The moduli space of even surfaces of general type with $K^2 = 8$, $p_g = 4$ and $q = 0$

Fabrizio Catanese [a,*], Wenfei Liu [b], Roberto Pignatelli [c]

[a] *Lehrstuhl Mathematik VIII, Mathematisches Institut der Universität Bayreuth, NWII, Universitätstr. 30, 95447 Bayreuth, Germany*
[b] *Fakultät für Mathematik, Universität Bielefeld, Universitätsstr. 25, 33615 Bielefeld, Germany*
[c] *Dipartimento di Matematica, Università di Trento, via Sommarive 14, 38123 Trento, Italy*

Abstract

Even surfaces of general type with $K^2 = 8$, $p_g = 4$ and $q = 0$ were found by Oliverio [35] as complete intersections of bidegree $(6, 6)$ in a weighted projective space $\mathbb{P}(1, 1, 2, 3, 3)$.

In this article we prove that the moduli space of even surfaces of general type with $K^2 = 8$, $p_g = 4$ and $q = 0$ consists of two 35-dimensional irreducible components intersecting in a codimension one subset (the first of these components is the closure of the open set considered by Oliverio). All the surfaces in the second component have a singular canonical model, hence we get a new example of a generically nonreduced moduli space.

Our result gives a posteriori a complete description of the half-canonical rings of the above even surfaces. The method of proof is, we believe, the most interesting part of the paper. After describing the graded ring of a cone we are able, combining the explicit description of some subsets of the moduli space, some deformation theoretic arguments, and finally some local algebra arguments, to describe the whole moduli space.

This is the first time that the classification of a class of surfaces is done using moduli theory: up to now first the surfaces were classified, on the basis of some numerical inequalities, or other arguments, and later on the moduli spaces were investigated.
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Résumé

Oliverio [35] a construit des surfaces pairs de type général avec $K^2 = 8$, $p_g = 4$ et $q = 0$ comme intersections complètes de bidegré $(6, 6)$ dans un espace projectif pondéré $\mathbb{P}(1, 1, 2, 3, 3)$.

Dans cet article, on montre que l'espace des modules des surfaces pairs de type général avec $K^2 = 8$, $p_g = 4$ et $q = 0$ se compose de deux composantes irréductibles de dimension 35, qui se coupent dans un sous-ensemble de codimension 1.

Le premier de ces composants est la fermeture de l'ensemble ouvert considéré par Oliverio. Toutes les surfaces du deuxième composant ont un modèle canonique singulier. Par conséquent, on obtient un nouvel exemple d'un espace de modules génériquement non réduit.

Notre résultat donne a posteriori une description complète des anneaux demi-canoniques des surfaces pairs au-dessus. Le procédé de demonstration est, croyons-nous, la partie la plus intéressante de l'article. Après avoir décrit l'anneau gradué d'un

* Corresponding author.

E-mail addresses: fabrizio.catanese@uni-bayreuth.de (F. Catanese), liuwenfei@math.uni-bielefeld.de (W. Liu), roberto.pignatelli@unitn.it (R. Pignatelli).

cône, combinant la description explicite de certains sous-ensembles de l'espace des modules, quelques arguments de théorie de déformation, et enfin quelques arguments d'algèbre locale, on peut décrire l'espace des modules complets.

C'est la première fois que la classification d'une classe de surfaces est effectuée à l'aide de la théorie des modules : jusqu'à présent les surfaces ont été classées d'abord sur la base de certaines inégalités numériques, ou d'autres arguments, et les espaces de modules ont été étudiés par la suite.

1. Introduction

Algebraic surfaces with geometric genus $p_g = 4$ have been a very natural object of study since Noether's seminal paper [34] in the 19-th century. Because their canonical map has image Σ_1 which most of the times is a surface in the projective 3-dimensional space $\mathbb{P}^3$, defined therefore by a single polynomial equation.

In the 20-th century new examples of such surfaces were found by several authors [31,21,20,29], and a substantial part of Chapter VIII of Enriques's book [20] is devoted to the discussion and the proposal of several constructions, in the range $4 \leqslant K^2 \leqslant 10$. New examples were then found in [9] and [15] (see also [10]). Nowadays the investigation of such surfaces is an interesting chapter of the theory of surfaces with small invariants, encompassing (easier) existence questions and (harder) investigation of moduli spaces.

By the inequalities of Noether and Bogomolov–Miyaoka–Yau, minimal surfaces of general type with $p_g = 4$ satisfy $4 \leqslant K^2 \leqslant 45$. Only recently the upper bound $K^2 = 45$ was shown to be achieved [3], while the first historical examples of surfaces which we mentioned above are surfaces with $4 \leqslant K^2 \leqslant 7$; by the work of Ciliberto and Catanese, [15] and [12], existence is known for each $4 \leqslant K^2 \leqslant 28$.

Irregular surfaces with $p_g = 4$ were later investigated in [14]: in this case $K^2 \geqslant 8$ since, by [18], one has $K^2 \geqslant 2p_g$ for irregular surfaces[1]; while $K^2 \geqslant 12$ if the canonical map has degree 1.

Surfaces with $p_g = 4$ and $K^2 = 4$ were classified by Noether and Enriques, but it took the work of Horikawa and Bauer [26–28,2] to finish the classification of the surfaces with $p_g = 4$ and $4 \leqslant K^2 \leqslant 7$ (necessarily regular). These are 'essentially' classified, in the sense that the moduli space is shown to be a union of certain (explicitly described) locally closed subsets: but there is missing complete knowledge of the incidence structure of these subsets of the moduli space. We refer to the survey [6] for a good account of the range $4 \leqslant K^2 \leqslant 7$, and to [11] for a previous more general survey (containing the construction of several new examples).

Minimal surfaces with $K^2 = 8$, $p_g = 4$, $q = 0$ have been the object of further work by several authors [15,16,35]. The surfaces constructed by Ciliberto have a birational canonical map, are not even, and have a trivial torsion group $H_1(S, \mathbb{Z})$ (unlike the ones considered in [16]); the ones constructed by Oliverio are simply connected (see [19]), and they are even (meaning that the canonical divisor is divisible by two: i.e., the second Stiefel Whitney class $w_2(S) = 0$, equivalently, the intersection form is even).

Therefore, for $K^2 = 8$, $p_g = 4$, $q = 0$ there are at least three different topological types [35, Remark 5.4], contrasting the situation for (minimal) surfaces with $p_g = 4$, $K^2 \leqslant 7$ which, when they have the same K^2, are homeomorphic to each other. Recently Bauer and the third author [7] classified minimal surfaces with $K^2 = 8$, $p_g = 4$, $q = 0$ whose canonical map is composed with an involution (while examples with canonical map of degree three are given in [32]).

Their work shows that the moduli space of minimal surfaces with $K^2 = 8$, $p_g = 4$, $q = 0$ has at least four irreducible components: and a new fifth one is described in the present paper.

Therefore the classification of minimal surfaces with $K^2 = 8$, $p_g = 4$, $q = 0$ seems a very challenging problem, yet not completely out of reach.

The present article provides a first step in this direction, classifying all the even surfaces and completely describing the corresponding subset $M^{ev}_{8,4,0}$ of the moduli space.

This is our main result: denote by $M^{ev}_{8,4,0}$ the moduli space of even surfaces of general type with $K^2 = 8$, $p_g = 4$ and $q = 0$. We show that $M^{ev}_{8,4,0}$, which a priori consists of several connected components of the whole moduli space

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[1] That the case $K^2 = 8$, $p_g = 4$ and $q = 1$ actually occurs is shown by the family of double covers of the product $E \times \mathbb{P}^1$, where E is an elliptic curve and the branch divisor has numerical type $(4, 6)$.

$M_{8,4,0}$, is indeed a single connected component of the moduli space $M_{8,4,0}$. Oliverio [35] found out that, if $|K_S|$ is base point free (this condition determines an open set of the moduli space) the half-canonical ring $R(S, L)$ realizes the canonical model X of the surface X as a $(6, 6)$ complete intersection in $\mathbb{P}(1^2, 2, 3^2)$. Since conversely such complete intersections having at worst Du Val singularities (i.e., rational double points) yield such canonical models, one gets as a result that this open set is an irreducible unirational open set of dimension 35 in the moduli space $M^{ev}_{8,4,0}$, hence it gives rise to an irreducible component which we denote by $M_{\mathcal{F}}$ (here $\mathcal{F}$ stands for "free" canonical system). We treat here the case where $|K_S|$ has base points and, completing Oliverio's result, we obtain the following

Theorem 1.1. *The moduli space $M^{ev}_{8,4,0}$ consists of two 35-dimensional irreducible components $M_{\mathcal{F}}$ and $M_{\mathcal{E}}$, such that the general points of $M_{\mathcal{F}}$ correspond to surfaces with base point free canonical systems, while all points of $M_{\mathcal{E}}$ correspond to surfaces whose canonical system has base points. Moreover $M_{\mathcal{F}}$ and $M_{\mathcal{E}}$ intersect in a codimension one irreducible subset.*

From the above theorem we derive a complete description of our surfaces in terms of explicit equations.

Corollary 1.2. *The semicanonical rings of even surfaces with $p_g = 4$, $K^2 = 8$ admit three types of presentations: the Oliverio presentation as complete intersections (see Theorem 3.4), the extrasymmetric presentation (see Proposition 5.8), and a special (M, V) format presentation (see Proposition 5.14).*

The subscript $\mathcal{E}$ in $M_{\mathcal{E}}$ stands for 'extrasymmetric', since the general points of $M_{\mathcal{E}}$ correspond to surfaces whose half-canonical ring admits an 'extrasymmetric' presentation.

Let us also point out that we are describing the Gieseker moduli space, and that in fact all the surfaces in the component $M_{\mathcal{E}}$ have a node, hence the Kuranishi family for S is nonreduced at each point of this component.

Our proof starts with Reid's method of infinitesimal extension of hyperplane sections (cf. [38]), which is the algebraic counterpart (in terms of graded rings) of the inverse of the classical geometrical method of sweeping the cone: taking the projective cone $\mathrm{Cone}(X)$ with vertex P over a projective variety, any hyperplane section of $\mathrm{Cone}(X)$ not passing through P is isomorphic to X, and one can make it degenerate to a hyperplane section of the cone passing through P, which is the cone $\mathrm{Cone}(H \cap X)$ with vertex P over the hyperplane section of X, $H \cap X$. Viewing the process in the inverse way, one may see X as a deformation of $\mathrm{Cone}(H \cap X)$ and indeed Schlessinger, Mumford and Pinkham [39,33,36] set up the theory of deformations of varieties with a $\mathbb{C}^*$ action to analyse this situation.

The advantage in the surface case is that the hyperplane section is a curve C and the graded ring of the cone over C is much more tractable than in the higher dimensional case.

It is so once more in our special situation. In order to describe the graded ring $R(S, L)$ associated to a half-canonical divisor L, we first calculate (see Proposition 4.4), in the case where the canonical system has base points, the quotient ring R associated to the restriction of $R(S, L)$ to a smooth curve C in $|L|$. Then we would like to recover $R(S, L)$ as an extension ring, which of course can be viewed as a deformation of the cone C_R associated to the graded ring R.

We can find some of these extension rings using two different "formats", an old one and a new one; the old one consists in writing the relations in R in terms of Pfaffians of certain extrasymmetric skew-symmetric matrices (see Example 5.3), while the new one is more complicated (see Example 5.11). These formats produce in a natural way two families of such surfaces embedded in a weighted projective space of dimension 6 (see Propositions 5.8 and 5.14) via their half-canonical rings.

As written in the abstract, our method gives a posteriori only a complete description of the half-canonical rings of the above surfaces (this was first achieved by the second author via heavy computer-aided calculations which are impossible to be reproduced in a paper).

We then prove that these two families fill the moduli space via a crucial study of the local deformation space of the cone C_R, obtained by first studying the infinitesimal deformations of first and second order. Then, using some local algebra arguments, we show that there cannot be higher order obstructions.

An interesting novel feature is that the deformation space has embedding codimension two and is not a local complete intersection.

2. Preliminaries

This section collects some notions and facts that will be used in the sequel.

2.1. Notation

The varieties that we consider in this paper are defined over the complex numbers $\mathbb{C}$.

Throughout the article S shall be the minimal model of a surface of general type with $K_S^2 = 8$, $p_g(S) = 4$ and $q(S) = 0$, and we shall assume that S is *even*, which means that there exists a divisor class L such that $2L \equiv K_S$, $\equiv$ denoting, as classically, linear equivalence. The first simple observation is that, if S is even, then S is minimal, since for an exceptional curve of the first kind E one would have $-1 = E \cdot K_S = 2E \cdot L$, a contradiction.

Observe moreover that L is not a priori unique, since the class of L is determined up to addition of a 2-torsion divisor class, and these form a finite group (only a posteriori we shall see that the class of L is unique, since all our surfaces will be shown to be deformation equivalent to a weighted complete intersection, hence they are all simply connected).

