ON RIGID COMPACT COMPLEX SURFACES AND MANIFOLDS

INGRID BAUER AND FABRIZIO CATANESE

Contents

1. Introduction

The present investigation originated from some natural questions concerning the series of families of surfaces exhibited in [CD16] to provide counterexamples to a question posed by Fujita in 1982 (see also [CD13], [CD14]). The families depend on some integer invariants, the main one being an arbitrary integer n coprime with 6, and for $n = 5$ they were first constructed in [BC08]: we shall refer to them as BCD-surfaces.

Date: September 27, 2016.

2010 Mathematics Subject Classification: 14B12, 14J15, 14J29, 14J80, 14E20, 14F17, 14D07, 32G05, 32Q55, 32J15.

Keywords: Rigid complex manifolds, branched or unramified coverings, deformation theory; projective classifying spaces.

The present work took place in the framework of the ERC-2013-Advanced Grant - 340258-TADMICAMT

We would like to thank Elisabetta Colombo, Paola Frediani and Alessandro Ghigi for useful conversations; also Philippe Eyssidieux for pointing out the reference to Zheng's paper, thus preventing the second author and Alessandro Ghigi from duplicating existing work.

In case $n = 5$, the surfaces S are ball quotients, hence they possess a Kähler metric with strongly negative curvature tensor, their universal covering $\tilde{S}$ is diffeomorphic to the Euclidean space $\mathbb{R}^4$, $\tilde{S}$ is a Stein manifold, and the surfaces S are rigid in all possible senses (see the definitions given in section one).

It is natural to ask whether similar properties hold for the other BCD surfaces, in particular to ask about their rigidity. In fact, we prove the following

Theorem 1.1. *The BCD surfaces are infinitesimally rigid and rigid. As a consequence, since their Albanese map is a semistable fibration $\alpha : S \to B$ onto a curve B of genus $b := \frac{1}{2}(n-1)$, and with fibres of genus $g = (n-1)$, we get a rigid curve B inside the moduli stack $\overline{\mathfrak{M}_{n-1}}$ of stable curves of genus $(n-1)$.*

An interesting feature of the fibration is that all fibres are smooth, except three fibres which are the union of two smooth curves of genus b intersecting transversally in one point, so that the Jacobians of all the fibres are principally polarized Abelian varieties, and B yields a complete curve inside $\mathfrak{A}_{n-1}$; it is an interesting question whether this curve is also rigid inside $\mathfrak{A}_{n-1}$ (see [Moo10], [CFG15] and [FGP15] for related questions).

Suspicion of rigidity came from the observation that all deformations of BCD also have an Albanese map which is a fibration onto a curve of genus b with exactly three singular fibres, of the same type as described above. The proof of rigidity however follows another path: first of all we observe that BCD surfaces admit as a finite unramified covering the Hirzebruch-Kummer coverings $HK_{CQ}(n)$, the minimal resolution of a covering of the plane with Galois group $(\mathbb{Z}/n)^5$, and branched on a complete quadrangle CQ (CQ is the union of the six lines in $\mathbb{P}^2$ joining four points in linear general position). Then one observes easily (proposition 2.5) that if we have a finite unramified $Y \to X$, and Y is rigid, then a fortiori X is rigid too.

Hence theorem 1.1 is implied by the following stronger

Theorem 1.2. *The Hirzebruch Kummer surfaces $HK_{CQ}(n)$ are infinitesimally rigid and rigid for all $n \in \mathbb{N}, n \geq 4$.*

The proof of the above theorem occupies the main body of the paper, is mainly based first on the fact that our surfaces are finite Galois coverings of the Del Pezzo surface of degree 5, which is the blow up of $\mathbb{P}^2$ in four points in general linear position, and which is the moduli space for ordered 5-tuples of points in $\mathbb{P}^1$, and as such it admits a biregular action by the symmetric group $\mathfrak{S}_5$. The other ingredients are Pardini 's formulae for direct image of sheaves under Abelian coverings, and then residue sequences associated to sheaves of logarithmic differential forms: these lead to difficult calculations which can be handled using symmetry (by the semidirect product of $(\mathbb{Z}/n)^5$ with $\mathfrak{S}_5$) and the very explicit descriptions of the Picard group of the Del Pezzo surface.

Afterwards, it became only natural to put this result in perspective: what do we know in general about rigid complex surfaces, and about rigid compact complex manifolds?

For curves, the answer is easy: the only rigid curve is $\mathbb{P}^1$, so Kodaira dimension $0, 1$ is excluded.

For complex surfaces, we can use the Enriques-Kodaira classification to show that again Kodaira dimension $0, 1$ is excluded, and more precisely we have

Theorem 1.3. *Let S be a smooth compact complex surface, which is rigid. Then either*

(1) S is a minimal surface of general type, or
(2) S is a Del Pezzo surface of degree $d \geq 5$ (i.e., one of the following surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$, $\mathbb{F}_1 = S_8$, S_7, S_6, S_5; where S_{9-r} is the blow-up of $\mathbb{P}^2$ in r points which are in general linear position).
(3) S is an Inoue surface of type S_M or $S^{(-)}_{N,p,q,r}$ (cf. [Ino74]).

Surfaces in classes (2) and (3) are infinitesimally rigid. Rigid surfaces in class (1) are also globally rigid, as well as those in (3), but the only rigid surface in class (2) is the projective plane $\mathbb{P}^2$.

Hence, as already observed, for surfaces rigidity implies that the Kodaira dimension is either $-\infty$ or maximal, equal to 2 (S is of general type), so that here Kodaira dimension $\mathrm{kod} = 0, 1$ is excluded.

We then show that a similar phenomenon is not true in higher dimension $n \geq 3$

Theorem 1.4. *For each $n \geq 3$ and for each $k = -\infty, 0, 2, \ldots n$ there is a rigid projective variety X of dimension n and Kodaira dimension $\mathrm{kod}(X) = k$.*

The construction of these examples (the case $n = 3, k = 0$ is due to Beauville) is not so difficult, since essentially rigidity is preserved by products and by rigid unramified quotients. It seems likely that the exception $\mathrm{kod} = 1$ should not occur, at least for n large; but we postpone the answer to this question to a future time.

Global rigidity for rigid varieties of general type is a consequence of the existence of moduli spaces, while in the $\mathrm{kod} = -\infty$ case it becomes rather complicate for $n \geq 3$, as shown by the work of Siu [Siu89], [Siu91], Hwang [Hwa95] and Hwang-Mok [HM98], essentially only the case of $\mathbb{P}^n$ and of the hyperquadric Q^n being solved.

Theorem 1.3 shows therefore that the problem of classifying rigid surfaces reduces to the same question for surfaces of general type. Here our new examples add to a not so long list:

(1) *ball quotients*: for these the universal covering is the two-dimensional complex ball $\mathbb{B}_2 \subset \mathbb{C}^2$, and they are pluri-rigid ([Siu80], [Mos73]), i.e. rigid in any possible way, as it happens for the
(2) irreducible bi-disk quotients: for these the universal covering of S is $\mathbb{B}_1 \times \mathbb{B}_1 \cong \mathbb{H} \times \mathbb{H}$, where $\mathbb{H}$ is the upper half plane, and the fundamental group $\pi_1(S) = \Gamma$ has dense image for any of the two projections $\Gamma \to \mathrm{PSL}(2, \mathbb{R})$([JY85], [Mok88]).

(3) *Beauville surfaces*: these are the rigid unramified quotients of products of curves ([Cat00]). They are infinitesimally rigid, strongly rigid but not étale rigid, which means that they have a finite unramified covering which is not rigid.
(4) *Mostow-Siu surfaces*, [MS80]; these are pluririgid, since they have a metric with strongly negative curvature.
(5) some *Kodaira fibrations* constructed by Catanese-Rollenske [CR09].

All these examples have in common the feature that their universal covering is diffeomorphic to $\mathbb{R}^4$, so they are classifying spaces $K(\pi, 1)$ of some finitely generated group π.

The previous observations lead to two quite interesting questions:

Question 1.5.

A) *Does there exist an infinitesimally rigid surface of general type which is not a $K(\pi, 1)$?*
B) *Does there exist a rigid, but not infinitesimally rigid surface of general type?*

For question A), by the Lefschetz hyperplane section theorem, it would suffice to find a rigid ample divisor in a rigid threefold which is a $K(\pi, 1)$. For question B), a natural approach, due to the result of Burns and Wahl [BW74], would be to find a minimal surface of general type S whose canonical model is singular and rigid.

We pose now several questions, hoping that the readers will find them interesting.

The subject of arrangements of lines in $\mathbb{P}^2$ has attracted a lot of attention of algebraic geometers after the work of Hirzebruch ([Hir83]) which provided explicit examples of ball quotients as Hirzebruch-Kummer coverings branched on rigid arrangements of lines (among these, the most famous are, beyond the complete quadrangle, the Hesse configuration $(9_4, 12_3)$ of 12 lines joining pairs of flexpoints of a smooth cubic curve, and its dual configuration $(12_3, 9_4)$ of the 9 lines dual to the flexpoints).

Natural questions are:

Question 1.6.

I) *For which rigid configuration $\mathcal{C}$ of lines in $\mathbb{P}^2$ is the associated Hirzebruch Kummer covering $HK_{\mathcal{C}}(n)$ rigid for $n >> 0$?*
II) *For which rigid configuration $\mathcal{C}$ of lines in $\mathbb{P}^2$ is the associated Hirzebruch Kummer covering $HK_{\mathcal{C}}(n)$ a $K(\pi, 1)$ for $n >> 0$?*
III) *For which rigid configuration $\mathcal{C}$ of lines in $\mathbb{P}^2$ does the associated Hirzebruch Kummer covering $HK_{\mathcal{C}}(n)$ possess a Kähler metric of negative sectional curvature $n >> 0$?*
IV) *For which rigid configuration $\mathcal{C}$ of lines in $\mathbb{P}^2$ is the associated Hirzebruch Kummer covering $HK_{\mathcal{C}}(n)$ étale rigid for $n >> 0$?*

Observe that if III) has a positive answer, then also II), by the Cartan-Hadamard theorem. Moreover existence of a strongly negative metric ([Siu80], [MS80]) implies étale rigidity.

For the surfaces $HK_{CQ}(n)$ the answer to II), III) and to strong and étale rigidity follows, in case that 5 divides n, from the work of Fangyang Zheng [Zhe99] who extended the Mostow-Siu technique to the case of normal crosisings. The case of other integers $n \geq 4$ is open.

Panov [Pan11] asserts (without giving full details, hence without specifying explicitly the meaning of $n >> 0$) a positive answer to II) for the surfaces $HK_{CQ}(n)$ and other examples by Hirzebruch: his method consists in finding polyhedral metrics of negative curvature. So the following question is not yet settled:

Question 1.7. *Are the surfaces $HK_{CQ}(n)$, for $n \geq 5$, $K(\pi, 1)$ spaces (or even for $n \geq 4$)?*

Of course one could ask a similar question also for non rigid configurations.

For rigid configurations the philosophy that for $n >> 0$ the deformations of $HK_{\mathcal{C}}(n)$ should correspond to the ones of the configuration is based on the following partly heuristic argument (a weaker result, i.e. up to taking the product with a smooth manifold, was used by Vakil in [Vak06] for some special configurations).

Assume that a point of the configuration P has valency $v_P \geq 3$ (i.e., at least 3 lines of the configuration pass through P: then the point P has to be blown up and projection from P induces on $HK_{\mathcal{C}}(n)$, for $n \geq 4$, a fibration over a curve B_P of genus ≥ 2. The existence of this fibration is a topological property of the surface S which is the minimal resolution of the singular Abelian covering of the plane; hence this fibration is stable under deformation, and any deformation embeds in a product of generalized Fermat curves. Also the number of singularities on the fibres of each such fibration is a topological invariant, and if the components of singular fibres are stable by deformation (this is true for BCD surfaces, since there is only one non separating vanishing cycle), then the exceptional curves would be stable under deformation, and the question would be reduced to proving that the equisingular deformations of the finite $(\mathbb{Z}/n)^r$ covering of $\mathbb{P}^2$ are trivial.

The middle step seems to be the most difficult one, and that's why in this paper we are obliged to a rather computationally involved proof; another reason for this is that the easy criterion given by Pardini (corollary 5.1 ii) of [Par91]) does not apply, since it is easy to show that, Y being the Del Pezzo surface of degree 5, there are plenty of characters χ of $G = (\mathbb{Z}/n)^5$ for which $H^1(\Theta_Y(-L_\chi) \neq 0$. Hence proving that all deformations are natural is a question of the same order of difficulty of proving rigidity.

We feel somehow that our results are like the tip of the iceberg, and to illustrate this philosophy we describe in the last section a new series of rigid line configurations, which is in some way the most natural construction (possibly known outside of algebraic geometry?): we call this the nth iterated Campedelli Burniat configuration $\mathcal{C}_{CB}(n)$, since for $n = 1$ it was used by Campedelli, and later by

Burniat. For $n = 0$ $\mathcal{C}_{CB}(n)$ is the complete quadrangle, and in the iterative steps we apply a contraction sending the 'external triangle' to the one with vertices the midpoints of the sides.

For many other known configurations, we defer to Hirzebruch's summary [Khi85] of Hofer's thesis, and to the book [BHH87].

This vast material offers ample source of examples in order to test the above questions in many concrete cases.

2. Rigidity

We start recalling the basic notions of *rigidity* for compact complex manifolds X of complex dimension n.

Definition 2.1.

(1) Two compact complex manifolds X and X' are said to be *deformation equivalent* if and only if there is a proper smooth holomorphic map
$$f\colon \mathfrak{X} \to \mathcal{B}$$
where $\mathcal{B}$ is a connected (possibly not reduced) complex space and there are points $b_0, b_0' \in \mathcal{B}$ such that the fibres $X_{b_0} := f^{-1}(b_0), X_{b_0'} := f^{-1}(b_0')$ are respectively isomorphic to X, X' ($X_{b_0} \cong X, X_{b_0'} \cong X'$).

