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Faithful actions of the absolute Galois group on connected components of moduli spaces

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Abstract We use a canonical procedure associating to an algebraic number a first a hyperelliptic curve C_a, and then a triangle curve (D_a, G_a) obtained through the normal closure of an associated Belyi function. In this way we show that the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of isomorphism classes of marked triangle curves, and on the set of connected components of marked moduli spaces of surfaces isogenous to a higher product (these are the free quotients of a product $C_1 \times C_2$ of curves of respective genera $g_1, g_2 \geq 2$ by the action of a finite group G). We show then, using again the surfaces isogenous to a product, first that it acts faithfully on the set of connected components of the moduli space of surfaces of general type (amending an incorrect proof in a previous arXiv version of the paper); and then, as a consequence, we obtain our main result: for each element $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, not in the conjugacy class of complex conjugation, there exists a surface of general type X such that X and the Galois conjugate surface X^σ have nonisomorphic fundamental groups. Using polynomials with only two

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critical values, we can moreover exhibit infinitely many explicit examples of such a situation.

1 Introduction

In the 60's J. P. Serre showed in [19] that there exists a field automorphism $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and a variety X defined over $\bar{\mathbb{Q}}$ such that X and the Galois conjugate variety X^σ have non isomorphic fundamental groups, in particular they are not homeomorphic.

In this note we give new examples of this phenomenon, using the so-called 'surfaces isogenous to a product' whose weak rigidity was proven in [7] (see also [8]) and which by definition are quotients of a product of curves $(C_1 \times C_2)$ of respective genera at least 2 by the free action of a finite group G.

One of our main results is a strong sharpening of the phenomenon discovered by Serre: observe in this respect that, if $\mathfrak{c}$ denotes complex conjugation, then X and $X^{\mathfrak{c}}$ are diffeomorphic.

Theorem 1.1 (= Theorem 6.8.) *If* $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *is not in the conjugacy class of* $\mathfrak{c}$*, then there exists a surface isogenous to a product* X *such that* X *and the Galois conjugate variety* X^σ *have non-isomorphic fundamental groups.*

Moreover, we give some faithful actions of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, related among them.

The following results are based on the concept of a (symmetry-) G-marked variety. A G-marked variety is a triple (X, G, η) where X is a projective variety, and $\eta\colon G \to \mathrm{Aut}(X)$ is an injective homomorphism (one says also that we have an effective action of the group G on X): here two such triples (X, G, η), (X', G, η') are isomorphic iff there is an isomorphism $f\colon X \to X'$, such that f carries the first action η to the second one η' (i.e., such that $\eta' = \mathrm{Ad}(f) \circ \eta$, where $\mathrm{Ad}(f)(\phi) := f\phi f^{-1}$). A particular case of marking is the one where $G \subset \mathrm{Aut}(X)$ and η is the inclusion: in this case we may denote a marked variety simply by the pair (X, G).

Theorem 1.2 (= Theorem 4.6.) *There is an action of* $\mathrm{Aut}(\mathbb{C})$ *on the set of isomorphism classes of marked triangle curves obtained by taking the conjugate by an automorphism* $\sigma \in \mathrm{Aut}(\mathbb{C})$*. This action factors through an action of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*. If* $a \in \bar{\mathbb{Q}} \setminus \mathbb{Z}$ *and the isomorphism class of the marked triangle curve* (D_a, G_a) *is fixed by* $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*, then* $\sigma(a) = a$*; in particular, the above action is faithful.*

Theorem 1.3 (= Theorem 5.8.) *The absolute Galois group* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on the set of connected components of the (coarse) moduli space of étale marked surfaces of general type.*

With a rather elaborate strategy we can then show the stronger result:

Theorem 1.4 (= Theorem 6.4.)*The absolute Galois group* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on the set of connected components of the (coarse) moduli space* $\mathfrak{M}$ *of surfaces of general type.*

Our method is closely related to the theorem of Belyi [5], which motivated the so-called theory of 'dessins d' enfants' (see [17]); these are, in view of Riemann's existence theorem (generalized by Grauert and Remmert in [15]), a combinatorial way to look at the monodromies of algebraic functions with only three branch points. We make in this paper an essential use of Belyi functions [5] and of their functoriality. Our point of view is however more related to the normal closure of Belyi functions, the so called marked triangle curves, i.e., pairs (C, G) with $G \subset \mathrm{Aut}(C)$ such that the quotient $C/G \cong \mathbb{P}^1$ and the quotient map is branched exactly in three points.

In the first section we describe a simple but canonical construction which, for each choice of an integer $g \geq 3$, associates to a complex number $a \in \mathbb{C} \setminus \mathbb{Q}$ a hyperelliptic curve C_a of genus g, and in such a way that $C_a \cong C_b$ iff $a = b$.

In the later sections we construct the associated triangle curves (D_a, G_a) and prove the above theorems.

It would be interesting to obtain similar types of results, for instance forgetting about the markings in the case of triangle curves, or even using only Beauville surfaces (these are the surfaces isogenous to a product which are rigid: see [7] for the definition of Beauville surfaces and [2,3,8] for further properties of these).

Theorem 6.4 was announced by the second author at the Alghero Conference 'Topology of algebraic varieties' in September 2006, and asserted (with an incorrect proof) in the previous ArXiv version of the paper [4]. The survey article [10] was then written after we realized of the gap in the proof. The results of the present article give in our opinion support to some conjectures made previously (see [4,10]).

The main new input of the present paper is the systematic use of twists of the second component of an action on a product $C_1 \times C_2$ by an automorphism of the group G and the discovery that this leads to an injective homomorphism of the Kernel $\mathfrak{K}$ of the action (on the set of connected components $\pi_0(\mathfrak{M})$ of the moduli space of surfaces of general type) into some Abelian group of the form $\oplus_G(Z(\mathrm{Out}(G))$, Z denoting the centre of a group. Then we use a known result (cf. [12]) that $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ does not contain any nontrivial normal abelian subgroup.

Observe that Robert Easton and Ravi Vakil (in [11]), with a completely different type of examples, obtained a result which is weaker than Theorem 6.4, they showed indeed that the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ operates faithfully on the set of irreducible components of the moduli spaces of surfaces of general type.

In the last section we use Beauville surfaces and polynomials with two critical values in order to produce infinitely many explicit and simple examples of pairs of surfaces of general type with nonisomorphic fundamental groups which are conjugate under the absolute Galois group (observe in particular that the two fundamental groups have then isomorphic profinite completions).

2 Very special hyperelliptic curves

Fix a positive integer $g \in \mathbb{N}$, $g \geq 3$, and define, for any complex number $a \in \mathbb{C} \setminus \{-2g, 0, 1, \dots, 2g-1\}$, C_a as the hyperelliptic curve of genus g which is the smooth complete model of the affine curve of equation

$$w^2 = (z-a)(z+2g)\Pi_{i=0}^{2g-1}(z-i).$$

C_a is the double covering of $\mathbb{P}^1_{\mathbb{C}}$ branched over $\{-2g, 0, 1, \dots, 2g-1, a\} \subset \mathbb{P}^1_{\mathbb{C}}$. We can define more generally, for each $a \in \mathbb{C}$, the complete curve C_a as the subscheme of the weighted projective plane $\mathbb{P}(1, 1, g+1)$, with coordinates (z_0, z_1, w), defined by the equation

$$w^2 = (z_1 - az_0)(z_1 + 2gz_0)\Pi_{i=0}^{2g-1}(z_1 - iz_0).$$

C_a is smooth if and only if $a \in \mathbb{C} \setminus \{-2g, 0, 1, \dots, 2g-1\}$. We shall call such a smooth curve C_a a very special hyperelliptic curve.

Proposition 2.1 *(1) Consider two complex numbers a, b such that $a \in \mathbb{C} \setminus \mathbb{Q}$: then $C_a \cong C_b$ if and only if $a = b$.*
(2) Assume now that $g \geq 6$ and let $a, b \in \mathbb{C} \setminus \{-2g, 0, 1, \dots, 2g-1\}$ be two complex numbers. Then $C_a \cong C_b$ if and only if $a = b$.

Proof One direction being obvious, assume that $C_a \cong C_b$.

Then the isomorphism between C_a and C_b induces a projectivity $\varphi : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ making the two sets with $2g+2$ elements $B_a := \{-2g, 0, 1, \dots, 2g-1, a\}$ and $B_b := \{-2g, 0, 1, \dots, 2g-1, b\}$ projectively equivalent over $\mathbb{C}$ (the latter set B_b has also cardinality $2g+2$ since $C_a \cong C_b$ and C_a smooth implies that also C_b is smooth).

In fact, the projectivity $\varphi : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ taking B_a to B_b is defined over $\mathbb{Q}$, since there are three rational numbers which are carried into three rational numbers (because $g \geq 2$).

(1) Since $a \notin \mathbb{Q}$ it follows that $\varphi(a) \notin \mathbb{Q}$ hence $\varphi(a) = b \notin \mathbb{Q}$ and φ maps $B := \{-2g, 0, 1, \dots 2g-1\}$ to B, and in particular φ has finite order.

Since φ yields an automorphism of $\mathbb{P}^1_{\mathbb{R}}$, it either leaves the cyclic order of $(-2g, 0, 1, \ldots, 2g-1)$ invariant or reverses it, and since $g \geq 3$ we see that there are three consecutive integers such that φ maps them to three consecutive integers. Therefore φ is either an integer translation, or an affine symmetry of the form $x \mapsto -x+2n$, where $2n \in \mathbb{Z}$. In the former case $\varphi = id$, since it has finite order, and it follows in particular that $a = b$. In the latter case it must be $2g + 2n = \varphi(-2g) = 2g - 1$ and $2n = \varphi(0) = 2g - 2$, and we derive the contradiction $-1 = 2n = 2g - 2$.

(2) Since we dealt with the case $a \notin \mathbb{Q}$ in 1), and by symmetry with the case $b \notin \mathbb{Q}$, we may and do assume that $a, b \in \mathbb{Q}$.

Step I) We prove first that $\varphi(x) = \pm x + r$ where $r \in \mathbb{Z}$.

