{K}}$$

Then for $\gamma\in SU(n,n)$ we have

$$\begin{aligned} \gamma^*\tilde{\psi} &= \frac{det(\mathrm{d}\gamma Z)^n}{\gamma^*\mathrm{K}}\\ &= \frac{(det(CZ+D)^{-2}\cdot det(\mathrm{d}Z))^n}{det(CZ+D)^{-2n}\mathrm{K}}\\ &= \frac{(det(CZ+D)^{-2})^n}{det(CZ+D)^{-2n}}\tilde{\psi}\\ &= \tilde{\psi} \end{aligned}$$

This shows that $\tilde{\psi}$ is $SU(n,n)$-invariant. Hence it is Aut($I_{n,n}$)-semi-invariant.

Domains of type II_{2k}.
This is the subdomain of $I_{2k,2k}$ given by the skew-symmetric matrices.
Here $\tilde{\psi}$ is given by

$$\tilde{\psi} = \frac{det(\mathrm{d}Z)^{\frac{2k-1}{2}}}{\mathrm{K}}$$

The Jacobian determinant of an isometry γ is given by

$$\gamma^*K = det(CZ+D)^{-(2k-1)}\mathrm{K}.$$

So

$$\gamma^*\tilde{\psi} = \frac{det(\mathrm{d}\gamma Z)^{\frac{2k-1}{2}}}{\gamma^* K}$$
$$= \frac{(det(CZ+D)^{-2} \cdot det(\mathrm{d}Z))^{\frac{2k-1}{2}}}{det(CZ+D)^{-(2k-1)} K}$$
$$= \tilde{\psi}$$

This shows that $\tilde{\psi}$ is invariant by $\mathrm{Aut}^0(II_{2k})$. Actually, in this case $\tilde{\psi}$ is invariant by $\mathrm{Aut}(II_{2k})$ since this last group is connected by [6, page 152].

Domains of type III_{2k+1}.
This is the subdomain of $I_{2k+1,2k+1}$ given by the symmetric matrices.
Here $\tilde{\psi}$ is given by

$$\tilde{\psi} = \frac{det(\mathrm{d}Z)^{k+1}}{\mathrm{K}}$$

The Jacobian determinant of an isometry γ is given by

$$\gamma^*\mathrm{K} = det(CZ+D)^{-2(k+1)}\mathrm{K}.$$

So

$$\gamma^*\tilde{\psi} = \frac{det(\mathrm{d}\gamma Z)^{k+1}}{\gamma^*\mathrm{K}}$$
$$= \frac{(det(CZ+D)^{-2} \cdot det(\mathrm{d}Z))^{k+1}}{det(CZ+D)^{-2(k+1)}\mathrm{K}}$$
$$= \tilde{\psi}$$

This shows that $\tilde{\psi}$ is invariant by $\mathrm{Aut}^0(III_{2k+1})$. Actually, in this case $\tilde{\psi}$ is invariant by $\mathrm{Aut}(III_{2k+1})$ since this last group is connected by [6, page 152].

Domains of type IV_{2k}, the so called Lie Balls.
The Cartan–Harish-Chandra realization of a domain of type IV_n in $\mathbb{C}^n$ is the subset D defined by the inequalities (compare [12], page 527)

$$|z_1^2 + z_2^2 + \cdots + z_n^2| < 1,$$
$$1 + |z_1^2 + z_2^2 + \cdots + z_n^2|^2 - 2\left(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2\right) > 0.$$

When $n = 2k$ our tensor $\tilde{\psi}$ is given by

$$\tilde{\psi} = \frac{(\mathrm{d}z_1 \odot \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \odot \mathrm{d}z_n)^{\frac{n}{2}}}{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \cdots \wedge \mathrm{d}z_n}.$$

Indeed, this follows from the proof of Theorem 1.2 and the fact that the Koecher norm function of D is $(\sum_{j=1}^{n} z_j^2)^{\frac{n}{2}}$ (see [20, page 183]).

Here we describe an alternative way to get the explicit formula for $\tilde{\psi}$, due to the second referee.

The compact dual of D is the hyperquadric $Q^n \subset \mathbb{P}^{n+1}$ defined by the polynomial $\sum_{j=0}^{n-1} X_j^2 - X_n^2 - X_{n+1}^2$. Notice that $\mathrm{SO}_0(n, 2) \subset \mathrm{Aut}(Q^n)$. The Borel embedding $j : D \to Q^n$ is given by

$$j(z_1, \cdots, z_n) = [2z_1 : 2z_2 : \cdots : 2z_n : \mathrm{i}(\Lambda - 1) : \Lambda + 1],$$

where $\Lambda := z_1^2 + \cdots + z_n^2$. The map j identifies the domain D with the SO_0(n,2)-orbit of the point $[0 : 0 : \cdots : 1 : \mathrm{i}] \in Q^n$, i.e., $D \cong \mathrm{SO}_0(n, 2)/\mathrm{SO}(n) \times \mathrm{SO}(2)$.