Hence we shall throughout consider a pair (S, L) as above, observing that deformations of the pair (S, L) correspond to deformations of S up to an étale base change.

Given a projective algebraic variety Y and a line bundle L,

$$R(Y, L) := \bigoplus_{n \geqslant 0} H^0(Y, nL)$$

is the graded ring of sections of the pair (X, L).

The canonical model of such a surface of general type S is the projective spectrum $X := \mathrm{Proj}(\mathcal{R}(S, K_S))$ of the canonical ring of S; X is also the projective spectrum $= \mathrm{Proj}(\mathcal{R}(S, L))$ of the semicanonical ring associated to the class of L, and is the only birational model of S with ample canonical class and at most rational double points as singularities. The above (graded) rings are all finitely generated $\mathbb{C}$-algebras. Observe that the line bundle L descends to a line bundle on X by the results of Artin in [1].

Observe that any deformation of S yields a family of relative canonical algebras and (up to étale base change) a family of relative half-canonical algebras. In particular, by the semicontinuity theorem, a minimal system of (homogeneous) generators for the semicanonical ring $\mathcal{R}(S, L)$ yields an embedding of X into a weighted projective space $\mathbb{P}(d_1, d_2, \ldots, d_h)$, and locally a deformation of S yields a family of projectively normal subschemes of $\mathbb{P}(d_1, d_2, \ldots, d_h)$.

By abuse of notation, we denote sometimes an element of a polynomial ring and its image in some quotient ring by the same symbol.

$\mathbb{C}[\epsilon] := \mathbb{C}[t]/(t^2)$ is the ring of dual numbers, so that ϵ is meant to be a first order parameter, i.e., ϵ has degree 0 and $\epsilon^2 = 0$. Given a $\mathbb{C}$-algebra R, $R[\epsilon] := R \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]$.

Finally, let $M = (m_{ij})$ be a 4×4 skew-symmetric matrix. Then recall that the Pfaffian of M is

$$\mathrm{pf}(M) = m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}.$$

2.2. Deformation of closed subschemes and graded rings

As we shall sometimes not only use the analytical theory of deformations, we observe the connection between Hilbert schemes and deformation theory, which is central in our arguments.

Theorem 2.1 (*Ideal–variety correspondence*). *There is a natural bijection between the set of closed subschemes of a weighted projective space and the set of saturated homogeneous ideals of the (weighted) polynomial ring, which associates any closed subscheme to its saturated homogeneous ideal.*

Proof. For closed subschemes of usual projective space, see for example [24, Example II.5.10]; for the weighted case, see [19, 3.1.2(iv)]. □

Theorem 2.2. *The deformation functor of closed subschemes in weighted projective space satisfies Schlessinger's conditions H_0, H_ϵ, $\bar{H}$, H_f, and hence is prorepresentable.*

Proof. This is [40, Proposition 3.2.1]. □

Let X be a closed subscheme of a weighted projective space $\mathbb{P}$. Let $\mathbb{A} = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring of $\mathbb{P}$ and $I \subset \mathbb{A}$ the saturated homogeneous ideal of X.

Theorem 2.3. *If depth $\mathbb{A}/I \geqslant 2$, then the deformation theory of the embedded projective scheme X is the same as that of the corresponding graded ring $\mathbb{A}/I$.*

Proof. For the case of usual projective space, we refer to [25, Proposition 8.8]; for the weighted case, one can adapt the same proof as [25, Proposition 8.8]. □

Thus in good situations, in particular in ours, we can study the deformations of weighted projective schemes via the deformations of their graded rings, indeed the parameter spaces we shall use will be locally dominating the Hilbert scheme.

Theorem 2.4 *(Hilbert scheme). There is a natural scheme parametrizing the set of closed subschemes of a fixed weighted projective space having a given Hilbert polynomial.*

Proof. This result is due to Grothendieck [22], and has been generalized in the multigraded case by [23]. □

The role of the Hilbert scheme becomes more apparent when we shall take the deformation theory of the cone $\text{Cone}(H \cap X)$ over the hyperplane section of the canonical model X in its half-canonical embedding in a weighted projective space.

Here we shall use the results of Schlessinger and Pinkham [39,36], in particular Pinkham's result:

Theorem 2.5 *(Hilbert scheme and deformations). The natural morphism of the Hilbert scheme to the Kuranishi space of $\text{Cone}(H \cap X)$ is smooth.*

It is important to remark that, whereas the Kuranishi family is versal at any point, it is only semi-universal at the point corresponding to $\text{Cone}(H \cap X)$: this is due to the $\mathbb{C}^*$ action which stabilizes the cone $\text{Cone}(H \cap X)$ but not its small deformations.

Observe finally that the local structure of the Gieseker moduli space is the quotient of the Kuranishi space of X by the finite group $\text{Aut}(X)$ (see [13] as a general reference), hence, in order to study the irreducible components of the moduli space, openness questions are reduced to the study of the Kuranishi space, in turn this is locally dominated by the Hilbert scheme (or any parameter space dominating the latter).

3. Oliverio's surfaces

Oliverio [35] studied the even surfaces of general type with $K_S^2 = 8$, $p_g = 4$ and $q = 0$ whose canonical system is base point free, showing that their canonical models are the general complete intersections of bidegree $(6, 6)$ in the weighted projective space $\mathbb{P}(1^2, 2, 3^2)$.

Let S be an even surface of general type with $K_S^2 = 8$, $p_g = 4$ and $q = 0$. Let L be a half-canonical divisor, that is, $2L = K_S$. We recall one more preliminary result.

Lemma 3.1.

(i) *For any $k \in \mathbb{Z}$, $h^1(S, kL) = 0$.*
(ii) *$h^0(S, L) = 2$, $h^0(S, 2L) = 4$, $h^0(S, kL) = k^2 - 2k + 5$ for $k \geqslant 3$.*

Proof. This is the content of Lemma 2.1 and Lemma 2.2 of [35]. □

Remark 3.2. Our standard notation shall be that x_1, x_2 be a basis of $H^0(S, L)$, while y completes $w_0 := x_1 x_2$, $w_1 := x_1^2$, $w_2 := x_2^2$ to a basis of $H^0(S, 2L)$, z_1, z_2 complete

$$x_1^3, \ x_1^2 x_2, \ x_1 x_2^2, \ x_2^3, \ yx_1, \ yx_2$$

to a basis of $H^0(S, 3L)$.

Observe that, since the canonical map of S cannot be composed with a pencil (see [41, Corollary 1 of Theorem 5.1] or [37, Theorem 3.4] for the case of genus 2 fibrations, [8] for the higher genus case), this implies that $\phi_K(S)$ is a quadric cone $\{w \mid w_1 w_2 = w_0^2\}$.

As a converse to the above remark, we show a result of independent interest:

Proposition 3.3. *Let S be a minimal surface with $K_S^2 = 8$, $p_g = 4$ and $q = 0$ and assume that the image of the canonical map $\phi_K(S)$ is a quadric cone. Then, denoting by L the strict transform (closure of the pull back of the Cartier locus) of a line on the quadric cone, we have $K_S = 2L + F$ where*

1. *either $F = 0$ and S is even, or*
2. *$L^2 = 1$, $K_S \cdot L = 3$, $F \cdot L = 1$, $|L|$ is a pencil of curves of genus 3 having a simple base point, and the canonical map has degree 3, or*
3. *$L^2 = 0$, $K_S \cdot L = F \cdot L = 2$, $|L|$ is a base point free pencil of curves of genus $g = 2$, and the degree of ϕ_K is equal to 2, or*
4. *$L^2 = 0$, $K_S \cdot L = F \cdot L = 4$, hence $|L|$ is a base point free pencil of curves of genus $g = 3$ and the degree of ϕ_K is equal to 4.*

Proof. As usual write $K_S = 2L + F$, and observe that K_S and L are nef divisors. Then

$$8 = K_S^2 = 2LK_S + FK_S = 4L^2 + 2LF + FK_S \geqslant 4L^2 + 2LF \geqslant 4L^2.$$

The above shows that $L^2 \in \{0, 1, 2\}$ and that FK_S is even.

If $L^2 = 2$, then $FK_S = 2LF = F^2 = 0$: hence first F is a sum of (-2) curves, and then we get $F = 0$ since the intersection form is strictly negative definite on the set of divisors which are sums of (-2) curves.

If $L^2 = 1$, then $K_S L$ is odd, at least 3, but $K_S L = 2 + LF$, whence by the above inequality $LF = 1$ and $K_S F = 2$, $F^2 = 0$. Hence $|L|$ is a pencil of curves of genus 3, and the degree of ϕ_K is then the intersection number $K_S L = 3$.

If instead $L^2 = 0$, then $LK_S = LF$ is even and $\geqslant 2$, so there are only the possibilities $LK_S = LF = 2$, or $LK_S = LF = 4$. In the latter case $FK_S = 0$, thus F is a sum of (-2) curves, in the former we have $4 = FK_S = 4 + F^2$, hence $F^2 = 0$, $FK_S = 4$. Again the degree of ϕ_K equals the intersection number $K_S L$. □

Observe that examples of all the above cases are known (see [32,7]).

We turn now to a slight improvement of Oliverio's result concerning the first case of Proposition 3.3.

Theorem 3.4. *The canonical models X of even surfaces S of general type with $K_S^2 = 8$, $p_g = 4$ and $q = 0$ whose canonical system is base point free are exactly the complete intersections of type $(6, 6)$ in the weighted projective space $\mathbb{P}(1^2, 2, 3^2)$ with at worst rational double points as singularities. In particular, there are coordinates x_1, x_2, y, z_1, z_2 such that X is defined by equations of the type*

$$f = z_1^2 + z_2 A(x_1, x_2, y) + F(x_1, x_2, y) = 0,$$
$$f' = z_2^2 + z_1 A'(x_1, x_2, y) + F'(x_1, x_2, y) = 0. \tag{$**$}$$

These surfaces form an open set in an irreducible unirational component of dimension 35 of the moduli space of surfaces of general type.

Proof. Oliverio [35, Theorem 5.2] proves that minimal even surfaces of general type with $K_S^2 = 8$, $p_g = 4$ and $q = 0$ whose canonical system is base point free have a canonical model X which is a complete intersection of type $(6, 6)$ in the weighted projective space $\mathbb{P}(1^2, 2, 3^2)$ (hence X has at worse rational double points as singularities).

Conversely, if $X = \{f = f' = 0\}$ is a complete intersection of two sextics in $\mathbb{P}(1^2, 2, 3^2)$ with only rational double points as singularities, then $K_X = \mathcal{O}_X(2)$, $K_X^2 = 8$, $p_g = 4$ and $q = 0$ and the only nontrivial thing to show is that the Weil divisor L such that $2L \equiv K_X$ is indeed a Cartier divisor (then the minimal model S is an even surface) and that the induced linear system is base point free.

To this purpose it suffices to show that $|K_X| = |2L|$ and $|3L|$ are base point free: because then $\mathcal{O}_X(2)$ and $\mathcal{O}_X(3)$ are invertible sheaves, hence also $\mathcal{O}_X(1)$ is invertible. Notice the surjection $H^0(\mathcal{O}_\mathbb{P}(2)) \to H^0(\mathcal{O}_X(K_X))$ which implies that the base locus of $|K_X|$ is the locus of zeros of all degree two monomials.

Choose weighted homogeneous coordinates x_1, x_2, y, z_1, z_2 in $\mathbb{P} := \mathbb{P}(1^2, 2, 3^2)$ and write

$$f = q(z_1, z_2) + g, \qquad f' = q'(z_1, z_2) + g'$$

with $g, g' \in (x_1, x_2, y)$ and q, q' quadratic forms in z_1, z_2. Assume by contradiction that the base locus of $|K_X|$ is not empty, i.e., that $X \cap \{x_1 = x_2 = y = 0\} \neq \emptyset$. Then q and q' have a common factor, say z_1, and the canonical system of the surface has a base point at $Q = [0, 0, 0, 0, 1]$.

Note that Q is a singular point of $\mathbb{P}(1^2, 2, 3^2)$, a quotient singularity of type $\frac{1}{3}(1, 1, 2, 3)$ with Zariski tangent space of dimension 7, whose basis is given by the monomials $x_1^i x_2^{3-i}$, $x_i y$, z_1. Since $\mathcal{O}_{\mathbb{P}(1^2,2,3^2)}(6)$ is Cartier, the Zariski tangent space of X at Q has then dimension at least $7 - 2 = 5$, contradicting the assumption on the singularities of X (they have Zariski tangent dimension at most 3).

Similarly, if $|3L|$ were not base point free, then X would contain the point $Q = [0, 0, 1, 0, 0]$, a quotient singularity of type $\frac{1}{2}(1, 1, 1, 1)$ with Zariski tangent space of dimension 10: this again contradicts the fact that the singularities of X have Zariski tangent dimension at most 3.