Observe in fact that in the set B_b each triple of consecutive points (for the cyclic order) is a triple of consecutive integers, if no element in the triple is $-2g$ or b. This excludes at most six triples. Keep in mind that $a \in B_a$ and consider all the triples of consecutive integers in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$: at most two such triples are not a triple of consecutive points of B_a [if $i < a < i+1$, then the triples are $(i-1, i, i+1)$ and $(i, i+1, i+2)$]. We conclude that there is a triple of consecutive integers in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ mapping to a triple of consecutive integers under φ (we have in fact at least $10 - 2 = 8$ such triples which are triples of consecutive points of B_a and we exclude at most 6 image triples of consecutive points in B_b which are not a triple of consecutive integers). Then either φ is an integer translation $x \mapsto x + n$, or it is a symmetry $x \mapsto -x + 2n$ with $2n \in \mathbb{Z}$.

Claim II: $\varphi(x) = \pm x + r \Rightarrow a = b$.

First proof of claim II:

If $\varphi(x) = x + r$, then B_b contains $\{r, \ldots, (2g-1) + r\}$ and either

(1) $r = 0$ and $\varphi = id$ (hence $a = b$)
(2) $r = 1$, B_b contains $2g$ and $-2g + 1$, a contradiction, or
(3) $r = -1$, B_b contains -1 and $-2g - 1$, a contradiction.

Similarly, if $\varphi(x) = -x + r$, B_b contains $\{r - (2g-1), \ldots, r\}$ and either

(1) $r = 2g - 1$ and $a = b = 4g - 1$ or
(2) $r = 2g$, B_b contains $2g$ and $4g$, a contradiction, or
(3) $r = 2g - 2$, B_b contains -1 and $4g - 2$, a contradiction.

□

Second proof of claim II:

In both cases the intervals equal to the respective convex spans of the sets B_a, B_b are sent to each other by φ, in particular the length is preserved and the extremal points are permuted. If $a \in [-2g, 2g-1]$ also $b \in [-2g, 2g-1]$ and in the translation case $r = 0$, so that $\varphi(x) = x$ and $a = b$. We see right away that φ cannot be a symmetry, because only two points belong to the left half of the interval.

If $a < -2g$ the interval has length $2g - 1 - a$, if $a > 2g - 1$ the interval has length $2g + a$. Hence, if both $a, b < -2g$, since the length is preserved, we find that $a = b$; similarly if $a, b > 2g - 1$.

By symmetry of the situation, we only need to exclude the case $a < -2g$, $b > 2g - 1$: here we must have $2g - 1 - a = 2g + b$, i.e., $a = -b - 1$. If φ is a symmetry then $b = \varphi(a)$, and we derive the same contradiction as in 1), since then $\varphi(-2g) = 2g + r = 2g - 1 \Rightarrow r = -1$, hence $\varphi(0) = -1 \notin B_b$, absurd. If instead we have a translation $\varphi(x) = x + r$, since $\varphi(2g - 1) = b$, $\varphi(2g - 2) = 2g - 1$, it follows that $r = 1$, and then $\varphi(-2g) = -2g + 1$ gives a contradiction. □

We shall assume from now on that $a, b \in \bar{\mathbb{Q}} \setminus \mathbb{Q}$ and that there is a field automorphism $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(a) = b$. (Obviously, for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ different from the identity, there are $a, b \in \bar{\mathbb{Q}} \setminus \mathbb{Q}$ with $\sigma(a) = b$ and $a \neq b$.)

The following is an application of Belyi's celebrated theorem asserting that an algebraic curve C can be defined over $\bar{\mathbb{Q}}$ if and only if it admits a Belyi function, i.e., a holomorphic function $f : C \to \mathbb{P}^1$ whose only critical values are in the set $\{0, 1, \infty\}$. The main assertion concerns the functoriality of a certain Belyi function.

Proposition 2.2 *Let $a \in \bar{\mathbb{Q}}$, let $P \in \mathbb{Q}[x]$ be the minimal polynomial of a, and consider the field $L := \mathbb{Q}[x]/(P)$. Let C_x be the hyperelliptic curve over $Spec(L)$ defined by the equation*

$$w^2 = (z_1 - xz_0)(z_1 + 2gz_0)\Pi_{i=0}^{2g-1}(z_1 - iz_0).$$

Then there is a rational function $F_x : C_x \to \mathbb{P}^1_L$ such that, for each $a \in \mathbb{C}$ with $P(a) = 0$, the rational function F_a (obtained under the specialization $x \mapsto a$) is a Belyi function for C_a.

Proof Let $f_x : C_x \to \mathbb{P}^1_L$ be the hyperelliptic quotient map, branched in $\{-2g, 0, 1, \ldots, 2g - 1, x\}$. Then $P \circ f_x$ has as critical values:

- the images of the critical values of f_x under P, which are in $\mathbb{Q}$,
- the critical values y of P, i.e. the zeroes of the discriminant $h_1(y)$ of $P(z) - y$ with respect to the variable z.

Since h_1 has degree $deg(P) - 1$, we obtain, inductively as in [5], $\tilde{f}_x := h \circ P \circ f$ whose critical values are all contained in $\mathbb{Q} \cup \{\infty\}$ (see [21] for more details). If we take any root a of P, then obviously $\tilde{f}_a$ has the same critical values as $\tilde{f}_x$.

Let now $r_1, \ldots, r_n \in \mathbb{Q}$ be the (pairwise distinct) finite critical values of $\tilde{f}_x$. We set:

$$y_i := \frac{1}{\Pi_{j \neq i}(r_i - r_j)}.$$

Let $N \in \mathbb{N}$ be a positive integer such that $m_i := N y_i \in \mathbb{Z}$. Then the rational function

$$g(t) := \Pi_i (t - r_i)^{m_i} \in \mathbb{Q}(t)$$

is ramified at most in ∞ and $r_1, \ldots r_n$. In fact, $g'(t)$ vanishes at most when $g(t) = 0$ or at the points where the logarithmic derivative $G(t) := \frac{g'(t)}{g(t)} = \sum_i m_i (\frac{1}{t - r_i})$ has a zero. However, $G(t)$ has simple poles at the n points $r_1, \ldots, r_n$ and by the choice made we claim that it has a zero of order n at ∞.

In fact, consider the polynomial $\frac{1}{N} G(t) \Pi_i (t - r_i)$, which has degree $\leq n - 1$ and equals

$$\sum_i y_i \Pi_{j \neq i}(t - r_j) = \sum_i \Pi_{j \neq i}(t - r_j) \frac{1}{\Pi_{j \neq i}(r_i - r_j)}.$$

It takes value 1 in each of the points $r_1, \ldots r_n$, hence it equals the constant 1.

It follows that the critical values of $g \circ \tilde{f}_x$ are at most $0, \ \infty, \ g(\infty)$.

We set $F_x := \Phi \circ g \circ \tilde{f}_x$ where Φ is the affine map $z \mapsto g(\infty)^{-1} z$, so that the critical values of F_x are equal to $\{0, 1, \infty\}$. It is obvious by our construction that for any root a of P, F_a has the same critical values as F_x, in particular, F_a is a Belyi function for C_a. □

Since in the sequel we shall consider the normal closure (we prefer here, to avoid confusion, not to use the term 'Galois closure' for the geometric setting) $\psi_a \colon D_a \to \mathbb{P}^1_{\mathbb{C}}$ of each of the functions $F_a \colon C_a \to \mathbb{P}^1_{\mathbb{C}}$, we recall in the next section the 'scheme theoretic' construction of the normal closure.

3 Effective construction of normal closures

In this section we consider algebraic varieties over the complex numbers, endowed with their Hausdorff topology, and, more generally, 'good' covering spaces (i.e., between topological spaces which are locally arcwise connected and semilocally simply connected).

Lemma 3.1 *Let $\pi : X \to Y$ be a finite 'good' unramified covering space of degree d between connected spaces X and Y.*

Then the normal closure Z of $\pi: X \to Y$ *(i.e., the minimal unramified covering of Y factoring through* π*, and such that there exists an action of a finite group G with* $Y = Z/G$*) is isomorphic to any connected component of*

$$W := W_\pi := (X \times_Y \ldots \times_Y X) \backslash \Delta \subset X^d \backslash \Delta,$$

where $\Delta := \{(x_1, \ldots, x_d) \in X^d | \exists i \neq j \ , \ x_i = x_j\}$ *is the big diagonal (also called 'fat diagonal').*

Proof Choose base points $x_0 \in X$, $y_0 \in Y$ such that $\pi(x_0) = y_0$ and denote by F_0 the fibre over y_0, $F_0 := \pi^{-1}(\{y_0\})$.

We consider the monodromy $\mu\colon \pi_1(Y, y_0) \to \mathfrak{S}_d = \mathfrak{S}(F_0)$ of the unramified covering π. The monodromy of $\phi\colon W \to Y$ is induced by the diagonal product monodromy $\mu^d : \pi_1(Y, y_0) \to \mathfrak{S}(F_0^d)$, such that, for $(x_1, \ldots, x_d) \in F_0^d$, we have $\mu^d(\gamma)(x_1, \ldots, x_d) = (\mu(\gamma)(x_1), \ldots, \mu(\gamma)(x_d))$.

Observe that the points of $W \cap F_0^d$ are just sequences of d distinct points $(x_1, \ldots, x_d)$ in F_0: hence, once we choose a bijection of F_0 with the set $\{1, 2, \ldots, d\}$, i.e., a base point $(\xi_1, \ldots, \xi_d) \in W \cap F_0^d$, then to $(x_1, \ldots, x_d)$ we associate the permutation such that $\xi_i \mapsto x_i, \forall i$. We obtain in this way an identification of $W \cap F_0^d$ with $\mathfrak{S}_d$.

It follows that the monodromy of $\phi\colon W \to Y$, $\mu_W : \pi_1(Y, y_0) \to \mathfrak{S}(\mathfrak{S}_d)$ is given by left translation $\mu_W(\gamma)(\tau) = \mu(\gamma) \circ (\tau)$.

