GENUS STABILIZATION FOR THE COMPONENTS OF MODULI SPACES OF CURVES WITH SYMMETRIES

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Abstract. In a previous paper, [CLP12], we introduced a new homological invariant ε for the faithful action of a finite group G on an algebraic curve.

We show here that the moduli space of curves admitting a faithful action of a finite group G with a fixed homological invariant ε, if the genus g' of the quotient curve satisfies $g' >> 0$, is irreducible (and non empty iff the class satisfies the 'admissibility' condition).

We achieve this by showing that the stable classes are in bijection with the set of admissible classes ε.

1. Introduction

The main purpose of this article is the determination of the irreducible components of the moduli spaces of curves admitting a given symmetry group G and to set up the stage for the investigation of their homological stabilization.

Let us first introduce the framework into which the stabilization theorem proven in this paper fits.

Let $g > 1$ be a positive integer, and let C be a projective curve of genus g. Consider a subgroup G of the group of automorphisms of C, and let $\gamma \in G \leq \mathrm{Aut}(C)$: then, since C is a $K(\Pi_g, 1)$ space, the homotopy class of $\gamma\colon C \to C$ is determined by

$$\pi_1(\gamma)\colon \pi_1(C, y_0) \to \pi_1(C, \gamma \cdot y_0)$$

and actually the topological type of the action is completely determined by its action on the fundamental group, up to inner automorphisms. Hence we get a group homomorphism

$$\rho\colon G \to \frac{\mathrm{Aut}(\Pi_g)}{\mathrm{Inn}(\Pi_g)} = \mathrm{Out}(\Pi_g)\,.$$

This homomorphism ρ is injective by Lefschetz' lemma ($\gamma \underset{h}{\sim} id_C \Rightarrow \gamma = id_C$) asserting that, for $g \geq 2$, only the identity transformation is homotopic to the identity. Moreover, since holomorphic maps are orientation preserving, we have actually

$$\rho\colon G \to \mathrm{Out}^+(\Pi_g) = \mathrm{Map}_g \cong \frac{\mathrm{Diff}^+(C)}{\mathrm{Diff}^0(C)},$$

hence indeed ρ determines the differentiable type of the action.

When we speak of the moduli space of curves admitting an action by a finite group G of a fixed topological type we fix $\rho(G)$ up to the action of $\mathrm{Aut}(G)$ on the source and the adjoint action of the mapping class group Map_g on the target. In order to be more precise, let us recall the description of the moduli spaces of curves via Teichmüller theory.

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Fix M to be the underlying oriented differentiable manifold of C, and let $\mathcal{CS}(M)$ the space of complex structures on M: then the moduli space of curves $\mathfrak{M}_g$, and Teichmüller space $\mathcal{T}_g$ are defined as:

$$\mathfrak{M}_g := \mathcal{CS}(M)/\mathrm{Diff}^+(M),\ \mathcal{T}_g := \mathcal{CS}(M)/\mathrm{Diff}^0(M),\ \Rightarrow \mathfrak{M}_g = \mathcal{T}_g/\mathrm{Map}_g.$$

Teichmüller's theorem says that $\mathcal{T}_g \subset \mathbb{C}^{3g-3}$ is an open subset homeomorphic to a ball. Moreover Map_g acts properly discontinuously on $\mathcal{T}_g$, but not freely.

As a corollary, the rational cohomology of the moduli space is calculated by group cohomology: $H^*(\mathfrak{M}_g, \mathbb{Q}) \cong H^*(\mathrm{Map}_g, \mathbb{Q})$.

Harer ([Harer85]) showed that these cohomology groups stabilize with g, while the ring structure of the cohomology of the "stable mapping class group" is described by a conjecture of Mumford, proven by Madsen and Weiss ([M-W07]).

Definition 1.1. *$\mathcal{T}_{g,\rho} := \mathcal{T}_g^{\rho(G)}$ is the fixed-point locus of $\rho(G)$ in $\mathcal{T}_g$.*

The first author ([Cat00]) proved an analogue of Teichmüller's theorem, that $\mathcal{T}_{g,\rho}$ is homeomorphic to a ball. Hence we get other subvarieties of the moduli space according to the following definition:

Definition 1.2. *$\mathfrak{M}_{g,\rho}$ is the (irreducible closed subvariety) image of $\mathcal{T}_{g,\rho}$ in $\mathfrak{M}_g$. That is, $\mathfrak{M}_{g,\rho}$ is the quotient of $\mathcal{T}_{g,\rho}$ by the normalizer of $\rho(G) \leq \mathrm{Map}_g$.*

Instead the marked moduli space $\mathfrak{M}\mathfrak{M}_{g,\rho}$ is the quotient of $\mathcal{T}_{g,\rho}$ by the centralizer of $\rho(G) \leq \mathrm{Map}_g$ (here the action of G is given, together with a marking of G, and we do not allow to change the given action of G up to an automorphism in $Aut(G)$.

The above description of the space of curves admitting a given topological action of a group G is quite nice, but not completely explicit. One can make everything more explicit via a more precise understanding through geometry: in this way one can find discrete invariants of the topological type of the action.

Consider the quotient curve $C' := C/G$, let $g' := g(C')$ be its genus , let $\mathcal{B} = \{y_1, \dots, y_d\}$ be the branch locus, and let m_i be the multiplicity of y_i.

$C \to C'$ is determined (by virtue of Riemann's existence theorem) by the monodromy: $\mu\colon \pi_1(C' \setminus \mathcal{B}, y_0) \to G$. We have:

$$\pi_1(C' \setminus \mathcal{B}, y_0) \cong \Pi_{g',d} := \langle \gamma_1, \dots, \gamma_d, \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'} | \prod_{i=1}^{d} \gamma_i \prod_{j=1}^{g'} [\alpha_j, \beta_j] = 1 \rangle.$$

Therefore the datum of μ is equivalent to the datum of a Hurwitz generating system, i.e. of a vector

$$v := (c_1, \dots, c_d, a_1, b_1, \dots, a_{g'}, b_{g'}) \in G^{d+2g'}$$

s.t.

i) G is generated by the entries $c_1, \dots, c_d, a_1, b_1, \dots, a_{g'}, b_{g'}$,
ii) $c_i \neq 1_G$, $\forall i$, and
iii)

$$\prod_{i=1}^{d} c_i \prod_{j=1}^{g'} [a_j, b_j] = 1.$$

Riemann's existence theorem shows that the irreducible components of the moduli space corresponding to curves with a given group G of automorphisms (i.e., G acts faithfully, or, equivalently, the monodromy μ is surjective!) are determined by the possible topological types of the branched covering $C \to C' = C/G$, which are in bijection with the points of the orbit space

$$\mathrm{Epi}(\Pi_{g',d}, G)/(\mathrm{Map}_{g',d} \times \mathrm{Aut}(G))$$

(here 'Epi' stands for 'Set of Epimorphisms').

Relation of the two approaches through the orbifold fundamental group π_1^{orb}. Let X be a "good" topological space and let $\tilde{X}$ be the universal cover of X. Assume that G is a finite group acting effectively on X. Then G can be lifted to a discontinuous group $\tilde{G}$ of homeomorphisms of $\tilde{X}$ in such a way that the quotients $\tilde{X}/\tilde{G}$ and X/G are homeomorphic. $\tilde{G}$ is called the **orbifold fundamental group** of the G-action and it is an extension of $\pi_1(X)$ by G, i.e., we have a short exact sequence

$$1 \to \pi_1(X) \to \tilde{G} =: \pi_1^{\mathrm{orb}}(X/G) \to G \to 1\,.$$

In our situation, $X = C$, and its universal cover is the upper half plane $\tilde{X} = \mathbb{H}$, hence $\pi_1^{\mathrm{orb}}(X/G)$ is a Fuchsian group.

We have:

$$C = \mathbb{H}/\Pi_g\,, \qquad C' = \mathbb{H}/\pi_1^{\mathrm{orb}}, \qquad \pi_1^{\mathrm{orb}} = \Pi_{g',d}/\langle\langle \gamma_1^{m_1}, \dots, \gamma_d^{m_d} \rangle\rangle\,.$$

where, as usual, $\langle\langle a_1, \dots, a_d \rangle\rangle$ denotes the subgroup normally generated by the elements $a_1, \dots, a_d$.

The above exact sequence yields, via conjugation acting on the normal subgroup $\pi_1(X) \cong \Pi_g$, a homomorphism $\rho\colon G \to Out(\Pi_g)$.

Fix now d points $y_1, \dots, y_d$ on the oriented differentiable manifold M' underlying C', and set $\mathcal{B} := \{y_1, \dots, y_d\}$. One can then define the d-marked Teichmüller space as the quotient

$$\mathcal{T}_{g',d} := \mathcal{C}S(M')/\mathrm{Diff}^0(M', \mathcal{B}).$$

We have thus a covering map of connected spaces $\mathcal{T}_{g',d} \to \mathcal{T}_{g,\rho}$; since $\mathcal{T}_{g,\rho}$ is a ball the above map is a homeomorphism (hence in particular also the topological type ρ determines the monodromy μ, as can be proven directly).

We can now describe the numerical and the homological invariants of ρ (equivalently, of μ or of v).

The first numerical invariant is called (by different authors) the **branching numerical function**/Nielsen class/ν-type.

Definition 1.3. *Let* $\mathrm{Conj}(G)$ *be the set of (nontrivial) conjugacy classes in the group* G*: then we let* $\nu\colon \mathrm{Conj}(G) \to \mathbb{N}$, $\nu(\mathcal{C}) := |\{i | c_i \in \mathcal{C}\}|$, *be the function which counts how many local monodromy elements (these are defined only up to conjugation) are in a fixed conjugacy class.*

This invariant was first introduced by Nielsen ([Niel37]) who proved that ν determines ρ if G is cyclic.