Therefore the quadratic forms q, q' have no common factor, and we can change coordinates so that the equations of X take the desired form (**), where however $A(0, 0, 1) \neq 0$ or $A'(0, 0, 1) \neq 0$:

$$f = z_1^2 + z_2 A(x_1, x_2, y) + F(x_1, x_2, y) = 0, \qquad f' = z_2^2 + z_1 A'(x_1, x_2, y) + F'(x_1, x_2, y) = 0.$$

These equations naturally exhibit X as a 4-to-1 cover of the quadric cone, and show that our surfaces form an open set in an irreducible unirational component of dimension 35 of the moduli space (since we have 6 affine parameters for A, A', 15 projective parameters for F, F' and we divide out by a group of dimension 7, the group of automorphisms of the quadric cone $\mathbb{P}(1, 1, 2)$; at any rate, the dimension follows also from [35, Corollary 5.3]). $\quad\square$

4. The hyperplane section $C \in |L|$

Let S be an even surface of general type with $K_S^2 = 8$, $p_g = 4$ and $q = 0$ and let L as above be a half-canonical divisor. We view the half-canonical curves as hyperplane sections of a suitable embedding of the canonical model into a weighted projective space, and we describe the associated graded ring.

Lemma 4.1. *If $|K_S|$ is not base point free, then*

(i) *$|K_S|$ has two base points Q, Q' with Q' infinitely near to Q;*
(ii) *$|L|$ has two base points Q, Q'' with Q'' infinitely near to Q and $Q'' \neq Q'$;*
(iii) *a general curve in $|L|$ is smooth of genus 4.*

Proof. These results are in Lemma 3.1, Lemma 3.3 and Theorem 4.1 of [35]. $\quad\square$

Let us fix a smooth curve $C \in |L|$. Then $Q \in C$ and $\mathcal{O}_C(L) = \mathcal{O}_C(2Q)$ by Lemma 4.1. Let x_1 be a section in $H^0(S, L)$ with $\text{div}(x_1) = C$. Using the long exact cohomology sequence associated to the exact sequence

$$0 \to \mathcal{O}_S\big((k-1)L\big) \xrightarrow{x_1} \mathcal{O}_S(kL) \to \mathcal{O}_C(2kQ) \to 0,$$

we have, by Lemma 3.1(i), the exact sequence

$$0 \to H^0\big(S, (k-1)L\big) \xrightarrow{x_1} H^0(S, kL) \to H^0(C, 2kQ) \to 0 \tag{1}$$

and it follows that

$$R(S, L)/(x_1) = R(C, 2Q) = R(C, Q)^{(2)} \subset R(C, Q),$$

where $R(C, Q)^{(2)} := \bigoplus_k R(C, Q)_{2k}$ is the even part of $R(C, Q)$. In other words, $\mathrm{Proj}\, R(C, 2Q)$ turns out to be a weighted hyperplane section of $\mathrm{Proj}\, R(S, L)$.

Lemma 4.2.

$$R(C, Q) = \mathbb{C}[\xi, \eta, \zeta]/(p),$$

where $\deg(\xi, \eta, \zeta) = (1, 3, 5)$ *and* p *is a weighted polynomial of degree* 15. *Moreover one can, up to automorphisms of* $\mathbb{P}(1, 3, 5)$, *assume that*

$$p = \zeta^3 - \eta^5 + \xi p',$$

where p' *is some suitable weighted polynomial of degree* 14.

Proof. We first calculate $h^0(C, mQ)$ for all $m \geqslant 0$. By the exact sequence (1), we have

$$h^0(C, 2kQ) = h^0(S, kL) - h^0\big(S, (k-1)L\big),$$

for any $k \geqslant 0$. Together with Lemma 3.1, this yields in particular

$$h^0(C, \mathcal{O}_C) = h^0(C, 2Q) = 1, \qquad h^0(C, 4Q) = 2, \qquad h^0(C, 6Q) = 4,$$

which in turns implies $h^0(C, Q) = 1$, $h^0(C, 5Q) = 3$.

Since Q is a base point of $|K_S|$, it is also a base point of $|4Q| = |K_{S|C}| = |K_S|_{|C}$, which implies that

$$h^0(C, 3Q) = h^0(C, 4Q) = 2.$$

For $m \geqslant 7$, we have $h^0(C, mQ) = m - 3$ by the Riemann–Roch theorem.

Now take a nonzero section $\xi \in H^0(C, Q)$. Then

$$H^0(C, Q) = \langle \xi \rangle, \qquad H^0(C, 2Q) = \big\langle \xi^2 \big\rangle$$

and ξ has only a simple zero at Q. There are sections $\eta \in H^0(C, 3Q) \setminus \xi H^0(C, 2Q)$ and $\zeta \in H^0(C, 5Q) \setminus \xi H^0(C, 4Q)$.

Since both η and ζ do not vanish at Q, there exist nonzero $a, b \in \mathbb{C}$ such that $a\eta^5 - b\zeta^3$ vanishes at Q. Therefore there is a polynomial p' in ξ, η, ζ of degree 14 such that

$$a\eta^5 - b\zeta^3 = \xi p'.$$

Up to rescaling the generators, we have $p = \eta^5 - \zeta^3 + \xi p' = 0$.

Now, ξ, η, ζ give a morphism into $\mathbb{P}(1, 3, 5)$. Therefore the image is an irreducible curve and there are no other relations than p holding among the three elements ξ, η, ζ.

In other words we get by pull back an injective ring homomorphism from $\mathbb{C}[\xi, \eta, \zeta]/(p)$ to $R(C, Q)$. Since, by the first part of our proof, both rings have the same Hilbert function, they are isomorphic. $\square$

Remark 4.3. Conversely a general curve $C = \{p = 0\}$ of degree 15 in $\mathbb{P}(1, 3, 5)$ is smooth of genus $1 + \frac{15(15-9)}{2 \cdot 15} = 4$ and $R(C, Q)$ is naturally isomorphic to $\mathbb{C}[\xi, \eta, \zeta]/p$, where $\{Q\} := C \cap (\xi = 0)$ is a Weierstraß point whose semigroup is generated by 3 and 5; the proof of Lemma 4.2 shows that every smooth curve of genus 4 with a Weierstraß point of this form arises in this way.

The smooth curves of degree 15 in $\mathbb{P}(1, 3, 5)$ form a linear system of dimension 12. Since $\dim \mathrm{Aut}\, \mathbb{P}(1, 3, 5) = 5$, they form a subvariety of dimension 7 in the moduli space of curves of genus 4; note that it is a divisor in the locus of the curves whose canonical image is contained in a quadric cone, which are those possessing only one g_3^1.

Proposition 4.4.

$$R(C, 2Q) = \mathbb{C}[x_2, y, z_1, z_2, u, v]/I,$$

where $\deg(x_2, y, z_1, z_2, u, v) = (1, 2, 3, 3, 4, 5)$, *and I is generated by the equations*

$$
\begin{aligned}
f_1 &= x_2 z_2 - y^2 & \deg 4, \\
f_2 &= x_2 u - y z_1 & \deg 5, \\
f_3 &= y u - z_1 z_2 & \deg 6, \\
f_4 &= x_2 v - z_1^2 & \deg 6, \\
f_5 &= y v - z_1 u & \deg 7, \\
f_6 &= z_2 v - u^2 & \deg 8, \\
f_7 &= z_1 A - y B + x_2 D & \deg 8, \\
f_8 &= u A - z_2 B + y D & \deg 9, \\
f_9 &= v A - u B + z_1 D & \deg 10.
\end{aligned}
$$

Here A, B, D are general polynomials of respective degrees 5, 6 and 7. Up to automorphisms, one can assume $A = v$, $B = z_2^2$.

Proof. We have shown that the graded ring $R(C, Q)$ corresponds to a projectively normal embedding in the weighted projective plane $\mathbb{P}(1, 3, 5)$. Therefore the subring $R(C, 2Q)$ is the even degree part of $R(C, Q)$, a quotient of the graded ring of the Veronese embedding of the plane $\mathbb{P}(1, 3, 5)$.

In other words, since the three generators ξ, η, ζ of $R(C, Q)$ have odd degrees, the even part $R(C, 2Q)$ is generated by the six products

$$
\begin{aligned}
x_2 &:= \xi^2, & y &:= \xi\eta, & z_1 &:= \xi\zeta, \\
z_2 &:= \eta^2, & u &:= \eta\zeta, & v &:= \zeta^2.
\end{aligned}
$$

These in fact define a closed embedding $\varphi : \mathbb{P}(1, 3, 5) \to \mathbb{P}(1, 2, 3^2, 4, 5)$. Generators of the ideal of $\varphi(\mathbb{P}(1, 3, 5))$ are the 2×2 minors of the 3×3 symmetric matrix

$$
\begin{pmatrix}
x_2 & y & z_1 \\
y & z_2 & u \\
z_1 & u & v
\end{pmatrix}.
$$

Note that C is the curve defined by $p = 0$ in $\mathbb{P}(1, 3, 5)$. So the ideal I of $\varphi(C)$ is generated by the defining equations of $\varphi(\mathbb{P}(1, 3, 5))$ plus ξp, ηp, ζp.

In view of Lemma 4.2, we can write these three homogeneous polynomials in terms of x_2, y, z_1, z_2, u, v:

$$
\begin{aligned}
\xi p_{15} &= z_1 v - y z_2^2 + x_2 D, \\
\eta p_{15} &= u v - z_2^3 + y D, \\
\zeta p_{15} &= v^2 - z_2^2 u + z_1 D. \qquad \square
\end{aligned}
$$

Note that $R(C, 2Q)$ is Cohen–Macaulay. In fact for any smooth projective curve C and an ample line bundle H, the graded ring $R(C, H)$ is Cohen–Macaulay (see [25, Proposition 8.6] and its proof).

5. Two families of surfaces

Recall that $R := R(S, L)/(x_1) = R(C, 2Q)$, where x_1 is an element of $H^0(S, L)$ defining the curve C. The hyperplane section principle [38, Proposition 1.2] gives the following, which is the explicit counterpart of the existence of a flat 1-dimensional family induced by the function x_1:

  (i) $R(S, L)$ needs exactly one more generator, namely x_1, and the other generators are lifted from R;
 (ii) the relations $F_1, \ldots, F_9$ among the generators of $R(S, L)$ are liftings of $f_1, \ldots, f_9$;
(iii) moreover the first syzygies among the f_i are lifted to those among the F_i.

Point (iii) is the tricky part of the principle, and is where "formats" are useful in order to write explicitly a flat family having as basis a locally closed set of an affine space. A format is simply a way to write an ideal in such a way that the obvious first syzygies are all the first syzygies (in other words, one produces automatically a flat family).

We will describe two formats. Each of them will produce a family of minimal surfaces of general type with $p_g = 4$, $q = 0$, $K^2 = 8$ and even canonical divisor.

5.1. The extrasymmetric format

This format was first introduced by M. Reid and D. Dicks (see [38], and [4,5] for further applications and a discussion).

Let us explain now what is an extrasymmetric format: we consider two vector spaces V, W of dimension 3, and the Pfaffian $\mathcal{P}$ locus in

$$\Lambda^2(V \oplus W),$$

corresponding to the cone over the Grassmann variety $G(1, 5)$.

$\mathcal{P}$ has codimension 6 and is arithmetically Gorenstein.

Even if the Pfaffian $\mathcal{P}$ locus in $\Lambda^2(V \oplus W)$ has codimension 6, we obtain a codimension 4 locus if we consider the tensors of the form:

$$\begin{pmatrix} B & D = CA \\ -{}^tD & -{}^tABA \end{pmatrix},$$

where C is symmetric (and B is skew-symmetric).

Then for these triples of matrices (A, B, C) the Pfaffian locus reduces codimension to 4.

Remark 5.1. In fact, for A invertible, w.l.o.g. we assume $A = I$, and the matrix is equivalent to

$$\begin{pmatrix} 0 & C + B \\ -C + B & 0 \end{pmatrix}$$

and we find the geometry of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$.

More precisely, we consider a 6×6 skew-symmetric *extrasymmetric* matrix

$$\tilde{N} = \begin{pmatrix} n_1 & n_2 & n_3 & n_4 & n_5 \\ & n_6 & n_7 & n_8 & n_4 \\ & & n_9 & an_7 & an_3 \\ & & & bn_6 & bn_2 \\ & & & & abn_1 \end{pmatrix}$$

and let $\mathcal{I}_\mathcal{E} \subset \tilde{\mathbb{A}}_\mathcal{E} := \mathbb{C}[n_1, n_2, \ldots, n_9, a, b]$ be the ideal generated by the 4×4 Pfaffians of $\tilde{N}$. A minimal system of generators of $\mathcal{I}_\mathcal{E}$ is given by 9 of these Pfaffians, the last 6 being just repetitions of simple multiples of them. These 9 generators are yoked by exactly 16 independent syzygies, which we can explicitly compute (see also [38, 5.5]).

