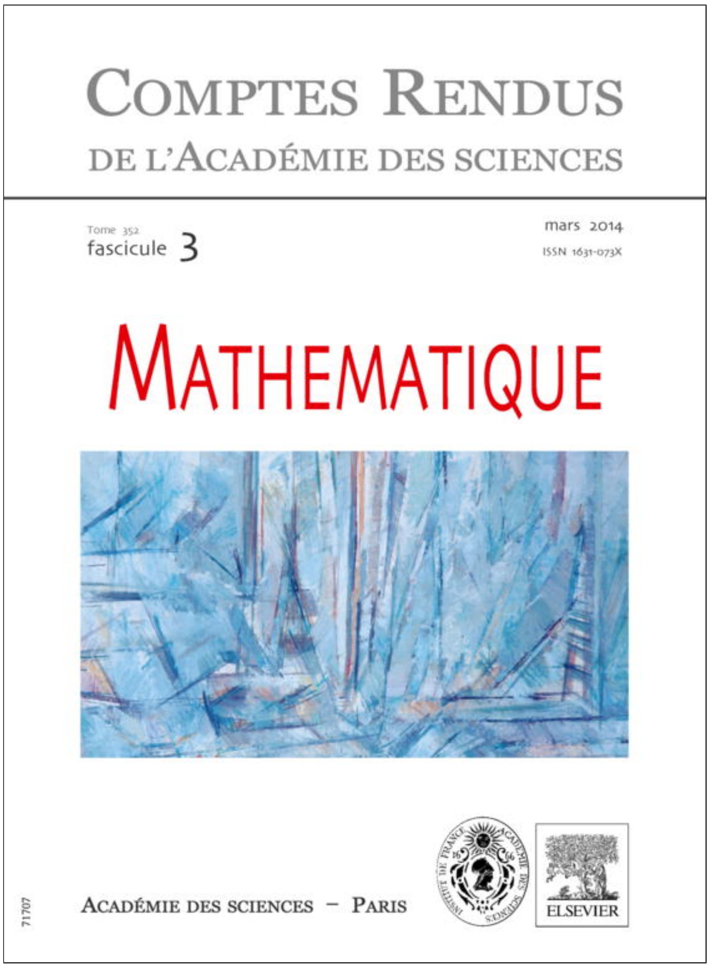
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The direct image of the relative dualizing sheaf needs not be semiample

L'image directe du faisceau dualisant relatif n'est pas nécessairement semi-ample

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A R T I C L E   I N F O

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A B S T R A C T

We provide details for the proof of Fujita's second theorem and prove that for a Kähler fibre space $f : X \to B$ over a smooth projective curve B, the direct image of the relative dualizing sheaf $V := f_* \omega_{X/B}$ is the direct sum of an ample and a unitary flat bundle. We also show that V needs not be semiample, which is our main result.

R É S U M É

Nous donnons des détails sur la démonstration du second théorème de Fujita et nous montrons que l'image directe du fibré canonique relatif $V := f_* \omega_{X/B}$ d'une fibration $f : X \to B$ sur une courbe B est la somme directe d'un fibré vectoriel ample et d'un fibré vectoriel unitairement plat si l'espace total X est une variété kählérienne compacte. Nous montrons en outre que V n'est en général pas semi-ample, ce qui constitue notre résultat principal.

1. Introduction

An important progress in classification theory was stimulated by a theorem of Fujita, who showed [3] that if X is a compact Kähler manifold and $f : X \to B$ is a fibration onto a smooth projective curve B (i.e., f has connected fibres), then the direct image of the relative dualizing sheaf $V := f_* \omega_{X|B}$ is a numerically semipositive vector bundle on B (over a curve, this is equivalent to saying that the bundle is nef). In this note, which is an abridged version of the article [1], we study further properties of V, related to semipositivity.

Recall that a vector bundle V on a curve is numerically semipositive if and only if every quotient bundle Q of V has degree $\deg(Q) \geqslant 0$, and V is ample if and only if every quotient bundle Q of V has degree $\deg(Q) > 0$ ([9], Theorem 2.4, cf. [1], Prop. 7, see also [15]). In the note [4], Fujita announced the following stronger result (in fact, a flat unitary bundle is numerically positive, cf. [1], Thm. 9):

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Theorem 1.1 *(Fujita's second theorem). Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B, and consider the direct image sheaf $V := f_*\omega_{X|B}$. Then V splits as a direct sum $V = A \oplus Q$, where A is an ample vector bundle and Q is a unitary flat bundle.*[1]