The (semi-)classical homological invariant is instead defined as follows.

Let $H := \langle\langle c_1, \dots, c_d \rangle\rangle$ be the subgroup normally generated by the local monodromies (local generators of the isotropy subgroups). Consider the quotient group $G'' := G/H$.

Then the covering $C \to C' = C/G$ factors through $C'' = C/H \to C'$, which is unramified and with group G''. The monodromy $\mu'' \colon \Pi_{g'} \to G''$ corresponds to a homotopy class of a continuous map $m'' : C' \to \mathrm{K}(G'', 1)$. Passing to homology we have

$$H_2(m'') = H_2(\mu'') \colon H_2(\Pi_{g'}, \mathbb{Z}) = H_2(C', \mathbb{Z}) \to H_2(G'', \mathbb{Z}) .$$

One defines then $h(v)$ as the image of the fundamental class of C', the generator $[C']$ of $H_2(C', \mathbb{Z})$ determined by the complex orientation:

$$h(v) := H_2(\mu'')[C'] \in H_2(G'', \mathbb{Z}) .$$

More generally Edmonds ([Edm I, Edm II]) showed that ν and h determine ρ for G abelian, and that, if moreover G is split-metacyclic and the action is free (i.e., $G = G''$), then h determines ρ.

In our recent paper [CLP12] we considered the case $G = D_n$ of the dihedral group of order $2n$. After showing that in this case (ν, h) does not determine ρ, we introduced therefore a finer homological invariant $\varepsilon \in G_\Gamma$, where Γ is the union of the conjugacy classes of the local monodromies c_i, and G_Γ is a group constructed from the given group G and the given subset Γ. The main result of [CLP12] was to prove that in the case of the dihedral group our invariant ε determines the class of ρ.

We do not recall here the definition of the group G_Γ and of the invariant $\varepsilon(v)$, since the whole section two is devoted to these definitions and their topological interpretation (which was not contained in [CLP12]). It suffices here to observe that this new invariant encodes in particular all the classical numerical and homological invariants.

It would be a too nice and simple world if this invariant would do the job for any group. But this invariant, which in the unramified case coincides with the classical invariant in $H_2(G, \mathbb{Z})$, does not distinguish irreducible components in general (see [DT06]).

It only does so if the genus g' of the quotient curve is large enough. The crucial point, as in Harer's theorem, is the concept of stabilization.

<u>Direct stabilization:</u> μ stabilizes by extending $\mu(\alpha_{g'+1}) = \mu(\beta_{g'+1}) = 1 \in G$. In other words, we add a handle to the quotient curve C', such that on it the monodromy is trivial.

<u>Stabilization:</u> is defined as the equivalence relation generated by direct stabilization.

In the étale case it was shown that the homology invariant is a full 'stable' invariant.

Theorem 1.4 (Dunfield-Thurston)**.** *In the unramified case ($d = 0$), for $g' >> 0$, the equivalence classes are in bijection with*

$$\frac{H_2(G, \mathbb{Z})}{\mathrm{Aut}(G)} .$$

The proof of the above theorem is based on the interpretation of second homology as bordism, and on Livingston 's theorem ([L85]) showing that two unramified monodromies having the same homology class in $H_2(G, \mathbb{Z})$ are stably equivalent. A very suggestive proof of Livingston 's theorem, based on the concept of a relative Morse function with increasing Morse indices, is given in [DT06], while an algebraic proof was given by Zimmermann in [Z87].

In the ramified case, the situation is much more complicated, and it turns out to be safer to rely on the algebraic technique of Zimmermann in order to set up a secure, even if technical, proof of the following main theorem:

Theorem 1.5. *For $g' >> 0$, the equivalence classes are in bijection with the set of admissible classes ε.*

In the above theorem, the condition of admissibility is the simple translation of the condition that the product of the local monodromies $c_1 \dots c_d$ must be a product of commutators.

In a sequel to this paper we shall prove another stabilization, which we call branching stabilization, and which generalizes the following result (see [FV91])

Theorem 1.6 (Conway -Parker). *In the case $g' = 0$, let $G = F/R$ where F is a free group, and assume that $H_2(G, \mathbb{Z}) \cong \frac{[F,F]\cap R}{[F,R]}$ is generated by commutators. Then there is an integer N such that if the numerical function ν takes values $\geq N$,*

then there is only one equivalence class with the given numerical function ν.

We shall get an analogous result for any genus g', using our fuller homological invariant ε. In the course of proving branching stabilization, we shall also give a different proof of our genus stabilization result, using a variant of the semi-group and of the group introduced by Conway and Parker.

Finally, we want to mention the interesting question of determining the class of groups for which no stabilization is needed, thus extending to other groups the result we obtained for the dihedral groups.

2. The ε-invariant

In this Section we first review the definition of the ε-invariant of Hurwitz vectors which was introduced in [CLP12, Sec. 3], then we give a topological interpretation of this invariant as a class in a certain relative homology group of G modulo an equivalence relation. Although we don't use explicitly this topological interpretation in the proof of the main theorem, many of the technical results which we prove throughout the paper are just algebraic reformulations of simple geometrical statements whose understanding is easier using a topological view of the ε-invariant.

Let us first recall the following

Definition 2.1. Let G be a finite group, and let $g', d \in \mathbb{N}$. A *g', d-Hurwitz vector in G* is an element $v \in G^{d+2g'}$, the Cartesian product of G $(d + 2g')$-times. A g', d-Hurwitz vector in G will also be denoted by

$$v = (c_1, \dots, c_d\,; \, a_1, b_1, \dots, a_{g'}, b_{g'})\,.$$

For any $i \in \{1, \dots, d + 2g'\}$, the i-th component v_i of v is defined as usual. The *evaluation* of v is the element

$$ev(v) := \prod_1^d c_j \cdot \prod_1^{g'} [a_i, b_i] \in G\,.$$

A *Hurwitz generating system of length $d + 2g'$ in G* is a g', d-Hurwitz vector v in G such that the following conditions hold:

(i) $c_i \neq 1$ for all i;

(ii) G is generated by the components v_i of v;

(iii) $\prod_1^d c_j \cdot \prod_1^{g'} [a_i, b_i] = 1$.

We denote by $HS(G; g', d) \subset G^{2g'+d}$ the set of all Hurwitz generating systems in G of type (g', d) (hence of length $d + 2g'$).

Definition 2.2. Let G be a finite group.

¿From now on, $F := \langle \hat{g} \,|\, g \in G \rangle$ shall be the free group generated by the elements of G. Let $R \trianglelefteq F$ be the normal subgroup of relations, so that $G = \frac{F}{R}$.

For any union of non-trivial conjugacy classes $\Gamma \subset G$, define

$$R_\Gamma := \langle\langle [F,R], \hat{a}\hat{b}\hat{c}^{-1}\hat{b}^{-1} \,|\, \forall a,c \in \Gamma, b \in G \ s.t.\ ab = bc \rangle\rangle\,,$$
$$G_\Gamma := \frac{F}{R_\Gamma}\,.$$

The map $\hat{a} \mapsto a$, $\forall a \in G$, induces a group homomorphism $\alpha \colon G_\Gamma \to G$ whose kernel shall be denoted by $K_\Gamma := Ker(\alpha)$.

By [CLP12, Lemma 3.2], $K_\Gamma = \frac{R}{R_\Gamma}$ is contained in the centre of G_Γ.
Finally we set:

Definition 2.3. Given a g', d-Hurwitz vector

$$v = (c_1, \dots, c_d; a_1, b_1, \dots, a_{g'}, b_{g'})$$

in G (cf. Definition 2.1), its *tautological lift*, $\hat{v}$, is the g', d-Hurwitz vector in G_Γ defined by

$$\hat{v} := (\widehat{c_1}, \dots, \widehat{c_d}; \widehat{a_1}, \widehat{b_1}, \dots, \widehat{a_{g'}}, \widehat{b_{g'}})$$

whose components are the tautological lifts of the components of v.

Given a g', d-Hurwitz vector v in G with $c_i \neq 1$, $\forall i = 1, \dots d$, we denote by Γ_v the union of all the conjugacy classes of G containing at least one c_i.

For any Hurwitz generating system v (or Hurwitz vector satisfying (iii)), set

$$\varepsilon(v) := \prod_1^d \widehat{c_j} \cdot \prod_1^{g'} [\widehat{a_i}, \widehat{b_i}] \in K_{\Gamma_v}\,,$$

to be the evaluation of the tautological lift $\hat{v}$ of v in G_{Γ_v} (cf. Definition 2.1).

By [CLP12, Lemma 3.5], any automorphism $f \in \mathrm{Aut}(G)$ induces an isomorphism $f_\Gamma \colon K_\Gamma \to K_{f(\Gamma)}$ in a natural way, such that $\varepsilon(f(v)) = f_\Gamma(\varepsilon(v))$, where $\Gamma = \Gamma_v$. Therefore the map

$$\varepsilon \colon HS(G; g', d) \to \coprod_\Gamma K_\Gamma\,, \quad v \mapsto \varepsilon(v) \in K_{\Gamma_v}$$

where $\coprod_\Gamma K_\Gamma$ denotes the disjoint union of the Γ's, descends to a map

$$\tilde{\varepsilon} \colon HS(G; g', d)/_{\mathrm{Aut}(G)} \to \left(\coprod_\Gamma K_\Gamma\right)/_{\mathrm{Aut}(G)}\,.$$

The key point here is that $\tilde{\varepsilon}$ is invariant under the action of the mapping class group $Map(g', d)$ [CLP12, Prop. 3.6], hence giving a well defined map

$$\hat{\varepsilon} \colon \left(HS(G; g', d)/_{\mathrm{Aut}(G)}\right)/_{Map(g',d)} \to \left(\coprod_\Gamma K_\Gamma\right)/_{\mathrm{Aut}(G)}\,. \tag{1}$$

One of the main results of [CLP12] says that, when $G = D_n$, $\hat{\varepsilon}$ is injective (*loc. cit.* Thm. 5.1). With the aid of this we reached a classification of the orbits of Hurwitz generating systems for D_n under the action of $Map(g', d)$ modulo automorphisms in $\mathrm{Aut}(G)$.