Definition 5.2. Let $\mathbb{A}$ be any weighted polynomial ring and consider a ring homomorphism $\varphi : \tilde{\mathbb{A}}_\mathcal{E} \to \mathbb{A}$; then the ideal I generated by $\varphi(\mathcal{I}_\mathcal{E})$ is generated by the 4×4 Pfaffians of the 6×6 skew-symmetric matrix $\varphi(\tilde{N})$, obtained by $\tilde{N}$ by substituting to each entry its image. We will say that $\varphi(\tilde{N})$ is an extrasymmetric format for the quotient ring $\mathbb{A}/I$.

Example 5.3. Computing the 4×4 Pfaffians of the matrix

$$
N = \begin{pmatrix}
A & B & z_1 & y & x_2 \\
 & D & u & z_2 & y \\
 & & v & u & z_1 \\
 & & & 0 & 0 \\
 & & & & 0
\end{pmatrix},
$$

the reader can check that it is an extrasymmetric format for R. Here $\mathbb{A} = \mathbb{C}[x_2, y, z_1, z_2, u, v]$ with the grading given in Proposition 4.4.

Let us consider, in Example 5.3, $\tilde{\mathbb{A}}_{\mathcal{E}}$ graded by the grading making φ a graded homomorphism. Since φ is surjective, it yields an isomorphism of graded rings $\mathbb{A} \cong \tilde{\mathbb{A}}_{\mathcal{E}} / \ker \varphi$ and

$$
\ker \varphi = \left(a - 1, b, n_1 - \tilde{A}(n_i), n_2 - \tilde{B}(n_i), n_6 - \tilde{D}(n_i) \right),
$$

where $\tilde{A}, \tilde{B}, \tilde{D}$ are obtained by A, B, D replacing the variables x_2, y, z_1, z_2, u, v by $n_5, n_4, n_3, n_8, n_7, n_9$ respectively.

Consider $\tilde{R} = \tilde{\mathbb{A}}_{\mathcal{E}} / \mathcal{I}_{\mathcal{E}}$, and write $\tilde{f}_1, \ldots, \tilde{f}_9$ for the nine Pfaffians of $\tilde{N}$ generating $\mathcal{I}_{\mathcal{E}}$. Here we can arrange the indices so that $\varphi(\tilde{f}_i) = f_i$ for $1 \leqslant i \leqslant 9$. Note that $R = \tilde{R} \otimes_{\tilde{\mathbb{A}}_{\mathcal{E}}} \mathbb{A}$ is such that $Spec(R)$ a codimension five complete intersection in $Spec(\tilde{R})$.

Lemma 5.4. $Tor_1^{\tilde{\mathbb{A}}_{\mathcal{E}}}(\mathbb{A}, \tilde{R}) = 0$.

Proof. This follows since $\mathbb{A} \cong \tilde{\mathbb{A}}_{\mathcal{E}} / \ker \varphi$ and $\ker \varphi$ is generated by a regular sequence, see [30]. $\square$

Corollary 5.5. *The first syzygy module of R is a reduction of the one of $\tilde{R}$.*

Proof. Let $\tilde{L}_\bullet \to \tilde{R} \to 0$ be a free resolution of $\tilde{R}$ over $\tilde{\mathbb{A}}_{\mathcal{E}}$. By Lemma 5.4, $\mathbb{A} \otimes_{\tilde{\mathbb{A}}_{\mathcal{E}}} \tilde{L}_\bullet \to \mathbb{A} \otimes_{\tilde{\mathbb{A}}_{\mathcal{E}}} \tilde{R} \to 0$ is exact at $\mathbb{A} \otimes_{\tilde{\mathbb{A}}_{\mathcal{E}}} \tilde{L}_1$, which implies the corollary. $\square$

Therefore, to calculate the syzygy module of R, it suffices to work out that of $\tilde{R}$.

Corollary 5.6. *The syzygies*

$$
\begin{aligned}
& \sigma_1: \ -z_1 f_1 + y f_2 - x_2 f_3 = 0 && \deg 7, \\
& \sigma_2: \ -u f_1 + z_2 f_2 - y f_3 = 0 && \deg 8, \\
& \sigma_3: \ z_1 f_2 - y f_4 + x_2 f_5 = 0 && \deg 8, \\
& \sigma_4: \ v f_1 + z_1 f_3 - z_2 f_4 + y f_5 = 0 && \deg 9, \\
& \sigma_5: \ v f_1 - u f_2 + y f_5 - x_2 f_6 = 0 && \deg 9, \\
& \sigma_6: \ v f_2 - u f_4 + z_1 f_5 = 0 && \deg 10, \\
& \sigma_7: \ -u f_3 + z_2 f_5 - y f_6 = 0 && \deg 10, \\
& \sigma_8: \ -v f_3 + u f_5 - z_1 f_6 = 0 && \deg 11, \\
& \sigma_9: \ B f_1 - A f_2 - y f_7 + x_2 f_8 = 0 && \deg 10, \\
& \sigma_{10}: \ -B f_2 + A f_4 + z_1 f_7 - x_2 f_9 = 0 && \deg 11, \\
& \sigma_{11}: \ D f_1 - A f_3 - z_2 f_7 + y f_8 = 0 && \deg 11, \\
& \sigma_{12}: \ B f_3 - A f_5 - z_1 f_8 + y f_9 = 0 && \deg 12, \\
& \sigma_{13}: \ -D f_2 + A f_5 + u f_7 - y f_9 = 0 && \deg 12, \\
& \sigma_{14}: \ D f_3 - A f_6 - u f_8 + z_2 f_9 = 0 && \deg 13,
\end{aligned}
$$

$$\sigma_{15}: \quad -Df_4 + Bf_5 + vf_7 - z_1 f_9 = 0 \qquad \text{deg } 13,$$

$$\sigma_{16}: \quad Df_5 - Bf_6 - vf_8 + uf_9 = 0 \qquad \text{deg } 14$$

generate all the syzygies between $f_1, \ldots, f_9$.

Proof. This follows by Corollary 5.6 and by the computation of the relations among the Pfaffians of $\tilde{N}$ done in [38, 5.5]. □

This is useful because we can then easily construct flat deformations of this ring. Indeed, if we lift the matrix to a bigger ring $\mathbb{B}$, we will get automatically a new ideal in $\mathbb{B}$ generated by lifts of I, and also the first syzygies of R will automatically lift, yielding flatness. More formally

Corollary 5.7. *Consider the map* $\varphi: \mathbb{A}_{\mathcal{E}} \to \mathbb{A}$ *in Example* 5.3, *a surjective ring homomorphism* $\pi: \mathbb{B} \to \mathbb{A}$, *and a ring homomorphism* $\psi: \mathbb{A}_{\mathcal{E}} \to \mathbb{B}$ *such that* $\varphi = \pi \circ \psi$ *(i.e. "ψ lifts φ").*
Let $F_1, \ldots, F_9$ *be the nine Pfaffians of* $\tilde{N}$ *such that* $\pi(F_i) = f_i$. *Then every relation among the* f_i *lifts to a relation among the* F_i.

Proof. Each relation σ_j is obtained by applying φ to a relation $\tilde{\sigma}_j$ among the $\tilde{f}_j$. Applying ψ to the same relations will give the required lifts. □

By the hyperplane section principle, we can use the above to construct a family of surfaces.

Proposition 5.8. *Consider the extrasymmetric matrix*

$$\mathcal{N} = \begin{pmatrix} \mathcal{A} & \mathcal{B} & z_1 & y & x_2 \\ & \mathcal{D} & u & z_2 & y \\ & & v & u & z_1 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix},$$

where $\mathcal{A}, \mathcal{B}, \mathcal{D}$ *are weighted polynomials of respective degrees* 5, 6 *and* 7 *in the graded polynomial ring* $\mathbb{C}[x_1, x_2, y, z_1, z_2, u, v]$ *with weights* $(1, 1, 2, 3, 3, 4, 5)$, *and let* $X \subset \mathbb{P}(1^2, 2, 3^2, 4, 5)$ *be given by the vanishing of the* 4×4 *Pfaffians of* $\mathcal{N}$.
Then, for general choice of $\mathcal{A}, \mathcal{B}, \mathcal{D}$, X *is a surface with at worse rational double points as singularities. In this case* $K_X^2 = 8$, $p_g(X) = 4$, $q = 0$, $K_X = \mathcal{O}_X(2)$ *and* $|K_X|$ *is not base point free. We obtain in this way a* 35-*dimensional unirational family* $M_{\mathcal{E}}$ *in the moduli space of surfaces of general type.*

Proof. Varying $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ we let X move in a fixed threefold, the cone over the weighted Veronese surface (its ideal is generated by the 2×2 minors of the 3×3 submatrix on the top-right corner of $\mathcal{N}$) which has a single not quasismooth point, with coordinates $[1, 0, 0, 0, 0, 0, 0]$. The same point is also the only base point of the linear system. By Bertini's theorem, the general surface X is nonsingular away from that point. If the coefficient of x_1^7 in $\mathcal{D}$ does not vanish, then this point is a node of X.

By Corollary 5.7 and its proof there is a Gorenstein symmetric free resolution of the ideal of X which lifts a resolution of the ideal of the curve $C = \{x_1 = 0\} \cap X$ in $\mathbb{P}(1, 2, 3^2, 4, 5)$: both are images by a suitable ring map of the Gorenstein symmetric resolution of $\mathcal{I}_{\mathcal{E}}$. Since $K_C = \mathcal{O}_C(3)$, then $K_X = \mathcal{O}_S(2)$, and it follows immediately that the invariants are as stated.

We show that the canonical system of S has a base point. Indeed, in the 2-plane $\{x_1 = x_2 = y = z_1 = 0\}$ the equations reduce to asking that the rank of the matrix

$$\begin{pmatrix} z_2 & u & \mathcal{A} \\ u & v & \mathcal{B} \end{pmatrix}$$

is not 2. Such a determinantal condition defines a locus of codimension at most 2, and with nontrivial cohomology class, hence not empty.

To compute the dimension of the family, we note that, on X, $z_2 = \frac{y^2}{x_2}$, $u = \frac{yz_1}{x_2}$, $v = \frac{z_1^2}{x_2}$. Then, forgetting the variables z_2, u and v, we get a projection map $\pi: X \dashrightarrow \mathbb{P}(1^2, 2, 3)$ which is birational onto its image, a surface Y of degree 10 whose equation is general in the ideal

$$\left(y^5, x_2 y^3, x_2 y^2 z_1, x_2 y z_1^2, x_2 z_1^3, x_2^2 y, x_2^2 z_1, x_2^3\right). \tag{2}$$

These surfaces Y belong to a family depending on 47 free parameters, so $Y = \pi(X)$ varies in a 46-dimensional family. We have to subtract from this dimension the dimension of the subgroup of $\operatorname{Aut} \mathbb{P}(1^2, 2, 3)$ preserving the ideal (2).

Note that $Y = \pi(X)$ has a point of multiplicity $\geqslant 3$ at the point p of coordinates $[x_1, x_2, y, z_1] = [1, 0, 0, 0]$.

The subgroup of automorphisms of $\operatorname{Aut} \mathbb{P}(1^2, 2, 3)$ preserving the ideal (2) is exactly, as one can verify, the isotropy group of p, a group of dimension 11. Finally we obtain $46 - 11 = 35$. $\quad\square$

5.2. The MV format

Consider a 5×5 skew-symmetric matrix $\tilde{M}$ and a vector $\tilde{V}$ as follows

$$\tilde{M} = \begin{pmatrix} m_{12} & m_{13} & m_{14} & m_{15} \\ & m_{23} & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & m_{45} \\ & & & \end{pmatrix}, \qquad \tilde{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix},$$

and let $\mathcal{I}_\mathcal{V} \subset \tilde{\mathbb{A}}_\mathcal{V} := \mathbb{C}[m_{12}, \ldots, m_{45}, v_1, \ldots, v_5]$ be the ideal generated by the 4×4 Pfaffians of $\tilde{M}$, and by the entries of $\tilde{M}\tilde{V}$.