If we denote by $G := \mu(\pi_1(Y, y_0)) \subset \mathfrak{S}_d$ the monodromy group, it follows right away that the components of W correspond to the cosets $G\tau$ of G. Thus all the components yield isomorphic covering spaces. □

The theorem of Grauert and Remmert [15] allows to extend the above construction to yield normal closures of finite morphisms between normal algebraic varieties. [1]

Corollary 3.2 *Let* $\pi : X \to Y$ *be a finite morphism between normal projective varieties, let* $B \subset Y$ *be the branch locus of* π *and set* $X^0 := X \setminus \pi^{-1}(B)$*,* $Y^0 := Y \setminus B$*.*

If X is connected, then the normal closure Z of π *is isomorphic to any connected component of the closure of* $W^0 := (X^0 \times_{Y^0} \ldots \times_{Y^0} X^0)\backslash\Delta$ *in the normalization* W^n *of* $W := \overline{(X \times_Y \ldots \times_Y X) \setminus \Delta}$.

Proof The irreducible components of W correspond to the connected components of W^0, as well as to the connected components Z of W^n. So, our component Z is the closure of a connected component Z^0 of W^0. We know

[1] A referee pointed out that the geometric construction of the normal closure is also carefully described in Proposition 5.3.9 of [20].

that the monodromy group G acts on Z^0 as a group of covering transformations and simply transitively on the fibre of Z^0 over y_0: by normality the action extends biholomorphically to Z, and clearly $Z/G \cong Y$. □

4 Faithful action of the absolute Galois group on the set of marked triangle curves (associated to very special hyperelliptic curves)

Let a be an algebraic number, $g \geq 3$, and consider as in Sect. 2 the hyperelliptic curve C_a of genus g defined by the equation

$$w^2 = (z-a)(z+2g)\Pi_{i=0}^{2g-1}(z-i).$$

Let $F_a \colon C_a \to \mathbb{P}^1$ be the Belyi function constructed in Proposition 2.2 and denote by $\psi_a \colon D_a \to \mathbb{P}^1$ the normal closure of C_a as in Corollary 3.2.

Remark 4.1 (1) We denote by G_a the monodromy group of D_a and observe that there is a subgroup $H_a \subset G_a$ acting on D_a such that $D_a/H_a \cong C_a$.

(2) Observe moreover that the degree d of the Belyi function F_a depends not only on the degree of the field extension $[\mathbb{Q}(a) : \mathbb{Q}]$, but much more on the height of the algebraic number a; one may give an upper bound for the order of the group G_a in terms of these.

The pair (D_a, G_a) that we get is a so-called triangle curve (see [8], page 539). We need here the following refined definition:

Definition 4.2 (1) A *G-marked variety* is a triple (X, G, η) where X is a projective variety and $\eta \colon G \to \mathrm{Aut}(X)$ is an injective homomorphism

(2) equivalently, a marked variety is a triple (X, G, α) where $\alpha \colon X \times G \to X$ is a faithful action of the group G on X

(3) Two marked varieties (X, G, α), (X', G, α') are said to be *isomorphic* if there is an isomorphism $f \colon X \to X'$ transporting the action $\alpha \colon X \times G \to X$ into the action $\alpha' \colon X' \times G \to X'$, i.e., such that

$$f \circ \alpha = \alpha' \circ (f \times \mathrm{id}) \Leftrightarrow \eta' = \mathrm{Ad}(f) \circ \eta, \quad \mathrm{Ad}(f)(\phi) := f \phi f^{-1}.$$

(4) If G is a subset of $\mathrm{Aut}(X)$, then the natural marked variety is the triple (X, G, i), where $i \colon G \to \mathrm{Aut}(X)$ is the inclusion map, and shall sometimes be denoted simply by the pair (X, G).

(4) A marked curve (D, G, η) consisting of a smooth projective curve of genus g and a faithful action of the group G on D is said to be a *marked triangle curve of genus* g if $D/G \cong \mathbb{P}^1$ and the quotient morphism $p \colon D \to D/G \cong \mathbb{P}^1$ is branched in three points.

Remark 4.3 Observe that:

(1) we have a natural action of $\mathrm{Aut}(G)$ on marked varieties, namely

$$\psi(X, G, \eta) := (X, G, \eta \circ \psi^{-1}).$$

The corresponding equivalence class of a G-marked variety is defined to be a *G-(unmarked) variety.*

(2) The action of the group $\mathrm{Inn}(G)$ of inner automorphisms does not change the isomorphism class of (X, G, η) since, for $\gamma \in G$, we may set $f := (\eta(\gamma))$, $\psi := \mathrm{Ad}(\gamma)$, and then $\eta \circ \psi = \mathrm{Ad}(f) \circ \eta$, since $\eta(\psi(g)) = \eta(\gamma g \gamma^{-1}) = \eta(\gamma)\eta(g)(\eta(\gamma)^{-1}) = \mathrm{Ad}(f)(\eta(g))$.

(3) In the case where $G = \mathrm{Aut}(X)$, we see that $\mathrm{Out}(G)$ acts simply transitively on the isomorphism classes of the $\mathrm{Aut}(G)$-orbit of (X, G, η).

Consider now our triangle curve D_a: we showed in proposition 2.2 that the three branch points in $\mathbb{P}^1$ are $\{0, 1, \infty\}$ and we may choose a monodromy representation

$$\mu\colon \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to G_a$$

corresponding to the normal ramified covering $\psi_a\colon D_a \to \mathbb{P}^1$. Denote further by $\tau_0,\ \tau_1,\ \tau_\infty$ the images of geometric loops around $0,\ 1,\ \infty$. Then G_a is generated by $\tau_0,\ \tau_1,\ \tau_\infty$ and $\tau_0 \cdot \tau_1 \cdot \tau_\infty = 1$. By Riemann's existence theorem the datum of these three generators of the group G_a determines a marked triangle curve (see for instance [7], page 24, or [8], Definition 2.1 on page 544, or, for a more extensive treatment, Section 2 of [2]).

We recall the operation of conjugating a variety by a field automorphism.

Remark 4.4 (1) $\sigma \in \mathrm{Aut}(\mathbb{C})$ acts on $\mathbb{C}[z_0, \dots z_n]$, by sending the element $P(z) = \sum_{I=(i_0,\dots,i_n)} a_I z^I$ to

$$\sigma(P)(z) := \sum_{I=(i_0,\dots,i_n)} \sigma(a_I) z^I.$$

(2) Let X be a projective variety

$$X \subset \mathbb{P}^n_{\mathbb{C}},\ X := \{z | f_i(z) = 0\ \forall i\}.$$

The action of σ extends coordinatewise to $\mathbb{P}^n_{\mathbb{C}}$, and carries X to the set $\sigma(X)$ which is another variety, denoted X^σ, and called the *conjugate variety*. In fact, since $f_i(z) = 0$ if and only if $\sigma(f_i)(\sigma(z)) = 0$, one has that

$$X^\sigma = \{w | \sigma(f_i)(w) = 0\ \forall i\}.$$

(3) if $f: X \to Y$ is a morphism, its graph Γ_f being a subscheme of $X \times Y$, we get a conjugate morphism $f^\sigma: X^\sigma \to Y^\sigma$.
(4) if $G \subset \mathrm{Aut}(X)$, and $i: G \to \mathrm{Aut}(X)$ is the inclusion, then σ determines another marked variety $(X^\sigma, G, \mathrm{Ad}(\sigma) \circ i)$, image of (X, G, i).

In other words, we have $G^\sigma \subset \mathrm{Aut}(X^\sigma)$ in such a way that, if we identify G with G^σ via $\mathrm{Ad}(\sigma)$, then $(X/G)^\sigma \cong X^\sigma/G$.

In a similar concrete way we can define the σ-conjugate variety X^σ of a quasi-projective variety, except that, in order to show that the definition is independent of the given embedding, one is forced to give the following more abstract definition matching with the previously given one for quasi-projective varieties.

Definition 4.5 Let $\sigma : \mathbb{C} \to \mathbb{C}$ be a field automorphism. We introduce then the functor from the category of complex varieties to itself defined by

$$X^\sigma := X \otimes_{\mathbb{C},\sigma} \mathbb{C},$$

which satisfies the following properties:

(i) if X is defined over a subfield $k_0 \subset \mathbb{C}$, i.e., there is a k_0-scheme X_0 such that $X \cong X_0 \otimes_{k_0} \mathbb{C}$, then X^σ depends only on the restriction $\sigma|_{k_0}$ and if moreover σ is the identity on k_0, then X^σ is canonically isomorphic to X;
(ii) the formation of X^σ is compatible with products;
(iii) for a group action $G \times X \to X$ with G finite (hence defined over $\mathbb{Q}$ as an algebraic group) then $G^\sigma = G$ canonically and applying the conjugation functor gives a conjugate action $G \times X^\sigma = G^\sigma \times X^\sigma \to X^\sigma$.

To any algebraic number $a \notin \mathbb{Z}$ there corresponds, through a canonical procedure (depending on an integer $g \geq 3$), a marked triangle curve (D_a, G_a). We can prove now the following:

Theorem 4.6 *There is an action of* $\mathrm{Aut}(\mathbb{C})$ *on the set of isomorphism classes of marked triangle curves obtained by taking the conjugate by an automorphism* $\sigma \in \mathrm{Aut}(\mathbb{C})$. *This action factors through an action of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. *If* $a \in \bar{\mathbb{Q}} \setminus \mathbb{Z}$ *and the isomorphism class of the marked triangle curve* (D_a, G_a) *is fixed by* $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*, then* $\sigma(a) = a$*; in particular, the above action is faithful.*

Proof Let (D, G) be a marked triangle curve, and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$: extend σ to $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ and take the transformed curve D^σ and the transformed graph of the action, a subset of $D^\sigma \times D^\sigma \times G$.

Since there is only a finite number of isomorphism classes of such pairs (D, G) of a fixed genus g and with fixed group G, it follows that D is defined over $\bar{\mathbb{Q}}$ and the chosen extension of σ does not matter up to isomorphism.