Fujita sketched the proof, but referred to a forthcoming article concerning the positivity of the so-called local exponents which however did not appear since. A first purpose of this article is to outline in Section 2 the missing details for the proof of the second theorem of Fujita, which are fully given in [1]. It is important to have in mind Fujita's second theorem in order to understand the question posed by Fujita in 1982 ([10], Problem 5): *Is the direct image $V := f_*\omega_{X|B}$ semi-ample*? In our particular case, where $V = A \oplus Q$ with A ample and Q unitary flat, it simply means that the representation of the fundamental group $\rho : \pi_1(B) \to U(r, \mathbb{C})$ associated with the flat bundle Q has finite image ([1], Thm. 9). The second aim of this article is to outline the proof of [1], Thm. 3, stating that this question has a negative answer:

Theorem 1.2. *There exists a surface X endowed with a fibration $f : X \to B$ onto a curve B of genus $\geqslant 3$, and with fibres of genus 6, such that $V := f_*\omega_{X|B}$ splits as a direct sum $V = A \oplus Q_1 \oplus Q_2$, where the summands Q_i $(i = 1, 2)$ are flat unitary rank-2 bundles having infinite monodromy group and where A is ample. In particular, V is not semi-ample.*

2. Fujita's second theorem

Let B be a smooth complex projective curve. A holomorphic vector bundle over it is identified with its sheaf of holomorphic sections. Assume now that $f : X \to B$ is a fibration of a compact Kähler manifold X over B, and consider the invertible sheaf $\omega := \omega_{X|B} = \mathcal{O}_X(K_X - f^*K_B)$. By Hironaka's theorem, there is a sequence of blow ups with smooth centres $\pi : \hat{X} \to X$ such that $\hat{f} := f \circ \pi : \hat{X} \to B$ has the property that all singular fibres F are such that $F = \sum_i m_i F_i$, and $F_{\mathrm{red}} = \sum_i F_i$ is a normal crossing divisor. Since $\pi_*\mathcal{O}_{\hat{X}}(K_{\hat{X}}) = \mathcal{O}_X(K_X)$, we obtain $\hat{f}_*\omega_{\hat{X}|B} = \hat{f}_*\mathcal{O}_{\hat{X}}(K_{\hat{X}} - \hat{f}^*K_B) = f_*\mathcal{O}_X(K_X - f^*K_B) = f_*\omega_{X|B}$. Therefore, we shall assume that all the reduced fibres of f are normal crossing divisors. By [12], there exists a cyclic Galois covering of B, $B' \to B = B'/G$, such that the normalization X'' of the fibre product $B' \times_B X$ admits a resolution $X' \to X''$ such that the resulting fibration $f' : X' \to B'$ has all the fibres which are reduced and normal crossing divisors. It is proved in [1], Prop. 13, that the sheaf $V' := f'_*\omega_{X'|B'}$ is a subsheaf of the sheaf $u^*(V)$, where $V := f_*\omega_{X|B}$, and the cokernel $u^*(V)/V'$ is concentrated on the set of points corresponding to singular fibres of f'. In particular, since V and V' are semipositive by Fujita's first theorem, *if V' satisfies the property that for each degree 0 quotient bundle Q' of V' then there is a splitting $V' = E' \oplus Q'$ for the projection $p : V' \to Q'$ and Q' is unitary flat, then V' splits as the direct sum $V' = A \oplus Q$, where A is an ample vector bundle and Q is flat unitary bundle, and the same conclusion holds also for V* (cf. [1], Prop. 13).

Theorem 2.1. *(See Fujita, [4].) Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B, and consider the direct image sheaf $V := f_*\omega_{X|B}$. Then V splits as a direct sum $V = A \oplus Q$, where A is an ample vector bundle and Q is a unitary flat bundle.*