(2) Two compact complex manifolds X and X' are said to be *direct deformation of each other* if and only if there is a proper smooth holomorphic map
$$f\colon \mathfrak{X} \to \mathcal{B}$$
as in (1), but where moreover $\mathcal{B}$ is assumed to be irreducible.

(3) Equivalently, two compact complex manifolds X and X' are *direct deformation of each other* if and only if there is a proper smooth holomorphic map
$$f\colon \mathfrak{X} \to \Delta$$
where $\Delta \subset \mathbb{C}$ is the unit disk, and where X, respectively X', are isomorphic to fibres of f.

(4) Equivalently, deformation equivalence is the equivalence relation generated by the relation of direct deformation. This means that two compact complex manifolds X and X' are *deformation equivalent* if and only if there is a sequence of compact complex manifolds $(X_i)_{i\in\{0,1,\dots,k\}}$ such that $X_0 = X$, $X_k = X'$ and X_i is a direct deformation of X_{i-1}.

(5) A compact complex manifold X is said to be *globally rigid* if for any compact complex manifold X', which is deformation equivalent to X, we have an isomorphism $X \cong X'$.

(6) A compact complex manifold X is instead said to be *(locally) rigid* (or just *rigid*) if for each deformation of X,
$$f\colon (\mathfrak{X}, X) \to (\mathcal{B}, b_0)$$

there is an open neighbourhood $U \subset \mathcal{B}$ of b_0 such that $X_t := f^{-1}(t) \cong X$ for all $t \in U$.

(7) A compact complex manifold X is said to be *infinitesimally rigid* if

$$H^1(X, \Theta_X) = 0,$$

where Θ_X is the sheaf of holomorphic vector fields on X.

(8) X is said to be *strongly rigid* if the set of compact complex manifolds Y which are homotopically equivalent to X, $\{Y | Y \sim_{h.e.} X\}$ consists of a finite set of isomorphism classes of globally rigid varieties.

(9) X is said to be *étale rigid* if every étale (finite unramified) cover Y of X is rigid (we can obviously combine this concept with the previous ones, and speak of étale globally rigid, étale infinitesimally rigid, ...

Remark 2.2. 1) If X is infinitesimally rigid, then X is also locally rigid. This follows by the Kodaira-Spencer-Kuranishi theory, since $H^1(X, \Theta_X)$ is the Zariski tangent space of the germ of analytic space which is the base $\mathrm{Def}(X)$ of the Kuranishi semiuniversal deformation of X. So, if $H^1(X, \Theta_X) = 0$, $\mathrm{Def}(X)$ is a reduced point and all deformations are induced by the trivial deformation. In other words, the condition of infinitesimal rigidity is equivalent to the condition that every deformation of X, when restricted to a suitable neighbourhood U of b_0, be isomorphic to the trivial deformation $X \times U$.

2) More generally, the definitions are so given that obviously strong rigidity implies global rigidity, which in turn implies local rigidity, as well as étale rigidity implies local rigidity.

3) The Fischer-Grauert theorem ([FG65])says conversely that if $\mathcal{B}$ is reduced, then the condition of local rigidity yields triviality of the family over a suitable neighbourhood of b_0.

4) Moreover, the Kuranishi theorem ([Kur62], [Kur65]) implies that the number of moduli of X, defined as $m(X) := \dim \mathrm{Def}(X)$, satisfies

$$m(X) = \dim \mathrm{Def}(X) \geq h^1(X, \Theta_X) - h^2(X, \Theta_X).$$

Hence, if $\mathrm{Def}(X)$ is reduced, and $m(X) \geq 1$, then necessarily X is not locally rigid. More generally, if $\mathrm{Def}(X)$ is reduced, the Kuranishi family is universal if $h^0(X, \Theta_X) = 0$ or $h^0(X, \Theta_{X_t})$ is a locally constant function for $t \in \mathrm{Def}(X)$, [Wav69].

5) For $n := \dim X = 1$, all the notions of rigidity are equivalent and it is well known that the only rigid curve is $\mathbb{P}^1$.

6) The following well known examples (see [Cat83]) illustrate the difference between global and infinitesimal rigidity. The Segre-Hirzebruch surface $\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ has a smooth Kuranishi space which is the germ at the origin of the vector space

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{C}^{n-1} \text{ for } n \geq 1, \ = 0 \text{ for } n = 0.$$

The family parametrizes extensions

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to V \to \mathcal{O}_{\mathbb{P}^1}(n) \to 0,$$

and the surfaces in the deformation are all the surfaces of the form $\mathbb{F}_{n-2k}$, $n \geq 2k$. Hence $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$ and $\mathbb{F}_1$ are infinitesimally rigid, but not globally rigid.

The following is a useful general result:

Theorem 2.3. *A compact complex manifold X is rigid, if and only if the Kuranishi space* $\mathrm{Def}(X)$ *(base of the Kuranishi family of deformations) is 0-dimensional.*

In particular, if $X = S$ is a smooth compact complex surface and

$$10\chi(\mathcal{O}_S) - 2K_S^2 + h^0(X, \Theta_S) > 0,$$

then S is not rigid.

Proof. The 'if' part being obvious by the versality of the Kuranishi family, we show the 'only if' part.

Without loss of generality, we can assume that for all $t \in \mathrm{Def}(X)$ $X_t \cong X$. Let $B \subset \mathrm{Def}(X)$ be the reduced subspace $B := \mathrm{Def}(X)_{red}$. Then, by the theorem of Fischer- Grauert [FG65] it follows that the pull back of the Kuranishi family to B is trivial, isomorphic then to $B \times X$.

If we assume that B is not a point, then there is $t \in B$ such that the derivative of the inclusion map $i : B \to \mathrm{Def}(X)$ is non zero. Hence the Kuranishi family is not semiuniversal in the point $i(t)$, contradicting Corollary 1 of [Mee11], which asserts that if $h^0(X_t, \Theta_{X_t})$ is constant, then the Kuranishi family is semiuniversal (versal in the author's unusual terminology) at each point.

In the case of surfaces, we use the Kuranishi inequality and Riemann-Roch:

$$\begin{aligned} \dim \mathrm{Def}(S) \geq h^1(\Theta_S) - h^2(\Theta_S) = -\chi(\Theta_S) + h^0(S, \Theta_S) = \\ = 10\chi(\mathcal{O}_S) - 2K_S^2 + h^0(X, \Theta_S). \end{aligned} \tag{2.1}$$

□

Now, before we dwell upon the analysis of rigidity in complex dimension 2, let us make a few easy but important observations. First of all, if X or Y are not rigid, then the same holds for the product $X \times Y$. We can moreover say when is $X \times Y$ infinitesimally rigid.

Proposition 2.4. *Let X and Y be infinitesimally rigid compact complex manifolds. Then $X \times Y$ is infinitesimally rigid if and only if*

$$h^0(X, \Theta_X) \cdot h^1(Y, \mathcal{O}_Y) = h^0(Y, \Theta_Y) \cdot h^1(X, \mathcal{O}_X) = 0.$$

Proof. The result is an easy consequence of the Künneth formula by which

$$H^1(X \times Y, \Theta_{X \times Y}) \cong$$

$$\cong H^1(X, \Theta_X) \oplus (H^0(X, \Theta_X) \otimes H^1(Y, \mathcal{O}_Y)) \oplus (H^1(X, \mathcal{O}_X) \otimes H^0(Y, \Theta_Y)) \oplus H^1(Y, \Theta_Y).$$

□

Proposition 2.5. *If $p : Z \to X$ is étale, i.e. a finite unramified holomorphic map between compact complex manifolds, then the infinitesimal rigidity of Z implies the infinitesimal rigidity of X. Moreover, if Z is rigid, then also X is rigid.*

Proof. For the first assertion, simply observe that $H^1(Z, \Theta_Z) = H^1(X, p_*(\Theta_Z)) = 0$, and that $p_*(\Theta_Z) = p_*(p^*\Theta_X)) = \Theta_X \otimes (p_*\mathcal{O}_Z)$ has Θ_X as a direct summand.

Proof of the second assertion: we have seen in theorem 2.3 that rigidity is equivalent to the condition that $\mathrm{Def}(X)$ is zero dimensional.

If X is not rigid, we pass to an unramified covering W of Z, which is a Galois cover of X.

In this way there is a subgroup $H \subset G$ such that $Z = W/H$, while $X = W/G$.

We use now a result from [Cat88]:

$$\mathrm{Def}(X) = \mathrm{Def}(W)^G, \ \mathrm{Def}(Z) = \mathrm{Def}(W)^H \Rightarrow \mathrm{Def}(X) \subset \mathrm{Def}(Z).$$

We conclude: if $\mathrm{Def}(Z)$ has dimension 0, i.e. it is a (possibly non reduced) point, a fortiori $\mathrm{Def}(X)$ is a point.

□

Remark 2.6. 1) The preceding propositions show that in dimension strictly higher than 2 it is easy to construct many examples of infinitesimally rigid varieties by taking étale quotients of products of infinitesimally rigid varieties. More generally, if G is a finite group acting on X, Y in such a way that both actions are rigid (this is a weaker notion than the rigidity of X, Y), then necessarily the quotient $W : (X \times Y)/G$ is infinitesimally rigid (W is a manifold if the diagonal action of G on $X \times Y$ is free).

2) The example of Beauville surfaces, the rigid quotients $(X \times Y)/G$, where X and Y are curves of respective genera greater or equal to 2 shows that the converse to proposition 2.5 does not hold. Beauville surfaces are rigid, globally rigid, strongly rigid [BCG08], but not étale rigid.

Theorem 2.7. *Let S be a smooth compact complex surface, which is (locally) rigid. Then either*

(1) S is a minimal surface of general type, or
(2) S is a Del Pezzo surface of degree $d \geq 5$, $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$, $\mathbb{F}_1 = S_8$, or S_7, S_6, S_5; where S_{9-r} is the blow-up of $\mathbb{P}^2$ in r points which are in general linear position.
(3) S is an Inoue surface of type S_M or $S^{(-)}_{N,p,q,r}$ (cf. [Ino74]).
(4) Rigid surfaces in class (1) are also globally rigid, surfaces in class (3) are infinitesimally and globally rigid, surfaces in class (2) are infinitesimally rigid, but the only rigid surface in class (2) is the projective plane $\mathbb{P}^2$.

Remark 2.8. For surfaces of general type it is expected to find examples which are rigid, but not infinitesimally rigid: such an example would be the one of a minimal surface S such that its canonical model $X(S)$ is infinitesimally rigid and singular (see [BW74]).

Proof. Let us begin with the last statement, (4).

1) A locally rigid minimal surface of general type is also globally rigid, due to the existence of a global moduli space for surfaces of general type [Gie77].

2) A Del Pezzo surface S of degree $d \geq 5$ is infinitesimally rigid, but S is globally rigid if and only if $S \cong \mathbb{P}^2$. This was already observed for $\mathbb{F}_0, \mathbb{F}_1$. For S_{9-r} with $r \geq 2$ it suffices to move the second point of the blow up until it becomes infinitesimally near to the first, so that we get Y with $-K_Y$ not ample. Since $-K_S$ is ample, Y is a deformation of S which is not isomorphic to S.

3) The infinitesimal rigidity of the Inoue surfaces in (3) was shown in [Ino74]. For their global rigidity, observe that the Inoue surfaces, belonging to three classes, $S^{(-)}_{N,p,q,r}$, S_M or $S^{(+)}_{N,p,q,r}$, are characterized ([Ino74], [Tel94]) by the condition that $b_1(S) = 1, b_2(S) = 0$ and that they contain no curves; the condition $b_1(S) = 1, b_2(S) = 0$ is stable by deformation and singles out also the minimal Hopf surfaces. The fundamental group $\pi_1(S)$ is also invariant by deformation; it suffices then to show that the fundamental groups of surfaces in class (3) cannot be equal to the fundamental groups of a Hopf surface or of an Inoue surface of type $S^{(+)}_{N,p,q,r}$. This is easy for the case of Hopf surfaces, which have a finite unramified covering which is a primary Hopf surface, diffeomorphic to $S^3 \times S^1$: hence for a Hopf surface $\pi_1(S)$ has cohomological dimension 1 (Γ has cohomological dimension n if $H^i(\Gamma, \mathbb{Q}) = 0 \ \forall i > n$). While the universal covering of a Inoue surface is $\mathbb{H} \times \mathbb{C}$, hence for an Inoue surface $H^i(\pi_1(S), \mathbb{Q}) = H^i(S, \mathbb{Q})$ and $\pi_1(S)$ has cohomological dimension 4.

In the case of $S^{(+)}_{N,p,q,r}$ it is quicker to use a deformation theoretic argument. In fact for a surface of type $S^{(+)}_{N,p,q,r}$ $H^i(S, \Theta) = \mathbb{C}$ for $i = 0, i = 1$, and $H^2(S, \Theta) = 0$ (implying that the Kuranishi family has a basis $\mathrm{Def}(S)$ smooth of dimension 1 containing only surfaces of the same type), while for surfaces in (3) $H^i(S, \Theta) = \mathbb{C}$ for $i = 0, 1, 2$: hence, by semicontinuity of the dimension of $H^i(S, \Theta) = \mathbb{C}$ for $i = 0, i = 1$, an Inoue surface as in (3) cannot be a limit of a family of Inoue surfaces of type $S^{(+)}_{N,p,q,r}$.

Let us now proceed with the proof of the main statements, (1)-(3).

a) Let S be a rigid smooth compact complex surface and assume that S is not minimal. Let

$$S = S_0 \to S_1 \to \ldots \to S_k = S'$$

be a sequence of point blow-ups such that S' is minimal. By Kodaira's theorem (cf. page 86 of [Kod63]), if S_i is rigid, then also the pair (S_{i+1}, p_{i+1}) (where $S_i \to S_{i+1}$ is the blow-up in $p_{i+1} \in S_{i+1}$) is rigid.