For later use, let us recall the following definition [CLP12, Def. 3.11].

Definition 2.4. Let $\Gamma \subset G$ be a union of non-trivial conjugacy classes of G. We define

$$H_{2,\Gamma}(G) = \ker\left(G_\Gamma \to G \times G_\Gamma^{ab}\right),$$

where $G_\Gamma \to G \times G_\Gamma^{ab}$ is the morphism with first component α (defined in Definition 2.2) and second component the natural epimorphism $G_\Gamma \to G_\Gamma^{ab}$.

Notice that

$$H_2(G,\mathbb{Z}) \cong \frac{R \cap [F,F]}{[F,R]} \cong \ker\left(\frac{F}{[F,R]} \to G \times G_\emptyset^{ab}\right).$$

In particular, when $\Gamma = \emptyset$, $H_{2,\Gamma}(G) \cong H_2(G,\mathbb{Z})$.

By [CLP12, Lemma 3.12] we have that the morphism

$$R \cap [F,F] \to \frac{R}{R_\Gamma}, \quad r \mapsto rR_\Gamma$$

induces a surjective group homomorphism

$$H_2(G,\mathbb{Z}) \to H_{2,\Gamma}(G). \tag{2}$$

2.1. **A topological interpretation of $\varepsilon(v)$.** For a finite group G, let BG be the CW-complex defined as follows (see [W, Ch. V, 7.] for more details). The 0-skeleton BG^0 consists of one point. The 1-skeleton

$$BG^1 = \bigvee_{g \in G} S_g^1$$

is a wedge of circles indexed by $g \in G$ meeting in BG^0. Then the fundamental group has a canonical isomorphism $\pi_1(BG^1) \cong F$, the isomorphism being given by sending a generator of $\pi_1(S_g^1)$ to $\hat{g}$. In this way we get an epimorphism $\pi_1(BG^1) \to G$ whose kernel is identified with R by the previous isomorphism. For any $r \in R$, let $h_r\colon S^1 \to BG^1$ be the continuous map such that the image of a chosen generator of $\pi_1(S^1)$ under $(h_r)_*$ is $r \in \pi_1(BG^1)$. Using h_r we attach the 2-cell E_r^2 to BG^1. This gives the 2-skeleton BG^2, with the property that $\pi_1(BG^2) \cong G$. The 3-skeleton BG^3 is defined by attaching 3-cells to BG^2 in such a way that $\pi_2(BG^3) = 0$ and, by induction, the $(n+1)$-skeleton BG^{n+1} is defined similarly in order to get $\pi_n(BG^{n+1}) = 0$, $n \geq 2$. The CW-complex BG is the inductive limit $BG := \cup\{BG^n \mid n \geq 0\}$ thus obtained. By construction BG is an Eilenberg-Mac Lane space of type $(G,1)$, i.e. a $K(G,1)$-space. Furthermore there is a principal G-bundle $EG \to BG$ with EG contractible.

Lemma 2.5. *Let G be a finite group and let BG be the CW-complex defined above. There is an isomorphism*

$$\frac{R}{[F,R]} \cong H_2(BG, BG^1)$$

such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & H_2(BG) \arrow[r] \arrow[d] & H_2(BG,BG^1) \arrow[r] \arrow[d] & H_1(BG^1) \arrow[r] \arrow[d] & H_1(BG) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{R\cap[F,F]}{[F,R]} \arrow[r] & \frac{R}{[F,R]} \arrow[r] & F^{ab} \arrow[r] & G^{ab} \arrow[r] & 0
\end{tikzcd}$$

where all the homology groups are with integer coefficients, the horizontal sequences are exact (the upper one being part of the homology sequence of the pair (BG, BG^1)) and the vertical arrows are isomorphisms (the one on the left is given by Hopf's theorem, see [H42]*, the two on the right are the canonical isomorphisms $H_1 \cong \pi_1^{ab}$).*

Proof. Recall that the homology of a CW-complex $K = \{K^n\}_{n\in\mathbb{N}}$ can be computed as follows (cf. [M, Ch. IX]). Define

$$\begin{aligned} &C_n(K) := H_n(K^n, K^{n-1})\,,\\ &\partial_n \colon C_n(K) \to C_{n-1}(K)\,, \quad \partial_n = j_{n-1}\circ\partial_*\,, \end{aligned}$$

where $\partial_* \colon H_n(K^n, K^{n-1}) \to H_{n-1}(K^{n-1})$ is the boundary operator of the long exact sequence of the pair (K^n, K^{n-1}) and $j_{n-1}\colon H_{n-1}(K^{n-1}) \to H_{n-1}(K^{n-1}, K^{n-2})$ is the homomorphism induced by the inclusion map. Then $\{C_\bullet(K), \partial_\bullet\}$ is a complex and $H_n(K) \cong H_n(C_\bullet(K))$.

We regard BG^1 as a 1-dimensional CW-complex, so the above construction gives the complexes $C_\bullet(BG)$ and $C_\bullet(BG^1)$. Notice that the inclusion $BG^1 \to BG$ gives an injective map $C_\bullet(BG^1) \to C_\bullet(BG)$ and define

$$C_\bullet(BG, BG^1) := \frac{C_\bullet(BG)}{C_\bullet(BG^1)}\,.$$

Since $C_n(K)$ is the free group with basis in 1-1 correspondence with the n-cells of K, we obtain the diagram

(3)
$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C_3(BG) \arrow[r] \arrow[d] & C_3(BG, BG^1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C_2(BG) \arrow[r] \arrow[d] & C_2(BG, BG^1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1(BG^1) \arrow[r] \arrow[d] & C_1(BG) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_0(BG^1) \arrow[r] \arrow[d] & C_0(BG) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where all rows are exact. Using the isomorphisms $H_n(K) \cong H_n(C_\bullet(K))$, the long exact homology sequence of the pair (BG, BG^1) and the long exact homology sequence associated to the previous diagram, we obtain isomorphisms

$$H_n(C_\bullet(BG, BG^1)) \cong H_n(BG, BG^1)\,, \quad \forall n\,.$$

Consider now the morphism $C_2(BG) \to \frac{R}{[F,R]}$ induced by sending any element of $C_2(BG)$ to the homotopy class of its boundary. By (3) we have isomorphisms $C_n(BG, BG^1) \to C_n(BG)$, $n \geq 2$, and so we obtain a group homomorphism:

(4)
$$C_2(BG, BG^1) \to \frac{R}{[F,R]}\,.$$

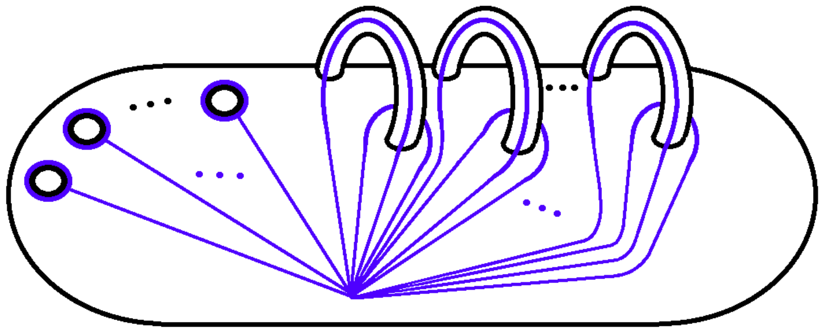

FIGURE 1.

By [H42, Satz I.] it follows that the kernel of (4) is $\partial\left(C_3(BG, BG^1)\right) = \partial\left(C_3(BG)\right)$. So we get the homomorphism

$$H_2(BG, BG^1) \to \frac{R}{[F,R]}\,. \tag{5}$$

By construction, the diagram in the statement commutes and (5) is an isomorphism by the 5-lemma. $\square$

Let now $p\colon C \to C'$ be a G-covering branched at $y_1, \dots, y_d \in C'$. Fix once and for all a point $y_0 \in C' \setminus \{y_1, \dots, y_d\}$ and a geometric basis $\gamma_1, \dots, \gamma_d, \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'}$ of $\pi_1(C' \setminus \{y_1, \dots y_d\}, y_0)$. The monodromy of p evaluated at the chosen geometric basis gives the Hurwitz generating system $v \in HS(G; g', d)$, well defined up to conjugation, which in turn determines p (by Riemann's existence theorem). For $\Gamma = \Gamma_v$, the ε-invariant of v is an element of

$$K_\Gamma := \frac{R}{R_\Gamma} = \left(\frac{R}{[F,R]}\right) \Big/ _{\langle\langle \hat{a}\hat{b}\hat{c}^{-1}\hat{b}^{-1} \,|\, a \in \Gamma\,,\, ab = bc \rangle\rangle}\,.$$

If p is unramified, then, under the identification $\frac{R \cap [F,F]}{[F,R]} \cong H_2(BG, \mathbb{Z})$, $\varepsilon(v) \in \frac{R\cap[F,F]}{[F,R]}$ coincides with the image $Bp_*[C'] \in H_2(BG, \mathbb{Z})$ of the fundamental class of C' under the morphism induced in homology by a classifying map $Bp\colon C' \to BG$ of p. We want to extend this topological interpretation of the ε-invariant to the ramified case.