Finally, apply the action of σ to the triangle curve (D_a, G_a) and assume that the isomorphism class of (D_a, G_a) is fixed by the action. This means then, setting $b := \sigma(a)$, that there is an isomorphism

$$f : D_a \to D_b = D_{\sigma(a)} = D_a^{\sigma}$$

such that $\mathrm{Ad}(f) = \mathrm{Ad}(\sigma)$.

In other words, σ identifies G_a with G_b by our assumption and the two actions of G_a on D_a and D_b are transported to each other by f. □

Lemma 4.7 $\mathrm{Ad}(\sigma)(H_a) = H_b$.

Proof of the Lemma Let K be the Galois closure of the field L introduced in proposition 2.2 (K is the splitting field of the field extension $\mathbb{Q} \subset L$), and view L as embedded in $\mathbb{C}$ under the isomorphism sending x to a.

Consider the curve $\hat{C}_x$ obtained from C_x by scalar extension $\hat{C}_x := C_x \otimes_L K$. Let also $\hat{F}_x := F_x \otimes_L K$ be the corresponding Belyi function with values in $\mathbb{P}^1_K$.

Apply now the effective construction of the normal closure of Sect. 3: hence, taking a connected component of $(\hat{C}_x \times_{\mathbb{P}^1_K} \ldots \times_{\mathbb{P}^1_K} \hat{C}_x) \setminus \Delta$ we obtain a curve D_x defined over K.

Note that D_x is not geometrically irreducible, but, once we tensor with $\mathbb{C}$, it splits into several components which are Galois conjugate and which are isomorphic to the conjugates of D_a.

Apply now the Galois automorphism σ to the triple $D_a \to C_a \to \mathbb{P}^1$. Since the triple is induced by the triple $D_x \to C_x \to \mathbb{P}^1_K$ by taking a tensor product $\otimes_K \mathbb{C}$ via the embedding sending x to a, and the morphisms are induced by the composition of the inclusion $D_x \subset (C_x)^d$ with the coordinate projections, respectively by the fibre product equation, it follows from Proposition 2.2 that σ carries the triple $D_a \to C_a \to \mathbb{P}^1$ to the triple $D_b \to C_b \to \mathbb{P}^1$. □

Since $\mathrm{Ad}(f) = \mathrm{Ad}(\sigma)$, under the isomorphism f the subgroup H_a corresponds to the subgroup H_b (i.e., $\mathrm{Ad}(f)(H_a) = H_b$).

We infer that, since $C_a = D_a/H_a$, $C_b = D_b/H_b$, f induces an isomorphism of C_a with C_b.

By Proposition 2.2 we conclude that $a = b$.

If we want to interpret our argument in terms of Grothendieck's étale fundamental group, we define $C_x^0 := F_x^{-1}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$, and accordingly $\hat{C}_x^0$ and D_x^0.

There are the following exact sequences for the Grothendieck étale fundamental group (compare Theorem 6.1 of [16]):

$$1 \to \pi_1^{alg}(D_a^0) \to \pi_1^{alg}(D_x^0) \to \mathrm{Gal}(\bar{\mathbb{Q}}/K) \to 1$$
$$1 \to \pi_1^{alg}(C_a^0) \to \pi_1^{alg}(\hat{C}_x^0) \to \mathrm{Gal}(\bar{\mathbb{Q}}/K) \to 1$$
$$1 \to \pi_1^{alg}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}) \to \pi_1^{alg}(\mathbb{P}^1_K \setminus \{0, 1, \infty\}) \to \mathrm{Gal}(\bar{\mathbb{Q}}/K) \to 1$$

where H_a and G_a are the respective factor groups for the (vertical) inclusions of the left hand sides, corresponding to the first and second sequence, respectively to the first and third sequence.

On the other hand, we also have the exact sequence

$$1 \to \pi_1^{alg}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}) \to \pi_1^{alg}(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}) \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to 1.$$

The finite quotient G_a of $\pi_1^{alg}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\})$ (defined over K) is sent by $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to another quotient, corresponding to $D_{\sigma(a)}$, and the subgroup H_a, yielding the quotient C_a, is sent to the subgroup $H_{\sigma(a)}$.

Remark 4.8 Assume that the two triangle curves D_a and $D_b = D_{\sigma(a)}$ are isomorphic through a complex isomorphism $f : D_a \to D_b$ (but without that necessarily $(f, \mathrm{Ad}(\sigma))$ yields an isomorphism of marked triangle curves $(D_a, G_a), (D_b, G_b)$).

We define $\psi : G_a \to G_a$ to be equal to $\psi := \mathrm{Ad}(\sigma^{-1} \circ f)$.

Then $\mathrm{Ad}(f) = \mathrm{Ad}(\sigma) \circ \psi$ and applying to $y \in D_b$, $y = f(x)$ we get

$$\mathrm{Ad}(f)(g)(y) = (\mathrm{Ad}(\sigma) \circ \psi)(g)(y) \Leftrightarrow f(g(x)) = (\mathrm{Ad}(\sigma) \circ \psi)(g)(f(x)).$$

Identifying G_a with G_b under $\mathrm{Ad}(\sigma)$, one can interpret the above formula as asserting that f is only 'twisted' equivariant ($f(g(x))$“ $=''$ $\psi(g)(f(x))$.

Proposition 4.9 *Assume that the two triangle curves D_a and $D_b = D_{\sigma(a)}$ are isomorphic under a complex isomorphism f and that the above automorphism $\psi \in \mathrm{Aut}(G)$ such that $f(g(x)) = (\mathrm{Ad}(\sigma) \circ \psi)(g)(f(x))$ is inner.*

Then $C_a \cong C_b$, hence $a = b$.

Proof If ψ is inner, then the marked triangle curves $(D_a, G_a) = (D_a, G_a, i_a)$ (i_a being the inclusion map of $G_a \subset \mathrm{Aut}(D_a)$), and its transform by σ, $(D_b, G_a^\sigma) = (D_b, G_a, \mathrm{Ad}(\sigma) \circ i_a)$ are isomorphic.

Then the argument of Theorem 4.6 implies that $C_a \cong C_b$, hence $a = b$. □

We pose here the following conjecture, which is a strengthening of the previous Theorem 4.6.

Conjecture 4.10 (Conjecture 2.13 in [10]) *The absolute Galois group* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on the set of isomorphism classes of (unmarked) triangle curves.*

The following definition will be useful in the proof of Theorem 6.4.

Definition 4.11 Let $a \in \bar{\mathbb{Q}}$ be an algebraic number and let (D_a, G_a) be the associated marked triangle curve obtained by the canonical procedure above (depending on an integer $g \geq 3$). Then $(D_a, \mathrm{Aut}(D_a))$ is called the *fully marked* triangle curve associated to a (it is a triangle curve by rigidity, see [8], page 545).

Remark 4.12 If we consider instead of (D_a, G_a) the fully marked triangle curve $(D_a, \mathrm{Aut}(D_a))$ we have also the subgroup $H_a \leq \mathrm{Aut}(D_a)$ such that $D_a/H_a = C_a$, where C_a is the very special hyperelliptic curve associated to the algebraic number a.

The same proof as the proof of Theorem 4.6 gives

Proposition 4.13 *The action of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *on the set of isomorphism classes of marked triangle curves, obtained by taking the conjugate by an automorphism* σ*, preserves fully marked triangle curves. The action on the set of isomorphism classes of fully marked triangle curves is faithful.*

5 Connected components of moduli spaces associated to very special hyperelliptic curves

Let us recall now the basic definitions underlying our next construction: the theory of surfaces isogenous to a product, introduced in [7] (see also [8]), and which holds more generally for varieties isogenous to a product.

Definition 5.1 (1) A surface *isogenous to a (higher) product* is a compact complex projective surface S which is a quotient $S = (C_1 \times C_2)/G$ of a product of curves of resp. genera $g_1, g_2 \geq 2$ by the free action of a finite group G. It is said to be *unmixed* if the embedding $i: G \to \mathrm{Aut}(C_1 \times C_2)$ takes values in the subgroup $\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$ (which has index at most two, see e.g. [7], cor. 3.9).

(2) A *Beauville surface* is a surface isogenous to a (higher) product which is *rigid*, i.e., it has no nontrivial deformation. In the unmixed case, S is Beauville if and only if, for $i = 1, 2$, (C_i, G) is a triangle curve (see for instance [8], page 552).

(3) An *étale marked surface* is a triple (S', G, η) where S' is a smooth projective surface such that the action of G is without fixed points. An étale marked surface can also be defined as a quintuple (S, S', G, η, F) where $\eta: G \to \mathrm{Aut}(S')$ is a faithful free action, and $F: S \to S'/G$ is an isomorphism.

Observe that, once a base point $y \in S$ is fixed, a surjection of the fundamental group $r: \pi_1(S, y) \to G$ determines an étale marked surface. Conversely, the marking provides the desired surjection r. Moving the base point y around amounts to replacing r with the composition $r \circ \mathrm{Ad}(\gamma)$, for $\gamma \in \pi_1(S, y)$. However,

$$(r \circ \mathrm{Ad}(\gamma))(\delta) = r(\gamma\delta\gamma^{-1}) = (\mathrm{Ad}(r(\gamma)) \circ r)(\delta).$$

Therefore we see that, if we want to get rid of the dependence on the base point, we have to divide by the group $\mathrm{Inn}(G)$ acting on the left.

In this case an associated subgroup of the covering (a subgroup of $\pi_1(S, y)$ of the form $p_*(\pi_1(S', z))$, $p\colon S' \to S$ being the projection, and where $p(z) = y$) is a normal subgroup, independent of the choice of a base point z above y; however, the corresponding isomorphism of the quotient group $\pi_1(S, y)/p_*(\pi_1(S', z))$ with G changes with y, and as a result the epimorphism r is modified by an inner automorphism of G. Moreover the action of nontrivial elements in $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ may transform the marking into a non isomorphic one.

Remark 5.2 Consider the coarse moduli space $\mathfrak{M}_{x,y}$ of canonical models of surfaces of general type X with $\chi(\mathcal{O}_X) = x$, $K_X^2 = y$. Gieseker [13] proved that $\mathfrak{M}_{x,y}$ is a quasi-projective variety.