Varying the point $p_{i+1} \in S_{i+1}$ we see that

$$\dim \mathrm{Aut}(S') \geq 2k.$$

In particular there are two linearly independent global holomorphic vector fields on S' and we get an exact sequence

$$0 \to \mathcal{O}_{S'}(D_1) \oplus \mathcal{O}_{S'}(D_2) \to \Theta_{S'} \to \mathcal{F} \to 0,$$

where $D_1, D_2 \geq 0$ are effective divisors on S' and $\mathcal{F}$ is a coherent sheaf with $\dim \operatorname{Supp}(\mathcal{F}) \leq 1$.

Taking the determinant, this implies that $-K_{S'} \geq D_1 + D_2$. Therefore either

i) $-K_{S'}$ is a strictly effective divisor, or
ii) $K_{S'} \equiv 0$.

Case i) bifurcates:

- (i-1) if S' is algebraic, then S' is ruled (since $K_{S'}H < 0$ for H very ample)
- (i-2) if S' is not algebraic, then S' is a surface of class VII_0, in particular $b_1(S') = 1, q(S') = 1, p_g(S') = 0 \Rightarrow \chi(S') = 0, b_2(S') = e(S') = -K_{S'}^2$.

In case (ii) S' is either a K3-surface, or a complex torus, or a Kodaira surface. But all these surfaces are not rigid: K3 surfaces have $\chi(S') = 2$ hence we can apply theorem 2.3. That tori and Kodaira surfaces are not rigid is well known (for Kodaira surfaces, look at [Kod64]).

In case (i-2) again theorem 2.3 applies as soon as $b_2(S') > 0$, since $10\chi(S') - 2K_{S'}^2 = 2b_2(S') > 0$.

If instead S' is a surface of class VII_0 with $b_2(S') = 0$, rigidity for S (again by theorem 2.3) implies that $k = 0$. Finally, if $k = 0$ then $S = S'$ and S is a Hopf surface or an Inoue surface ([Tel94]). By [Ino74] an Inoue surface is rigid if and only if it is of type S_M or of type$S^{(-)}_{N,p,q,r}$.

Hopf surfaces are not rigid by [Kod68], [Kat75] and [Dab82].

Thus we have seen that if S rigid and non minimal then each of its minimal models S' is ruled.

Now, a minimal ruled surface S' is either $\mathbb{P}^2$ or is a $\mathbb{P}^1$-bundle over a curve C of genus $g = g(C)$.

Claim. S' must be regular.

Proof of the claim: Observe that a minimal ruled surface S' has

$$K_{S'}^2 = 8(1-g), \quad \chi(\mathcal{O}_{S'}) = 1 - g.$$

This implies that

$$0 = \dim \operatorname{Def}(S) \geq 6g - 6 + 2k.$$

Therefore either $g = 0$ or $g = 1$ and $k = 0$.

There are now two ways to proceed. The first argument uses [Sei92], Lemma 2 and Lemma 3 which says that $h^0(\Theta_{S'}) \geq 1$ for $g = 1$ and $k = 0$: this is a contradiction since then $\dim \operatorname{Def}(S') \geq h^0(\Theta_{S'}) \geq 1$.

The second argument is more geometric: every ruled surface is obtained from the product $\mathbb{P}^1 \times C$ via a sequence of elementary transformations. Now, if the genus g of C is ≥ 1, then C is not rigid, moreover we can perform all the required elementary transformations that lead to S when we deform C: hence we can arbitrarily deform the Albanese variety C of S and S is not rigid.

$\square$

Therefore S' is either $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, in particular $\chi(S') = \chi(S) = 1$. Since S is assumed to be rigid, this implies that $10 \leq 2K_S^2$, i.e. $K_S^2 \geq 5$.

Claim. S is a Del Pezzo surface, i.e., $-K_S$ is very ample.

But this follows since if one blows up special points in $\mathbb{P}^1 \times \mathbb{P}^1$ or in $\mathbb{P}^2$, this contradicts the rigidity of S.

b) We can now assume S to be minimal.

If $\text{kod}(S) = -\infty$ and S is not ruled, then S is of type VII_0, a case which we already treated.

Hence the theorem will be proven once we can rule out the case where S is minimal of Kodaira dimension 0 or 1. Assume S to be minimal with $\text{kod}(S) = 0, 1$. Then $K_S^2 = 0$ and $\chi(S) \geq 0$. Since $10\chi(S) - 2K_S^2 = 10\chi(S) \leq 0$ we have $\chi(S) = 0$. This rules out, as done earlier, $K3$-surfaces and Enriques surfaces. Moreover, we have already observed that Abelian surfaces and Kodaira surfaces are not rigid; the same holds for hyperelliptic surfaces.

Hence we are reduced to the case $\text{kod}(S) = 1$, i.e. properly elliptic surfaces.

Assume that $\text{kod}(S) = 1$ and consider for a suitable $m >> 0$ the m-th canonical map

$$\varphi = \varphi_{|mK_S|} \colon S \to B.$$

By the formula of Zeuthen-Segre, since $\chi(S) = 0 \Rightarrow e(S) = 0$, we see that the singular fibres are multiples of smooth elliptic curves, and φ is isotrivial. Then there exists a finite Galois covering $B' \to B$ such that the normalization S' of the fibre product $B' \times_B S$ is isomorphic to a product $B' \times E$, where E is an elliptic curve.

Now $S = S'/G$, where G acts on B' with quotient B, and G acts on E by translations (since all the fibres of φ have no rational component). This shows that we can freely deform the elliptic curve E, hence S is not rigid.

$\square$

3. Examples

Remark 3.1. The up to now known examples of rigid surfaces of general type are the following:

(1) the so-called *ball quotients*: these are the smooth projective surfaces S whose universal covering is the two-dimensional complex ball $\mathbb{B}_2$; they are infinitesimally rigid, strongly rigid and étale rigid.
(2) irreducible bi-disk quotients, i.e. the universal covering of S is $\mathbb{B}_1 \times \mathbb{B}_1 \cong \mathbb{H} \times \mathbb{H}$, where $\mathbb{H}$ is the upper half plane, moreover if we write $S = \mathbb{H} \times \mathbb{H}/\Gamma$ the fundamental group Γ has dense image for any of the two projections $\Gamma \to \text{PSL}(2, \mathbb{R})$; they are infinitesimally rigid, strongly rigid and étale rigid.
(3) *Beauville surfaces*; they are infinitesimally rigid, strongly rigid but not étale rigid.

(4) *Mostow-Siu surfaces*, [MS80]; these are infinitesimally rigid, strongly rigid and étale rigid.
(5) some *Kodaira fibrations* (Catanese-Rollenske) [CR09]; these are rigid and strongly rigid, infinitesimal rigidity and étale rigidity is not proven in [CR09] but could be true.

Definition 3.2. A compact complex manifold is called a *projective classifying space* if its universal covering $\tilde{X}$ is contractible.

Remark 3.3. 1) All known examples of rigid surfaces of general type are projective classifying spaces.

2) The examples (1)-(3), and (5) are strongly rigid.

In the previous section we have seen that if $n = \dim(X) = 2$ and X is rigid, then either the Kodaira dimension of X is $-\infty$, or $Kod(X) = n$ (X is of general type).

It is an interesting question whether we can say anything about the Kodaira dimension of rigid manifolds of a given dimension n.

Next, we show that we can obtain, for $n \geq 4$, rigid generalized hyperelliptic manifolds (they have Kodaira dimension $\mathrm{kod}(X) = 0$).

Theorem 3.4. *Let E be the Fermat (equianharmonic) elliptic curve, with the standard action of $G := (\mathbb{Z}/3)^2$:*

$$E := \{(x : y : z) \in \mathbb{P}^2 | x^3 + y^3 + z^3 = 0\},$$

let

$$e_1(x : y : z) := (\epsilon x : y : z),\ e_2(x : y : z) := (x : \epsilon y : z),\ \epsilon := \exp(\frac{2}{3}\pi i).$$

Define $e_3 := -e_1 - e_2, e_4 := e_1 - e_2$.

Consider the following automorphisms of G, defined uniquely by the conditions:

- $\psi_1(e_1) := e_1,\ \psi_1(e_2) := e_2,$
- $\psi_2(e_1) := e_1,\ \psi_2(e_2) := -e_2,$
- $\psi_3(e_1) := e_4,\ \psi_3(e_2) := e_3,$
- $\psi_i(e_1) := e_1,\ \psi_i(e_2) := -e_4$ *for $i \geq 4$, and*

let G act on E^n by:

$$g(p_1, \dots, p_n) := (\psi_1(g)(p_1), \psi_2(g)(p_2), \dots, \psi_n(g)(p_n)).$$

Then G acts on E^n freely for $n \geq 4$, and $X := (E^n)/G$ is a rigid compact complex manifold with Kodaira dimension equal to zero, strongly rigid exactly for $n = 4$.

Proof. Observe that the elements in G which have fixed points on E are just the multiples of the vectors $e_1, e_2, e_3 := -e_1 - e_2$, whereas the non zero multiples of e_4 act freely.

It suffices to show that the action on E^4 is free, because then the action on E^n is a fortiori free for $n \geq 4$.

If $g \in G \setminus \{0\}$ is a multiple of e_1, then $\psi_3(g)$ is a multiple of e_4, hence g acts freely; (nontrivial) multiples of e_4 act freely since $\psi_1(e_4) = e_4$, for multiples of e_2 this follows from $\psi_4(e_2) = e_4$, and for multiples of e_3 since $\psi_2(e_3) = -e_4$.

Hence the action is free and X is a complex manifold.

We claim that X is infinitesimally rigid. For this, let $Z := E_1 \times \ldots \times E_n$, where $E_i \cong E$ for all i, and G acts via $\psi_i(g)$ on E_i.

Then by the Künneth formula we have

$$H^1(\Theta_Z) = \Big(\bigoplus_{i=1}^n H^1(\Theta_{E_i})\Big) \oplus \Big(\bigoplus_{i=1}^n \Big(H^0(\Theta_{E_i}) \otimes (\oplus_{j\neq i} H^1(\mathcal{O}_{E_j}))\Big)\Big).$$

We have

$$H^1(\Theta_X) = H^1(\Theta_Z)^G = \Big(\bigoplus_{i=1}^n H^1(\Theta_{E_i})^G\Big) \oplus \Big(\bigoplus_{i=1}^n \Big(H^0(\Theta_{E_i}) \otimes (\oplus_{j\neq i} H^1(\mathcal{O}_{E_j}))\Big)^G\Big).$$

Now, $H^1(\Theta_{E_i})^G = 0$, while for the other terms we define φ to be the character e_1^*; it is the character of the representation on $H^0(\Omega^1_{E_1})$.

Then the character of $H^0(\Theta_{E_i}) \otimes H^1(\mathcal{O}_{E_j})$ equals $-\varphi \circ \psi_i - \varphi \circ \psi_j$; it is therefore nontrivial if the character $\varphi \circ \psi_i + \varphi \circ \psi_j$ is nontrivial, which follows by our assumption. In fact the characters in question are just:

$$\varphi \circ \psi_1 \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = m_1, \ \varphi \circ \psi_2 \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = m_2, \ \varphi \circ \psi_3 \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = m_1 - m_2,$$

and

$$\varphi \circ \psi_i \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = m_1 + m_2, \ \forall i \geq 4.$$

Therefore $H^1(\Theta_X) = 0$, whence X is infinitesimally rigid.

Moreover, any Y homotopically equivalent to X has an unramified G-cover W which has the same integral cohomology algebra of a complex torus, hence (see [Cat15] for this and other assertions) is a complex torus with an action of the group G.

Let $\Lambda := \pi_1(W) \cong \pi_1(E^n)$: the G-action on Λ is induced by conjugation via the exact sequence

$$1 \to \Lambda \to \pi_1(Y) \cong \pi_1(X) \to G \to 1.$$

The action of G on Λ is a direct sum $\Lambda = \oplus_1^4 \Lambda_i$, where Λ_i is a free $R_3 := \mathbb{Z}[x]/(x^2 + x + 1)$-module, of rank respectively 1,1,1,$n-3$, on which G acts via the surjection $\mathbb{Z}[G] \to R_3$ corresponding to the character $\phi_i : G \to \mathbb{Z}/3$ with kernel generated by $\psi_i^{-1}(e_4)$.

Since for each $i \neq j \in \{1, 2, 3, 4\}$ there is a $g \in G$ such that $\psi_i(g)$ has fixed points on E, whereas $\psi_j(g)$ acts freely, we see (as in [BC12], or [Cat15] page 389) that W splits as a direct sum $W_1 \times W_2 \times W_3 \times W_4$ of three elliptic curves and one torus, corresponding to the complex subspaces $\Lambda_i \otimes \mathbb{R}$ of the complex vector space $\Lambda \otimes \mathbb{R}$.

Because the W_i are stable for the G action, and Λ_i is a free R_3- module, it follows that W_i is the Fermat elliptic curve for $i = 1, 2, 3$, and also for $i = 4$ if $n = 4$.

In the case where $n > 4$, we can take a family of complex structures on $\Lambda_4 \otimes \mathbb{C} = H_1 \oplus H_2$ (here H_1 is the eigenspace for the character ϕ_4, H_2 is the eigenspace for the character $\overline{\phi_4}$,), such that $H^{1,0}$ is the direct sum of an arbitrary k-dimensional subspace of H_1 with an arbitrary subspace of H_2 of dimension $n - 3 - k$.

For $1 < k < n - 3$ we obtain a family of non rigid manifolds, since then

$$H^1(\Theta_W)^G = ((H^{1,0})^\vee \otimes \overline{H^{1,0}})^G \supset \mathrm{Hom}(H_1, \overline{H_2})^G \neq 0.$$

□

In the following we show that for $n \geq 3$ there are examples of rigid projective manifolds X of Kodaira dimension $2, \dots, n$.