The second fundamental form σ of Q^n in $\mathbb{P}^{n+1}$ is a section of $S^2(\Omega^1_{Q^n}) \otimes N$, where N is the normal bundle of $Q^n \subset \mathbb{P}^{n+1}$. It is well-known that N, as a sheaf on Q^n, is $\mathcal{O}_{Q^n}(2)$. Since the canonical sheaf K_{Q^n} is $\mathcal{O}_{Q^n}(-n)$, for $n = 2k$ we have $K_{Q^n} = \mathcal{O}(-2)^{\otimes k} = (N^\vee)^{\otimes k}$. So $\sigma^k := \sigma \otimes \cdots \otimes \sigma$ (k copies) gives a section of $S^n(\Omega^1_{Q^n})(-K_{Q_n})$ whose restriction to $j(D)$ induces $\tilde{\psi} = j^*\sigma^k$ on D.

Indeed, we have, up to a constant depending on the chosen isomorphism $K_{Q^n} = \mathcal{O}(-2)^{\otimes k} = (N^\vee)^{\otimes k}$,

$$\begin{aligned} j^*\sigma^k &= j^* \frac{\left((\mathrm{d}X_0)^2 + \cdots + (\mathrm{d}X_{n-1})^2 - (\mathrm{d}X_n)^2 - (\mathrm{d}X_{n+1})^2\right)^k}{\mathrm{d}X_0 \wedge \cdots \wedge \mathrm{d}X_{n-1}} \\ &= \frac{\left((\mathrm{d}2z_1)^2 + \cdots + (\mathrm{d}2z_n)^2 - (\mathrm{d}\mathrm{i}(\Lambda-1))^2 - (\mathrm{d}(\Lambda+1))^2\right)^k}{\mathrm{d}2z_1 \wedge \cdots \wedge \mathrm{d}2z_n} \\ &= \frac{4^k\left((\mathrm{d}z_1)^2 + \cdots + (\mathrm{d}z_n)^2\right)^k}{2^n \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n} \\ &= \tilde{\psi} \end{aligned}$$

5 Proof of the Kazhdan's type corollary

Consider the conjugate variety X^σ: since K_X is ample we may assume that X is projectively embedded by $H^0(X, \mathcal{O}_X(mK_X))$.

σ carries X to X^σ and K_X to K_{X^σ}, hence also X^σ has ample canonical divisor.

Consider a slope zero tensor ψ on X: then ψ^σ is also a slope zero tensor, and moreover σ sends the ring of polynomial functions on the tangent space TX_p to the corresponding ring of polynomial functions on the tangent space $TX^\sigma_{\sigma(p)}$: hence the degrees and multiplicities of the irreducible factors of ψ_p are the same as the degrees and multiplicities of the irreducible factors of $\psi_{\sigma(p)}$.

We conclude then immediately by the last assertion of our main Theorems 1.2 and 1.3 that the universal covering of X^σ is $\tilde{X}$.

6 Examples

Assume that the polynomial ψ_p associated to a semi special tensor is a square $\psi_p = N^2$, where N is irreducible (the more general case where N is squarefree follows then right away).

Then the universal covering $\tilde{X}$ is an irreducible symmetric tube domain such that $d/r = 2$.

It follows from Theorem 2.3 that $\tilde{X}$ is either $I_{2,2}$ or III_3. In particular X has either dimension 4 or 6.

Proposition 6.1 *Assume that K_X is ample and X admits a semispecial tensor ψ.*

If the multiplicities of the divisor associated to $f =: \psi_p$ are at most two then $\tilde{X}$ is a product of one-dimensional disks, of domains of type $I_{2,2}$ or of type III_3.

Moreover, if all multiplicities are two then the number of factors of f and the dimension n of X determine $\tilde{X}$.

Proof The hypotheses imply that the polynomial $f = \psi_p$ can be factorized as

$$f = c \prod_{j=1}^{p} N_j^{e_j}$$

where $e_j \leq 2$. If $e_j = 1$ then the corresponding factor is a disk.

If $e_j = 2$ by the previous observation the corresponding factor is is either $I_{2,2}$ or III_3, and this shows the first assertion.

The hypothesis of the second statement is that

$$f = c \prod_{j=1}^{p} N_j^2$$

Let us denote by a the number of times that $I_{2,2}$ occurs in $\tilde{X}$ and by b the number of times that III_3 occurs in $\tilde{X}$.

Then

$$\begin{cases} 4a + 6b = n = dim(X) \\ a + b = p \end{cases}$$

Hence, knowing p and n, we know a, b and also $\tilde{X}$. □

7 Slope zero tensors of higher degree

Let's treat first the case where $\tilde{X}$ is an irreducible symmetric bounded domain of tube type of dimension n and rank r, but where we consider more generally the sheaf $S^k(\Omega^1_{\tilde{X}})(-mK_{\tilde{X}})$, k, m being positive integers.