By definition, any loop γ_i of the geometric basis consists of a path $\tilde{\gamma}_i$ from y_0 to a point z_i near y_i and of a small loop around y_i. Let Σ be the Riemann surface (with boundary) obtained from C' after removing the open discs surrounded by these small loops. Fix once and for all a CW-decomposition of Σ as follows. The 0-skeleton Σ^0 consists of the point y_0 and, for any $i = 1, \dots, d$, the intersection z_i between $\tilde{\gamma}_i$ and the small circle of γ_i around y_i. The 1-skeleton Σ^1 is given by the geometric basis and the 2-skeleton Σ^2 consists of one cell (see Figure 1).

The restriction p_Σ of $p\colon C \to C'$ to $p^{-1}(\Sigma)$ is an unramified G-covering of Σ and hence corresponds to a continuous map $Bp_\Sigma\colon \Sigma \to BG$, well defined up to homotopy. Let $Bp_1\colon \Sigma \to BG$ be a cellular approximation of Bp_Σ. Since Bp_1 can be regarded as a map of pairs $Bp_1\colon (\Sigma, \partial\Sigma) \to (BG, BG^1)$, the push-forward of the fundamental (orientation) class $[\Sigma, \partial\Sigma]$ gives an element

$$Bp_{1*}[\Sigma, \partial\Sigma] \in H_2(BG, BG^1) = \frac{R}{[F, R]}\,.$$

This element depends on the chosen cellular approximation Bp_1 of Bp_Σ, but its image in K_Γ does not (see Lemma 2.6).

In order to compute $Bp_{1*}[\Sigma, \partial\Sigma]$ it is useful to recall the construction of Bp_1. As before, let

$$v = (c_1, \dots, c_d, a_1, b_1, \dots, a_{g'}, b_{g'}) \in G^{d+2g'}$$

be a Hurwitz generating system of p with respect to the given geometric basis. Consider the relation

$$r := \widehat{c_1} \cdot \ldots \cdot \widehat{c_d} \cdot \prod_{j=1}^{g'} [\widehat{a_j}, \widehat{b_j}] \in \pi_1(BG^1)$$

and a continuous map $h_r\colon \Sigma^1 \to BG^1$ representing r. By the construction of BG there is a 2-cell E_r attached along h_r. Choose a homeomorphism $E_r \cong \Sigma^2$. So we get a continuous map $Bp_1\colon \Sigma \to BG$. To prove that Bp_1 is homotopic to Bp, we need to prove that the pull-back $Bp_1^*(EG \to BG)$ under Bp_1 of $EG \to BG$ is $p_\Sigma\colon p^{-1}(\Sigma) \to \Sigma$. This follows from the fact that $EG \to BG$ is the universal cover and from the fact that the monodromy vector of $Bp_1^*(EG \to BG)$ with respect to the given geometric basis coincides with v up to conjugation. Moreover, by the proof of Lemma 2.5, it follows that under the isomorphism $H_2(BG, BG^1) \cong \frac{R}{[F,R]}$

$$Bp_{1*}[\Sigma, \partial\Sigma] = \prod_1^d \widehat{c_i} \cdot \prod_1^{g'} [\widehat{a_j}, \widehat{b_j}] \in \frac{R}{[F, R]}\,. \tag{6}$$

Lemma 2.6. *Let $\Gamma = \Gamma_v \subset G$, where v is the Hurwitz generating system of p with respect to the given geometric basis. Then the image of $Bp_{1*}[\Sigma, \partial\Sigma]$ in K_Γ does not depend on the cellular approximation Bp_1 of Bp. Denote this element $Bp_*[\Sigma, \partial\Sigma] \in K_\Gamma$. Then, by* (6) *we have:*

$$Bp_*[\Sigma, \partial\Sigma] = \varepsilon(v) \in K_\Gamma\,.$$

Proof. Let $Bp_2\colon (\Sigma, \partial\Sigma) \to (BG, BG^1)$ be another cellular approximation of Bp_Σ. Then there exists an homotopy $F\colon I \times \Sigma \to BG$ such that $F(0, x) = Bp_1(x)$ and $F(1, x) = Bp_2(x)$, $\forall\, x \in \Sigma$, where $I = [0, 1]$. Without loss of generality we can assume that F is cellular with respect to the standard CW-decomposition of $I \times \Sigma$ ([W, (4.7)]):

$$(I \times \Sigma)^n = \bigcup_{i=0}^{n} I^i \times \Sigma^{n-i}\,,$$

where I is seen as a 1-dimensional CW-complex with $I^0 = \{0\} \cup \{1\}$.

Consider now the chain homotopy

$$\varphi_n\colon C_n(BG) \to C_{n+1}(BG)$$

between the chain maps $Bp_{1\#}$ and $Bp_{2\#}$ associated to F. Then we have (cf. [M, (7.4.1)]):

$$Bp_{1\#} - Bp_{2\#} = \partial_{n+1} \circ \varphi_n + \varphi_{n-1} \circ \partial_n\,.$$

¿From this it follows that

$$Bp_{1*}[\Sigma, \partial\Sigma] - Bp_{2*}[\Sigma, \partial\Sigma] = [\varphi_1(\partial\Sigma)] \in H_2(BG, BG^1)\,.$$

Notice that

$$[\varphi_1(\partial\Sigma)] = \left(F_{|I\times\partial\Sigma}\right)_* [I \times \partial\Sigma, \partial(I \times \partial\Sigma)] \in H_2(BG, BG^1)\,.$$

The claim now follows from the fact that $I \times \partial\Sigma$ is the union of cylinders $I \times S^1$ and

$$\left(F_{|I\times\partial\Sigma}\right)_* [I \times S^1, \partial(I \times S^1)] = \hat{a}\hat{b}\hat{c}^{-1}\hat{b}^{-1}\,,$$

where $\hat{a}$ (resp. $\hat{c}$) is the image of the fundamental class of $\{0\} \times S^1$ (resp. $\{1\} \times S^1$) under F, $\hat{b}$ is the image of $I \times \{z_i\}$ under F, for some $z_i \in \Sigma^0$. Notice that, since $\left(F_{|I\times\partial\Sigma}\right)_* [I \times S^1, \partial(I \times S^1)]$ is the class of a 2-cell, $\hat{a}\hat{b}\hat{c}^{-1}\hat{b}^{-1}$ must be a relation for G. □

3. The main Theorem

In this Section we prove our main result. Roughly speaking, it says that for g' sufficiently large the map given by the $\hat{\varepsilon}$-invariant (1) is injective, its image is independent of g' and coincides with the classes of admissible ν-types. We begin then with the following

Definition 3.1. Let $v \in HS(G; g', d)$ and let $\nu(v) \in \bigoplus_{\mathcal{C}} \mathbb{Z}\langle\mathcal{C}\rangle$ ($\mathcal{C}$ runs over the set $Conj(G)$ of conjugacy classes of G) be the vector whose $\mathcal{C}$-component is the number of v_j, $j \leq d$, which belong to $\mathcal{C}$.
The map

$$\nu\colon HS(G; g', d) \to \bigoplus_{\mathcal{C}} \mathbb{Z}\langle\mathcal{C}\rangle \tag{7}$$

obtained in this way induces a map

$$\tilde{\nu}\colon HS(G; g', d)/_{\mathrm{Aut}(G)} \to \left(\bigoplus_{\mathcal{C}} \mathbb{Z}\langle\mathcal{C}\rangle\right)/_{\mathrm{Aut}(G)}$$

which is $Map(g', d)$-invariant, therefore we get a map

$$\hat{\nu}\colon \left(HS(G; g', d)/_{\mathrm{Aut}(G)}\right)/_{Map(g',d)} \to \left(\bigoplus_{\mathcal{C}} \mathbb{Z}\langle\mathcal{C}\rangle\right)/_{\mathrm{Aut}(G)}\,.$$

For any $v \in HS(G; g', d)$, we call $\hat{\nu}(v)$ the (unmarked) ν-**type** of v (the marked version $\nu(v)$ is called shape in [FV91]).

Remark 3.2. Let $v \in HS(G; g', d)$ and let $\Gamma_v \subset G$ be the union of the conjugacy classes of the v_j, $j \leq d$. The abelianization $G^{ab}_{\Gamma_v}$ of G_{Γ_v} can be described as follows:

$$G^{ab}_{\Gamma_v} \cong \bigoplus_{\mathcal{C}\subset\Gamma} \mathbb{Z}\langle\mathcal{C}\rangle \bigoplus_{g\in G\setminus\Gamma_v} \mathbb{Z}\langle g\rangle\,,$$

where $\mathcal{C}$ denotes a conjugacy class of G. Moreover $\nu(v)$ coincides with the vector whose $\mathcal{C}$-components are the corresponding components of the image in $G^{ab}_{\Gamma_v}$ of $\varepsilon(v) \in G_{\Gamma_v}$ under the natural homomorphism $G_{\Gamma_v} \to G^{ab}_{\Gamma_v}$. It follows that ν in (7) factors as

$$\nu = A \circ \varepsilon$$

where

$$A\colon \coprod_{\Gamma} K_\Gamma \to \bigoplus_{\mathcal{C}} \mathbb{Z}\langle\mathcal{C}\rangle$$

is induced from the abelianization $G_\Gamma \to G^{ab}_\Gamma$. To take into account the automorphisms of G one defines similarly $\widehat{A}$ in such a way that $\widehat{\nu} = \widehat{A} \circ \widehat{\varepsilon}$.