Theorem 3.5. *For $n \geq 3$ there are rigid n-dimensional manifolds of Kodaira dimension k, for each k with $2 \leq k \leq n$.*

Proof. Consider the group

$$G := < x, y, z, w, t | x^3, y^3, z^3, w^3, t^3, y^x = yz, z^x = zw, z^y = zt > .$$

This group has order 3^5 and has abelianization $G/[G, G] = (\mathbb{Z}/3\mathbb{Z})^2$. More precisely, G sits inside an exact sequence

$$0 \to (\mathbb{Z}/3\mathbb{Z})^3 \to G \to G/[G, G] = (\mathbb{Z}/3\mathbb{Z})^2 \to 0,$$

where $\varphi\colon G \to G/[G, G]$ satisfies $\varphi(x) = (1, 0) = e_1$, $\varphi(y) = (0, 1) = e_2$ and $\varphi(z) = \varphi(w) = \varphi(t) = 0$.

By [BBF12] G admits a Beauville structure of type $(3, 3, 9), (3, 3, 9)$ given by $(x, y, (xy)^{-1}), (xt, y^2w, (xty^2w)^{-1})$, i.e., we get two triangle curves $\lambda_i\colon C_i \to C_i/G \cong \mathbb{P}^1$.

Let $S = (C_1 \times C_2)/G$ be the above described Beauville surface and let E be the Fermat elliptic curve with the action defined in theorem 3.4 and consider e.g. the (non faithful) G-action

$$g(x) := \varphi(g)(x)$$

on E. Then obviously the diagonal action of G on $C_1 \times C_2 \times (C_2)^{k-2} \times E^{n-k}$ is free, and the quotient is a projective manifold of dimension n and Kodaira dimension k. We claim that X is infinitesimally rigid. The proof is the same as in the proof of the infinitesimal rigidity in theorem 3.4. □

Remark 3.6. The above manifolds are strongly rigid for $n - k = 1$, as it follows by the same proof as in theorem 3.4. Twisting the action on the Fermat elliptic curves as in loc. cit. we can show strong rigidity for $n - k \leq 4$.

Remark 3.7. That there do exist rigid threefolds of Kodaira dimension 0 was shown by Beauville: in [Bea83], page 5, he constructed a rigid Calabi-Yau m-fold, $m = 3, 4, 6$, as the minimal resolution of a quotient $E^m/(\mathbb{Z}/m)$ where $\mathbb{Z}/m$ acts on E by multiplication $z \to \eta z$, η being a primitive mth root of unity, and diagonally on E^m.

4. Results related to the geometry of the Del Pezzo surface of degree 5

Consider the smooth Del Pezzo surface Y of degree 5, which is the blow up $Y :=\hat{\mathbb{P}}^2(p_1, p_2, p_3, p_4)$ of the plane in four points $p_1, p_2, p_3, p_4 \in \mathbb{P}^2$ in general position.

Y contains exactly 10 lines which are in bijection with (unordered) pairs $\{i, j\} \subset \{1, 2, 3, 4, 5\}$ in the following classical way:

- E_{i5} is the exceptional curve over p_i, $i \in \{1, 2, 3, 4\}$;
- E_{ij} is the strict transform of the line in $\mathbb{P}^2$ passing through p_h and p_k, where $\{i, j, h, k\} = \{1, 2, 3, 4\}$.

Note that by a slight abuse of notation we may also write E_{ij} with $i > j$ instead of $E_{\{i,j\}}$, just assuming silently that $E_{ij} = E_{ji}$.

The following is the intersection behaviour of the 10 lines. We have, for $i, j, h, k \in \{1, 2, 3, 4, 5\}$:

$$E_{ij}E_{hk} = \begin{cases} -1, & \text{if } \ |\{i,j\} \cap \{h,k\}| = 2, \\ 0, & \text{if } \ |\{i,j\} \cap \{h,k\}| = 1, \\ 1, & \text{if } \ |\{i,j\} \cap \{h,k\}| = 0. \end{cases}$$

Remark 4.1.

1) The symmetric group $\mathfrak{S}_5$ acts on Y by permutation of the indices $\{1, \dots, 5\}$. This action induces a permutation of the five pencils on Y which yield conic bundle structures:

- $X_i := |L - p_i|$, $i \in \{1, 2, 3, 4\}$, whose general element is the strict transform of a line in $\mathbb{P}^2$ passing through p_i;
- $X_5 := |2L - p_1 - p_2 - p_3 - p_4|$, whose elements are the strict transforms of the conics passing through p_1, p_2, p_3, p_4.

2) Each of the five pencils X_i has three reducible fibers corresponding to the three $(2, 2)$ partitions of $\{1, 2, 3, 4, 5\} \setminus \{i\}$; e.g. for $i = 1$:

$$L - p_1 \equiv E_{34} + E_{25} \equiv E_{24} + E_{35} \equiv E_{23} + E_{45}.$$

3) $\mathrm{Pic}(Y) \cong \mathbb{Z}^5$ is generated by the ten lines E_{ij}, $1 \leq i < j \leq 5$ and the relations are generated by the linear equivalences associated to the pencils: for each subset $I := \{i_1, i_2, i_3, i_4\}$ of cardinality 4 of $\{1, 2, 3, 4, 5\}$ we have

$$E_{i_1i_2} + E_{i_3i_4} \equiv E_{j_1j_2} + E_{j_3j_4}, \ \text{ for } \ \{j_1, j_2, j_3, j_4\} = I.$$

In the sequel we are going to study the linear independence in $\mathrm{Pic}(Y)$ of a subset of the set of 10 lines E_T for $T = \{t_1, t_2\} \subset \{1, 2, 3, 4, 5\}$. In fact, we prove the following:

Proposition 4.2. *Let $T_1, \dots, T_k \subset \{1, 2, 3, 4, 5\}$ be k pairwise different subsets of cardinality 2. Then we have:*

(1) $\mathrm{rk}\langle E_{T_1}, \dots, E_{T_6}\rangle < 5 \iff$ *either* $\exists\ j \in \{1,2,3,4,5\}$ *s.th.* $T_1, \dots T_6 \subset \{1,2,3,4,5\} \setminus \{j\}$*, or there exist* $i \neq j$ *such that* $\{T_1, \dots, T_6\}$ *is the set of pairs which intersect the subset* $\{i, j\}$ *in exactly one element. In both cases:* $\mathrm{rk}\langle E_{T_1}, \dots, E_{T_6}\rangle = 4$*;*

(2) $\mathrm{rk}\langle E_{T_1}, \dots, E_{T_5}\rangle = 4 \iff$ *we have two reducible fibres of a pencil* $X_j \iff \exists\ j \in \{1,2,3,4,5\}$ *s.th (after maybe renumbering the* T_i*'s)* $j \notin T_1, T_2, T_3, T_4$ *and* $T_1 \cup T_2 = T_3 \cup T_4 = \{1,2,3,4,5\} \setminus \{j\}$*;*

(3) $\mathrm{rk}\langle E_{T_1}, \dots, E_{T_k}\rangle = 5$*, if* $k \geq 7$*.*

Proof. We start the proof with a list of elementary observations.

Remark 4.3.

i) If $T_1, \dots T_6 \subset \{1,2,3,4,5\} \setminus \{j\}$, then $E_{T_1}, \dots, E_{T_6}$ are the six irreducible components of the three reducible fibres of the pencil X_j hence by Zariski's lemma these six irreducible curves generate a subgroup of rank 4 in $\mathrm{Pic}(Y)$. Moreover, for any $T_7 \neq T_1, \dots, T_6$, we have $\mathrm{rk}\langle E_{T_1}, \dots, E_{T_7}\rangle = 5$ since T_7 has non zero intersection with the fibre of X_j.

ii) If there is a $j \in \{1,2,3,4,5\}$ s.th. $j \in T_1, T_2, T_3, T_4$, then $E_{T_1}, E_{T_2}, E_{T_3}, E_{T_4}$ are pairwise disjoint and linearly independent in $\mathrm{Pic}(Y)$. If $T_5 \neq T_1, T_2, T_3, T_4$, then $\mathrm{rk}\langle E_{T_1}, \dots, E_{T_5}\rangle = 5$, since the intersection matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

has rank 5.

iii) If $T_1, T_2, T_3, T_4 \subset \{1,2,3,4,5\} \setminus \{j\}$, then we have two cases:

a) $\mathrm{rk}\langle E_{T_1}, \dots, E_{T_4}\rangle = 3 \iff$ none of the four curves E_{T_i} is disjoint from the others $\iff$ we have the components of two reducible fibres of the pencil $X_j \iff$ for each $T = \{i_1, i_2\} \in \{T_1, T_2, T_3, T_4\}$, also $T' = \{1,2,3,4,5\} \setminus \{j, i_1, i_2\} \in \{T_1, T_2, T_3, T_4\}$;

b) $\mathrm{rk}\langle E_{T_1}, \dots, E_{T_4}\rangle = 4 \iff$ there is an $i \in \{1,2,3,4\}$ (equivalently, there are two such indices i) such that E_{T_i} is disjoint from the others.

In case b): if T_5 is not a component of a reducible fibre of the pencil X_j, then $\mathrm{rk}\langle E_{T_1}, \dots, E_{T_5}\rangle = 5$.

Observe that i) shows one direction of (1). Assume now that there are $T_1, \dots, T_6 \subset \{1,2,3,4,5\}$ such that $\mathrm{rk}\langle E_{T_1}, \dots, E_{T_6}\rangle < 5$: then by ii) we know that for each $j \in \{1,2,3,4,5\}$, j is not contained in four of the T_i' s. We can clearly assume that we are not in case i), hence for each $j \in \{1,2,3,4,5\}$ there is an $i \in \{1,2,3,4,5,6\}$ such that $j \in T_i$. An easy counting argument shows that there is a $j \in \{1,2,3,4,5\}$ such that $j \in T_{i_1}, T_{i_2}, T_{i_3}$, for three distinct such T_i's.

Without loss of generality we can assume that $j = 1$ and

$$T_1 = \{1, 2\},\ T_2 = \{1, 3\},\ T_3 = \{1, 4\}.$$

Moreover, since there is an i such that $5 \in T_i$, we can assume that $T_4 = \{2, 5\}$. Observe now that $\mathrm{rk}\langle E_{T_1}, \ldots, E_{T_4}\rangle = 4$ and we have a section T_3 of X_4, one reducible fibre $T_2 + T_4$, and anothere component of a reducible fibre.

If T_5 is a component of the third reducible fibre ($T_5 = \{1, 5\}$ or $T_5 = \{2, 3\}$), then we get $\mathrm{rk}\langle E_{T_1}, \ldots, E_{T_5}\rangle = 5$, whereas if $T_5 = \{3, 5\}$ the rank remains 4, similarly if $T_5 = \{4, 5\}$ since then we have two reducible fibres of the pencil X_3.

We get $\mathrm{rk}\langle E_{T_1}, \ldots, E_{T_5}\rangle = 5$ for $T_5 = \{2, 4\}$ (consider the pencil X_3), and for $T_5 = \{3, 4\}$ (intersecting with E_{12} and E_{34} we see that if E_{24} were linearly dependent of the other four, we would have

$$E_{24} = -E_{12} + 2E_{14} + aE_{13} + bE_{25},$$

contradicting that the intersection number of E_{24} with a fibre of X_4 is 1.

The conclusion is that $\mathrm{rk}\langle E_{T_1}, \ldots, E_{T_6}\rangle = 5$ unless there is $j \notin T_i\ \forall i = 1, \ldots, 6$ or unless, up to symmetry, we have the six curves

$$T_1 = \{1, 2\},\ T_2 = \{1, 3\},\ T_3 = \{1, 4\}, T_4 = \{2, 5\},\ T_5 = \{3, 5\},\ T_6 = \{4, 5\}.$$

This shows (1) and (3).

In order to show (2), observe that by ii) each $j \in \{1, 2, 3, 4, 5\}$ is contained in at most three of the T_i's, and it at least one, else we have the reducible fibres of a pencil X_j, and we are done. We can therefore either assume that we are again in the situation above

$$T_1 = \{1, 2\},\ T_2 = \{1, 3\},\ T_3 = \{1, 4\},\ T_4 = \{2, 5\},$$

or each j belongs to exactly two T_i' s.

In the former case $\mathrm{rk}\langle E_{T_1}, \ldots, E_{T_5}\rangle = 5$ for any choice of T_5 different from $\{3, 5\}, \{4, 5\}$, or we have two reducible fibres of the pencil X_4, respectively of the pencil X_3, which is our assertion.

In the latter case without loss of generality we can assume that

$$T_1 = \{1, 2\},\ T_2 = \{2, 3\},\ T_3 = \{3, 4\},\ T_4 = \{4, 5\},\ T_5 = \{5, 1\},$$

i.e., we have a pentagon and the intersection matrix has rank 5.

□

Consider

$$D\colon = \bigcup_{\{i,j\} \subset \{1,2,3,4,5\}} E_{ij} \subset Y$$

and let e_1, e_2, e_3, e_4, e_5 be a basis of $(\mathbb{Z}/n\mathbb{Z})^5$.

Definition 4.4. Given a normal complex space Y and a closed analytic subspace D containing $Sing(Y)$, the *Kummer covering of exponent* n of Y branched on D (also called the *maximal Abelian covering of exponent* n of Y branched on D) is the ramified locally finite covering $\pi : X \to Y$ with X a normal analytic space such that the restriction of π to $Y \setminus D$ is the Galois unramified covering associated to the monodromy homomorphism

$$\varphi\colon H_1(Y \setminus D, \mathbb{Z}) \to H_1(Y \setminus D, \mathbb{Z}) \otimes \mathbb{Z}/n\mathbb{Z}.$$

In the case where Y is the Del Pezzo surface of degree 5, and D is the union of the 10 lines of Y, we shall speak of the *Hirzebruch-Kummer covering of exponent n associated to the complete quadrangle.*

In this special case

$$\varphi\colon H_1(Y \setminus D, \mathbb{Z}) \to H_1(Y \setminus D, \mathbb{Z}) \otimes \mathbb{Z}/n\mathbb{Z} \cong H_1(Y \setminus D, \mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^5,$$

Observe that $H_1(Y \setminus D, \mathbb{Z}) \cong \mathbb{Z}^5$ is generated by 10 elements $\epsilon_{ij} = \epsilon_{ji}$ where ϵ_{ij} is the class of a small loop around the line E_{ij}.