This gives 10 polynomials, which form a minimal system of generators of $\mathcal{I}_\mathcal{V}$:

$$g_1: \ -m_{23}m_{45} + m_{24}m_{35} - m_{34}m_{25},$$
$$g_2: \ m_{13}m_{45} - m_{14}m_{35} + m_{34}m_{15},$$
$$g_3: \ -m_{12}m_{45} + m_{14}m_{25} - m_{24}m_{15},$$
$$g_4: \ m_{12}m_{35} - m_{13}m_{25} + m_{23}m_{15},$$
$$g_5: \ -m_{12}m_{34} + m_{13}m_{24} - m_{23}m_{14},$$
$$g_6: \ m_{12}v_2 + m_{13}v_3 + m_{14}v_4 + m_{15}v_5,$$
$$g_7: \ m_{12}v_1 - m_{23}v_3 - m_{24}v_4 - m_{25}v_5,$$
$$g_8: \ m_{13}v_1 + m_{23}v_2 - m_{34}v_4 - m_{35}v_5,$$
$$g_9: \ m_{14}v_1 + m_{24}v_2 + m_{34}v_3 - m_{45}v_5,$$
$$g_{10}: \ m_{15}v_1 + m_{25}v_2 + m_{35}v_3 + m_{45}v_4$$

yoked (cf. [17, pages 20 and 21]) by 16 independent first syzygies (i.e., relations):

$$m_{12}g_2 + m_{13}g_3 + m_{14}g_4 + m_{15}g_5 = 0,$$
$$-m_{12}g_1 + m_{23}g_3 + m_{24}g_4 + m_{25}g_5 = 0,$$
$$-m_{13}g_1 - m_{23}g_2 + m_{34}g_4 + m_{35}g_5 = 0,$$
$$-m_{14}g_1 - m_{24}g_2 - m_{34}g_3 + m_{45}g_5 = 0,$$
$$-m_{15}g_1 - m_{25}g_2 - m_{35}g_3 - m_{45}g_4 = 0,$$
$$v_5g_4 - v_4g_5 - m_{23}g_6 + m_{13}g_7 - m_{12}g_8 = 0,$$
$$-v_5g_3 + v_3g_5 - m_{24}g_6 + m_{14}g_7 - m_{12}g_9 = 0,$$
$$-v_5g_2 + v_2g_5 + m_{34}g_6 - m_{14}g_8 + m_{13}g_9 = 0,$$
$$-v_5g_1 + v_1g_5 - m_{34}g_7 + m_{24}g_8 - m_{23}g_9 = 0,$$
$$v_4g_3 - v_3g_4 - m_{25}g_6 + m_{15}g_7 - m_{12}g_{10} = 0,$$

$$-v_3 g_2 + v_2 g_3 + m_{45} g_6 - m_{15} g_9 + m_{14} g_{10} = 0,$$
$$v_4 g_1 - v_1 g_4 - m_{35} g_7 + m_{25} g_8 - m_{23} g_{10} = 0,$$
$$-v_3 g_1 + v_1 g_3 - m_{45} g_7 + m_{25} g_9 - m_{24} g_{10} = 0,$$
$$v_2 g_1 + -v_1 g_2 - m_{45} g_8 + m_{35} g_9 - m_{34} g_{10} = 0,$$
$$v_4 g_2 - v_2 g_4 + m_{35} g_6 - m_{15} g_8 + m_{13} g_{10} = 0,$$
$$v_1 g_6 + v_2 g_7 + v_3 g_8 + v_4 g_9 + v_5 g_{10} = 0.$$

Remark 5.9. In the previous cases we had a codimension 4 Gorenstein subscheme of a weighted projective space defined by an ideal with 9 generators: the ideals $I \subset \mathbb{A}$ and $\mathcal{I}_{\mathcal{E}} \subset \mathbb{A}_{\mathcal{E}}$.

Here we need 10 generators. Moreover, the locus has codimension 5: indeed, the 5 Pfaffians describe a codimension 3 Gorenstein subscheme, and at the general point of it $\tilde{M}$ has rank 2, so the latter 5 polynomials give locally just two conditions.

The important point for us is that the number of first syzygies is 16, as in the previous cases.

Definition 5.10. Let $\mathbb{A}$ be any weighted polynomial ring, consider a ring homomorphism $\varphi \colon \tilde{\mathbb{A}}_{\mathcal{V}} \to \mathbb{A}$, and set $M := \varphi(\tilde{M})$, $V = \varphi(\tilde{V})$. Let I be the ideal generated by $\varphi(\mathcal{I}_{\mathcal{V}})$; I is generated by the 4×4 Pfaffians of M and by $MV = 0$. In this situation we will say that (M, V) is an MV format for the quotient ring $\mathbb{A}/I$.

Example 5.11. We write an MV format for our ring R.

Again we choose the graded ring $\mathbb{A} = \mathbb{C}[x_2, y, z_1, z_2, u, v]$. By Proposition 4.4 we can assume $A = v$. Then the pair of matrices (M, V), where M is antisymmetric and V is a vector, and with

$$M = \begin{pmatrix} v & u & z_2 & D \\ & z_1 & y & B \\ & & 0 & v \\ & & & u \end{pmatrix}, \qquad V = \begin{pmatrix} x_2 \\ -y \\ z_1 \\ 0 \\ 0 \end{pmatrix}$$

is an MV format for R.

Indeed if we compute the image of the 10 generators of $\mathcal{I}_{\mathcal{V}}$ we get exactly (up to a sign) the polynomials f_i: $\varphi(g_1) = f_5, \varphi(g_2) = -f_6, \varphi(g_3) = -f_8, \varphi(g_4) = f_9, \varphi(g_5) = f_3, \varphi(g_6) = -f_5, \varphi(g_7) = -f_4, \varphi(g_8) = -f_2, \varphi(g_9) = -f_1, \varphi(g_{10}) = -f_7$. The polynomial which we obtain twice is f_5, which is produced twice by M_1, the first row of M: $-f_5$ equals both $M_1 V$ $(= \varphi(g_6))$ and the Pfaffian of M which ignores it $(-\varphi(g_1))$.

Lemma 5.12. *The map $\varphi \colon \tilde{\mathbb{A}}_{\mathcal{V}} \to \mathbb{A}$ given by Example 5.11 maps the 16 relations among the 10 generators of $\mathcal{I}_{\mathcal{V}}$ onto a generating system of the relations between $f_1, \ldots, f_9$.*

Proof. This is a straightforward computation, comparing the images of these relations with the relations in Corollary 5.6. $\square$

In this case a general lift of φ will not produce a flat family, because the ideal of the general fibre will need 10 generators. Still, a useful weaker statement holds. It says that we obtain a flat family if we keep the equation saying that the Pfaffian g_1 obtained by deleting the first row and column of M equals the opposite of the product $g_6 := M_1 V$.

Corollary 5.13. *Consider the map $\varphi \colon \tilde{\mathbb{A}}_{\mathcal{V}} \to \mathbb{A}$ in Example 5.11, a surjective ring homomorphism $\pi \colon \mathbb{B} \to \mathbb{A}$, and a ring homomorphism $\psi \colon \mathbb{A}_{\mathcal{V}} \to \mathbb{B}$ such that $\varphi = \pi \circ \psi$ (i.e. "ψ lifts φ"). Assume moreover $\psi(g_6) = -\psi(g_1)$, i.e. that $\psi(\tilde{M}_1 \tilde{V})$ equals the image by ψ of the Pfaffian obtained by deleting the first row and column of $\tilde{M}$.*

Then $\{\psi(g_i)\}$ has cardinality 9: denote its elements by $\pm F_1, \ldots, \pm F_9$ so that $\pi(F_i) = f_i$ (so, e.g., $F_6 = -\varphi(g_2)$).
Then every relation among the f_i lifts to a relation among the F_i.

Proof. ϕ maps the 16 generating relations among the g_i to a set of generating relations among the f_i; ψ maps the same relations to relations among their lifts, the F_i. Since a generating system of relations lift, every relation does. $\square$

This format produces naturally a family of surfaces.

Proposition 5.14. *Consider, in the graded polynomial ring* $\mathbb{C}[x_1, x_2, y, z_1, z_2, u, v]$ *with weights* $(1, 1, 2, 3, 3, 4, 5)$, *a number* $c_0 \in \mathbb{C}$, *three general homogeneous polynomials* $\mathcal{D}$, $\mathcal{B}$, *and* l *of respective degrees* 7, 6 *and* 1 *of the form*

$$l = c_1 x_1 + c_2 x_2,$$
$$\mathcal{B} = v\mathcal{B}_v + u\mathcal{B}_u + z_2 \mathcal{B}_{z_2} + z_1 \mathcal{B}_{z_1} + y\mathcal{B}_y + \mathcal{B}_x,$$
$$\mathcal{D} = v\mathcal{D}_v + u\mathcal{D}_u + z_2 \mathcal{D}_{z_2} + z_1 \mathcal{D}_{z_1} + y\mathcal{D}_y + \mathcal{D}_x.$$

Then consider the pair of matrices $(\mathcal{M}, \mathcal{V})$ (*cf. Example* 5.11) *with*

$$\mathcal{M} = \begin{pmatrix} v & u & z_2 & \mathcal{D} \\ & z_1 & y & \mathcal{B} \\ & & l & v + l\mathcal{B}_y - c_0\mathcal{D}_y \\ & & & u - l\mathcal{B}_{z_1} + c_0\mathcal{D}_{z_1} \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} x_2 \\ -y + l\mathcal{B}_v - c_0\mathcal{D}_v \\ z_1 + l\mathcal{B}_u - c_0\mathcal{D}_u \\ l\mathcal{B}_{z_2} - c_0\mathcal{D}_{z_2} \\ c_0 \end{pmatrix}.$$

Assume moreover that $c_0\mathcal{D}_x = l\mathcal{B}_x$.

Let $X \subset \mathbb{P}(1^2, 2, 3^2, 4, 5)$ *be the zero locus of the ideal generated by the* 4×4 *Pfaffians of* $\mathcal{M}$ *and by the entries of* $\mathcal{M}\mathcal{V}$. *We get in this way a reducible family of surfaces with reducible base* $\mathcal{T}$. *The open subset* $\{c_0 \neq 0\} \subset \mathcal{T}$ *is irreducible, as well as its closure* $\mathcal{T}_1$.

Then, for a general choice of $\mathcal{D}$, $\mathcal{B}$, l *and* c_0 *in* $\mathcal{T}_1$, X *is a surface with at worse Du Val singularities (rational double points). If* X *has Du Val singularities, then* X *is the canonical model of a surface of general type and, if* S *is the minimal model of* X, *then* $K_S^2 = 8$, $p_g(S) = 4$, $q = 0$, $K_S = \mathcal{O}_S(2)$, *and* S *is an even surface.*

The case $c_0 \neq 0$ *gives exactly all the surfaces with base point free canonical system, described in Theorem* 3.4. $\mathcal{T}_1$ *gives a 35-dimensional irreducible locally closed set* $M_{\mathcal{F}}$, *in the moduli space of surfaces of general type, which contains the set of Oliverio surfaces. Moreover* $M_{\mathcal{F}} \cap M_{\mathcal{E}}$ *is irreducible of dimension* 34.

Remark 5.15. First of all, we may write the polynomials $\mathcal{B}_v, \ldots$ uniquely if we require that $\mathcal{B}_x$ is a polynomial only in the variables x_1, x_2, $\mathcal{B}_y$ is a polynomial only in the variables x_1, x_2, y, and so on, following the increasing weight order x_1, x_2, y, z_1, z_2, u, v.

The parameter space $\mathcal{T}$ is reducible, since in fact the equation

$$c_0\mathcal{D}_x = l\mathcal{B}_x \quad \Leftrightarrow \quad c_0\mathcal{D}_x = (c_1 x_1 + c_2 x_2)\mathcal{B}_x$$

is equivalent to requiring either that $c_0 \neq 0$ and then $\mathcal{D}_x = c_0^{-1}(c_1 x_1 + c_2 x_2)\mathcal{B}_x$, or that $c_0 = c_1 = c_2 = 0$, or $c_0 = \mathcal{B}_x = 0$.

The closure of the irreducible affine set $\mathcal{T} \cap \{c_0 \neq 0\}$ shall be denoted by $\mathcal{T}_1$, while $\mathcal{T}_2 := \{c_0 = c_1 = c_2 = 0\}$, $\mathcal{T}_3 := \{c_0 = \mathcal{B}_x = 0\}$.

On $\mathcal{T}_2$, $\mathcal{B}_x$ and $\mathcal{D}_x$ are arbitrary, hence $\mathcal{T}_2 \cap \mathcal{T}_1$ is the subset where $\mathcal{B}_x$ divides $\mathcal{D}_x$. Similarly, on $\mathcal{T}_3$, $l = (c_1 x_1 + c_2 x_2)$, $\mathcal{D}_x$ are arbitrary, hence $\mathcal{T}_3 \cap \mathcal{T}_1$ is the subset where $l = (c_1 x_1 + c_2 x_2)$ divides $\mathcal{D}_x$. The intersection of the three components $\mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathcal{T}_3$ is easily seen to be equal to $\{c_0 = c_1 = c_2 = \mathcal{B}_x = 0\}$.

In the above theorem we consider only the first irreducible component $\mathcal{T}_1$, the closure of $\{c_0 \neq 0\}$. By the forthcoming Theorem 6.6 all the other surfaces shall belong to $M_{\mathcal{F}} \cup M_{\mathcal{E}}$.