We denote by $\mathfrak{M}$ the disjoint union $\cup_{x,y\geq 1}\mathfrak{M}_{x,y}$, and we call it the *moduli space of surfaces of general type*.

Fix a finite group G and consider the moduli space $\hat{\mathfrak{M}}^G_{x,y}$ for étale marked surfaces (X, X', G, η, F), where the isomorphism class $[X] \in \mathfrak{M}_{x,y}$.

This moduli space $\hat{\mathfrak{M}}^G_{x,y}$ is empty if there is no surjection $r : \pi_1(X, y) \to G$, otherwise we obtain that $\hat{\mathfrak{M}}^G_{x,y}$ is a finite étale covering space of $\mathfrak{M}_{x,y}$ with fibre over X equal to the quotient set

$$\mathrm{Epi}(\pi_1(X, y), G)/\mathrm{Inn}(G).$$

By the theorem of Grauert and Remmert [15] $\hat{\mathfrak{M}}^G_{x,y}$ is a quasi-projective variety.

The following theorem concerning surfaces isogenous to a product is a minor amendment to Theorem 3.3 of [8], and a rephrasing of Theorem 5.19 of [9]):

Theorem 5.3 *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product. Then any surface X with the same topological Euler number and the same fundamental group as S is diffeomorphic to S. The corresponding subset $\mathfrak{M}^{top}_S = \mathfrak{M}^{diff}_S$ of the moduli space, corresponding to surfaces orientedly homeomorphic, resp. orientedly diffeomorphic to S, is a union of connected components of $\mathfrak{M}$. It is either irreducible or it consists of two irreducible connected components which are exchanged by complex conjugation. The whole* *subset corresponding to surfaces X with the same topological Euler number and the same fundamental group as S consists of at most four irreducible connected components (corresponding to the choices of possibly replacing a factor C_i by its complex conjugate).*

Remark 5.4 The irreducible connected components corresponding to surfaces X with the same topological Euler number and the same fundamental group as S are precisely the irreducible connected components containing $S = (C_1 \times C_2)/G$, respectively $\bar{S}$, respectively $(\bar{C_1} \times C_2)/G$, respectively $(C_1 \times \bar{C_2})/G = \overline{(\bar{C_1} \times C_2)/G}$, where $\bar{X}$ denotes the complex conjugate of X.

These components can coincide (as it is the case when the group G is trivial).

The irreducible connected components containing S, respectively $\bar{S}$, have as union $\mathfrak{M}_S^{top} = \mathfrak{M}_S^{diff}$.

If S is a Beauville surface (i.e., S is rigid) then we have at most four points and it follows that S is defined over $\bar{\mathbb{Q}}$, whence the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the discrete subset of the moduli space $\mathfrak{M}$ of surfaces of general type corresponding to Beauville surfaces.

The following conjecture, suggested by the argument used in order to prove our main theorem, is quite tempting.

Conjecture 5.5 (Conjecture 2.11 in [10]) *The absolute Galois group* Gal $(\bar{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on the discrete subset of the moduli space* $\mathfrak{M}$ *of surfaces of general type corresponding to Beauville surfaces.*

5.1 Construction of certain families of surfaces isogenous to a product

Fix now an integer $g \geq 3$, and another integer $g' \geq 2$.

Consider now all the algebraic numbers $a \notin \mathbb{Q}$ and all the possible smooth complex curves C' of genus g', observing that the fundamental group of C' is isomorphic to the standard group

$$\pi_{g'} := \left\langle \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'} | \Pi_{i=1}^{g'} [\alpha_i, \beta_i] = 1 \right\rangle.$$

Since $g' \geq 2$ and, as we saw in section three, G_a is 2−generated there are plenty of epimorphisms (surjective homomorphisms) $\mu \colon \pi_{g'} \to G_a$. For instance it suffices to consider the epimorphism $\theta \colon \pi_{g'} \to \mathbb{F}_{g'}$ from $\pi_{g'}$ to the free group $\mathbb{F}_{g'} := \langle \lambda_1, \ldots, \lambda_{g'} \rangle$ in g' letters given by $\theta(\alpha_i) = \theta(\beta_i) = \lambda_i, \forall$ $1 \leq i \leq g'$, and to compose θ with the surjection $\phi \colon \mathbb{F}_{g'} \to G_a$, given by $\phi(\lambda_1) = \tau_0$, $\phi(\lambda_2) = \tau_1$, and $\phi(\lambda_i) = 1$ for $3 \leq i \leq g'$.

Consider all the possible epimorphisms $\mu\colon \pi_{g'} \to G_a$. Each such μ gives a normal unramified covering $D' \to C'$ with monodromy group G_a.

Definition 5.6 Let $\mathfrak{N}_a$ be the subset of the moduli space of surfaces of general type given by surfaces isogenous to a product of unmixed type $S \cong (D_a \times D')/G_a$, where D_a, D' are as above (and the group G_a acts by the diagonal action).

From Theorem 5.3 follows:

Proposition 5.7 *(i) For each $a \in \bar{\mathbb{Q}}$, $\mathfrak{N}_a$ is a union of connected components of the moduli space of surfaces of general type.*
(ii) The absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *acts on the set of connected components of the moduli space $\mathfrak{M}$ of surfaces of general type.*
(iii) Moreover, for $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, $\sigma(\mathfrak{N}_a) = \mathfrak{N}_{\sigma(a)}$.

Proof (i) Since D_a is a triangle curve, the pair (D_a, G_a) is rigid, whereas, varying C' and μ, we obtain the full union of the moduli spaces for the pairs (D', G_a), corresponding to the possible free topological actions of the group G_a on a curve D' of genus $|G_a|(g'-1)+1$.

Thus, according to Theorem 3.3 of [8], the family of surfaces $S \cong (D_a \times D')/G_a$ obtained varying C' and μ gives a union of connected components of the moduli space $\mathfrak{M}$ of surfaces of general type.

(ii) Choose now the canonical model X of a surface of general type S and apply the field automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ to a point of the Hilbert scheme corresponding to the m-canonical image of S (if $m \geq 5$ the corresponding surface is isomorphic to X, cf. [6], later m shall be taken sufficiently large). We obtain a surface which we denote by X^σ, and whose minimal model is S^σ.

Since the subscheme of the Hilbert scheme corresponding to m-pluricanonical embedded surfaces is defined over $\mathbb{Q}$, it follows that the action on the set of connected components of this subscheme, which is in bijection with the set of connected components of the moduli space (since for $m >> 0$ $\mathfrak{M}$ is the GIT quotient of the above subscheme, see [13]), depends only on the image $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (as observed in (i) of 4.5).

(iii) Choose now a surface S as above (thus, $[S] \in \mathfrak{N}_a$) and apply the field automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ to S obtaining S^σ.

By taking the fibre product of σ with $D_a \times D' \to S$ it follows that S^σ has an étale covering with group G_a which is the product $(D_a)^\sigma \times (D')^\sigma$.

Recall that $(C_a)^\sigma = C_{\sigma(a)}$ (since $\sigma(a)$ corresponds to another embedding of the field L into $\mathbb{C}$), and recall the established equality for Belyi maps $(F_a)^\sigma = F_{\sigma(a)}$, which implies $(D_a)^\sigma = D_{\sigma(a)}$.

On the other hand, the quotient of $(D')^\sigma$ by the action of the group G_a has genus equal to the dimension of the space of invariants $\dim(H^0(\Omega^1_{(D')^\sigma})^{G_a})$, but this dimension is the same as $g' = \dim(H^0(\Omega^1_{D'})^{G_a})$. Hence the action of G_a on $(D')^\sigma$ is also free (by Hurwitz' formula), and we have shown that S^σ is a surface whose moduli point is in $\mathfrak{N}_{\sigma(a)}$. □

We prove now an intermediate result.

Theorem 5.8 *The absolute Galois group* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on the set of connected components of the (coarse) moduli space of étale marked surfaces isogenous to a higher product.*

Proof Given $a \in \bar{\mathbb{Q}}$, consider a connected component $\hat{\mathfrak{N}}_a^\rho$ of the space of G_a-étale-marked surfaces of general type $\mathfrak{M}^G_{x,y}$ corresponding to a fixed surjection

$$\mu\colon \pi_{g'} \to G := G_a.$$

To the monodromy μ and to an isomorphism $\pi_1(C', y) \cong \pi_{g'}$, where C' is a curve of genus g', $y \in C'$, corresponds an unramified G-covering $C_2 \to C'$, and we denote by g_2 the genus of C_2, thus $g_2 - 1 = |G|(g' - 1)$.

The homomorphism μ has then a kernel isomorphic to π_{g_2}, and conjugation by elements of $\pi_{g'}$ determines a homomorphism

$$\rho\colon G \to \mathrm{Out}^+(\pi_{g_2}) = \mathrm{Map}_{g_2},$$

where $\mathrm{Out}^+(\pi_{g_2}) \subset \mathrm{Out}(\pi_{g_2})$ is the index two subgroup of automorphism classes whose action on the abelianization $\mathbb{Z}^{2g_2}$ of π_{g_2} has determinant $= 1$.

The homomorphism ρ is called the topological type of the action of G, and is well defined, up to conjugation in the mapping class group Map_{g_2}.

Our theorem follows now from the following

Main Claim: if $\hat{\mathfrak{N}}_a^\rho = \sigma(\hat{\mathfrak{N}}_a^\rho)$, then necessarily $a = \sigma(a)$.
Proof of the main claim: our assumption says that there are two curves C, C' of genus g', and two respective covering curves C_2, C_2', with group G_a and monodromy type μ (equivalently, with topological type ρ of the action of G_a), such that there exists an isomorphism

$$f : D_a^\sigma \times C_2^\sigma \to D_a \times C_2'$$

commuting with the action of G_a on both surfaces.

By the rigidity lemma 3.8 of [7], f is of product type, and since one action is not free while the other is free, we obtain that $f = f_1 \times f_2$, where $f_1 : D_a^\sigma \to D_a$ commutes with the G_a action.