The relations satisfied by these generators are the linear combinations of the $\mathfrak{S}_5$-transforms of the relation

$$\epsilon_{12} = \epsilon_{34} + \epsilon_{35} + \epsilon_{45}. \tag{4.1}$$

In terms of these generators, the homomorphism φ is concretely given by

$$\begin{aligned} &\epsilon_{13} \mapsto e_5, \quad \epsilon_{14} \mapsto e_1, \quad \epsilon_{23} \mapsto e_4, \\ &\epsilon_{24} \mapsto e_2, \quad \epsilon_{34} \mapsto e_3, \end{aligned}$$

Remark 4.5. Then

- $\epsilon_{12} \mapsto -(e_1 + \ldots + e_5)$;
- $\epsilon_{45} \mapsto -(e_1 + e_2 + e_3)$;
- $\epsilon_{15} \mapsto e_2 + e_3 + e_4$;
- $\epsilon_{25} \mapsto e_1 + e_3 + e_5$;
- $\epsilon_{35} \mapsto -(e_3 + e_4 + e_5)$.

Remark 4.6. For $\sigma \in (\mathbb{Z}/n\mathbb{Z})^5$ denote by D_σ the union of the components of D having σ as local monodromy.

In our case either $D_\sigma = 0$ or D_σ is irreducible, and consists of the unique curve E_{ij} such that $\varphi(\epsilon_{ij}) = \sigma$.

Remark 4.7. The Hirzebruch-Kummer covering of exponent n associated to the complete quadrangle shall here for brevity be referred to as the $HK(n)$-surface: it has a group of automorphisms which is the semidirect group of $G := (\mathbb{Z}/n\mathbb{Z})^5$ with $\mathfrak{S}_5$.

In the case $n = 2$ we obtain a K3 surface, which was investigated by van Geemen, van Luijk and others [FGvGvL13]; the case $n = 3$ was investigated by Roulleau in [Rou11].

We shall prove now the following:

Proposition 4.8. *Consider for $n \geq 4$, the Kummer covering of exponent n of Y branched in D. Let $\psi \in ((\mathbb{Z}/n\mathbb{Z})^5)^*$ be a character of $G := (\mathbb{Z}/n\mathbb{Z})^5$.*

1) If $n \neq 4, 6$, then

$$\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} < 5 \iff \{D_\sigma : \psi(\sigma) \neq -1\} = \{A, B_1, B_2, B_3\},$$

where A, B_i are irreducible components of D with $AB_i = 1$, $B_iB_j = 0$ for $i \neq j$.

2) If $n = 6$, then $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} < 5$ if and only if

- *$\{D_\sigma : \psi(\sigma) \neq -1\} = \{A, B_1, B_2, B_3\}$, where A, B_i are irreducible components of D with $AB_i = 1$, $B_iB_j = 0$ for $i \neq j$, or*
- *$\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{ij} : j \in \{1, 2, 3, 4, 5\} \setminus \{i\}\}$, for some i.*

3) If $n = 4$, then $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} < 5$ if and only if one of the following conditions is satisfied:

- *$\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{T_1}, \ldots, E_{T_5}\}$, where there is a j such that $T_i \in \{1, \ldots, 5\} \setminus \{j\}$;*
- *$\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{T_1}, E_{T_2}, E_{T_3}\}$, where there are i, j such that $T_1, T_2, T_3 \subset \{1, \ldots, 5\} \setminus \{i, j\}$.*

Proof. 1) Let $\psi = (a_1, \ldots, a_5) \in ((\mathbb{Z}/n\mathbb{Z})^5)^*$ be a character of $G := (\mathbb{Z}/n\mathbb{Z})^5$, such that $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} \leq 4$. Then by Proposition 4.2 we know that

$$|\{D_\sigma : \psi(\sigma) \neq -1\}| \leq 6,$$

and

a) if $|\{D_\sigma : \psi(\sigma) \neq -1\}| = 6$, then $\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{T_1}, \ldots, E_{T_6}\}$, where either
   a.1) $T_1, \ldots T_6 \subset \{1, 2, 3, 4, 5\} \setminus \{j\}$ for some j;
   a.2) there are $1 \leq i < j \leq 5$ such that the T_h' s are the subsets of cardinality two with $|T_h \cap \{i, j\}| = 1$

b) if $|\{D_\sigma : \psi(\sigma) \neq -1\}| = 5$, then $\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{T_1}, \ldots, E_{T_5}\}$, where there is a j such that (after possibly renumbering the T_h's)

$$T_1 \cup T_2 = T_3 \cup T_4 = \{1, \ldots, 5\} \setminus \{j\}.$$

We consider the following three cases:

I) $|\{D_\sigma : \psi(\sigma) \neq -1\}| \leq 4$;
II) $|\{D_\sigma : \psi(\sigma) \neq -1\}| = 5$;
III) $|\{D_\sigma : \psi(\sigma) \neq -1\}| = 6$.

I) Assume that $\{D_\sigma : \psi(\sigma) = -1\} = \{E_{T_1}, \ldots, E_{T_k}\}$ with $k \geq 6$. Then we distinguish two cases:

Case 1: there is a $j \in \{1, \ldots, 5\}$ such that $T_1, \ldots, T_6 \subset \{1, \ldots, 5\} \setminus \{j\}$. Without loss of generality we can assume that $j = 5$, hence

$$\{E_{T_1}, \ldots, E_{T_6}\} = \{E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34}\}.$$

$\psi(E_{T_i}) = -1$ implies that $a_1 = \ldots = a_5 = -1$, whence

$$-1 = \psi(E_{12}) = -(a_1 + \ldots + a_5) = 5,$$

and this leads to a contradiction for $n \neq 6$.

Case 2: for each $j \in \{1, \ldots, 5\}$ there is an $i \in \{1, \ldots, 6\}$ such that $j \in T_i$.

2-1) Assume here that there is a j which is contained in four T_i's; then, without loss of generality, we can assume $j = 1$ and $T_1 = \{1,2\}$, $T_2 = \{1,3\}$, $T_3 = \{1,4\}$, $T_4 = \{1,5\}$. But then $\psi(E_{T_i}) = -1$ implies $a_5 = a_1 = a_2 + a_3 + a_4 = -1$ and we get for $n \neq 4$ a contradiction since

$$-1 = \psi(E_{12}) = -(a_1 + \ldots + a_5) = 3.$$

2-2) Assume that each j belongs to at least one and at most three T_h's. Therefore, again by an easy counting argument, we see that there is a $j \in \{1,2,3,4,5\}$ such that $j \in T_{i_1}, T_{i_2}, T_{i_3}$. Without loss of generality we may assume that $j = 3$ and $T_1 = \{1,3\}$, $T_2 = \{2,3\}$, $T_3 = \{3,4\}$ and since there is a k such that $5 \in T_k$ we may assume that $T_4 = \{4,5\}$. Then $\psi(E_{T_i}) = -1$ for $i = 1,2,3,4$ implies $a_5 = a_4 = a_3 = -(a_1 + a_2 + a_3) = -1$. Using now that $[a_1 + a_2] = [1 - a_3] = 2$, hence $a_2 = [2 - a_1]$ we obtain:

$$\begin{array}{ll} \psi(E_{12}) = [-(a_1 + \ldots + a_5)] = 1 \neq -1, & \psi(E_{14}) = a_1, \\ \psi(E_{15}) = [a_2 + a_3 + a_4] = [-a_1], & \psi(E_{24}) = a_2 = [2 - a_1], \\ \psi(E_{25}) = [a_1 + a_3 + a_5] = [-a_2] = [a_1 - 2], & \psi(E_{35}) = [-(a_3 + a_4 + a_5)] = 3. \end{array}$$

In order that two of the above five values are equal to -1 (the first is never -1), since we have two pairs of opposite values, the only possibility is that $a_1 = \pm 1$. Wlog we may assume $a_1 = 1$. Then $\psi(E_{15}) = \psi(E_{25}) = -1$, $\psi(E_{14}) = \psi(E_{24}) = 1$ and when $n \neq 4$

$$\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{12}, E_{14}, E_{24}, E_{35}\},$$

which is exactly the situation described in 1).

II) $\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{T_1}, \ldots, E_{T_5}\}$ and we assume $\mathrm{rk}\langle E_{T_1}, \ldots, E_{T_5}\rangle < 5$. Then four of the E_{T_i}'s have to be as in Remark 4.3 iii), a), yielding two reducible fibres, and we can assume $T_1 = \{1,2\}$, $T_2 = \{4,5\}$, $T_3 = \{1,4\}$, $T_4 = \{2,5\}$. We have then two possibilities, according to a section or a component of the third reducible fibre:

a) $T_5 = \{3,5\}$, or

b) $T_5 = \{1,5\}$. In both cases $\psi(E_T) = -1$ for $T \in \{\{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}\}$ implies that $a_5 = a_4 = a_2 = a_3 = -1$. This shows that a) is impossible, since $\psi(E_{15}) = [a_2 + a_3 + a_4] = -3 \neq -1$; for $n \neq 4$ also b) is impossible, since $-1 = \psi(E_{35}) = [-(a_3 + a_4 + a_5)] = 3 \neq -1$.

III) $\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{T_1}, \ldots, E_{T_6}\}$ and we assume $\mathrm{rk}\langle E_{T_1}, \ldots, E_{T_6}\rangle < 5$. Therefore by Proposition 4.2, (1), we may assume that either (up to $\mathfrak{S}_5$-symmetry) we have

$$\{D_\sigma : \psi(\sigma) = -1\} = \{E_{23}, E_{34}, E_{24}, E_{15}\},$$

or

$$\{D_\sigma : \psi(\sigma) = -1\} = \{E_{13}, E_{34}, E_{23}, E_{35}\}.$$

The first alternative cannot occur, since then $a_2 = a_3 = a_4 = -1$ hence $a_2 + a_3 + a_4 = -3 \neq -1$. The second alternative can only occur for $n = 4$, since we would have $-1 = a_5 = a_3 = a_4 = -(a_3 + a_4 + a_5) = 3$.

Hence assertion 1) is proven.

2) For $n = 6$ the only additional case is I), case 1: here $\psi = (5,5,5,5,5)$ and we get $\{D_\sigma : \psi(\sigma) = -1\} = \{E_{ij} : j \in \{1,2,3,4,5\} \setminus \{5\}\}$. This proves 2).

3) For $n = 4$ we need to treat several possibilities:

- I) Case 2-1
- I) Case 2-2
- II) possibility b
- III) second alternative.

I) Case 2-1: $T_1 = \{1,2\}$, $T_2 = \{1,3\}$, $T_3 = \{1,4\}$, $T_4 = \{1,5\}$.

$\psi(E_{T_i}) = -1$ implies $a_5 = a_1 = a_2 + a_3 + a_4 = -1$ and $-1 = \psi(E_{12}) = -(a_1 + \ldots + a_5) = 3$. Then we get the values for the components of the 3 reducible fibres:

- $\psi(\epsilon_{25}) = -2 + a_3$, $\psi(\epsilon_{34}) = a_3$,
- $\psi(\epsilon_{35}) = -2 + a_2$, $\psi(\epsilon_{24}) = a_2$,
- $\psi(\epsilon_{45}) = -2 + a_4$, $\psi(\epsilon_{23}) = a_4$.

Since two of the above values must be $= -1$, we can wlog assume that $a_2 = a_3 = 1$, hence also $a_4 = 1$ and we obtain the second exceptional case in the statement 3)

$$\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{23}, E_{24}, E_{34}\}.$$

I) Case 2-2: the previous analysis shows that for $n = 4$

$$\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{12}, E_{14}, E_{24}\}.$$

II) b: here $\{D_\sigma : \psi(\sigma) = -1\} = \{E_{13}, E_{23}, E_{24}, E_{34}, E_{35}\}$.

$\psi(E_T) = -1$ for $T \in \{\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}$ implies that $a_5 = a_4 = a_2 = a_3 = -1$, then we have also $\psi(E_{35}) = [-(a_3 + a_4 + a_5)] = 3 = -1$. Since we assume that $|\{D_\sigma : \psi(\sigma) = -1\}| = 5$, and the other values are $1, a_1, 2 - a_1, a_1 - 2, -a_1$ we get $\psi = (a_1, -1, -1, -1, -1)$ with $a_1 = 0, 2$. In both cases

$$\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{12}, E_{14}, E_{15}, E_{25}, E_{45}\}.$$

III) second alternative:

$$\{D_\sigma : \psi(\sigma) = -1\} = \{E_{13}, E_{34}, E_{23}, E_{35}\}.$$

We show now that this cannot occur, since then we would first of all have $a_5 = a_3 = a_4 = -(a_3 + a_4 + a_5) = -1$.

Since moreover we assume that $|\{D_\sigma : \psi(\sigma) = -1\}| = 4)$ and the other six values are:

$$a_1, a_1 - 2, a_2, a_2 - 2, 1 - a_1 - a_2, -1 - a_1 - a_2,$$

we obtain that $a_1, a_2 \neq 1, -1$. Also, we should have $a_1 + a_2 \neq 0, 2$, hence we derive a contradiction.

□

5. Invariants of Hirzebruch-Kummer coverings associated to the complete quadrangle

The next two sections shall lead to the proof of the following main result:

Theorem 5.1. *Let $\pi\colon S \to Y$ be the surface $HK(n)$, the Kummer covering of exponent $n \geq 3$ of the Del Pezzo surface Y of degree 5, branched on the divisor $D \in |-2K_Y|$, union of the 10 lines of Y. Then S is a smooth surface of general type with K_S ample, and*

$$H^1(S, \Theta_S) = 0$$

for $n \geq 4$. Hence S is infinitesimally and globally rigid for $n \geq 4$.