Proof. One can verify that the assumption $c_0\mathcal{D}_x = l\mathcal{B}_x$ boils down to the fact that the two equations produced by the first row coincide (i.e., the Pfaffian of the minor of $\mathcal{M}$ where one erases the first row and column equals the first entry of $\mathcal{M}\mathcal{V}$).

Then, by Corollary 5.13 we have a flat family with base $\mathcal{T}_1$, giving an irreducible locally closed set of the moduli space, which we denote by $M_{\mathcal{F}}$.

We consider the subset $M_{\mathcal{E}\mathcal{F}}$ of the moduli space of surfaces of general type given by the image of $\{c_0 = l = 0\} \cap \mathcal{T}_1$.

We have that $M_{\mathcal{E}\mathcal{F}} \subset M_{\mathcal{E}}$: to write a surface in $M_{\mathcal{E}\mathcal{F}}$ in the format of Proposition 5.8 it suffices to take $\mathcal{A} = v$.

We deduce then the existence of a surface X with at most rational double points as singularities in $M_{\mathcal{E}\mathcal{F}}$ as in the proof of Proposition 5.8. By flatness and Proposition 5.8, its minimal resolution has $K_S^2 = 8$, $p_g(S) = 4$, $q(S) = 0$ and K_S is the pull back of $K_X = \mathcal{O}_X(2)$.

We compute the dimension of $M_{\mathcal{E}\mathcal{F}}$, forgetting the variables z_2, u and v and taking the associated projection as in the proof of Proposition 5.8, and then using $z_2 = \frac{y^2}{x_2}$, $u = \frac{yz_1}{x_2}$, $v = \frac{z_1^2}{x_2}$. The image of S is a surface Σ of degree 10 general in the ideal

$$\left(y^5, x_2y^3, x_2y^2z_1, x_2yz_1^2, x_2z_1^3, x_2^2y^2, x_2yz_1, x_2z_1^2, x_2^2x_1^6y, x_2^3\right).$$

Comparing with the ideal (2), we have only 46 parameters: the monomial we are missing is $x_2^2x_1^5z_1$. Arguing as in the proof of Proposition 5.8, $\dim M_{\mathcal{E}\mathcal{F}} = 45 - 11 = 34 = \dim M_{\mathcal{E}} - 1$.

When $c_0 \neq 0$, the two equations of smaller degree eliminate u and v, embedding the surface as a complete inter-section of type $(6,6)$ in $\mathbb{P}(1^2, 2, 3^2)$, so, by Theorem 3.4, the canonical system is base point free.

Conversely, all the isomorphism classes of canonical models of such complete intersection surfaces are here. Indeed, choosing for simplicity $c_0 = c_1 = 1$, $c_2 = 0$, $\mathcal{D}_x = x_1\mathcal{B}_x$ (to ensure that we are in $\mathcal{T}_1$) we get

$$u = -\mathcal{D}_{z_1} + x_1(-\mathcal{D}_u + \mathcal{B}_{z_1}) - y\mathcal{D}_v + x_1^2\mathcal{B}_u + x_1y\mathcal{B}_v + x_1z_1 + x_2z_2 - y^2,$$
$$v = \mathcal{D}_y + x_1\mathcal{D}_{z_2} - x_1\mathcal{B}_y - x_1^2\mathcal{B}_{z_2} - z_1\mathcal{D}_v + x_1z_1\mathcal{B}_v + x_2u - yz$$

and X becomes the complete intersection in $\mathbb{P}(1^2, 2, 3^3)$ of the two sextics which are obtained eliminating u, v in the sextics

$$yu - z_1z_2 - x_1v,$$
$$\mathcal{B} - y\mathcal{D}_{z_2} + x_1y\mathcal{B}_{z_2} - z_1\mathcal{D}_u + x_1z_1\mathcal{B}_u - x_2v + z_1^2.$$

It follows that we get all pencils of sextics (with base locus a surface with at most rational double points of singular-ities) containing a sextic in the ideal generated by x_1, y and z_1z_2. On the other hand, if we cannot find such a sextic in the pencil even after a projective coordinate change, then, arguing as in the proof of Theorem 3.4, the canonical system has a base point, a contradiction.

Therefore we have shown that the surfaces X_t in the family $M_{\mathcal{F}}$ admit a smooth deformation X_0; X_0 is an even surface, because it is smooth and $\mathcal{O}_X(K_{X_0}) \cong \mathcal{O}_{X_0}(2)$. Hence all the minimal models S_t of our canonical models X_t, being diffeomorphic to X_0, are even surfaces.

By Theorem 3.4 the surfaces with base point free canonical system form a 35-dimensional irreducible open set of the moduli space, so $\overline{M_{\mathcal{F}}}$ is an irreducible component of the moduli space, containing the set of Oliverio surfaces.

Finally, we have already proved that $M_{\mathcal{E}} \cap M_{\mathcal{F}}$ contains the irreducible family $M_{\mathcal{E}\mathcal{F}}$ of dimension 34. On the other hand, let X be a surface in $M_{\mathcal{F}}$ which is also in $M_{\mathcal{E}}$. Then K_X has base points, so $c_0 = 0$. Moreover, the equation of degree 4 must be of the form $WZ - Y^2$ for some forms W, Y, Z of respective degree 1, 2 and 3: this forces $c_1 = 0$, so $l = c_2x_2$. A long but straightforward computation shows that we can make a coordinate change so that the generators are still produced by matrices $\mathcal{M}$ and $\mathcal{V}$ as in the statement, but with $l = 0$. So $X \in M_{\mathcal{E}\mathcal{F}}$. $\quad\square$

6. Deformations of the cone and the moduli space

Our next goal is to prove that $M_{\mathcal{E}}$ and $M_{\mathcal{F}}$ fill $M_{8,4,0}^{ev}$. By Theorem 3.4 and Proposition 5.14 it suffices to restrict our considerations to surfaces S in $M_{8,4,0}^{ev}$ such that $|K_S| = |2L|$ is not base point free. It will be convenient to consider only the canonical models X of such surfaces, observing that $K_X = 2L$ and $R(X, L)$ is an *extension ring* of degree one [38] of the ring $R = R(C, 2Q)$, where C is a general curve in the pencil $|L|$ (this simply means that $R \cong R(X, L)/(x_1)$, where the coordinate x_1 has degree 1, which is the algebraic counterpart of the geometric process of taking a hyperplane section).

First of all, if no nonsingularity condition is set forth, a trivial extension ring of $R \cong \mathbb{A}/(f_1, \ldots, f_9)$, where $\mathbb{A}$ is the polynomial ring $\mathbb{C}[x_2, y, z_1, z_2, u, v]$, is given by the cone

$$C_R := \mathrm{Proj}\big(\mathbb{B}/(f_1, \ldots, f_9)\big),$$

where $\mathbb{B}$ is the polynomial ring $\mathbb{C}[x_1, x_2, y, z_1, z_2, u, v]$ and $\deg(x_1, x_2, y, z_1, z_2, u, v) = (1, 1, 2, 3, 3, 4, 5)$.

Every extension ring of R can be viewed as a deformation of C_R, since in both situations the issue is to lift the same generators $f_1, \ldots, f_9$ of the graded ideal and their first syzygies $\sigma_1, \ldots, \sigma_{16}$.

Pay attention that the ideal J of C_R is different from the one of R, since J is generated by $f_1, \ldots, f_9$ in the bigger polynomial ring $\mathbb{B}$.

An explicit calculation of the infinitesimal deformations of C_R occupies the main part of this section.

6.1. First order deformations of C_R

As usual, we begin by calculating the space T^1 of first order deformations: since we know that the Kuranishi family is parametrized by a complex analytic subspace of T^1.

A first order deformation of C_R (see for instance [39], Section 1) is an element of $Hom(J, \mathbb{B}/J)$ and can be therefore written in the following form:

$$F_i^{(1)} = f_i + \epsilon \cdot \sum_{k \geq 0} x_1^k f'_{i,k}, \quad 1 \leq i \leq 9, \tag{3}$$

where $f'_{i,k} \in \mathbb{A} = \mathbb{C}[x_2, y, z_1, z_2, u, v]$ which is viewed as a subring of the polynomial ring $\mathbb{B}$; so the $F_i^{(1)}$'s are elements in $\mathbb{B}[\epsilon]$.

A standard observation is that each $f'_{i,k}$ can be viewed as an element of R which is a quotient of $\mathbb{A}$. In fact, supposing that $F_i^{(1)}$ and $G_i^{(1)}$ define two first order deformations of C_R and that $F_i^{(1)} - G_i^{(1)}$ is in $\epsilon \cdot J$ for $1 \leq i \leq 9$: then they actually define the same first order deformation, since $F_i^{(1)}$'s and the $G_i^{(1)}$'s generate the same ideal of $\mathbb{B}[\epsilon]$.

For $1 \leq j \leq 16$, suppose the relations (first syzygies) σ_j are $\sum_{1 \leq i \leq 9} l_{ij} f_i = 0$ (see Corollary 5.6). Then the relations between the $F_i^{(1)}$ should be of the form

$$\sum_{1 \leq i \leq 9} (l_{ij} + \epsilon \cdot m_{ij}) F_i^{(1)} = 0,$$

where m_{ij} is an element of $\mathbb{B}$. The possibility of lifting the relations is equivalent to the condition that we get a homomorphism of J into $\mathbb{B}/J$ and yields the exact restrictions on the $f'_{i,k}$ in (3).

Lemma 6.1. *For the first order deformations, it suffices to lift the following five of the sixteen relations (first order syzygies):* $\sigma_1, \sigma_3, \sigma_5, \sigma_9, \sigma_{10}$.

Proof. Indeed, we have the following equivalences (mod J) between the first order syzygies

$$\begin{aligned}
z_1\sigma_2 &\equiv u\sigma_1, & y\sigma_{11} &\equiv v\sigma_2 + z_2\sigma_9, \\
y\sigma_4 &\equiv -z_1\sigma_2 + z_2\sigma_3, & x_2\sigma_{12} &\equiv -(z_2^2\sigma_1 + v\sigma_3 + z_1\sigma_9 + y\sigma_{10}), \\
z_1\sigma_6 &\equiv v\sigma_3, & z_1\sigma_{13} &\equiv v\sigma_6 + u\sigma_{10}, \\
x_2\sigma_8 &\equiv v\sigma_1 + z_1\sigma_5, & z_1\sigma_{14} &\equiv v\sigma_8 + u\sigma_{12}, \\
v\sigma_7 &\equiv u\sigma_8, & z_1\sigma_{15} &\equiv z_2^2\sigma_6 + v\sigma_{10}, \\
& & z_1\sigma_{16} &\equiv z_2^2\sigma_8 + v\sigma_{12}.
\end{aligned}$$

Since the variables $x_2, \dots, v$ are not zero-divisors in C_R, the syzygies $\sigma_1, \sigma_3, \sigma_5$ imply $\sigma_2, \sigma_4, \sigma_6, \sigma_7, \sigma_8$ mod J by the first column, and $\sigma_1, \dots, \sigma_{10}$ imply the remaining ones $\sigma_{11}, \dots, \sigma_{16}$ by the second column. $\square$

We can calculate the $f'_{i,k}$ separately, since they correspond to different degrees in x_1. Denote by V_k the space of first order deformations having degree k in x_1, and by V'_k the subspace of V_k induced by variations of the entries of the matrix N (while preserving the extrasymmetric format) in Example 5.3. Due to the previous observations, we see that the calculations of first order deformations essentially take place in the quotient ring $\mathbb{B}/J$.

Proposition 6.2.

(i) $V_k = V'_k$ for $k \geq 2$, that is, every first order deformation of C_R with degree 2 in x_1 is obtained from the extra-symmetric format.

(ii) $\dim V_1/V'_1 = \dim V_0/V'_0 = 1$, that is, in degrees 1 (resp. 0) in x_1, the first order deformations that are induced by the matrix format build a subspace of codimension 1.

Proof. For every $k \geqslant 0$, let $F_i^{(1)} = f_i + \epsilon x_1^k f_{i,k}'$, $1 \leqslant i \leqslant 9$ be a first order deformation of C_R, with degree k in x_1. We will compare V_k' and V_k for each $k \geqslant 0$.

Note that $\deg(f_1, \ldots, f_9) = (4, 5, 6, 6, 7, 8, 8, 9, 10)$ and $\deg f_{i,k}' = \deg f_i - k$. If $k > 10$, then $f_{i,k}' = 0$ by degree reason. If $k = 10$, then

$$f_{1,10}' = \cdots = f_{8,10}' = 0.$$

By the relation σ_{10}, we have

$$-z_2^2 f_{2,10}' + v f_{4,10}' + z_1 f_{7,10}' - x_2 f_{9,10}' = 0$$

and it follows that $f_{9,10}'$ is also 0. A similar argument shows that, if $k = 8$ or 9, then $f_{i,k}' = 0$ for $1 \leqslant i \leqslant 9$. Therefore $V_k = 0$ for $k \geqslant 8$ and also $V_k' = 0$ a fortiori.