Therefore the marked triangle curves (D_a, G_a, i_a) and $(D_a^\sigma, G_a, \mathrm{Ad}(\sigma)i_a)$ are isomorphic and by Theorem 4.6 we get $a = \sigma(a)$. □

6 What happens, if we forget the marking?

We shall assume throughout that we are given a nontrivial element $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and that $a \neq b$ are elements in $\bar{\mathbb{Q}}$ such that $b = \sigma(a)$.

Observe the following: σ acts nontrivially on the set $\pi_0(\mathfrak{M})$ of connected components of $\mathfrak{M}$ if we find a, b as above such that:

(**) $\mathfrak{N}_a$ and $\mathfrak{N}_b$ do not intersect.

Else, (**) does not hold for each a, $\mathfrak{N}_a$ and $\mathfrak{N}_b$ intersect and, by the structure theorem for surfaces isogenous to a product, we obtain that for all a the two triangle curves D_a and D_b are isomorphic.

Moreover, if σ acts as the identity on $\pi_0(\mathfrak{M})$ it acts as the identity on the subset $\pi_0(\mathfrak{N}_a)$ of $\pi_0(\mathfrak{M})$ (defined in 5.6) and whose points correspond to the connected components $\mathfrak{N}_a^\rho$ (set of isomorphism classes of surfaces $S = (D_a \times C_2)/G_a$, where the topological type of the action of G_a on C_2 is ρ).

6.1 Twisting a surface isogenous to a product

We can produce more connected components of the moduli space of surfaces isogenous to a product via the following construction.

Definition 6.1 Assume that (C_1, G, η_1), (C_2, G, η_2) are two marked curves, and consider their product $(C_1 \times C_2, G \times G, \eta_1 \times \eta_2)$. For each $\lambda \in \mathrm{Aut}(G)$ we consider the subgroup

$$G(\lambda) := \{(g, \lambda(g)) \subset G \times G\}.$$

We denote by S_λ the quotient $(C_1 \times C_2)/G(\lambda)$, reserving the notation S for the case $\lambda = Id$. In the case where the action of G on C_2 is free, then all the surfaces S_λ are surfaces isogenous to a product.

In particular, if S is a point in the connected component $\mathfrak{N}_a^\rho$, for each element $\lambda \in \mathrm{Aut}(G)$, the twisted surface S_λ is an element in the connected component $\mathfrak{N}_a^{\rho\lambda^{-1}}$ of the moduli space, in $\pi_0(\mathfrak{N}_a)$, corresponding to the epimorphism $\lambda \circ \mu$ instead of μ.

The action of the absolute Galois group on S_λ is induced by the one on $C_1 \times C_2$, so that $S_\lambda^\sigma = (C_1^\sigma \times C_2^\sigma)/\mathrm{Ad}(\sigma)(G(\lambda))$.

6.2 Automorphism of G_a associated to elements of the absolute Galois group acting trivially on $\pi_0(\mathfrak{M})$

Proposition 6.2 *Assume that σ acts as the identity on the subset $\pi_0(\mathfrak{N}_a)$ of $\pi_0(\mathfrak{M})$, in particular that $\sigma(\mathfrak{N}_a^\rho) = \mathfrak{N}_a^\rho$ for all topological types ρ of the action of G_a on C_2 (associated to a surjection $\mu : \pi_{g'} \to G_a =: G$).*

Let $S = (D_a \times C_2)/G_a$ be a point in $\mathfrak{N}_a^\rho$. Then there is an isomorphism $f_1 : D_a \to D_{\sigma(a)}$.

Define $\psi_1 := \mathrm{Ad}(f_1^{-1}\sigma)$: then ψ_1 lies in the centre $Z(\mathrm{Aut}(G))$ of $\mathrm{Aut}(G)$, in particular the class $[\psi_1] \in \mathrm{Out}(G)$ lies in the centre $Z(\mathrm{Out}(G))$.

Proof For each $\lambda \in \mathrm{Aut}(G)$, we have by our assumption an isomorphism of fundamental groups of $\pi_1(S_\lambda)$ with $\pi_1(S_\lambda^\sigma)$, induced by an isomorphism of S_λ with the conjugate $(S_\lambda'')^\sigma$ of another surface S_λ'' in the connected component of S_λ.

By the unicity of the minimal realisation of a surface isogenous to a product (see [7], prop. 3.13) this isomorphism lifts to an isomorphism of product type

$$f_1 \times f_2 : D_a \times C_2 \cong D_a^\sigma \times (C_2'')^\sigma = D_{\sigma(a)} \times (C_2'')^\sigma.$$

Notice that, for each λ, $(S_\lambda'')^\sigma$ is a quotient of $D_{\sigma(a)} \times (C_2'')^\sigma$.

Identifying $(S_\lambda'')^\sigma$ to $S_\lambda = (C_1 \times C_2)/G$ via the given isomorphism, we get that the Galois automorphism σ acts on $G \times G$ by a product automorphism $\psi_1 \times \psi_2$, where $\psi_1 := \mathrm{Ad}(f_1^{-1}\sigma)$ is uniquely defined, while ψ_2 is only defined up to an inner automorphism, corresponding to an automorphism of C_2 contained in G.

Since σ descends to the respective quotients, we must have:

$$(\psi_1 \times \psi_2)(G(\lambda)) = G(\lambda) \Leftrightarrow (\psi_1(g), \psi_2(\lambda(g))) \in G(\lambda)$$

$$\Leftrightarrow \psi_2(\lambda(g))) = \lambda(\psi_1(g)) \ \forall g \in G.$$

By setting $\lambda = Id$, we obtain $\psi_1 = \psi_2$, and using

$$\psi_2 \circ \lambda = \lambda \circ \psi_1, \ \forall \lambda,$$

we conclude that ψ_1 lies in the centre $Z(\mathrm{Aut}(G))$. □

Remark 6.3 Clearly, if the class $[\psi_1] \in \mathrm{Out}(G)$ is trivial, then the triangle curves (D_a, G) and (D_b, G) differ by an inner automorphism of G and we conclude by proposition 4.9 that $C_a \cong C_b$, hence $a = b$, a contradiction.

Therefore we may assume that the class $[\psi_1] \in Z(\mathrm{Out}(G))$ is nontrivial for each σ acting as the identity on $\pi_0(\mathfrak{N}_a)$.

We are now ready for the proof of

Theorem 6.4 *The absolute Galois group* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on the set of connected components of the (coarse) moduli space* $\mathfrak{M}$ *of surfaces of general type.*

There are two main intermediate results, which obviously together imply Theorem 6.4.

For the first we need a new definition.

Definition 6.5 Let $a \in \bar{\mathbb{Q}} \setminus \mathbb{Q}$ and define $\tilde{G}_a := \mathrm{Aut}(D_a)$.

Given a surjective homomorphism

$$\tilde{\mu}\colon \pi_{g'} \to \tilde{G}_a,$$

with topological type $\tilde{\rho}$, consider all the étale covering spaces $C_2 \to C_2/\tilde{G}_a = C'$ of curves C' of genus g' with the corresponding topological type $\tilde{\rho}$.

Consider then the connected component $\tilde{\mathfrak{N}}_a^{\tilde{\rho}}$ of the moduli space of surfaces of general type $\mathfrak{M}$ corresponding to surfaces isogenous to a product of the type

$$S = (D_a \times C_2)/\tilde{G}_a.$$

Proposition 6.6 *Let* $\mathfrak{K}$ *be the kernel of the action of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *on* $\pi_0(\mathfrak{M})$*. Then* $\mathfrak{K}$ *is an abelian subgroup.*

Proof We want to embed the kernel $\mathfrak{K}$ in an abelian group, e.g. in the direct product of groups of the form $Z(\mathrm{Out}(G))$, using Proposition 6.2.

Assume that σ lies in the kernel $\mathfrak{K}$. Then, for each algebraic number a, and every $\tilde{\rho}$ as above, σ stabilizes the connected component $\tilde{\mathfrak{N}}_a^{\tilde{\rho}}$.

Let us denote here for simplicity $G := \tilde{G}_a$.

Hence to σ we associate an element $[\psi_1] \in Z(\mathrm{Out}(G))$, which is nontrivial if and only if $\sigma(a) \neq a$.

Therefore it suffices to show that, for a fixed $a \in \bar{\mathbb{Q}}$, and $\tilde{\rho}\colon G \to \mathrm{Map}_{g_2} = \mathrm{Out}(\pi_{g_2})$, the map described in Proposition 6.2

$$\sigma \mapsto [\psi_1] \in Z(\mathrm{Out}(G))$$

is a homomorphism.

Observe that, in fact, there is a dependence of ψ_1 on σ and on the algebraic number a. To stress these dependences, we change the notation and denote the isomorphism ψ_1 corresponding to σ and a by $\psi_{\sigma,a}$, i.e.,

$$\psi_{\sigma,a}(g) = \Phi_1^{-1} \circ g^{\sigma} \circ \Phi_1 = \Phi_1^{-1} \circ \sigma g \sigma^{-1} \circ \Phi_1,$$

where $\Phi_1 \colon D_a \to D_{\sigma(a)}$ is the isomorphism (analogous to the previously considered f_1) induced by the fact that σ stabilizes the component $\tilde{\mathfrak{N}}_a^{\tilde{\rho}}$.

Since D_a is fully marked, whenever we take another isomorphism $\Phi \colon D_a \to D_{\sigma(a)}$ we have that $(\Phi)^{-1} \circ \Phi_1 \in \mathrm{Aut}(D_a) = G$. Therefore, since we work in $\mathrm{Out}(G_a)$, the class of $\psi_{\sigma,a}$ does not depend on the chosen isomorphism $\Phi \colon D_a \to D_{\sigma(a)}$.

Let now σ, τ be elements of $\mathfrak{K}$. We have then (working always up to inner automorphisms of G):

- $\psi_{\sigma,a} = \mathrm{Ad}(\varphi^{-1}\sigma)$, for any isomorphism $\varphi \colon D_a \to D_{\sigma(a)}$, and any algebraic number a;
- $\psi_{\tau,a} = \mathrm{Ad}(\Phi^{-1}\tau)$, for any isomorphism $\Phi \colon D_a \to D_{\tau(a)}$, and any algebraic number a;
- $\psi_{\tau\sigma,a} = \mathrm{Ad}(\Psi^{-1}\tau\sigma)$, for any isomorphism $\Psi \colon D_a \to D_{\tau\sigma(a)}$.