Consider the usual eigensheaf decomposition

$$\pi_* \mathcal{O}_S \cong \bigoplus_{\psi \in ((\mathbb{Z}/n\mathbb{Z})^5)^*} \mathcal{L}_\psi^{-1}.$$

In the remaining part of the section, for each character $\psi \in ((\mathbb{Z}/n\mathbb{Z})^5)^*$ we want to calculate the character sheaves $\mathcal{L}_\psi$ (for $\psi = 0$, $\mathcal{L}_0 = \mathcal{O}_Y$).

Remark 5.2. Viewing $Y \to \mathbb{P}^2$ as the blow up of the plane $\mathbb{P}^2$ in p_1, p_2, p_3, p_4, we denote by L the pull back of a line in $\mathbb{P}^2$ and by E_i (instead of E_{i5}) the exceptional curve over p_i. In this way we get a standard basis for $\mathrm{Pic}(Y)$.

We begin with the following lemma, where we denote by [] the remainder after division by n, $[] : \mathbb{Z}/n\mathbb{Z} \to \{0, \dots, n-1\}$, $a \mapsto [a]$.

Lemma 5.3. *For each character $\psi = (a_1, \dots a_5) \in ((\mathbb{Z}/n\mathbb{Z})^5)^*$ we have:*

$$n\mathcal{L}_\psi \equiv F(a_1, \dots, a_5)L - \lambda_1(a_1, \dots, a_5)E_1 - \\ - \lambda_2(a_1, \dots, a_5)E_2 - \lambda_3(a_1, \dots, a_5)E_3 - \lambda_4(a_1, \dots, a_5)E_4, \tag{5.1}$$

where

- $F(a_1, \dots, a_5) = [a_1] + [a_2] + [a_3] + [a_4] + [a_5] + [-(a_1 + \dots + a_5)]$,
- $\lambda_1(a_1, \dots, a_5) = [a_2] + [a_3] + [a_4] - [a_2 + a_3 + a_4]$,
- $\lambda_2(a_1, \dots, a_5) = [a_1] + [a_3] + [a_5] - [a_1 + a_3 + a_5]$,
- $\lambda_3(a_1, \dots, a_5) = [a_1] + [a_2] + [-(a_1 + \dots + a_5)] - [-(a_3 + a_4 + a_5)]$,
- $\lambda_4(a_1, \dots, a_5) = [a_4] + [a_5] + [-(a_1 + \dots + a_5)] - [-(a_1 + a_2 + a_3)]$,

Proof. As in [BC08], p. 392, we use the formula

$$n\mathcal{L}_\chi \equiv \sum_{1 \leq i < j \leq 5} \psi(\varphi(\epsilon_{ij}))E_{ij}.$$

The claim follows from writing the right hand side in terms of the basis L, E_1, E_2, E_3, E_4 of $\mathrm{Pic}(Y)$. □

Remark 5.4. We have the following:

(1) $F(a_1, \dots, a_5) \equiv 0 \mod n$, $0 \leq F(a_1, \dots, a_5) \leq 5n$,

(2) $F(a_1,\dots,a_5)=0 \iff (a_1,\dots,a_5)=0$,
(3) $\lambda_i(a_1,\dots,a_5)\equiv 0 \mod n$, $0\le \lambda_i(a_1,\dots,a_5)\le 2n$,
(4) $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 2F(a_1,\dots,a_5)-S(a_1,\dots,a_5)$,

where

$S(a_1,\dots,a_5) := [-(a_1+a_2+a_3)]+[a_2+a_3+a_4]+[a_1+a_3+a_5]+[-(a_3+a_4+a_5)]$.

Then $S(a_1,\dots,a_5)\equiv 0 \mod n$, $0\le S(a_1,\dots,a_5)\le 3n$.

Remark 5.5. If for $\psi=(a_1,\dots a_5)$ we have $\lambda_i(a_1,\dots a_5)=2n$, then $\psi(E_i)\neq n-1$.

Proposition 5.6. *Let S be the $HK(n)$ surface, and let $\pi\colon S\to Y$ be the Kummer covering of exponent n of Y branched in D, where $n\ge 3$. Then S is a smooth surface of general type with K_S ample.*

Moreover, we have:

$$K_S^2 = 5(n-2)^2 n^3, \quad e(S) = n^3(2n^2-10n+15).$$

Proof. The canonical divisor of S multiplied by n satisfies

$nK_S = \pi^*[nK_Y+(n-1)D] = -\pi^*(n-2)K_Y \Rightarrow K_S^2 = n^3(n-2)^2K_Y^2 = n^3(n-2)^2 5.$

In particular K_S is ample for $n\ge 3$.

We use then the additivity of the topological Euler Poincaré characteristic, obswrving that the ten lines meet in 15 points, and each intersects three other lines. Hence

$$7 = e(Y) = e(Y-D)+e(D^*)+15,$$

where D^* is the disjoint union of the ten lines, three times punctured, so that $e(D^*)=-10$ and $e(Y-D)=2$.

Using that unramified coverings of degree d multiply e by d, we get

$$e(S) = n^3[2n^2-10n+15].$$

□

For the proof of theorem 5.1 we use the following formulae by R. Pardini (cf. [Par91]). Recall that $\Omega^1_Y(\log D_j)_{j\in J}$ is the sheaf of meromorphic 1-forms generated as a sheaf of $\mathcal{O}_Y$-modules by Ω^1_Y and by $d(\log\delta_j)$, for each divisor $D_j = div(\delta_j)$, $j\in J$.

Proposition 5.7. *Let G be an Abelian group and let $\pi\colon S\to Y$ be a Galois cover with group G between compact smooth manifolds. Then for each character $\psi\in G^*$ we have*

$$\pi_*(\Omega^1_S\otimes\Omega^2_S)^\psi = \Omega^1_Y(\log D_\sigma : \sigma\in\mathcal{S}_\psi)(K_Y+\mathcal{L}_\psi),$$

where $\mathcal{S}_\psi := \{\sigma : \psi(\sigma)\neq n-\frac{n}{m(\sigma)} | m(\sigma)=\mathrm{ord}(\sigma)\}$.

Remark 5.8. In our situation: $\sigma \in G$ has always order $m(\sigma) = n$, hence

$$\mathcal{S}_\psi = \{\sigma : \psi(\sigma) \neq n-1\}.$$

Recall also once more that each $D_\sigma \neq 0$ is an irreducible (-1)-curve.

Observe that by Serre duality we have, for all characters ψ:

$$h^i(S, \Theta_S)^{-\psi} = h^{2-i}(\Omega^1_Y(\log D_\sigma : \psi(\sigma) \neq -1)(K_Y + \mathcal{L}_\psi)).$$

Since S is of general type, $h^0(S, \Theta_S) = 0$, hence in order to show that $h^1(S, \Theta_S) = 0$ it suffices to prove that $\forall \psi$:

$$h^0(\Omega^1_Y(\log D_\sigma : \psi(\sigma) \neq -1)(K_Y + \mathcal{L}_\psi)) = \chi(\Omega^1_Y(\log D_\sigma : \psi(\sigma) \neq -1)(K_Y + \mathcal{L}_\psi)).$$

Lemma 5.9. *For each $\psi \in ((\mathbb{Z}/n\mathbb{Z})^5)^*$ and each divisor Δ on Y we have:*

$$\chi(\Omega^1_Y(\log D_\sigma : \psi(\sigma) \neq -1)(\Delta)) = \Delta^2 - 5 + \sum_{\sigma : \psi(\sigma) \neq -1} (1 + D_\sigma \cdot \Delta). \tag{5.2}$$

Proof. Consider the exact residue sequence

$$0 \to \Omega^1_Y(\Delta) \to \Omega^1_Y(\log D_\sigma : \psi(\sigma) \neq -1)(\Delta) \to \bigoplus_{\sigma : \psi(\sigma) \neq -1} \mathcal{O}_{D_\sigma}(\Delta) \to 0. \tag{5.3}$$

An easy Chern classes calculation (since $\chi(\Omega^1_Y) = -5$) yields

$$\chi(\Omega^1_Y(\Delta)) = \Delta^2 - 5$$

moreover

$$\chi(\bigoplus_{\sigma : \psi(\sigma) \neq -1} \mathcal{O}_{D_\sigma}(\Delta)) = \sum_{\sigma : \psi(\sigma) \neq -1} (1 + D_\sigma \Delta)$$

hence the claim follows. □

Remark 5.10. Recall once more that the branch divisor D of the Kummer covering consists of 10 (-1)-curves which have the following property:

- if B is an irreducible component of D, then there exist exactly 3 irreducible components A_1, A_2, A_3 of D such that $A_1 \cdot B = A_2 \cdot B = A_3 \cdot B = 1$, all the other irreducible components of D are disjoint from B.

In the sequel we give a complete classification of the classes of the divisors $K_Y + \mathcal{L}_\psi$, $\psi \in ((\mathbb{Z}/n\mathbb{Z})^5)^*$ in the Neron-Severi group $NS(Y)$ of Y (here $NS(Y) = \mathrm{Pic}(Y)$).

5.1. $F(a_1, \ldots, a_5) = n$. Make first the obvious observation that $F(a_1, \ldots, a_5)$ is greater or equal to the sum of the positive summands yielding $\lambda_i(a_1, \ldots, a_5)$, $\forall i$. If there is an $i \in \{1, 2, 3, 4\}$ such that $\lambda_i = n$, then for the strict transforms of the three lines, say L_1, L_2, L_3, passing through p_i, we have $\psi(L_1) + \psi(L_2) + \psi(L_3) = n$, hence for the other three lines we have $\psi(L) = 0$ (since $F = n$). Therefore $\lambda_j = 0$ for $j \neq i$.

Otherwise $\lambda_i = 0$ for all i.

Therefore in this case we have the possibilities:

(1) $K_Y+\mathcal{L}_\psi \equiv -2L+E_i+E_j+E_k$, where $i,j,k \in \{1,2,3,4\}$ pairwise different;
(2) $K_Y+\mathcal{L}_\psi \equiv -2L+E_1+E_2+E_3+E_4$.

5.2. $F(a_1,\dots,a_5)=2n$. Assume there is an $i \in \{1,2,3,4\}$ such that $\lambda_i = 2n$, then the same argument as above shows that for the strict transforms of the three lines, say L_1, L_2, L_3, passing through p_i, we have $\psi(L_1)+\psi(L_2)+\psi(L_3)=2n$, while $\psi(L)=0$ for the strict transforms of the other three lines. Therefore $\lambda_j = 0$ for $j \neq i$ and we get the following possibility:

(3) $K_Y+\mathcal{L}_\psi \equiv -L-E_i+E_j+E_k+E_l$, where $\{i,j,k,l\}=\{1,2,3,4\}$.

We can now assume that $\lambda_i \leq n$ for all $i \in \{1,2,3,4\}$. Recall that

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 4n - S, \quad S \in \{0,n,2n,3n\}.$$

If $S=0$, then $\lambda_1=\lambda_2=\lambda_3=\lambda_4=n$, hence we have the possibility:

(4) $K_Y+\mathcal{L}_\psi \equiv -L$.

Remark 5.11. Observe that $S=0$ implies that $a_1=a_4$, $a_2=a_5$, $a_3=[-(a_1+a_2)]$ and $F=2([a_1]+[a_2]+[a_3])$.

If $S=n$, then $\lambda_1+\lambda_2+\lambda_3+\lambda_4=3n$, hence there is an $i \in \{1,2,3,4\}$ such that $\lambda_i=0$ and $\lambda_j=n$ for $j \neq i$. We have then:

(5) $K_Y+\mathcal{L}_\psi \equiv -L+E_i$ for some $i \in \{1,2,3,4\}$.

If $S=2n$, then $\lambda_1+\lambda_2+\lambda_3+\lambda_4=2n$, and we have then:

(6) $K_Y+\mathcal{L}_\psi \equiv -L+E_i+E_j$ for some $i \neq j \in \{1,2,3,4\}$.

Finally, if $S=3n$, then $\lambda_1+\lambda_2+\lambda_3+\lambda_4=n$, and we have:

(7) $K_Y+\mathcal{L}_\psi \equiv -L+E_i+E_j+E_k$ where $i \neq j \neq k \in \{1,2,3,4\}$.

5.3. $F(a_1,\dots,a_5)=3n$. Assume that $S=0$: then, by remark 5.11, we have:

$$3n = F = 2([a_1]+[a_2]+[a_3]) \implies [a_1]+[a_2]+[a_3]=\frac{3n}{2}.$$

If n is odd, this is a contradiction, whence we can assume that n is even. On the other hand, we have that $\lambda_i=n$ for all i, contradicting the fact that $\lambda_1+\lambda_2+\lambda_3+\lambda_4=6n$. Hence this case is not possible.

If we assume that there is an $i \in \{1,2,3,4\}$ such that $\lambda_i=2n$, then for the strict transforms of the three lines, say L_1,L_2,L_3, passing through p_i, we have $\psi(L_1)+\psi(L_2)+\psi(L_3) \geq 2n$, hence for the other three lines we have $\psi(L_4)+\psi(L_5)+\psi(L_6) \leq n$ (since $F=3n$). Therefore $\lambda_j \leq n$ for $j \neq i$.

$\underline{S=n}$: then $\lambda_1+\lambda_2+\lambda_3+\lambda_4=5n$, hence there is an $i \in \{1,2,3,4\}$ such that $\lambda_i=2n$ and $\lambda_j=n$ for $j \neq i$. Therefore we have the possibility:

(8) $K_Y+\mathcal{L}_\psi \equiv -E_i$ for $i \in \{1,2,3,4\}$.