Definition 3.3. An element

$$\nu = (n_{\mathcal{C}})_{\mathcal{C}} \in \bigoplus_{\mathcal{C} \neq \{1\}} \mathbb{Z}\langle \mathcal{C} \rangle$$

is **admissible** if the following equality holds in G^{ab} for its natural $\mathbb{Z}$-module structure:

$$\sum_{\mathcal{C}} n_{\mathcal{C}} \cdot [\mathcal{C}] = 0$$

where $[\mathcal{C}]$ denotes the class of any element of $\mathcal{C}$ in the abelianization of G.

Accordingly we say that $\widehat{\nu} \in (\bigoplus_{\mathcal{C}} \mathbb{Z}\langle \mathcal{C} \rangle) /_{\mathrm{Aut}(G)}$ is admissible if it is the class of an admissible element.

The main result of the paper is then the following

Theorem 3.4. *Let G be a finite group. Then for any $d \in \mathbb{N}$ there is an integer $s = s(d)$ such that*

$$\hat{\varepsilon}\colon \left(HS(G; g', d)/_{\mathrm{Aut}(G)}\right) /_{Map(g',d)} \to \left(\coprod_{\Gamma} K_{\Gamma}\right) /_{\mathrm{Aut}(G)}$$

is injective for any $g' > s$. Moreover, for any $g' > s$, the image of $\hat{\varepsilon}$ is independent of g' and it coincides with the pre-image under $\widehat{A}$ of the admissible $\widehat{\nu}$-types.

The main tool to understand the set $\left(HS(G; g', d)/_{\mathrm{Aut}(G)}\right) /_{Map(g',d)}$ is provided by the so-called *stabilization*, which we are going once more to review. For any Hurwitz generating system

$$v = (c_1, \dots, c_d, a_1, b_1, \dots, a_{g'}, b_{g'}) \in HS(G; g', d)$$

define the (h-)stabilization v^h of v inductively by

$$v^0 = v\,, \quad v^1 = (c_1, \dots, c_d, a_1, b_1, \dots, a_{g'}, b_{g'}, 1, 1)\,, \quad v^h = (v^{h-1})^1\,, \quad \forall\, h \in \mathbb{N}\,.$$

Topologically, if v corresponds to the monodromy $\mu\colon \pi_1(C') \to G$, then v^h corresponds to the monodromy $\mu^h\colon \pi_1(C' \# C'') \to G$ obtained by extending μ by 1 on the elements of $\pi_1(C'')$, where C'' is an oriented surface of genus h and $C' \# C''$ is the connected sum of C' with C''.

It is easy to see that stabilization satisfies the following properties: the ε-invariant does not change under stabilization,

$$\varepsilon(v) = \varepsilon(v^h)\,, \quad \forall\, v\,, h\,; \tag{8}$$

it respects the equivalence relation given by the actions of $\mathrm{Aut}(G)$ and $Map(g', d)$, therefore we get maps

(9)
$$\left(HS(G; g', d)/_{\mathrm{Aut}(G)}\right) /_{Map(g',d)} \to \left(HS(G; g'+h, d)/_{\mathrm{Aut}(G)}\right) /_{Map(g'+h,d)}\,, \quad \forall\, g', h, d$$
$$[v] \mapsto [v^h]\,.$$

Moreover, the ε-invariant is stably a fine invariant. This is the content of the following theorem that extends to the ramified case the analogous result of Livingston [L85] for non-ramified group actions (see also [DT06, Z87]).

Theorem 3.5. *Let G be a finite group and let $v, w \in HS(G; g', d)$ such that $\nu(v) = \nu(w)$ (in particular $\Gamma_v = \Gamma_w = \Gamma$). If*

$$\varepsilon(v) = \varepsilon(w) \in K_\Gamma \,,$$

then $\exists h \in \mathbb{N}$ such that the classes of v^h and w^h in $\left(HS(G; g'+h, d)/_{\mathrm{Aut}(G)}\right)/_{Map(g'+h,d)}$ coincide.

We postpone the proof of this theorem to the next section. Here we use it to give a proof of Thm. 3.4.

Proof of Thm. 3.4. First of all we have that, for $g' \geq |G|$, any $v \in HS(G; g'+1, d)$ is a stabilization, that is: the map

$$\begin{gathered} \varsigma_{g'} \colon HS(G; g', d))/_{Map(g',d)} \to HS(G; g'+1, d))/_{Map(g'+1,d)} \\ \text{induced by } v \mapsto v^1 \end{gathered}$$

is surjective. This follows from the proof of [DT06, Prop. 6.16], where the result is stated for free group actions, but the same proof works also for non-free actions.
As a consequence we have that (9) is surjective for $g' \geq |G|$.

Since the sets $\left(HS(G; g', d)/_{\mathrm{Aut}(G)}\right)/_{Map(g',d)}$ are finite sets, there exists an integer [1] $s = s(d)$ such that (9) is bijective for any $g' > s$, and $h \geq 1$.

Let now $g' > s$ and let $v, w \in HS(G; g', d)$ with $\varepsilon(v) = \varepsilon(w)$. From Thm. 3.5 there exists h with $[v^h] = [w^h] \in \left(HS(G; g'+h, d)/_{\mathrm{Aut}(G)}\right)/_{Map(g'+h,d)}$. Since stabilization is bijective in this range we get $[v] = [w] \in \left(HS(G; g', d)/_{\mathrm{Aut}(G)}\right)/_{Map(g',d)}$. Hence $\widehat{\varepsilon}$ is injective for $g' > s$.
The fact that $\mathrm{Im}(\widehat{\varepsilon})$ does not depend on g' now follows from (8).

We now prove that $\mathrm{Im}(\widehat{\varepsilon})$ is the preimage under $\widehat{A}$ of the admissible ν-types, for any $g' > s$. First notice that for any admissible $\nu = (n_\mathcal{C})_\mathcal{C} \in \bigoplus_\mathcal{C} \mathbb{Z}\langle \mathcal{C} \rangle$ there exists $v \in HS(G; s+1, d)$ with $\nu(v) = \nu$. Indeed for any $(c_1, \ldots, c_d)$ with $\nu(c_1, \ldots, c_d) = \nu$ the condition that ν is admissible implies that $c_1 \cdot \ldots \cdot c_d \in [G, G]$, hence it is a product of commutators,

$$c_1 \cdot \ldots \cdot c_d = \left(\prod_{j=1}^{r} [a_j, b_j] \right)^{-1} \,,$$

for some $r \in \mathbb{N}$. If $c_1, \ldots, c_d, a_1, b_1, \ldots, a_r, b_r$ do not generate G, we add pairs of the form $(g, 1)$, so we obtain a Hurwitz generating system $w \in HS(G; r, d)$ with $\nu(w) = \nu$. Moreover we can assume that $r > s$. Since $\varsigma_{g'}$ is surjective for $g' > s$, we get a Hurwitz system $v \in HS(G; s+1, d)$ whose stabilization is $Map(g', d)$-equivalent to w and the claim follows.
Finally we prove that for any $\xi \in K_\Gamma$, $\Gamma = \Gamma_v$, with $A(\xi) = \nu$ there exists $w \in HS(G; s+1, d)$ with $\varepsilon(w) = \xi$. Since $A(\xi) = A(\varepsilon(v)) = \nu$, $\varepsilon(v)^{-1} \cdot \xi \in H_{2,\Gamma}(G)$. By (2) there exists $\eta \in H_2(G, \mathbb{Z})$ which maps to $\varepsilon(v)^{-1} \cdot \xi$. Since bordism is the same as homology in dimension 2 (cf. also [DT06, Thm. 6.20]), there is a Hurwitz system $v' \in HS(G; h, 0)$ such that $\varepsilon(v') = \eta \in H_2(G, \mathbb{Z})$. Let $v'' \in HS(G; s+1+h, d)$ be the system whose first $d + 2(s+1)$ components coincide with those of v and the last $2h$ components are those of v', then we have: $\varepsilon(v'') = \xi$. By (8) the system $w \in HS(G; s+1, d)$ that maps to v'' under $\varsigma_{s+h} \circ \ldots \circ \varsigma_{s+1}$ satisfies $\varepsilon(w) = \xi$. This concludes the proof of the theorem.

□

[1] to prove that s can be explicitly given it would suffice to show that once $\varsigma_{g'}$ is bijective, then also $\varsigma_{g''}$ is bijective for $g'' > g'$.