We can choose $\Psi := \varphi^{\tau} \circ \Phi$, and then we see immediately that

$$\begin{aligned}\psi_{\tau\sigma,a} &= \mathrm{Ad}((\varphi^{\tau} \circ \Phi)^{-1}\tau\sigma) = \mathrm{Ad}(\Phi^{-1}(\varphi^{\tau})^{-1}\tau\sigma)\\ &= \mathrm{Ad}(\Phi^{-1}\tau\varphi^{-1}\tau^{-1}\tau\sigma) = \mathrm{Ad}(\Phi^{-1}\tau)\,\mathrm{Ad}(\varphi^{-1}\sigma) = \psi_{\sigma,a} \circ \psi_{\tau,a}. \qquad (1)\end{aligned}$$

This shows that the map

$$\mathfrak{K} \to \prod_{a \in \bar{\mathbb{Q}}} \prod_{\tilde{\rho}} Z(\mathrm{Out}(\tilde{G}_a)),$$

which is injective by Remark 6.3, is in fact a group homomorphism. Therefore $\mathfrak{K}$ is abelian (as a subgroup of an abelian group). □

Proposition 6.7 *Any abelian normal subgroup $\mathfrak{K}$ of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *is trivial.*

Proof Let $N \subset \bar{\mathbb{Q}}$ be the fixed subfield for the subgroup $\mathfrak{K}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

N is a Galois extension of the Hilbertian field $\mathbb{Q}$ and N, if $\mathfrak{K}$ is not trivial, is not separably closed. Hence, by proposition 16.11.6 of [12] then $\mathrm{Gal}(N) := \mathrm{Gal}(\bar{\mathbb{Q}}/N)$ is not prosolvable, in particular, it is not abelian. However, not only is in general the closure of a normal abelian subgroup in a topological group also normal and abelian, but also one sees right away that $\mathfrak{K} = \mathrm{Gal}(N)$ since $\mathfrak{K}$ is closed (as intersection of closed subgroups). Hence we derive a contradiction. □

Theorem 6.4 has the following consequence:

Theorem 6.8 *If $\sigma \in$* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *is not in the conjugacy class of complex conjugation $\mathfrak{c}$, then there exists a surface isogenous to a product X such that X and the Galois conjugate surface X^{σ} have non isomorphic fundamental groups.*

Proof By a Theorem of Artin (see cor. 9.3 in [18]) we know that any σ which is not in the conjugacy class of $\mathfrak{c}$ has infinite order.

By Theorem 6.4 the orbits of σ on the subset of $\pi_0(\mathfrak{M})$ corresponding to the union of the $\mathfrak{N}_a$'s have unbounded cardinality, otherwise there is a power of σ acting trivially, contradicting the statement of 6.4.

Take now an orbit with five elements at least: then we get surfaces $S_0, S_1 := S_0^\sigma, \ldots, S_4 := S_3^\sigma$, which belong to five different components. Since we have at most four different connected components where the fundamental group is the same, we conclude that there is an $i \leq 3$ such that $\pi_1(S_i) \neq \pi_1(S_{i+1})$. □

In a previous version of the paper we assumed that the following question has a positive answer.

Question 6.9 *Let S be a surface isogenous to a product and assume that S^σ has the same fundamental group, hence it is diffeomorphic to S. Is it true that S^σ is orientedly diffeomorphic to S? The reason to ask this question, also in greater generality under the assumption that X is diffeomorphic to X^σ, is that σ sends a cohomology group $H^i(X, \Omega_X^j)$ to the corresponding one for X^σ, thus preserving the Hodge summands of the cohomology.*

Observe now that X_a and $(X_a)^\sigma$ have isomorphic Grothendieck étale fundamental groups. In particular, the profinite completions of $\pi_1(X_a)$ and $\pi_1((X_a)^\sigma)$ are isomorphic. In the last section we shall give explicit examples where the actual fundamental groups are not isomorphic.

Another interesting consequence is the following. Observe that the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the set of connected components of the (coarse) moduli space of minimal surfaces of general type. Theorem 6.8 has as a consequence that this action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *does not* induce an action on the set of isomorphism classes of fundamental groups of surfaces of general type.

Corollary 6.10 $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ does not admit *an action on the set $\mathcal{G}$ of isomorphism classes of groups which is compatible with the map π_1 associating to an element $\mathcal{N} \in \pi_0(\mathfrak{M})$ the isomorphism class of the fundamental group $\pi_1(S)$ of the minimal model of a(ny) surface S of general type such that $[S] \in \mathcal{N}$ (i.e., an action such that $\sigma(\pi_1(S)) = \pi_1(S^\sigma)$).*

Proof In fact, complex conjugation does not change the isomorphism class of the fundamental group (X and $\bar{X}$ are diffeomorphic). Now, if we had such an action on the set of isomorphism classes of fundamental groups, then the whole normal closure $\mathfrak{H}$ of the $\mathbb{Z}/2$ generated by complex conjugation (the set of automorphisms of finite order, by the cited theorem of E. Artin, see corollary 9.3 in [18]) would act trivially.

By Theorem 6.8 the subgroup $\mathfrak{H}$ would then be equal to the union of these elements of order 2 in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. But a group where each element has order

≤ 2 is abelian, and again we would have a normal abelian subgroup, $\mathfrak{H}$, of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, contradicting 6.7. □

The above arguments show that the set of elements $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that for each surface of general type S and S^σ have isomorphic fundamental groups is indeed a subgroup where all elements of order two, in particular it is an abelian group of exponent 2.

Question 6.11 (Conjecture 2.5 in [10]) *Is it true that for each $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, different from the identity and from complex conjugation, there exists a surface of general type S such that S and S^σ have non isomorphic fundamental groups?*

It is almost impossible to calculate explicitly the fundamental groups of the surfaces constructed above, since one has to explicitly calculate the monodromy of the Belyi function of the very special hyperelliptic curves C_a.

Therefore we give in the next section explicit examples of pairs of rigid surfaces with non isomorphic fundamental groups which are Galois conjugate.

7 Explicit examples

In this section we provide, as we already mentioned, explicit examples of pairs of surfaces with non isomorphic fundamental groups which are conjugate under the absolute Galois group. Hence they have non isomorphic fundamental groups with isomorphic profinite completions (recall that the completion of a group G is the inverse limit

$$\hat{G} = lim_{K \trianglelefteq_f G}(G/K),$$

of the factors G/K, K being a normal subgroup of finite index in G).

The surfaces in our examples are rigid. In fact, we can prove the following

Theorem 7.1 *There exist Beauville surfaces which yield explicit examples of Galois conjugate surfaces with non-isomorphic fundamental groups (whose profinite completions are isomorphic).*

Before proving the above result by constructing explicitly two conjugate Beauville surfaces with non-isomorphic topological fundamental groups, we review briefly some facts concerning complex polynomials with two critical values {0, 1} (see [3] for an elementary treatment of what follows).

Let $P \in \mathbb{C}[z]$ be a polynomial with critical values {0, 1}.

In order not to have infinitely many polynomials with the same branching behaviour, one considers only *normalized polynomials*

$$P(z) := z^n + a_{n-2}z^{n-2} + \ldots a_0.$$

The condition that P has only $\{0, 1\}$ as critical values, implies, as we shall briefly recall, that P has coefficients in $\bar{\mathbb{Q}}$. Fix the types $(m_1, \ldots, m_r)$ and $(n_1, \ldots, n_s)$ of the cycle decompositions of the respective local monodromies around 0 and 1: we can then write our polynomial P in two ways, namely as:

$$P(z) = \prod_{i=1}^{r} (z - \beta_i)^{m_i},$$

and

$$P(z) = 1 + \prod_{k=1}^{s} (z - \gamma_k)^{n_k}.$$

We have the equations $F_1 = \sum m_i \beta_i = 0$ and $F_2 = \sum n_k \gamma_k = 0$ (since P is normalized). Moreover, $m_1 + \ldots + m_r = n_1 + \ldots + n_s = n = deg P$ and therefore, since $\sum_j (m_j - 1) + \sum_i (n_i - 1) = n - 1$, we get $r + s = n + 1$.

Since we have $\prod_{i=1}^{r} (z - \beta_i)^{m_i} = 1 + \prod_{k=1}^{s} (z - \gamma_k)^{n_k}$, comparing coefficients we obtain further $n - 1$ polynomial equations with integer coefficients in the variables β_i, γ_k, which we denote by $F_3 = 0, \ldots, F_{n+1} = 0$. Let $\mathbb{V}(n; (m_1, \ldots, m_n), (n_1, \ldots, n_s))$ be the algebraic set in affine $(n + 1)$-space defined by the equations $F_1 = 0, \ldots, F_{n+1} = 0$. Mapping a point of this algebraic set to the vector $(a_0, \ldots, a_{n-2})$ of coefficients of the corresponding polynomial P we obtain a set

$$\mathbb{W}(n; (m_1, \ldots, m_n), (n_1, \ldots, n_s))$$

(by elimination of variables) in affine $(n - 1)$ space. Both these are finite algebraic sets defined over $\mathbb{Q}$ since by Riemann's existence theorem they are either empty or have dimension 0.

Observe also that the equivalence classes of monodromies $\mu \colon \pi_1(\mathbb{P}^1 \setminus \{0.1.\infty\}) \to \mathfrak{S}_n$ correspond to the orbits of the group of n-th roots of 1 (we refer to [3] for more details).

Lemma 7.2

$$\mathbb{W} := \mathbb{W}(7; (2, 2, 1, 1, 1); (3, 2, 2))$$

is irreducible over $\mathbb{Q}$ *and splits into two components over* $\mathbb{C}$.

Proof This can easily be calculated by a MAGMA routine. □

The above lemma implies that $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts transitively on $\mathbb{W}$. Looking at the possible monodromies, one sees that there are exactly two real non equivalent polynomials.