$\underline{S = 2n}$: then $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4n$, hence either there are $i, j \in \{1, 2, 3, 4\}$ such that $\lambda_i = 2n$, $\lambda_j = 0$ and $\lambda_k = n$ for $k \neq i, j$, or $\lambda_i = n$ for all $i \in \{1, 2, 3, 4\}$. Therefore we have the possibilities:

(9) $K_Y + \mathcal{L}_\psi \equiv E_j - E_i$, for $i \neq j \in \{1, 2, 3, 4\}$;
(10) $K_Y + \mathcal{L}_\psi \equiv 0$.

$\underline{S = 3n}$: then $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 3n$, hence we have the possibilities:

(11) $K_Y + \mathcal{L}_\psi \equiv E_j + E_i - E_k$, for $i \neq j \neq k \in \{1, 2, 3, 4\}$;
(12) $K_Y + \mathcal{L}_\psi \equiv E_i$ for $i \in \{1, 2, 3, 4\}$.

5.4. $F(a_1, \ldots, a_5) = 4n$. If $S = 0$, then by remark 5.11, we have:

$$4n = F = 2([a_1] + [a_2] + [a_3]) \implies [a_1] + [a_2] + [a_3] = 2n.$$

Moreover, $a_1 = a_4$, $a_2 = a_5$, hence $\lambda_i = 2n$ for all $i \in \{1, 2, 3, 4\}$. This implies:

(13) $K_Y + \mathcal{L}_\psi \equiv L - E_1 - E_2 - E_3 - E_4$.

Assume that there is an $i \in \{1, 2, 3, 4\}$ such that $\lambda_i = 0$: then for the strict transforms of the three lines, say L_1, L_2, L_3, passing through p_i, we have $\psi(L_1) + \psi(L_2) + \psi(L_3) < n$, hence for the other three lines we have $3n - 3 \geq \psi(L_4) + \psi(L_5) + \psi(L_6) > 3n$ (since $F = 4n$), a contradiction. Therefore $\lambda_j \geq n$ for $i \in \{1, 2, 3, 4\}$.

$\underline{S = n}$: then $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 7n$, hence we have the possibility:

(14) $K_Y + \mathcal{L}_\psi \equiv L - E_i - E_j - E_k$ for $i \neq j \neq k \in \{1, 2, 3, 4\}$.

$\underline{S = 2n}$: then $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 6n$, hence we have:

(15) $K_Y + \mathcal{L}_\psi \equiv L - E_i - E_j$, for $i \neq j \in \{1, 2, 3, 4\}$.

$\underline{S = 3n}$: then $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 5n$, hence we have:

(16) $K_Y + \mathcal{L}_\psi \equiv L - E_i$ for $i \in \{1, 2, 3, 4\}$.

5.5. $F(a_1, \ldots, a_5) = 5n$. Assume that there is an $i \in \{1, 2, 3, 4\}$ such that $\lambda_i < 2n$: then for the strict transforms of the three lines, say L_1, L_2, L_3, passing through p_i, we have $\psi(L_1) + \psi(L_2) + \psi(L_3) < 2n$, hence for the other three lines we have $3n - 3 \geq \psi(L_4) + \psi(L_5) + \psi(L_6) > 3n$ (since $F = 5n$), a contradiction. Therefore $\lambda_j = 2n$ for $i \in \{1, 2, 3, 4\}$. Therefore we have:

(17) $K_Y + \mathcal{L}_\psi \equiv 2L - E_1 - E_2 - E_3 - E_4$ for $i \in \{1, 2, 3, 4\}$.

We shall use the following

Proposition 5.12. *Let $n = 5$, or $n > 6$. Then for $\psi \in ((\mathbb{Z}/n\mathbb{Z})^5)^*$ we have $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} < 5$ if and only if $\{D_\sigma : \psi(\sigma) \neq -1\} = \{A, B_1, B_2, B_3\}$, where B_1, B_2, B_3 are pairwise disjoint and $AB_i = 1$. In this case $K_Y + \mathcal{L}_\psi \equiv A$.*

Proof. The first part is Proposition 4.8. It remains to show that if $\{D_\sigma : \psi(\sigma) \neq -1\} = \{A, B_1, B_2, B_3\}$ then $K_Y + \mathcal{L}_\psi \equiv A$.

Using the $\mathfrak{S}_5$ symmetry, we may assume that A is the strict transform of one of the six lines in $\mathbb{P}^2$, $A = E_{34}$.

Then we have:

$$\begin{array}{lll} \psi(E_{14}) = a_1 = n-1, & \psi(E_{24}) = a_2 = n-1 & \psi(E_{23}) = a_4 = n-1, \\ \psi(E_{13}) = a_5 = n-1, & \psi(E_{45}) = [2-a_3] = n-1 & \psi(E_{35}) = [2-a_3] = n-1, \end{array}$$

whence $a_3 = 3$. This implies that $F(n-1, n-1, 3, n-1, n-1) = 4n$ and $\lambda_1 = \lambda_2 = 2n$, $\lambda_3 = \lambda_4 = n$. Hence $K_Y + \mathcal{L}_\psi \equiv E_{34}$.

□

Remark 5.13. 1) For later use, we work out also the calculation in the case where $A = E_{i5}$, wlog $i = 4$.

Here we have: $\psi(E_{13}) = \psi(E_{23}) = \psi(E_{12}) = n-1$, i.e. $a_1 = a_2 = a_3 = n-1$. Moreover,

$$\begin{array}{lll} \psi(E_{15}) = [a_2 + a_3 + a_4] = [a_4 - 2] = n-1 & \Longrightarrow & a_4 = 1, \\ \psi(E_{25}) = [a_1 + a_3 + a_5] = [a_5 - 2] = n-1 & \Longrightarrow & a_5 = 1, \\ \psi(E_{35}) = [-(a_3 + a_4 + a_5)] = n-1. & & \end{array}$$

Then $F(n-1, n-1, n-1, 1, 1) = 3n$, $\lambda_1 = \lambda_2 = \lambda_3 = n$ and $\lambda_4 = 0$. Therefore $K_Y + \mathcal{L}_\psi \equiv E_{45}$.

Observe that $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} < 5$ implies that $F(a_1, \dots, a_5) \in \{3n, 4n\}$.

2) If $n = 6$ and $\{D_\sigma : \psi(\sigma) = -1\} = \{E_{ij} : j \in \{1,2,3,4,5\} \setminus \{i\}\}$, for some i, whence we get $K_Y + \mathcal{L}_\psi \equiv X_i \equiv E_{jk} + E_{lm}$ for $\{j,k,l,m\} = \{1,2,3,4,5\} \setminus \{i\}$. In fact, by symmetry, we can wlog assume that

$$\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{15}, E_{25}, E_{35}, E_{45}\}, \; K_Y + \mathcal{L}_\psi \equiv X_5.$$

In this case $a_i = 5 \; \forall i$ and the calculation is straightforward.

3) If $n = 4$, then $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} < 5$ if and only if one of the following conditions is satisfied:

a) $\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{T_1}, \dots, E_{T_5}\}$, where there is a j such that $T_i \in \{1, \dots, 5\} \setminus \{j\}$;
b) $\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{T_1}, E_{T_2}, E_{T_3}\}$, where there are i, j such that $T_1, T_2, T_3 \subset \{1, \dots, 5\} \setminus \{i, j\}$.

In case b) we get $K + \mathcal{L}_\psi \equiv E_{ij}$. Assume in fact $i = 1, j = 2$; then $a_h = 3 \; \forall h \neq 3, a_3 = 1$, hence one easily sees that $K + \mathcal{L}_\psi \equiv E_{12}$.

In case a) instead, we have $K + \mathcal{L}_\psi \equiv A - B$, where $B \in \{E_{T_1}, \dots, E_{T_5}\}$, $A \in \{D_\sigma : \psi(\sigma) = -1\}$, $AB = 0$ and $A(E_{T_1} + \ldots + E_{T_5}) = 2$.

In fact, by symmetry, we can wlog assume that in case a)

$$\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{14}, E_{12}, E_{15}, E_{25}, E_{45}\};$$

then $a_h = 3 \; \forall h = 2,3,4,5, a_1 \neq 1, 3$. For $a_1 = 0$ we get $K_Y + \mathcal{L}_\psi \equiv E_{35} - E_{15}$, $a_1 = 2$ we get $K_Y + \mathcal{L}_\psi \equiv E_{34} - E_{45}$.

6. Cohomology of logarithmic differential forms

In this section we consider the following situation: let $P_1, \ldots, P_m$ be m distinct points in $\mathbb{P}^2$ and let

- $\pi \colon Y := \hat{\mathbb{P}}^2(P_1, \ldots, P_m) \to \mathbb{P}^2$ be the blow-up of the projective plane in $P_1, \ldots, P_m$;
- $D_1, \ldots, D_N \subset \mathbb{P}^2$ be smooth rational curves, such that
- $D_1 + \ldots + D_N$ has global normal crossings.

The aim of this paragraph is to prove the following general vanishing theorem:

Theorem 6.1. *Let $\mu, l, r, k \in \mathbb{N}$ such that*

$$0 \leq \mu \leq l \leq r \leq k \leq N,$$

and consider

- $A := D_{l+1} + \ldots + D_r + \ldots + D_k$,
- $B := D_1 + \ldots + D_\mu$,
- $D := A - B$,
- $\mathcal{F} := \Omega^1_Y(\log D_1, \ldots, \log D_r)(D)$.

Assume that

(1) $H^2(Y, \mathcal{F}) = 0$;
(2) for $l+1 \leq i \leq r$ *we have* $D_i(A - B) \geq -1$;
(3) for $l+1 \leq i \leq k$ *we have*

$$D_i(\sum_{\nu=1}^{k} D_\nu - B) \geq 1;$$

(4)

$$\mathrm{rk}\{D_i | 1 \leq i \leq k, D_i D_1 = \ldots = D_i D_\mu = 0\} \geq m + 1 - \mu + R,$$

where

$$R := \sum_{i=\mu+1, \exists j \in \{1,\ldots,\mu\}: D_j D_i \neq 0}^{k} (D_i(D_1 + \ldots + D_\mu) - 1).$$

Then $H^1(Y, \mathcal{F}) = 0$.

Proof. Consider

$$\mathcal{G} := \Omega^1_Y(\log D_1, \ldots, \log D_l)(D).$$

Then we have the exact residue sequence

$$0 \to \mathcal{G} \to \mathcal{F} \to \bigoplus_{i=l+1}^{r} \mathcal{O}_{D_i}(A - B) \to 0.$$

By (2), for all $l+1 \leq i \leq r$, we have that $\mathcal{O}_{D_i}(A - B) \cong \mathcal{O}_{\mathbb{P}^1}(n)$, where $n \geq -1$. Therefore by the long exact cohomology sequence it follows:

- $H^2(Y, \mathcal{G}) = 0$ (since $H^2(Y, \mathcal{F}) = 0$),

- if $H^1(Y,\mathcal{G})=0$, then $H^1(Y,\mathcal{F})=0$.

Hence it suffices to show that $H^1(Y,\mathcal{G})=0$.
Set now
$$\mathcal{G}' := \Omega^1_Y(\log D_1,\dots,\log D_k)(-B)$$
and consider the exact sequence (cf. e.g. [EV92], p. 13)
$$0 \to \mathcal{G}' \to \mathcal{G} \to \bigoplus_{i=l+1}^{k} \Omega^1_{D_i}(D_1+\ldots+D_k-B) \to 0.$$
Since $D_i \cong \mathbb{P}^1$, we have
$$\Omega^1_{D_i}(D_1+\ldots+D_k-B) \cong \mathcal{O}_{\mathbb{P}^1}(D_i(K_Y+D_i+D_1+\ldots+D_k-B)) \cong \mathcal{O}_{\mathbb{P}^1}(n),$$
where by (3), $n=-2+D_i(D_1+\ldots+D_k-B)\geq -1$.
Therefore it follows:

- $H^2(Y,\mathcal{G}')=0$ (since $H^2(Y,\mathcal{G})=0$),
- if $H^1(Y,\mathcal{G}')=0$, then $H^1(Y,\mathcal{G})=0$,

and it suffices to show that $H^1(Y,\mathcal{G}')=0$.
Using the analog of lemma 5.9 for m blow ups we calculate

$$(6.1)\quad \chi(\mathcal{G}') = B^2-m-1+\sum_{i=1}^{k}(1-BD_i) =$$
$$= (D_1+\ldots+D_\mu)^2-m-1+\sum_{i=1}^{k}(1-D_i(D_1+\ldots+D_\mu)) =$$
$$= -m-1+\mu+\sum_{i=\mu+1}^{k}(1-D_i(D_1+\ldots+D_\mu)) = (\sum_{i=\mu+1, D_1D_i=\ldots=D_\mu D_i=0}^{k} 1)-m-1+\mu-R.$$

On the other hand, consider the commutative diagram:

$$(6.2)\quad \begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ H^0(\mathcal{G}') \arrow[r] \arrow[d] & H^0(\Omega^1_Y(\log D_1,\dots,\log D_k)) \arrow[d] \\ \bigoplus_{i:D_iD_1=\ldots=D_iD_\mu=0} H^0(\mathcal{O}_{D_i}) \arrow[r] & \bigoplus_{i=1}^{k} H^0(\mathcal{O}_{D_i}) \arrow[d] \\ & H^1(\Omega^1_Y) \end{tikzcd}$$

Since the homomorphism in $H^1(\Omega^1_Y)$ takes the constant function equal to 1 on D_i to the Chern class of D_i, this implies that

(6.3)

$$h^0(Y,\mathcal{G}') \le (\sum_{i=1, D_1D_i=\ldots=D_\mu D_i=0}^{k} 1) - \mathrm{rk}\{D_i : D_iD_1 = \ldots D_iD_\mu = 0, 1 \le i \le k\} \le$$

$$\le (\sum_{i=1, D_1D_i=\ldots=D_\mu D_i=0}^{k} 1) - m - 1 + \mu - R = \chi(\mathcal{G}'),$$

where the last inequality holds by assumption (4).