Next we show that $V_k / V_k' = 0$ for every $2 \leqslant k \leqslant 7$. Since the calculations are similar, we treat only the case when $k = 7$ and leave the rest of the verifications to the reader.

Case $k = 7$: for degree reasons, one has $f_{1,7}' = \cdots = f_{4,7}' = 0$. The syzygy σ_3 implies that

$$z_1 f_{2,7}' - y f_{4,7}' + x_2 f_{5,7}' = 0$$

and it follows that $f_{5,7}' = 0$. In turn we have $f_{6,7}' = 0$ by σ_5.

Then σ_9 and σ_{10} yield

$$y f_{7,7}' = x_2 f_{8,7}', \qquad z_1 f_{7,7}' = x_2 f_{9,7}',$$

which implies that $(f_{7,7}', f_{8,7}', f_{9,7}') = (cx_2, cy, cz_1)$ with $c \in \mathbb{C}$, but this infinitesimal deformation is induced by varying one entry of the 6×6 antisymmetric matrix $N \colon D \mapsto D + \epsilon c x_1^7$.

For (ii) (resp. (iii)), we will show that, up to a scalar, there is exactly one first order deformation with degree 1 (resp. 0) in x_1 that cannot be induced by varying the entries of the matrix N.

Let us treat now the case $k = 1$: as before, we have that the degrees

$$\deg(f_{1,1}', \ldots, f_{9,1}') = (3, 4, 5, 5, 6, 7, 7, 8, 9).$$

Using the infinitesimal matrix entry changes of the form $z_2 \mapsto z_2 + \epsilon x_1 \cdot (\cdots)$ and $x_2 \mapsto x_2 + \epsilon a x_1$, we can assume $f_{1,1}' = c_1 z_1$. A similar change of the entry u allows us to assume that $f_{2,1}' = c_2 u$. Then the relation σ_1 gives

$$-z_1 f_{1,1}' + y f_{2,1}' - x_2 f_{3,1}' = 0$$

and we see that $f_{2,1}' = 0$, $f_{3,1}' = -c_1 v$. Now making an appropriate change at v we can assume $f_{4,1}' = c_{4,1} y z_2 + c_{4,2} v$. We have the following equations by σ_3, σ_5:

$$-y f_{4,1}' + x_2 f_{5,1}' = 0, \qquad v f_{1,1}' + y f_{5,1}' - x_2 f_{6,1}' = 0$$

and it is not hard to see that

$$f_{4,1}' = -c_1 y z_2, \qquad f_{5,1}' = -c_1 z_2^2, \qquad f_{6,1}' = -c_1 D.$$

Next, up to appropriate first order changes of v, z_2^2, D in the top left corner of N by multiples of x_1, we can assume $f_{7,1}' = c_7 z_2 u$. The relations σ_9, σ_{10} imply that

$$z_2^2 f_{1,1}' - y f_{7,1}' + x_2 f_{8,1}' = 0, \qquad v f_{4,1}' + z_1 f_{7,1}' - x_2 f_{9,1}' = 0$$

and we find that

$$c_7 = c_1, \qquad f_{8,1}' = f_{9,1}' = 0.$$

Summing up, we have the following first order deformation with degree 1 in x_1:

$$f_{1,1}' = c_1 z_1, \qquad f_{3,1}' = -c_1 v, \qquad f_{4,1}' = -c_1 y z_2,$$
$$f_{5,1}' = -c_1 z_2^2, \qquad f_{6,1}' = -c_1 D, \qquad f_{7,1}' = c_1 z_2 u, \qquad f_{2,1}' = f_{8,1}' = f_{9,1}' = 0,$$

which is evidently not induced by the entry changes of N.

Case $k = 0$: since the calculation is similar to the case $k = 1$, we leave it to the reader. The codimension of the space of first order deformations in degree 0 that come by entry changes of the matrix N is one and a basis of V_0/V_0' is represented by

$$
\begin{aligned}
&f_{1,0}' = -c_0 u, && f_{4,0}' = c_0 z_2 (d_{0,2} z_1 - z_2), && f_{7,0}' = -c_0 (D_y z_1 + d_{0,2} z_2 v), \\
&f_{2,0}' = -c_0 v, && f_{5,0}' = -c_0 (\delta x_2^7 + D_y y + D_{z_1} z_1), && f_{8,0}' = c_0 (\delta x_2^6 z_1 + D_{z_1} v), \\
&f_{3,0}' = 0, && f_{6,0}' = -c_0 (\delta x_2^6 y + D_y z_2 + D_{z_1} u), && f_{9,0}' = -c_0 D_y v,
\end{aligned}
$$

where we have decomposed D as $D = \delta x_2^7 + D_y y + D_{z_1} z_1 + d_{0,2} z_2 u$. $\quad\square$

Recall that the subspace of elements of nonpositive grading in the space of first order deformations $Hom(J, \mathbb{B}/J)$ yields the tangent space to the Hilbert scheme at the point corresponding to the cone C_R.

However, to calculate the tangent space to the Kuranishi family, we must consider isomorphism classes of first order deformations, i.e., we must divide by the subspace generated by the action of the Lie algebra of vector fields on the weighted projective space, the tangent space to the group of projective automorphisms.

We divide therefore by these infinitesimal coordinate changes and, using Proposition 6.2, we can assume that any first order deformation of the cone C_R is equivalent to one of the form

$$
F_i^{(1)} = f_i + \epsilon \left(f_{i,0}' + x_1 f_{i,1}' + \sum_{k \geq 0} x_1^k f_{i,k}'' \right), \quad 1 \leq i \leq 9, \tag{4}
$$

where $f_{i,0}'$ (resp. $f_{i,1}'$) is the deformation with degree 0 (resp. 1) in x_1 in the proof of Proposition 6.2 and the $f_i + \sum_{k \geq 0} x_1^k f_{i,k}''$ ($1 \leq i \leq 9$) are the 4×4 Pfaffians of the following antisymmetric matrix:

$$
\mathcal{N}^{(1)} = \begin{pmatrix}
\mathcal{A}^{(1)} & \mathcal{B}^{(1)} & z_1 & y & x_2 \\
 & \mathcal{D}^{(1)} & u & z_2 & y \\
 & & v & u & z_1 \\
 & & & 0 & 0 & z_1 \\
 & & & & 0
\end{pmatrix}, \tag{5}
$$

with

$$
\begin{aligned}
\mathcal{A}^{(1)} &= v + \epsilon \mathcal{A}' = v + \epsilon a_5 x_1^5, \\
\mathcal{B}^{(1)} &= z_2^2 + \epsilon \mathcal{B}' = z_2^2 + \epsilon (b_1 x_1 v + b_2 x_1^2 u + b_3 x_1^3 z_2 + b_6 x_1^6), \\
\mathcal{D}^{(1)} &= D + \epsilon \mathcal{D}' = \delta x_2^7 + D_y y + D_{z_1} z_1 + d_{0,2} z_2 u + \epsilon (\delta' x_2^7 + D_y' y + D_{z_1}' z_1 + d_{0,2}' z_2 u) \\
&\quad + \epsilon (d_{1,1} x_1 y u + d_{1,2} x_1 z_2^2 + d_{2,1} x_1^2 y z_2 + d_{2,2} x_1^2 v + d_3 x_1^3 u \\
&\quad + d_{4,1} x_1^4 z_1 + d_{4,2} x_1^4 z_2 + d_5 x_1^5 y + d_7 x_1^7),
\end{aligned}
$$

so that $\mathcal{A}^{(1)} - v = \epsilon \mathcal{A}'$, $\mathcal{B}^{(1)} - z_2^2 = \epsilon \mathcal{B}'$, $\mathcal{D}^{(1)} - D = \epsilon \mathcal{D}'$ are first order infinitesimals. Here we have thrown as many terms as possible from $\mathcal{A}^{(1)}$ and $\mathcal{B}^{(1)}$ to $\mathcal{D}^{(1)}$ using the equations $f_1 = \cdots = f_9 = 0$.

Remark 6.3. If we use neither the coordinate changes nor the entry changes of N in the proof and keep track of the free parameters, then we obtain all the dimensions:

- $\dim_{\mathbb{C}} V_k = 0$, for $k \geq 8$.
- $\dim_{\mathbb{C}} \{V_7, \ldots, V_0\} = \{1, 2, 4, 6, 11, 16, 23, 30\}$.

Since $V_k = Hom_R(J/J^2, R)_{-k}$ [38, Theorem 1.10], these dimensions can be calculated in Macaulay 2 for an explicitly assigned D as follows:

```
--We count here the dimensions of first order deformations
--of the graded ring R=C[x_2,y,z_1,z_2,u,v]/(f_1,...,f_9) with D=0.

AA=QQ[x_2,y,z_1,z_2,u,v,Degrees=>{1,2,3,3,4,5}];
f_1=x_2*z_2-y^2;
f_2=x_2*u-y*z_1;
f_3=y*u-z_1*z_2;
f_4=x_2*v-z_1^2;
f_5=y*v-z_1*u;
f_6=z_2*v-u^2;
f_7=z_1*v-y*z_2^2;
f_8=u*v-z_2^3;
f_9=v^2-z_2^2*u;
I=ideal(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9);
--the ideal of R;
I'=I/I^2;R=AA/I;
H=Hom(I',R);h=hilbertSeries (H,Order => 1)
--h is the Hilbert series of Hom_R(I/I^2,R) up to degree 0;
--the coefficients of h are exactly the dimensions of V_k, 0<=k<=7.
```

6.2. Lifting to higher orders

We shall now try to lift the first order deformations obtained in Section 6.1 to higher order. We shall do this in several steps, first of all we shall calculate the tangent cone to the base $\mathcal{B}$ of the Kuranishi family (equivalently, to the Hilbert scheme).

We shall in this way obtain some quadratic equations which a posteriori will be shown to yield the equations defining $\mathcal{B}$; but since we do not want to use computer assisted calculations, we shall proceed in steps, first showing that these equations define the tangent cone, then that these equations define $\mathcal{B}$ after a possible coordinate change, and only in the proof of the main theorem we shall see that these equations define $\mathcal{B}$ in the initially chosen coordinates.

Proposition 6.4. *If a first order deformation of C_R as defined in* (4) *lifts to a genuine deformation, then*

$$c_0 a_5 = c_1 a_5 = c_0 d_7 - c_1 b_6 \equiv 0 \quad (\mathrm{mod}\ \mathfrak{M}^3),$$

$\mathfrak{M}$ *being the maximal ideal of the origin in the vector space of first order deformations.*

Proof. Starting with $F_i^{(1)}$ ($1 \leqslant i \leqslant 9$), we can write a one parameter family of deformations of order n as

$$F_i^{(n)} = f_i + t f_i^{(1)} + \cdots + t^n f_i^{(n)}, \quad 1 \leqslant i \leqslant 9,$$

where $f_i^{(1)} = f_{i,0}' + x_1 f_{i,1}' + \sum_{k \geqslant 0} x_1^k f_{i,k}''$ is the part defining $F_i^{(1)}$ in (4) and t is an infinitesimal parameter of order n (i.e. $t^{n+1} = 0$). Therefore the $F_i^{(n)}$ are elements in $\mathbb{B}[t]/(t^{n+1})$. For $m \leqslant n$, there is a natural surjection of rings

$$\mathbb{B}[t]/(t^{n+1}) \to \mathbb{B}[t]/(t^{m+1}),$$

and we denote by $F_i^{(m)}$ the image of $F_i^{(n)}$ in $\mathbb{B}[t]/(t^{m+1})$. The first syzygies $\sigma_1, \ldots, \sigma_{16}$ between the f_i should lift to those between the $F_i^{(m)}$ for any $m \leqslant n$, so that the $F_i^{(m)}$ define a deformation of C_R of order m.

The relation σ_1 between f_i lifts to the second order as

$$-z_1 F_1^{(2)} + y F_2^{(2)} - x_2 F_3^{(2)} = t \big(c_1 x_1 F_4^{(2)} - c_0 F_5^{(2)} \big) - t^2 c_0^2 (D - d_{0,2} z_2 u) + t^2 c_1^2 x_1^2 y z_2$$
$$- t^2 c_0 c_1 d_{0,2} x_1 z_1 z_2 + t^2 \big(-z_1 f_1^{(2)} + y f_2^{(2)} + x_2 f_3^{(2)} \big)$$

and from this we see that $f_2^{(2)}$ does not contain x_1^5.