Assume that there exists a tensor $\tilde{\psi} \in H^0(\tilde{X}, S^k(\Omega^1_{\tilde{X}})(-mK_{\tilde{X}}))$ semi-invariant by the full automorphism group $\mathrm{Aut}(\tilde{X})$.

Then by the theorem of Korányi-Vági and its Corollary 2.2

$$\tilde{\psi}_x = N^a(z)(dz^{top})^{-m} \tag{2}$$

and

$$k = m \cdot n = r \cdot a \tag{3}$$

since $\tilde{\psi}$ is invariant by the diagonal subgroup $S = \{e^{i\theta} I_n\}$.

Conversely, if condition (3) holds then $\tilde{\psi}$ is semi-invariant by the full group of automorphisms (the proof is the same as in 4.1), hence $\tilde{\psi}$ descends to any Clifford-Klein form X of $\tilde{X}$: providing a section ψ of the sheaf $S^k(\Omega_X)(-mK_X) \otimes \eta$, where η is the sheaf corresponding to the signature character.

Let now $\tilde{X}$ be the product $\Omega_1 \times \cdots \times \Omega_h$ of the irreducible symmetric bounded domains of tube type of dimension n_j and rank r_j, $j = 1, \ldots, h$.

If $\tilde{\psi} \in H^0(\tilde{X}, S^k(\Omega^1_{\tilde{X}})(-mK_{\tilde{X}}))$ is invariant by $\mathrm{Aut}(\Omega_1) \times \cdots \times \mathrm{Aut}(\Omega_h)$ then

$$\psi_x = N_1^{a_1}(z_1) \ldots N_h^{a_h}(z_h)(dz_1^{top} \wedge \cdots \wedge dz_h^{top})^{-m}. \tag{4}$$

$k = m \cdot n$ and $a_j \cdot r_j = m \cdot n_j$ for $j = 1, \ldots, h$.

Conversely, if the numerical conditions $a_j \cdot r_j = m \cdot n_j$ hold for $j = 1, \ldots, h$, then the above formula for ψ_x defines a section of the sheaf $S^k(\Omega^1_{\tilde{X}})(-mK_{\tilde{X}})$ invariant by $\mathrm{Aut}(\Omega_1) \times \cdots \times \mathrm{Aut}(\Omega_h)$.

Now notice that, for any product $\Omega_1 \times \cdots \times \Omega_h$ of irreducible symmetric bounded domains of tube type of dimension n_j and rank r_j, we can **always** find integers $m, a_1, \ldots, a_j$ such that the numerical conditions $a_j \cdot r_j = m \cdot n_j$ hold for $j = 1, \ldots, h$.

By using the two-torsion invertible sheaf η corresponding to the signature (of $\mathcal{S} \subset \mathcal{S}_h$) we get a non zero section

$$\psi \in H^0(S^{mn}(\Omega^1_{\tilde{X}})(-mK_{\tilde{X}}) \otimes \eta).$$

If η is nontrivial, replace ψ by ψ^2: we obtain in this way a slope zero tensor. Hence

Theorem 1.3 *Let X be a compact complex manifold of dimension n.*
Then the following two conditions:

(1) *K_X is ample*
(2) *X admits a slope zero tensor $\psi \in H^0(S^{mn}(\Omega^1_X)(-mK_X))$, where m is a positive* integer;

hold if and only if $X \cong \Omega/\Gamma$, where Ω is a bounded symmetric domain of tube type and Γ is a cocompact discrete subgroup of $\mathrm{Aut}(\Omega)$ *acting freely.*

Moreover, the degrees and the multiplicities of the irreducible factors of the polynomial ψ_p determine uniquely the universal covering $\widetilde{X} = \Omega$.

In particular, for $m = 2$, we get that the universal covering $\widetilde{X}$ is a polydisk if and only if ψ_p is the square of a squarefree polynomial.

The proof is identical to the proof of Theorem 1.2 taken into account the observation made above (for the existence part) that it is possible to find the numbers $m, a_1, \cdots, a_j$ such that the numerical conditions $a_j \cdot r_j = m \cdot n_j$ holds for $j = 1, \ldots, h$.

Here is one more example.

Let X be a compact three-dimensional complex manifold with K_X ample and such that $\psi \in H^0(S^{3m}(\Omega^1_X)(-mK_X) \otimes \eta)$.

Then either $\tilde{X} = \mathbb{H} \times \mathbb{H} \times \mathbb{H}$ or $\tilde{X}$ is the Lie ball, i.e., the domain of type IV and dimension 3.

In this last case the sheaf $S^6(\Omega^1_X)(-2K_X)$ has a section.

Notice that the rank = 2 does not divide the dimension = 3 and that the divisor of the section is not reduced.

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