In both cases, which will be explicitly described later on, the two permutations, of types (2, 2) and (3, 2, 2), are seen to generate $\mathfrak{A}_7$ and the respective normal closures of the two polynomial maps are easily seen to give (we use here the fact that the automorphism group of $\mathfrak{A}_7$ is $\mathfrak{S}_7$) nonequivalent triangle curves D_1, D_2.

By Hurwitz's formula, we see that $g(D_i) = \frac{|\mathfrak{A}_7|}{2}(1 - \frac{1}{2} - \frac{1}{6} - \frac{1}{7}) + 1 = 241$.

Definition 7.3 Recall first that an (ordered) system of generators $(a_1, \ldots, a_m)$ of a group G is said to be *spherical* iff the product $a_1 \cdot \cdots \cdot a_m$ equals the identity element of G.

Let now (a_1, a_2, a_3) and (b_1, b_2, b_3) be two spherical systems of generators of a finite group G of the same signature, i.e., $\{ord(a_1), ord(a_2), ord(a_3)\} = \{ord(b_1), ord(b_2), ord(b_3)\}$. Then (a_1, a_2, a_3) and (b_1, b_2, b_3) are called *Hurwitz equivalent* iff they are equivalent under the equivalence relation generated by

$$(a_1, a_2, a_3) \equiv \left(a_2, a_2^{-1} a_1 a_2, a_3\right),$$

$$(a_1, a_2, a_3) \equiv \left(a_1, a_3, a_3^{-1} a_2 a_3\right).$$

It is well known that two such triangle curves are isomorphic, compatibly with the action of the group G, if and only if the two spherical systems of generators are *Hurwitz equivalent*.

Lemma 7.4 *There is exactly one Hurwitz equivalence class of triangle curves given by a* spherical system of generators of signature (5, 5, 5) *of* $\mathfrak{A}_7$.

Proof This is shown by an easy MAGMA routine. □

Remark 7.5 In other words, if D_1 and D_2 are two triangle curves given by spherical systems of generators of signature (5, 5, 5) of $\mathfrak{A}_7$, then D_1 and D_2 are not only isomorphic as algebraic curves, but they have the same action of G.

Let D be the triangle curve given by a(ny) spherical system of generators of signature (5, 5, 5) of $\mathfrak{A}_7$. Then by Hurwitz' formula D has genus 505.

Consider the two triangle curves D_1 and D_2 as in Lemma 7.2 and ensueing discussion. Clearly $\mathfrak{A}_7$ acts freely on $D_1 \times D$ as well as on $D_2 \times D$ and we obtain two non isomorphic Beauville surfaces $S_1 := (D_1 \times D)/G$, $S_2 := (D_2 \times D)/G$.

Remark 7.6 The following proposition yields the proof of Theorem 7.1.

Proposition 7.7 *(1) S_1 and S_2 have non-isomorphic fundamental groups.*
(2) There is a field automorphism $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $S_2 = (S_1)^\sigma$. In particular, the profinite completions of $\pi_1(S_1)$ and $\pi_1(S_2)$ are isomorphic.

Proof (1) Obviously, the two surfaces S_1 and S_2 have the same topological Euler characteristic. If they had isomorphic fundamental groups, by Theorem 3.3 of [8], S_2 would be the complex conjugate surface of S_1. In particular, D_1 would be the complex conjugate triangle curve of D_2: but this is absurd since we shall show, in the discussion following Remark 7.8, that both D_1 and D_2 are real triangle curves.
(2) We know that $(S_1)^\sigma = ((D_1)^\sigma \times (D)^\sigma)/G$. Since there is only one Hurwitz class of triangle curves given by a spherical system of generators of signature (5, 5, 5) of $\mathfrak{A}_7$, we have $(D)^\sigma \cong D$ (with the same action of G). □

We determine now explicitly the respective fundamental groups of S_1 and S_2.

In general, let $(a_1, \dots, a_n)$ and $(b_1, \dots, b_m)$ be two sets of spherical generators of a finite group G of respective order signatures $r := (r_1, \dots, r_n)$, $s := (s_1, \dots, s_m)$. We denote the corresponding 'polygonal' curves by D_1, resp. D_2.

Assume now that the diagonal action of G on $D_1 \times D_2$ is free. We get then the smooth surface $S := (D_1 \times D_2)/G$, isogenous to a product.

Denote by $T_r := T(r_1, \dots, r_n)$ the *polygonal group*

$$\left\langle x_1, \dots, x_{n-1} | x_1^{r_1} = \dots = x_{n-1}^{r_{n-1}} = (x_1 x_2 \dots x_{n-1})^{r_n} = 1 \right\rangle.$$

We have the exact sequence (cf. [7] cor. 4.7)

$$1 \to \pi_{g_1} \times \pi_{g_2} \to T_r \times T_s \to G \times G \to 1,$$

where g_i is the genus of D_i.

Let Δ_G be the diagonal in $G \times G$ and let H be the inverse image of Δ_G under $\Phi\colon T_r \times T_s \to G \times G$. We get the exact sequence

$$1 \to \pi_{g_1} \times \pi_{g_2} \to H \to G \cong \Delta_G \to 1.$$

Remark 7.8 $\pi_1(S) \cong H$ (cf. [7] cor. 4.7).

We choose now an arbitrary spherical system of generators of signature (5, 5, 5) of $\mathfrak{A}_7$, for instance ((1, 7, 6, 5, 4), (1, 3, 2, 6, 7), (2, 3, 4, 5, 6)). Note that we use here MAGMA's notation, where permutations act on the right (i.e., ab sends x to $(xa)b$).

Fig. 1 Monodromy correspondig to (2)

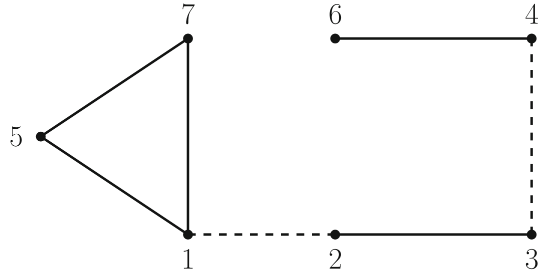
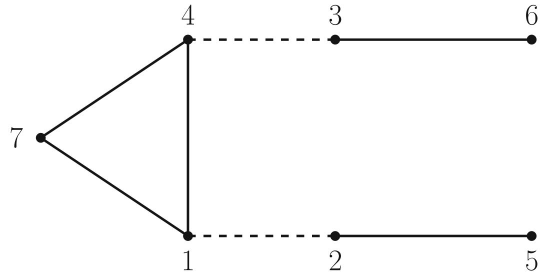


Fig. 2 Monodromy correspondig to (3)



A MAGMA routine shows that

$$((1,2)(3,4),(1,5,7)(2,3)(4,6),(1,7,5,2,4,6,3)) \tag{2}$$

and

$$((1,2)(3,4),(1,7,4)(2,5)(3,6),(1,3,6,4,7,2,5)) \tag{3}$$

are two representatives of spherical generators of signature (2, 6, 7) yielding two non isomorphic triangle curves C_1 and C_2, each of which is isomorphic to its complex conjugate. In fact, an alternative direct argument is as follows. First of all, C_i is isomorphic to its complex conjugate triangle curve since, for an appropriate choice of the real base point, complex conjugation sends $a \mapsto a^{-1}, b \mapsto b^{-1}$ and one sees that the two corresponding monodromies are permutation equivalent (see Figs. 1, 2).

Moreover, since $\mathrm{Aut}(\mathfrak{A}_7) = \mathfrak{S}_7$, if the two triangle curves were isomorphic, then the two monodromies would be conjugate in $\mathfrak{S}_7$. That this is not the case is seen again by Figs. 1 and 2.

The two corresponding homomorphisms $\Phi_1 \colon T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7$ and $\Phi_2 \colon T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7$ give two exact sequences

$$1 \to \pi_1(C_1) \times \pi_1(C) \to T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7 \to 1,$$

and

$$1 \to \pi_1(C_2) \times \pi_1(C) \to T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7 \to 1,$$

yielding two non isomorphic fundamental groups $\pi_1(S_1) = \Phi_1^{-1}(\Delta_{\mathfrak{A}_7})$ and $\pi_1(S_2) = \Phi_2^{-1}(\Delta_{\mathfrak{A}_7})$ fitting both in an exact sequence of type

$$1 \to \pi_{241} \times \pi_{505} \to \pi_1(S_j) \to \Delta_{\mathfrak{A}_7} \cong \mathfrak{A}_7 \to 1,$$

where $\pi_{241} \cong \pi_1(C_1) \cong \pi_1(C_2)$, $\pi_{505} = \pi_1(C)$.

Using the same method that we used for our main theorems, namely, using a surjection of a group $\pi_{g'} \to \mathfrak{A}_7$, $g' \geq 2$, we get infinitely many examples of pairs of fundamental groups which are nonisomorphic, but which have isomorphic profinite completions.

We obtain the following theorem, whose proof we omit (being based on arguments already given).

Theorem 7.9 *There is an infinite sequence of integers $g_1 < g_2 < \ldots < g_i < \ldots$, where each g_i is of the form $g_i = 1 + \frac{1}{2}7!(g_i' - 1)$, and for each g_i there is a pair of surfaces $S_1(g_i)$ and $S_2(g_i)$ isogenous to a product, such that*

- *the corresponding connected components $\mathfrak{N}(S_1(g_i))$ and $\mathfrak{N}(S_2(g_i))$ are different,*
- *there is a $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(\mathfrak{N}(S_1(g_i))) = \mathfrak{N}(S_2(g_i))$,*
- *$\pi_1(S_1(g_i))$ is non isomorphic to $\pi_1(S_2(g_i))$, but they have isomorphic profinite completions,*
- *the fundamental groups fit into an exact sequence*

$$1 \to \pi_{241} \times \pi_{g_i} \to \pi_1(S_j(g_i)) \to \mathfrak{A}_7 \to 1, \;\; j = 1, 2.$$

Remark 7.10 (1) Many more explicit examples as the one above (but with cokernel group different from $\mathfrak{A}_7$) can be obtained using polynomials with two critical values.

(2) A construction of polynomials with two critical values having a very large Galois orbit was proposed to us by Duco van Straten.

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