Proof. By the above discussion it suffices to prove the theorem in the semistable case. Let n be the dimension of X. Let V^* denote the restriction of V to the noncritical locus B^* of f and let $\mathcal{H}^* = (\mathcal{H}^*, \nabla, F)$ denote the variation of polarized Hodge structures underlying the local system $R^{n-1}f_*(\mathbb{C})$ such that $V^* = F^{n-1}(\mathcal{H}^*)$. Let $\mathcal{DH}$ be the canonical extension of $\mathcal{H}^*$ to B, characterized in the semistable case by the nilpotence of the residue matrices of ∇ at the singular points. By the results of Schmid [17], the Hodge filtration extends to a holomorphic filtration of $\mathcal{DH}$, also denoted by F, and it is proved in [11] (cf. also [14]) that $V = F^{n-1}(\mathcal{DH})$. The restriction to V^* of the polarization on $\mathcal{H}^*$ induces the structure of a Hermitian vector bundle on V^*. By [19], Prop. 4.4, for each singular point $s \in S := B \setminus B^*$, there exists a basis of V given by elements σ_j such that their norm in the flat metric outside the punctures grows at most logarithmically (cf. [8]). Hence, for each quotient bundle Q of V, with Q^* denoting the restriction of Q to B^*, the determinant $\det(Q)$ admits a metric h with growth at most logarithmic at the punctures $s \in S$. By [11], Lemma 5, and [16], Prop. 3.4, the degree $\deg(\det(Q))$ of Q is hence given by the integral of the first Chern form $c_1(\det(Q), h) = \Theta_h$ of the singular metric. One has (see [6], Lecture 2):

$$\Theta_{V^*} = \Theta_{\mathcal{H}^*}|_{V^*} + \bar{\sigma}^t\sigma = \bar{\sigma}^t\sigma,$$

with σ denoting the second fundamental form. Griffiths proves ([5], cf. [6], Corollary 5) that the curvature of the dual $(V^*)^\vee$ is semi-negative, since its local expression is of the form $\mathrm{i}h'(z)\,\mathrm{d}\bar{z} \wedge \mathrm{d}z$, where $h'(z)$ is a semipositive definite Hermitian matrix (cf. [1], Section 2, for a discussion on the various notions of curvature positivity). In particular, the curvature Θ_{V^*} of V^* is semipositive. The dual of the principle 'curvature decreases in Hermitian subbundles' [7] implies that the curvature of Q^* is also semipositive. Therefore we can conclude that, since $\deg(Q) = 0$, the quotient Q^* carries a flat connection. Moreover, using the Hermitian splitting, we can view Q^* as a subbundle of V^*. Since the local monodromy of Q^* at the

[1] We remark that, while unitary flatness of a bundle implies numerical semipositivity, flatness alone does not, as shown by the following result ([1], Thm. 4): *Let $f : X \to B$ be a Kodaira fibration, i.e., X is a surface and all the fibres of f are smooth curves not all isomorphic to each other. Then the direct image sheaf $V := f_*\omega_{X|B}$ has strictly positive degree hence $\mathcal{H} := R^1 f_*(\mathbb{C}) \otimes \mathcal{O}_B$ is a flat bundle which is not numerically semipositive.*

singular points $s \in S$ is unipotent (the fibration f being semistable) and moreover unitary, the local monodromy at each $s \in S$ is trivial. Hence we conclude that Q^* has a flat extension to B which we denote by $\hat{Q}$. This extension is tautologically the canonical extension of Q^* and hence we can view $\hat{Q}$ as a subbundle of $\mathcal{DH}$. Since $Q^* \subseteq F^{n-1}(\mathcal{H}^*)$, we have the inclusion $\hat{Q} \subset V = F^{n-1}(\mathcal{DH}) \subset \mathcal{DH}$, and we obtain a homomorphism $\psi : \hat{Q} \to Q$ composing the inclusion $\hat{Q} \to V$ with the surjection $V \to Q$. From the fact that ψ is an isomorphism over B^*, we infer that ψ is an isomorphism: since $\det(\psi)$ is not identically zero, and is a section of a degree zero line bundle. Hence we conclude that the composition of ψ^{-1} with the inclusion $\hat{Q} \to V$ gives then the desired splitting of the surjection $V \to Q$. □

3. A counterexample to Fujita's question

Consider the fibration of projective curves $\varphi : Y \to \mathbb{P}^1_{[x_0,x_1]} =: P$ defined by the minimal resolution of singularities of $\Sigma \to P$, where Σ is the singular μ_7-Galois cover of $\mathbb{P}^1_{[y_0:y_1]} \times P$ (μ_7 denoting the cyclic group of order 7), given by the equation:

$$z_1^7 = y_1 y_0 (y_1 - y_0)(x_0 y_1 - x_1 y_0)^4 x_0^3.$$

Let $P^* = P \setminus \{0, 1, \infty\}$ and let $\tilde{\varphi} : Y^* \to P^*$ denote the restriction of φ to $\varphi^{-1}(P^*) =: Y^*$. The group μ_7 acts fibrewise on the family and $V := \varphi_*(\omega_{Y/P})$ as well as $\mathcal{H}^* = R^1\tilde{\varphi}_*\mathbb{C}_{Y^*} \otimes \mathcal{O}_{P^*}$ splits according to the eigenspaces for the characters $\chi_j : \mu_7 \longrightarrow \mathbb{C}^*$, $\sigma \longmapsto e^{\frac{2\pi i j}{7}}$ $(j = 0, 1, \ldots, 6)$ (we shall denote by V_j, resp. $\mathcal{H}_j^*$, the χ_j-eigensheaf of V, resp. $\mathcal{H}^*$). The fibres $\mathcal{H}_j^*(x)$ of $\mathcal{H}_j^*$ over a point $x \in P^*$ are the vector spaces $H^1(C_x, \mathbb{C})^{\chi_j}$, which have dimension 2, and we have $V_j(x) = H^0(C_x, \Omega^1_{C_x})^{\chi_j} \subseteq \mathcal{H}_j^*(x)$ for $x \in P^*$. It is proven in [1] that in the case $j = 1$ there is a basis of $H^0(C_x, \Omega^1_{C_x})^{\chi_1}$ given by η and $y\eta$, where (in affine coordinates):

$$\eta = y^{-\frac{6}{7}}(y-1)^{-\frac{6}{7}}(x-y)^{-\frac{3}{7}}\,dy. \tag{1}$$

This implies that for any $x \in P^*$ there is an equality $V_1(x) = \mathcal{H}_1^*(x)$ which implies an equality of rank-2 vector bundles $\mathcal{H}_1^* = V_1^* := V_1|_{P^*}$ (cf. [2]). The Gauß–Manin connection ∇_1 on $\mathcal{H}_1^* = V_1^*$ (restriction of the Gauß–Manin connection on $\mathcal{H}^*$ to $\mathcal{H}_1^*$) is a flat connection whose local horizontal sections are integrals of the form $g(x) = \int \eta$ $(x \in P^*)$, where η is as in (1). By [13], pp. 163–169, the function $g(x)$ is a solution of the Gauß hypergeometric differential equation $D(\frac{8}{7}, \frac{3}{7}, \frac{9}{7})$ associated with the hypergeometric function ${}_2F_1(\frac{8}{7}, \frac{3}{7}, \frac{9}{7}; x)$. This implies that ∇_1 is isomorphic to the connection associated with $D(\frac{8}{7}, \frac{3}{7}, \frac{9}{7})$. The differential equation $D(\frac{8}{7}, \frac{3}{7}, \frac{9}{7})$ is non-resonant and hence irreducible. Therefore the monodromy group of ∇_1 is irreducible. Moreover, by the Riemann scheme of $D(\frac{8}{7}, \frac{3}{7}, \frac{9}{7})$ (computed as in [13], p. 164) the local monodromy of ∇_1 at the punctures $0, 1 \in P$ is a homology of order 7 and hence is of order 7 in the associated projective linear group. Hence, by the results of Schwarz [18], the monodromy of ∇_1 is infinite. Consider now a ramified covering $\psi : B \to P$, locally at each branch point $0, 1, \infty$ of type $x \mapsto x^7$, and let $\tilde{\psi} : B^* := \psi^{-1}(P^*) \to P^*$ denote the restriction of ψ to $\psi^{-1}(P^*)$. Let $f : X \to B$ be the minimal resolution of the fibre product $B \times_P Y \to B$. Again, the cyclic group μ_7 acts fibrewise on X and it follows fibre-by-fibre that the restriction of the χ_1-eigensheaf $(f_*\omega_{X/B})^{\chi_1}$ to B^* coincides with the pullback of the flat bundle $\tilde{\psi}^*(V_1^*)$. The fibration f has only three singular fibres, but around them the local monodromy of $(f_*\omega_{X/B})^{\chi_1}|_{B^*} = \tilde{\psi}^*(V_1^*)$ is trivial, because the local monodromy of ∇_1 at $0, 1, \infty$ is of order 7. Therefore the vector bundle $(f_*\omega_{X/B})^{\chi_1}|_{B^*}$ extends to a vector bundle $Q_1 \subseteq f_*\omega_{X/B}$ on B carrying a flat connection. But since the monodromy of ∇_1 is infinite, the monodromy of the flat connection on Q_1 is also infinite. Hence Q_1 is a flat (and unitary) summand in $f_*\omega_{X/B}$ with infinite monodromy. The same arguments can be carried out for the character χ_2, leading to another flat summand Q_2 in $f_*\omega_{X/B}$ having also infinite monodromy, and hence leading to the proof of Theorem 1.2.

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