The first syzygy σ_3 lifts as

$$
\begin{aligned}
z_1 F_2^{(2)} - y F_4^{(2)} + x_2 F_5^{(2)} ={}& t\big(c_0\big(d_{0,2} z_2 F_2^{(2)} - F_7^{(2)}\big) - c_1 x_1 z_2 F_1^{(2)}\big) \\
& + t^2\big(c_1^2 x_1^2 z_1 z_2 - c_0^2 D_y z_1 + c_0\big(z_1 \mathcal{A}' - y\mathcal{B}' + x_2 \mathcal{D}'\big)\big) \\
& + t^2\big(z_1 f_2^{(2)} - y f_4^{(2)} + x_2 f_5^{(2)}\big).
\end{aligned}
$$

Note that $t^2 c_0 z_1 \mathcal{A}'$ contains $t^2 c_0 a_5 x_1^5 z_1$ and $t^2 c_0 x_2 \mathcal{D}'$ contains $t^2 c_0 d_7 x_1^7 x_2$ (cf. (5)). Since $t^2 c_0 a_5 x_1^5 z_1$ could only be cancelled by $t^2 z_1 f_2^{(2)}$, but unfortunately $f_2^{(2)}$ does not contain x_1^5, $c_0 a_5$ must be 0. Besides $t^2 c_0 d_7 x_1^7 x_2$ can only be absorbed into $t^2 x_2 f_5^{(2)}$, so the coefficient of x_1^7 in $f_5^{(2)}$ is $-c_0 d_7$.

The relation σ_5 gives

$$
\begin{aligned}
v F_1^{(2)} - u F_2^{(2)} + y F_5^{(2)} - x_2 F_6^{(2)} ={}& t\big(c_1 x_1 F_7^{(2)} + c_0\big(D_y F_1^{(2)} + D_{z_1} F_2^{(2)}\big)\big) \\
& + t^2\big(c_0^2(u D_y + v D_{z_1}) + c_0 c_1 d_{0,2} x_1 z_2 v + c_1^2 x_1^2 z_2 u\big) \\
& - t^2 c_1 x_1\big(z_1 \mathcal{A}' - y\mathcal{B}' + x_2 \mathcal{D}'\big) \\
& + t^2\big(v f_1^{(2)} - u f_2^{(2)} + y f_5^{(2)} - x_2 f_6^{(2)}\big).
\end{aligned}
$$

The term $t^2 c_1 a_5 x_1^6 z_1$ in $t^2 c_1 x_1 z_1 \mathcal{A}'$ cannot be absorbed in any of $t^2 f_i^{(2)}$, $i = 1, 2, 5, 6$, hence $c_1 a_5 = 0$. On the other hand the term $t^2 c_1 b_6 x_1^7 y$ in $t^2 c_1 x_1 y \mathcal{B}'$ can only be absorbed into $t^2 y f_5^{(2)}$, so the coefficient of x_1^7 in $f_5^{(2)}$ is $-c_1 b_6$. Comparing with the coefficient of x_1^7 in $f_5^{(2)}$ determined by σ_3 above, we obtain $c_0 d_7 = c_1 b_6 = 0$. Summing up, we have the following restrictions on the coefficients:

$$
c_0 a_5 = 0, \qquad c_1 a_5 = 0, \qquad c_0 d_7 = c_1 b_6. \qquad \square
$$

For the reader's benefit, we observe that the above equations describe the algebraic set

$$
\{a_5 = c_0 d_7 - c_1 b_6 = 0\} \cup \{c_0 = c_1 = 0\},
$$

which is not a complete intersection since it has codimension two, while the space of quadrics containing it has dimension three.

6.3. The moduli space

Let's come back to our original problem about the moduli space of even surfaces with $K^2 = 8$, $p_g = 4$, $q = 0$.

In the next lemma we are essentially continuing with the previous calculations, except that we set for convenience

$$
a := a_5, \qquad d := d_7, \qquad b := b_6.
$$

Our purpose is to show that our previous equations, which were among the equations defining the tangent cone to the base of the Kuranishi family, are indeed the equations of the base of the Kuranishi family.

Lemma 6.5. *Set*

$$
f_0 := c_0 a, \qquad f_1 := c_1 a, \qquad g := c_0 d - c_1 b.
$$

Let $\mathcal{P}$ be the polynomial ring $\mathbb{C}[a, b, c_0, c_1, d, u_1, \ldots, u_m]$ and consider an ideal $J \subset \mathcal{P}$ such that J contains polynomials F_0, F_1, G such that

$$
F_0 \equiv f_0 \pmod{\mathfrak{M}^3}, \qquad F_1 \equiv f_1 \pmod{\mathfrak{M}^3}, \qquad G \equiv g \pmod{\mathfrak{M}^3},
$$

$\mathfrak{M}$ *being the maximal ideal of the origin.*

Let W the subscheme associated to J, and assume that W contains two distinct irreducible components of codimension 2. Then, up to an analytic change of coordinates, we may assume that

$$
(**) \quad J = (f_0, f_1, g), \qquad W = W_1 \cup W_2, \qquad W_1 = V(c_0, c_1), \qquad W_2 = V(a, g).
$$

In particular, W is schematically the union of two irreducible components of codimension 2 which are complete intersections.

Proof. We divide the proof in several steps.

Step 1. Observe that W is a subscheme of $V(I)$, where I is the ideal $I := (F_0, F_1, G)$.

W.L.O.G. we may replace J by I. In fact, once the final assertion $(**)$ is proven for I, W is a subscheme of $V(I) = W_1 \cup W_2$ containing two irreducible components of codimension 2, hence $W = V(I)$, and the proof is finished.

Step 2. Assume henceforth $J = I$. We observe first that W cannot have any component of codimension 1. Else there would be a function F dividing F_0, F_1, G: in particular the leading form f of F would divide f_0, f_1, g, therefore f would be a constant, and F would be a unit in the local ring of the origin.

Let $Z_i := V(F_i, G)$: the same argument shows that neither Z_0 nor Z_1 has a component of codimension 1, hence Z_0, Z_1 are complete intersections of codimension 2.

Step 3. Consider now the irreducible decomposition $W = \bigcup_i W_i$ of W into irreducible components. By what we have seen, each W_i has codimension either 2 or 3. We assume that W_1 and W_2 have codimension equal to 2.

Step 4. Consider now the tangent cone $\mathcal{C}$ of W at the origin. The irreducible decomposition $W = \bigcup_i W_i$ of W yields a decomposition $\mathcal{C} = \bigcup_i \mathcal{C}_i$. Observe moreover that $\mathcal{C}$ is a subscheme of

$$W' := V(f_0, f_1, g) = L \cup Q, \qquad L := V(c_0, c_1), \qquad Q := V(a, g).$$

We may assume W.L.O.G. that L is the tangent cone of W_1, and Q is the tangent cone of W_2

Step 5. Note that $W' = L \cup Q$ holds schematically, and we have a corresponding projective subscheme of codimension 2 and degree 3 of which $\mathcal{C}$ has a subscheme. Hence there are no other components W_i of codimension 2: these would contribute to a higher degree of the subscheme $\mathcal{C}$. Hence, if there are other components $W_3, \ldots$, they have codimension 3. We shall now show that these latter do not exist.

Step 6. Since the tangent cone L to W_1 is smooth, then W_1 is smooth at the origin, and, by a suitable local change of coordinates, we may assume that

$$W_1 = V(c_0, c_1).$$

Since moreover F_0, F_1, G are in the ideal (c_0, c_1), we obtain, after a suitable change of coordinates

$$F_0 = c_0 a + c_1 \beta_1, \qquad F_1 = c_1(a + \alpha) + c_0 \beta_0, \qquad G = c_0 d - c_1 b,$$

where α, β_0, β_1 have all order at least 2 at the origin. Moreover, we can assume (changing the coordinate a and adding possibly a multiple of G to F_0 and F_1) that

(6.1) the variables c_0 and b do not appear in β_1, and α,

(6.1) the variable c_1 does not appear in β_0.

Step 7. Consider now $W \setminus W_1$, and intersect with $c_0 = 0$: we obtain the algebraic set

$$c_0 = b = \beta_1 = (a + \alpha) = 0,$$

which must have codimension 3.

It follows that $(a + \alpha)$ divides β_1. We can therefore subtract a multiple of F_1 to F_0 and obtain that c_0 divides F_0.

Whence, we can finally assume that $F_0 = c_0 a$.

Step 8. Now, the components of $V(F_0, G)$ are $W_1 = L$, $Q = V(a, G)$ and $V(c_0, b)$.

But $W \cap V(c_0, b) = \{c_0 = b = c_1(a + \alpha) = 0\}$, which has codimension 3. Whereas $W \cap Q = V(a, G, c_1 \alpha + c_0 \beta_0)$; this component must have codimension 2, so it must be Q, and F_1 belongs to the ideal (a, G). We can subtract a multiple of G to F_1 hence we may obtain that a divides F_1, i.e., that a divides β_0 and α.

Step 9. We may now write

$$F_1 = a\big(c_1(1 + A) + c_0 B\big),$$

hence, subtracting a multiple of F_0 to F_1, we may assume that $B \equiv 0$.

Furthermore, multiplying by the unit $(1 + A)$ and its inverse the variables c_1 and b, we finally get new coordinates where

$$F_0 = c_0 a, \qquad F_1 = c_1 a, \qquad G = c_0 d - c_1 b.$$

This is exactly what we wanted to show. $\quad\square$

Theorem 6.6. *$M_{8,4,0}^{ev}$ is connected and its irredundant irreducible decomposition is $M_{8,4,0}^{ev} = M_{\mathcal{F}} \cup M_{\mathcal{E}}$.*

Proof. We already know that $M_{\mathcal{F}}$, $M_{\mathcal{E}}$ are irreducible unirational subsets of $M_{8,4,0}^{ev}$: indeed each of them is defined as the closure of a morphism from an open set of an affine space into the moduli space.

We also know that $M_{\mathcal{F}}$ is the closure of the open set $M_{\mathcal{F}}^0$ of $M_{8,4,0}^{ev}$ consisting of surfaces with base point free canonical system: hence $M_{\mathcal{F}}$ is clearly an irreducible component of $M_{8,4,0}^{ev}$, and it suffices to show that every surface not in $M_{\mathcal{F}}^0$ lies either in $M_{\mathcal{F}}$ or in $M_{\mathcal{E}}$, and that these two subsets do intersect. This will accomplish the proof.

Suppose then that S is an even surface with $K_S^2 = 8$, $p_g = 4$, $q = 0$ and that $|K_S|$ is not base point free. As before, let L be a half-canonical divisor, i.e., $K_S = 2L$. Let X be the canonical model of S.

Then X is a small deformation of the cone C_R over a smooth hyperplane section H of X, and the base $\mathfrak{B}$ of the Kuranishi family of C_R consists, by Propositions 5.8, 5.14, 6.4 and Lemma 6.5 of two irreducible components: we want to show that the open set $\mathfrak{B}'$ corresponding to the partial smoothings of C_R yielding surfaces with Du Val singularities remains connected, and that the points of one irreducible component of $\mathfrak{B}'$ correspond to canonical models in $M_{\mathcal{F}}$ and the points of the other to canonical models in $M_{\mathcal{E}}$, thus our statement shall follow.

We can translate everything back into the algebraic description of the extension rings of $R = R(C, 2Q)$ for a smooth $C \in |L|$ and $Q \in bs|L|$ (see Section 4).

The half-canonical ring $R(X, L)$ is an extension ring of $R = R(C, 2Q)$ hence it can be put in the form of (4), (we take now a small and general specialization $\epsilon \in \mathbb{C}$) and, by Proposition 6.4, $c_0 a_5 = c_1 a_5 = c_0 d_7 - c_1 b_6 = 0$, up to higher order terms. Let us note that, up to a coordinate change in R we can always assume that the coefficient of x_2^7 in D vanishes: just choose ζ and η with a common zero.

Now, if $c_0 = c_1 = 0$, the equations are in the extrasymmetric format of Proposition 5.8, so X is an element of $M_{\mathcal{E}}$. Else, $a_5 = 0$. Moreover, since we assumed the vanishing of the coefficient of x_2^7 in D, we can decompose $\mathcal{B}$ and $\mathcal{D}$ as in Proposition 5.14 with $\mathcal{B}_x = b_6 x_1^6$ and $\mathcal{D}_x = d_7 x_1^7$. Then $c_0 d_7 - c_1 b_6 = 0$ gives $c_1 x_1 \mathcal{B}_x = c_0 \mathcal{D}_x$.

It follows that $R(X, L)$ is as described in Proposition 5.14 with $l = c_1 x_1$, and therefore $X \in M_{\mathcal{F}}$.

We see then directly that the base $\mathfrak{B}$ of the Kuranishi family of C_R contains the subset $\mathfrak{B}'' := \{c_0 a_5 = c_1 a_5 = c_0 d_7 - c_1 b_6 = 0\}$, hence by Lemma 6.5 it equals $\mathfrak{B}$ and we have shown that X belongs to $M_{\mathcal{F}} \cup M_{\mathcal{E}}$.

Finally, the condition $M_{\mathcal{F}} \cap M_{\mathcal{E}} \neq 0$ was shown already in Proposition 5.14. $\square$

Acknowledgements

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