4. Proof of Theorem 3.5

Let $v, w \in HS(G, g', d)$ be two Hurwitz generating systems with $\nu(v) = \nu(w)$ and $\varepsilon(v) = \varepsilon(w)$. If

$$\begin{aligned} v =&(v_1, \dots, v_d\,;\, v_{d+1}, v_{d+2}, \dots, v_{d+2g'-1}, v_{d+2g'}) \quad \text{and} \\ w =&(w_1, \dots, w_d\,;\, w_{d+1}, w_{d+2}, \dots, w_{d+2g'-1}, w_{d+2g'})\,, \end{aligned}$$

then, without loss of generality (using braid group moves on the first d components) we may assume that v_i is conjugate to w_i, $i = 1, \dots, d$, and that the following equality holds:

$$\prod_{i=1}^{d} \widehat{w}_i \prod_{j=1}^{g'} [\widehat{w_{d+2j-1}}, \widehat{w_{d+2j}}] \quad \equiv \quad \prod_{i=1}^{d} \widehat{v}_i \prod_{j=1}^{g'} [\widehat{v_{d+2j-1}}, \widehat{v_{d+2j}}] \quad (\bmod\ R_\Gamma)\,. \tag{10}$$

Let us rewrite equation (10) modulo $[F, R]$. This means that there are relations $\widehat{x}_\ell \widehat{y}_\ell \widehat{z}_\ell^{\,-1} \widehat{y}_\ell^{\,-1} \in R$, $\ell = 1, \dots, N$, and with $x_\ell \in \Gamma$, such that

$$\prod_{i=1}^{d} \widehat{w}_i \prod_{j=1}^{g'} [\widehat{w_{d+2j-1}}, \widehat{w_{d+2j}}] \equiv \prod_{i=1}^{d} \widehat{v}_i \prod_{j=1}^{g'} [\widehat{v_{d+2j-1}}, \widehat{v_{d+2j}}] \prod_{\ell=1}^{N} \left(\widehat{x}_\ell \widehat{y}_\ell \widehat{z}_\ell^{\,-1} \widehat{y}_\ell^{\,-1} \right)^{\pm 1} (\bmod\ [F, R])\,. \tag{11}$$

Proposition 4.1. *Let $v, w \in HS(G, g', d)$ be two Hurwitz generating systems. Assume $v_i = w_i$, $\forall i = 1, \dots, d$, $ev(\hat{v}) \equiv ev(\hat{w})$ $(\bmod\ [F, R])$ (i.e., $N = 0$ in (11)), and $G = \langle v_{d+1}, \dots, v_{d+2g'} \rangle = \langle w_{d+1}, \dots, w_{d+2g'} \rangle$. Then v and w are stably equivalent.*

Proof. We have

$$ev(\hat{w}) = \prod_{i=1}^{d} \widehat{v}_i \prod_{j=1}^{g'} [\widehat{w_{d+2j-1}}, \widehat{w_{d+2j}}] \equiv ev(\hat{v}) = \prod_{i=1}^{d} \widehat{v}_i \prod_{j=1}^{g'} [\widehat{v_{d+2j-1}}, \widehat{v_{d+2j}}] \quad (\bmod\ [F, R])$$

hence

$$\prod_{j=1}^{g'} [\widehat{w_{d+2j-1}}, \widehat{w_{d+2j}}] \equiv \prod_{j=1}^{g'} [\widehat{v_{d+2j-1}}, \widehat{v_{d+2j}}] \quad (\bmod\ [F, R])\,. \tag{12}$$

This means that the right hand side of (12) differs from the left hand side by a product of commutators in $[F, R]$. Now, by Livingston's theorem [L85] (cf. the algebraic proof in [Z87]), it follows that we can realize these commutators by adding handles with trivial monodromies and acting with the mapping class group $Map_{g'}$. The mapping classes that are used in this procedure can be seen to be the restriction of mapping classes in $Map(g', d)$ that act as the identity on the first d components of the Hurwitz systems (see [Z87], 2.1–2.5). The proposition now follows from the similar result in the unramified case. $\square$

We prove Theorem 3.5 by showing that v and w are stably equivalent to Hurwitz generating systems for which the hypotheses of Proposition 4.1 are satisfied. This is achieved through two reduction steps: first we prove that, after stabilization, we can assume that $v_i = w_i$, $i = 1, \dots, d$, and $G = \langle v_{d+1}, \dots, v_{d+2g'} \rangle = \langle w_{d+1}, \dots, w_{d+2g'} \rangle$; then we stabilize further to obtain $N = 0$ in (11).

1st step: reduction to the case $v_i = w_i$**,** $i = 1, \dots, d$**,** $G = \langle v_{d+1}, \dots, v_{d+2g'} \rangle = \langle w_{d+1}, \dots, w_{d+2g'} \rangle$**.**

Proposition 4.2. *Let*

$$v = (c_1, \dots, c_d; a_1, b_1, \dots, a_{g'}, b_{g'}) \in HS(G; g', d)$$

be a Hurwitz generating system and let $g_1, \dots, g_d \in G$*. Set* $c'_i = g_i c_i g_i^{-1}$*. Then, there exists* $\varphi \in Map(g' + d, d)$ *such that*

$$\varphi \cdot v^d \quad = \quad (c'_1, \dots, c'_d\,;\, \lambda_1, \mu_1, \dots, \lambda_d, \mu_d, a_1, , b_1, \dots, a_{g'}, b_{g'})\,. \tag{13}$$

Here v^d *is obtained from* v *by adding* d *handles with trivial monodromies. Precise formulas for the* λ*'s and* μ*'s are given below.*

We use the following

Lemma 4.3. *For any*

$$(c_1, \dots, c_d\,;\, a_1, b_1, \dots, a_{g'}, b_{g'}) \in HS(G; g', d)$$

and for any $x \in G$*, we have:*

$$(c_1, \dots, c_d\,;\, 1, 1, a_1, b_1, \dots, a_{g'}, b_{g'}) \approx (c_1, \dots, c_d\,;\, x, 1, a_1, b_1, \dots, a_{g'}, b_{g'})\,.$$

Proof. A direct computation shows that, for any automorphism φ of the form 2.1–2.5 in [Z87], the map

$$\overline{\varphi}\colon \langle c_1, \dots, c_d; a_1, \dots, b_{g'} | \prod_1^d c_i \prod_1^{g'} [a_j, b_j] = 1 \rangle \to \langle c_1, \dots, c_d; a_1, \dots, b_{g'} | \prod_1^d c_i \prod_1^{g'} [a_j, b_j] = 1 \rangle$$

$$c_i \mapsto c_i, \quad a_j \mapsto \varphi(a_j), \quad b_j \mapsto \varphi(b_j)$$

defines an automorphism, so $\overline{\varphi} \in Map\,(C', \mathcal{B}; \{y_0\})$. Here $\mathcal{B}$ is the branch locus of $C \to C'$ and $Map\,(C', \mathcal{B}; \{y_0\})$ is the group of isotopy classes of diffeomorphisms $f\colon C' \to C'$ such that f preserves the orientation, $f(\mathcal{B}) = \mathcal{B}$, and $f(y_0) = y_0$.

By [Z87] Lemma 2.6 the claim is true for any $x \in \langle a_1, \dots, b_{g'} \rangle$, therefore it remains to prove that

$$(c_1, \dots, c_d\,;\, x, 1, a_1, b_1, \dots, a_{g'}, b_{g'}) \approx (c_1, \dots, c_d\,;\, c_i x, 1, a_1, b_1, \dots, a_{g'}, b_{g'}),$$

for any $i = 1, \dots, d$. Using the braid group, it is enough to prove the above equivalence when $i = d$: the result follows as a direct consequence of Proposition 6.2 of [CLP12], (i), with $\ell = 1$ (see figure 2), yielding the following useful transformation which leaves all the components of the Hurwitz vector unchanged except for

$$\begin{aligned} & a_1 \mapsto c_d a_1, \quad c_d \mapsto (c_d a_1 b_1 a_1^{-1}) c_d (c_d a_1 b_1 a_1^{-1})^{-1} \\ & \qquad \Leftrightarrow v_{d+1} \mapsto v_d v_{d+1}, \quad v_d \mapsto g v_d g^{-1} \\ & \qquad\qquad g := (v_d v_{d+1} v_{d+2} v_{d+1}^{-1}). \end{aligned} \tag{14}$$

□

Proof. (of Proposition 4.2) φ is the composition of the mapping classes corresponding to the following steps 2 – 7. Set $h_i = c_i^{-1} g_i$ and perform the following operations.

1. Add a trivial handle to v obtaining: $(c_1, \dots, c_d\,;\, 1, 1, a_1, b_1, \dots, a_{g'}, b_{g'})$.
2. Bring c_1 to the d-th position using the braid group:

$$(c_1, \dots, c_d\,;\, 1, 1, a_1, b_1, \dots, a_{g'}, b_{g'}) \approx (c_1 c_2 c_1^{-1}, \dots, c_1\,;\, 1, 1, a_1, b_1, \dots, a_{g'}, b_{g'})\,.$$

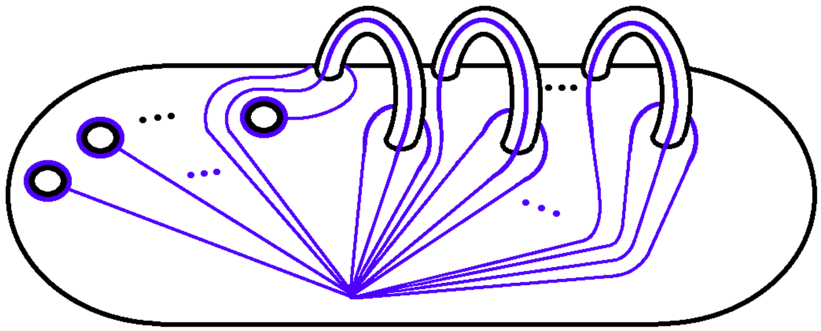

Figure 2.

3. Apply Lemma 4.3 with $x = h_1$:

$$(c_1c_2c_1^{-1}, \dots, c_1\,;\, 1, 1, a_1, b_1, \dots, a_{g'}, b_{g'}) \approx (c_1c_2c_1^{-1}, \dots, c_1\,;\, h_1, 1, a_1, b_1, \dots, a_{g'}, b_{g'})\,.$$

4. Change $h_1, 1$ to h_1, h_1 according to the automorphism of [Z87, 2.1.b)].
5. Apply again Proposition 6.2 of [CLP12] (i) with $\ell = 1$ (see (14) and figure 2 again):

$$\begin{aligned}
&(c_1c_2c_1^{-1}, \dots, c_1\,;\, h_1, h_1, a_1, b_1, \dots, a_{g'}, b_{g'})\\
\approx&(c_1c_2c_1^{-1}, \dots, (c_1h_1h_1h_1^{-1})c_1(c_1h_1h_1h_1^{-1})^{-1}\,;\, c_1h_1, h_1, a_1, b_1, \dots, a_{g'}, b_{g'})\\
=&(c_1c_2c_1^{-1}, \dots, g_ic_1g_i^{-1}\,;\, c_1h_1, h_1, a_1, b_1, \dots, a_{g'}, b_{g'})\\
=&(c_1c_2c_1^{-1}, \dots, c_1'\,;\, c_1h_1, h_1, a_1, b_1, \dots, a_{g'}, b_{g'})\,.
\end{aligned}$$

6. Use the braid group to move the last monodromy to the first position:

$$\begin{aligned}
&(c_1c_2c_1^{-1}, \dots, c_1'\,;\, c_1h_1, h_1, a_1, b_1, \dots, a_{g'}, b_{g'})\\
\approx\ &(c_1', \overline{c_2}, \dots, \overline{c_d}\,;\, c_1h_1, h_1, a_1, b_1, \dots, a_{g'}, b_{g'})
\end{aligned}$$

where $\overline{c_i}$ is a conjugate of c_i, $\forall i$.
7. Repeat the steps above for $\overline{c_2}$ and so on.

□

Remark 4.4. The condition $G = \langle v_{d+1}, \dots, v_{d+2g'}\rangle = \langle w_{d+1}, \dots, w_{d+2g'}\rangle$ in Prop. 4.1 can be achieved by using Lemma 4.3.

2nd step: reduction to the case $N = 0$. Let v and w be Hurwitz generating systems as in the beginning of the section. By the 1st step we assume that $v_i = w_i$, $i = 1, \dots, d$, and that $G = \langle v_{d+1}, \dots, v_{d+2g'}\rangle = \langle w_{d+1}, \dots, w_{d+2g'}\rangle$. By hypothesis we have:

$$ev(\hat{w}) \equiv ev(\hat{v}) \prod_{\ell=1}^{N} \left(\widehat{x}_\ell \widehat{y}_\ell \widehat{z}_\ell^{\,-1} \widehat{y}_\ell^{\,-1}\right)^{\sigma_\ell} (\,\mathrm{mod}\ [F, R])\,, \tag{15}$$

where $x_\ell y_\ell z_\ell^{-1} y_\ell^{-1} = 1$, $\sigma_\ell = \pm 1$, $x_\ell, z_\ell \in \Gamma$.

The main result of this subsection is the following

Proposition 4.5. *Let $v, w \in HS(G; g', d)$ be Hurwitz generating systems with $\nu(v) = \nu(w)$ and $\varepsilon(v) = \varepsilon(w)$. Assume further $v_i = w_i$, $i = 1, \ldots, d$, and $G = \langle v_{d+1}, \ldots, v_{d+2g'}\rangle = \langle w_{d+1}, \ldots, w_{d+2g'}\rangle$. Then there exist $h \in \mathbb{N}$, $\varphi, \psi \in Map(g' + h, d)$ such that*

$$\begin{aligned} ev(\widehat{\psi \cdot w^h}) &\equiv ev(\widehat{\varphi \cdot v^h}) \quad (\bmod\ [F, R])\,, \text{ and} \\ (\psi \cdot w^h)_i &= (\varphi \cdot v^h)_i\,, \quad \forall i = 1, \ldots, d\,. \end{aligned}$$

To prove Proposition 4.5, we first rewrite $\prod_{\ell=1}^{N} \left(\widehat{x_\ell}\widehat{y_\ell}\widehat{z_\ell}^{-1}\widehat{y_\ell}^{-1}\right)^{\sigma_\ell}$ in (15) as a product of commutators of the form $[\hat\eta, \hat\xi]$ where $\xi \in \Gamma$ and $[\eta, \xi] = 1$. To achieve this, we use the following identities.

Lemma 4.6. *Let $x, y, z, y_1, z_1 \in G$ be such that*

$$xyz^{-1}y^{-1} = zy_1{z_1}^{-1}{y_1}^{-1} = 1\,.$$

Then the following congruences hold, where as usual $\hat x, \hat y, \hat z, \widehat{y_1}, \widehat{z_1} \in F$ are the tautological lifts of x, y, z, y_1, z_1.

(i) $\widehat{xy}\widehat{z}^{-1}\widehat{y}^{-1} \equiv \widehat{y}^{-1}\widehat{xy}\widehat{z}^{-1} \quad (\bmod\ [F, R])$.

(ii) $(\widehat{xy}\widehat{z}^{-1}\widehat{y}^{-1})^{-1} = \widehat{yz}\widehat{y}^{-1}\widehat{x}^{-1} \equiv \widehat{zy}^{-1}\widehat{x}^{-1}\widehat{y} \quad (\bmod\ [F, R])$.

(iii) $(\hat x\hat y\hat z^{-1}\hat y^{-1})(\hat z\widehat{y_1}\widehat{z_1}^{-1}\widehat{y_1}^{-1}) \equiv \hat x\widehat{yy_1}\widehat{z_1}^{-1}\widehat{yy_1}^{-1} \quad (\bmod\ [F, R])$.

(iv) $\hat x\hat y^\sigma\hat z^{-1}\hat y^{-\sigma} \equiv \hat x\widehat{y^\sigma}\hat z^{-1}\widehat{y^\sigma}^{-1} \quad (\bmod\ [F, R])$*, where $\sigma = \pm 1$.*

Proof. (i) and (ii) follows from the fact that, if $r \in R$, then

$$r \equiv \widehat{y}^{-1} r \widehat{y}\ (\ \bmod\ [F, R]).$$

For (iv) we will use that $[F, R]$ is a normal subgroup of F.

(iii) Using (i) we have:

$$(\hat x\hat y\hat z^{-1}\hat y^{-1})(\hat z\widehat{y_1}\widehat{z_1}^{-1}\widehat{y_1}^{-1}) \equiv (\hat y^{-1}\hat x\hat y\hat z^{-1})(\hat z\widehat{y_1}\widehat{z_1}^{-1}\widehat{y_1}^{-1}) = \hat y^{-1}\hat x\hat y\widehat{y_1}\widehat{z_1}^{-1}\widehat{y_1}^{-1} \quad (\ \bmod\ [F, R])\,.$$

Moreover the above element is in R, since it is congruent to a product of two elements in R modulo $[F, R]$. Hence we have by the usual token:

$$\hat y^{-1}\hat x\hat y\widehat{y_1}\widehat{z_1}^{-1}\widehat{y_1}^{-1} \equiv \hat x\hat y\widehat{y_1}\widehat{z_1}^{-1}\widehat{y_1}^{-1}\hat y^{-1} \quad (\ \bmod\ [F, R])\,,$$

in fact the right hand side is just the conjugate of the left hand side by $\widehat{y}$. Finally

$$(\hat x\hat y\widehat{y_1}\widehat{z_1}^{-1}\widehat{y_1}^{-1}\hat y^{-1})(\hat x\widehat{yy_1}\widehat{z_1}^{-1}\widehat{yy_1}^{-1})^{-1} = (\hat x\hat y\widehat{y_1})[\widehat{z_1}^{-1}, \widehat{y_1}^{-1}\hat y^{-1}\widehat{yy_1}](\hat x\hat y\widehat{y_1})^{-1} \in [F, R]\,.$$

(iv) We have: $(\hat x\hat y^\sigma\hat z^{-1}\hat y^{-\sigma})(\hat x\widehat{y^\sigma}\hat z^{-1}\widehat{y^\sigma}^{-1})^{-1} = (\hat x\hat y^\sigma)[\hat z^{-1}, \hat y^{-\sigma}\widehat{y^\sigma}](\hat x\hat y^\sigma)^{-1} \in [F, R]$.

□

Lemma 4.7. *Let v, w be as in Proposition 4.5. Then there exist $M \in \mathbb{N}$ and, for $m = 1, \ldots, M$, elements $\xi_m \in \Gamma$ and $\eta_m \in G$ with $[\xi_m, \eta_m] = 1$, such that*

$$ev(\hat w) \equiv ev(\hat v)\prod_{m=1}^{M}[\widehat{\xi_m}, \widehat{\eta_m}] \quad (\bmod\ [F, R])\,.$$

Proof. Using Lemma 4.6 (ii) and (iv), rewrite (15) as

$$ev(\hat w) \equiv ev(\hat v)\prod_{\ell=1}^{N}\widehat{a_\ell}\widehat{b_\ell}\widehat{c_\ell}^{-1}\widehat{b_\ell}^{-1}(\bmod\ [F, R])\,, \tag{16}$$

where $a_\ell = x_\ell$, $b_\ell = y_\ell$, $c_\ell = z_\ell$, if $\sigma_\ell = 1$, and $a_\ell = z_\ell$, $b_\ell = y_\ell^{-1}$, $c_\ell = x_\ell$, if $\sigma_\ell = -1$.

Consider the image of (16) in the abelianized group F^{ab}. Since $v_i = w_i$, $i = 1, \dots, d$, we get:

$$\prod_{\ell=1}^{N} \widehat{a_\ell}\widehat{c_\ell}^{-1} = 1(\bmod [F,F]) .$$

Hence there exists a permutation $\tau \in \mathfrak{S}_N$, such that $c_\ell = a_{\tau(\ell)}$, for any $\ell = 1, \dots, N$.

Let us treat first the case where τ is a cycle of length N: then the set $\{a_{\tau^k(1)} \,|\, k \in \mathbb{N}\} = \{a_1, \dots, a_N\}$ and, since the product of the factors (mod $[F,R]$) is independent of the order,

$$\prod_{\ell=1}^{N} \widehat{a_\ell}\widehat{b_\ell}\widehat{c_\ell}^{-1}\widehat{b_\ell}^{-1} \equiv (\widehat{a_1}\widehat{b_1}\widehat{c_1}^{-1}\widehat{b_1}^{-1}) \prod_{k=1}^{N-1} \left(\widehat{c_{\tau^{k-1}(1)}}\widehat{b_{\tau^k(1)}}\widehat{c_{\tau^k(1)}}^{-1}\widehat{b_{\tau^k(1)}}^{-1} \right) (\bmod [F,R]) .$$

Setting $\xi_1 := a_1$ and $\eta_1 := \prod_{k=0}^{N-1} b_{\tau^k(1)}$, we obtain:

$$\prod_{\ell=1}^{N} \widehat{a_\ell}\widehat{b_\ell}\widehat{c_\ell}^{-1}\widehat{b_\ell}^{-1} \equiv [\widehat{\xi_1}, \widehat{\eta_1}] \quad (\bmod [F,R]) ,$$

where the equivalence follows from Lemma 4.6, (iii). Since the left hand side of the above equivalence is in R, it follows that $[\xi_1, \eta_1] = 1$.

The general case, where τ is a product of cycles of length $< N$, follows by induction. □

To complete the proof of Proposition 4.5, we need to know how $ev(\hat{v})$ changes under the action of the mapping class group, modulo $[F,R]$. Notice in fact that $ev(\hat{v})$ is $Map(g', d)$-invariant only modulo R_Γ.

Lemma 4.8. *Let $v \in HS(G; g', d)$ and let $\varphi \in Map(g', d)$. Then we have:*

(i) $ev(\widehat{\varphi \cdot v}) = ev(\hat{v})$, *if* $(\varphi \cdot v)_i = v_i$, $\forall i = 1, \dots, d$;

(ii) $ev(\widehat{\varphi \cdot v}) \equiv ev(\hat{v}) \left(\widehat{v_{i+1}}^{-1}\widehat{v_i}^{-1}\widehat{v_{i+1}}\widehat{v_{i+1}^{-1}v_iv_{i+1}} \right) \quad (\bmod [F,R])$, *if φ is the half-twist* $\sigma_i(v_i, v_{i+1}) = (v_{i+1}, v_{i+1}^{-1}v_iv_{i+1})$;

(iii) $ev(\widehat{\varphi \cdot v}) \equiv ev(\hat{v}) \left(\widehat{g}\widehat{v_d}^{-1}\widehat{g}^{-1}\widehat{gv_dg^{-1}} \right)$ (mod $[F,R]$), *if φ is as in* (14) *(Proposition 6.2 i) of* [CLP12] *with $\ell = 1$) and then* $g = v_dv_{d+1}v_{d+2}v_{d+1}^{-1}$.

Proof. The proof is similar to that of Proposition 3.6 of [CLP12] (invariance of the ε-invariant under $Map(g', d)$).

Let $\alpha \colon \frac{F}{[F,R]} \to G$ be the morphism $\hat{g} \mapsto g$. Then $\ker(\alpha) = \frac{R}{[F,R]}$ is central in $\frac{F}{[F,R]}$. We have that

$$\alpha((\widehat{\varphi \cdot v})_i) = (\varphi \cdot v)_i = \alpha((\varphi \cdot \hat{v})_i) , \quad \forall i = 1, \dots, d + 2g' .$$

Hence, there are $\zeta_i \in \frac{R}{[F,R]}$ such that

$$(\widehat{\varphi \cdot v})_i = (\varphi \cdot \hat{v})_i \cdot \zeta_i , \quad \forall i = 1, \dots, d + 2g' .$$

Since $\frac{R}{[F,R]} \leq \frac{F}{[F,R]}$ is central, we can replace $(\widehat{\varphi \cdot v})_i$ with $(\varphi \cdot \hat{v})_i$, $\forall i = d+1, \dots, d+2g'$, in the expression for $ev(\widehat{\varphi \cdot v})$. This is enough to prove the claim if φ acts as the identity on v_i, $i = 1, \dots, d$. Indeed, in this case, we have:

$$ev(\widehat{\varphi \cdot v}) = ev(\varphi \cdot \hat{v}) = ev(\hat{v}) ,$$

by the fact that evaluation is invariant under mapping classes. This proves (i).

(ii) follows by a direct computation.

Finally, consider the case where φ is as defined in (14) (Proposition 6.2 i) of [CLP12] with $\ell = 1$). We have that $(\varphi \cdot v)_d = g v_d g^{-1}$ and $(\varphi \cdot v)_i = v_i$, $i = 1, \ldots, d-1$, where $g \in G$ is given above. Let $x \in \frac{F}{[F,R]}$ such that $(\varphi \cdot \hat{v})_d = x \widehat{v_d} x^{-1}$. Since $\alpha(x) = g$, there is $\eta \in \frac{R}{[F,R]}$ – hence central – such that $x = \widehat{g}\eta$. It follows that

$$(\varphi \cdot \hat{v})_d = (\widehat{g}\eta)\widehat{v_d}(\widehat{g}\eta)^{-1} = \widehat{g}\widehat{v_d}\widehat{g}^{-1} .$$

Now notice that

$$\begin{aligned}
(\widehat{\varphi \cdot v})_d &= \widehat{g v_d g^{-1}} \\
&= \widehat{g}\widehat{v_d}\widehat{g}^{-1} \left(\widehat{g}\widehat{v_d}^{-1}\widehat{g}^{-1}\widehat{g v_d g^{-1}} \right) \\
&= (\varphi \cdot \hat{v})_d \left(\widehat{g}\widehat{v_d}^{-1}\widehat{g}^{-1}\widehat{g v_d g^{-1}} \right) ,
\end{aligned}$$

from which the claim follows. □

Proof of Proposition 4.5. Let $M \in \mathbb{N}$, $\xi_m \in \Gamma$ and $\eta_m \in G$ be as in Lemma 4.7. Then

$$ev(\hat{w}) \equiv ev(\hat{v}) \prod_{m=1}^{M} [\widehat{\xi_m}, \widehat{\eta_m}] \quad (\bmod [F,R]) . \tag{17}$$

Since $\xi_1 \in \Gamma$, there exists $k \in \{1, \ldots, d\}$ such that v_k is conjugate to ξ_1. Without loss of generality, assume $k = d$. Arguing as in Proposition 4.2, there exists $\varphi_1' \in Map(g'+1, d)$ such that

$$\begin{aligned}
&ev(\widehat{\varphi_1' \cdot w^1}) \equiv ev(\widehat{\varphi_1' \cdot v^1}) \prod_{m=1}^{M} [\widehat{\xi_m}, \widehat{\eta_m}] \quad (\bmod [F,R]) , \\
&(\varphi_1' \cdot w^1)_i = (\varphi_1' \cdot v^1)_i , \quad \forall i = 1, \ldots, d , \\
&(\varphi_1' \cdot w^1)_d = \xi_1 .
\end{aligned} \tag{18}$$

Equation (18) holds true because of (17), Lemma 4.8 and Lemma 4.3. Moreover there exists also $\varphi_1 \in Map(g'+2, d)$ such that $\varphi_1 \cdot v^2 = (\varphi_1' \cdot v^1)^1$, the vector obtained by adding one handle with trivial monodromies to $\varphi_1 \cdot v^1$.

By the argument of the proof of Lemma 4.3, we have that this vector is $Map(g'+2, d)$-equivalent to

$$(v_1, \ldots, v_{d-1}, \xi_1; \eta_1, 1, (\varphi_1' \cdot v^1)_{d+1}, (\varphi_1' \cdot v^1)_{d+2}, v_{d+1}, \ldots, v_{d+2g'}) .$$

using only transformations as in Lemma 4.8 i). Here we use the assumption that the v_j, $j > d$, generate G.

Using the automorphisms of [Z87, 2.1.a) and b)] the vector is further equivalent to

$$(v_1, \ldots, v_{d-1}, \xi_1; 1, \eta_1, (\varphi_1' \cdot v^1)_{d+1}, (\varphi_1' \cdot v^1)_{d+2}, v_{d+1}, \ldots, v_{d+2g'}) .$$

Now, by (14) (Proposition 6.2 (i) with $\ell = 1$ of [CLP12]) there exists $\varphi_2 \in Map(g'+2, d)$ such that

$$\begin{aligned}
&\varphi_2 \cdot (\varphi_1 \cdot v^2) \\
= \ &(v_1, \ldots, v_{d-1}, \xi_1\eta_1\xi_1\eta_1^{-1}\xi_1^{-1}; \xi_1, \eta_1, (\varphi_1' \cdot v^1)_{d+1}, (\varphi_1' \cdot v^1)_{d+2}, v_{d+1}, \ldots, v_{d+2g'}) \\
= \ &(v_1, \ldots, v_{d-1}, \xi_1; \xi_1, \eta_1, (\varphi_1' \cdot v^1)_{d+1}, (\varphi_1' \cdot v^1)_{d+2}, v_{d+1}, \ldots, v_{d+2g'}) ,
\end{aligned}$$

since $[\xi_1, \eta_1] = 1$ according to Lemma 4.7. The same property implies $ev(\widehat{\varphi_2 \cdot \varphi_1 \cdot v^2}) = ev(\widehat{\varphi_1 \cdot v^2})[\widehat{\xi_1}, \widehat{\eta_1}]$.

This proves the desired assertion if $M = 1$. The general case, where $M > 1$, is proven inductively along the same lines.

□

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