Since $h^2(Y,\mathcal{G}') = 0$ the inequality $h^0(Y,\mathcal{G}') \le \chi(\mathcal{G}') = h^0(Y,\mathcal{G}') - h^1(Y,\mathcal{G}')$, implies that $h^1(Y,\mathcal{G}') = 0)$ and the theorem is proven.

□

Remark 6.2. Note that the assumption "(1) $h^2(\mathcal{F}) = 0$" in the previous Theorem is always satisfied in our applications by Remark 5.8.

7. Proof of Theorem 5.1 for $n \neq 4, 6$

In this section we use the classification of all the possibilities for $K + \mathcal{L}_\psi$ in order to prove our main result. Let $\psi \in ((\mathbb{Z}/n\mathbb{Z})^5)^*$.

7.1. **$F(a_1, \ldots, a_5) = n$.** Observe that by remark 5.13 we know that $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} = 5$. We have two cases here:

(1) $K_Y + \mathcal{L}_\psi \equiv -2L + E_i + E_j + E_k$, where $i, j, k \in \{1, 2, 3, 4\}$ pairwise different;
(2) $K_Y + \mathcal{L}_\psi \equiv -2L + E_1 + E_2 + E_3 + E_4$.

In case (1) we can assume wlog that

$$K_Y + \mathcal{L}_\psi \equiv -2L + E_1 + E_2 + E_3 \equiv -X_5 - E_4 \equiv -E_{13} - E_{24} - E_{45}.$$

Hence $\mathcal{L}_\psi = L - E_4$, $\lambda_4 = n$ and $F(a_1, \ldots, a_5) = n$, therefore we have $\psi(E_{i4}) = 0$ $\forall i \le 3$, i.e. $a_1 = a_2 = a_3 = 0$, which implies also $\psi(E_{45}) = 0$. Since $[a_4] + [a_5] + [-(a_1 + \ldots + a_5)] = n$, we have $\psi(L) = -1$ for at most one of the strict transforms of lines in $\mathbb{P}^2$ contained in D; in case there is one such line, assume wlog $\psi(E_{23}) = a_4 = n - 1$ and for all the others $\psi \neq -1$. This implies that we have

$$\{D_\sigma : \psi(\sigma) = -1\} = \emptyset \quad \text{or} \quad \{\mathrm{D}_\sigma : \psi(\sigma) = -1\} = \{\mathrm{E}_{23}, \mathrm{E}_{15}\}.$$

We dispose of case (1) via the following

Proposition 7.1.

$$\begin{aligned}(7.1)\quad & h^1(Y, \Omega^1_Y(\log E_{ij} : \{i,j\} \subset \{1,2,3,4,5\})(-E_{13} - E_{24} - E_{45})) = \\ & = h^1(Y, \Omega^1_Y(\log E_{ij} : \{i,j\} \neq \{2,3\}, \{1,5\})(-E_{13} - E_{24} - E_{45})) = 0.\end{aligned}$$

Proof. We have to verify in both cases the assumptions of theorem 6.1. Note that (1) is automatically satisfied by remark 5.8 and (2), (3) are empty. Observe that the sets

$$\begin{aligned}(7.2)\quad &\{E_{ij} : \{i,j\} \subset \{1,2,3,4,5\}, E_{ij}E_{13} = E_{ij}E_{24} = E_{ij}E_{45} = 0\} = \\ &= \{E_{ij} : \{i,j\} \neq \{2,3\}, \{1,5\}, E_{ij}E_{13} = E_{ij}E_{24} = E_{ij}E_{45} = 0\} = \{E_{14}, E_{34}\}.\end{aligned}$$

have rank 2, hence (4) is satisfied, since, for all E_{ij} not a component of B, $E_{ij}B \leq 1$ (B consists of a reducible fibre plus a section of the pencil X_5).

This proves the claim.

□

In case (2) we have $K_Y + \mathcal{L}_\psi \equiv -X_5$. Since $\lambda_i = 0$ for all $i \in \{1,2,3,4\}$, if for the strict transform of a line L we have $\psi(L) = -1$, wlog $\psi(E_{14}) = a_1 = n-1$, we infer then that $a_2, a_3, a_5, [-(a_1 + \ldots + a_5)] = 0$, $a_4 = 1$. In this case we have then

$$\{D_\sigma : \psi(\sigma) = -1\} = \{E_{14}, E_{25}, E_{35}\}.$$

If instead for all lines L we have $\psi(L) \neq -1$ wlog we may assume (since $S(a_1, \cdots, a_5) = 2n$) that

$$\{D_\sigma : \psi(\sigma) = -1\} \subset \{E_{25}, E_{35}\}.$$

Since $-X_5 \equiv -B := -(E_{13} + E_{24})$, in both cases assumptions (1)-(3) of theorem 6.1 are satisfied. There remains to show (4). It is first of all immediate to see that $R = 0$ (B is a reducible fibre of X_5).

Observe that

$$\{E_{ij} : \{i,j\} \subset \{1,2,3,4,5\}, E_{ij}E_{13} = E_{ij}E_{24} = 0\} = \{E_{14}, E_{34}, E_{12}, E_{23}\}.$$

This set has rank 3, and if we remove from this set its intersection with $\{E_{25}, E_{35}\}$, respectively $\{E_{14}, E_{25}, E_{35}\}$, we obtain either the full set or the set $\{E_{34}, E_{12}, E_{23}\}$, which has rank 3. Whence (4) is satisfied in both cases.

7.2. $F(a_1, \ldots, a_5) = 2n$. Consider the first subcase which, up to symmetry of the four blown up points, can be written as

$$(3)\quad K_Y + \mathcal{L}_\psi \equiv -L - E_4 + E_1 + E_2 + E_3 \equiv -E_{j4} - E_{45} + E_{j5},$$

for each $j \in \{1,2,3\}$.

Since $\lambda_4 = 2n$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, we have $\psi(E_{23}) + \psi(E_{13}) + \psi(E_{12}) = 2n$ and $\psi(E_{14}) = \psi(E_{24}) = \psi(E_{34}) = 0$ (since $F = 2n$). I.e., $a_1 = a_2 = a_3 = 0$, $\psi(E_{45}) = 0$. Moreover, $[a_4] + [a_5] + [-(a_1 + \ldots + a_5)] = 2n$. Therefore we have the following possibilities:

i) $[a_4], [a_5], [-(a_1 + \ldots + a_5)] = [-(a_4 + a_5)] \neq -1$, hence $\psi(D_\sigma) \neq -1$ for all σ;

ii) wlog $[a_5], [-(a_4 + a_5)] \neq -1$, $[a_4] = -1$; hence $\{D_\sigma : \psi(\sigma) = -1\} = \{E_{23}, E_{15}\}$;

iii) wlog $[-(a_4+a_5)] \neq -1$, $[a_4]=[a_5]=-1$; hence $\{D_\sigma : \psi(\sigma)=-1\} = \{E_{23}, E_{13}, E_{15}, E_{25}\}$.

Observe that cases ii), iii) occur in particular for $n=5$:

ii) $\psi = (0,0,0,4,3,3)$,
iii) $\psi = (0,0,0,4,4,2)$.

Since for $n=5$ the Hirzebruch-Kummer covering is a ball quotient, and ball quotients are infinitesimally rigid by [CV60], p. 500.

$$\begin{aligned}(7.3)\quad & h^1(Y, \Omega^1_Y(\log D_\sigma : D_\sigma \neq E_{23}, E_{13}, E_{15}, E_{25})(-E_{j4}-E_{45}+E_{j5})) = \\ & = h^1(Y, \Omega^1_Y(\log D_\sigma : D_\sigma \neq E_{23}, E_{15})(-E_{j4}-E_{45}+E_{j5})) = 0.\end{aligned}$$

For i) consider instead the exact sequence

$$\begin{aligned}(7.4)\quad & 0 \to \mathcal{F}' := \Omega^1_Y(\log D_\sigma : D_\sigma \neq E_{23}, E_{15})(-E_{j4}-E_{45}+E_{j5}) \to \mathcal{F} := \\ & = \Omega^1_Y(\log D_\sigma : D_\sigma \neq 0)(-E_{j4}-E_{45}+E_{j5}) \to \\ & \to \mathcal{O}_{E_{23}}(-E_{j4}-E_{45}+E_{j5}) \oplus \mathcal{O}_{E_{15}}(-E_{j4}-E_{45}+E_{j5}) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to 0.\end{aligned}$$

Since $h^1(\mathcal{F}')=0$ (by ii)), it follows that $h^1(\mathcal{F})=0$, and we have proven case i).

The next case is:

(4) $K_Y + \mathcal{L}_\psi \equiv -L \equiv -X_i - E_{i5}$, $\forall i \in \{1,2,3,4\}$.

Here $\lambda_i = n$ $\forall i \Rightarrow S(a_1,\dots,a_5)=0$ hence by Remark 5.11, we have $a_1=a_4$, $a_2=a_5$, $a_3=[-(a_1+a_2)]$ and $[a_1]+[a_2]+[a_3]=n$. This implies that $\psi(E_{i5})=0$ for all i and we have two cases:

i) $[a_1],[a_2],[a_3] \neq -1$, hence $\psi(D_\sigma) \neq -1$ for all σ;
ii) wlog $[a_1]=-1$; $[a_2],[a_3] \neq -1$, hence $\{D_\sigma : \psi(\sigma)=-1\} = \{E_{23}, E_{14}\}$.

These cases occur in particular for $n=5$:

i) $\psi = (1,2,3,1,2)$,
ii) $\psi = (4,1,0,4,1)$.

Therefore, by the same argument as above, we get

$$\begin{aligned}(7.5)\quad & h^1(Y, \Omega^1_Y(\log D_\sigma : D_\sigma \neq 0)(-X_i - E_{i5})) = \\ & = h^1(Y, \Omega^1_Y(\log D_\sigma : D_\sigma \neq E_{23}, E_{14})(-X_i - E_{i5})) = 0.\end{aligned}$$

Consider the case

(5) $K_Y + \mathcal{L}_\psi \equiv -X_i$ for some $i \in \{1,2,3,4\}$.

Wlog we can assume $i=1$, i.e., $\lambda_1 = 0$ and $\lambda_j = n$ for $j \neq 1$. Then

$$K_Y + \mathcal{L}_\psi \equiv -X_1 \equiv -E_{23} - E_{45} \equiv -E_{24} - E_{35} \equiv -E_{34} - E_{25}.$$

According to theorem 6.1 it suffices to show (since then $A=0, R=0$):

- there is a decomposition $X_1 = D_1 + D_2$, such that $\psi(D_1), \psi(D_2) \neq n-1$,

- $M := \{D_\sigma : \psi(\sigma) \neq n-1, D_\sigma D_1 = D_\sigma D_2 = 0\}$ has rank 3.

Since $\lambda_1 = 0$, we have (assume as always $0 \le a_i < n$) $a_2 + a_3 + a_4 < n$. If one of the a_2, a_3, a_4 equals $n-1$, say $a_4 = n-1$, then $a_2 = a_3 = 0$. In any case, $\psi(E_{34}), \psi(E_{24}) \neq n-1$.

Since $S = n$ there exists at most one $i \in \{1,2,3,4\}$ such that $\psi(E_{i5}) = n-1$. This implies that either $\psi(E_{25}) \neq n-1$ or $\psi(E_{15}) \neq n-1$, hence $(D_1, D_2) = (E_{34}, E_{25})$ or $(D_1, D_2) = (E_{24}, E_{15})$ satisfy the first condition above. We can assume wlog that $\psi(E_{34}), \psi(E_{25}) \neq n-1$.

If $\psi(E_{24}), \psi(E_{35}), \psi(E_{23}), \psi(E_{45}) \neq n-1$, we are done since $\{E_{24}, E_{35}, E_{23}, E_{45}\} \subset M$, which implies that M has rank 3.

We show now that it suffices to prove that the case $\psi(E_{23}) = a_4 = n-1 = \psi(E_{45}) = n-1$ cannot occur. Since, if $\psi(E_{23}) \neq n-1$ and $\psi(E_{45}) = n-1$, then the set $\{E_{24}, E_{35}, E_{23}\} \subset M$, which has rank three. If instead $\psi(E_{23}) = a_4 = n-1$ and $\psi(E_{45}) \neq n-1$, we have already remarked that $a_2 = a_3 = 0$. Since $\lambda_3 = n$, $\psi(E_{35}) = n-1$ if and only if $a_1 + a_6 = 2n-1$, which is not possible. Hence $\{E_{24}, E_{35}, E_{45}\} \subset M$.

Let's show that $\psi(E_{23}) = a_4 = n-1 = \psi(E_{45}) = n-1$ cannot occur. Otherwise we have $a_4 + a_5 + a_6 = \lambda_4 + [\psi(E_{45})] = 2n-1$, whence $a_5 + a_6 = n$. $F = 2n$ implies $a_1 = 1$. On the other hand, since $\lambda_2 = \lambda_3 = n$ we have $a_1 + a_5 \ge n$ and $a_1 + a_6 \ge n$, a contradiction.

The next case is:

(6) $K_Y + \mathcal{L}_\psi \equiv -L + E_i + E_j$ for $i \neq j \in \{1,2,3,4\}$.

Wlog $i = 1, j = 2$, i.e., $K_Y + \mathcal{L}_\psi \equiv -L + E_1 + E_2 \equiv -E_{34}$.

Assume that $\psi(E_{34}) = a_3 = n-1$: then (since $\lambda_1 = \lambda_2 = 0$) $a_2 = a_4 = a_1 = a_5 = 0$, which contradicts $F(a_1, \dots, a_5) = 2n$. Therefore $\psi(E_{34}) \neq n-1$ and by theorem 6.1 it suffices to show: $M := \{D_\sigma : \psi(\sigma) \neq n-1, D_\sigma E_{34} = 0\}$ has rank at least 4.

Note that:

- $F = 2n \implies$ at most two of the E_{ij}'s, $\{i,j\} \subset \{1,2,3,4\}$ have $\psi(E_{ij}) = n-1$;
- $S = 2n \implies$ at most two of the E_{i5}'s, $i \in \{1,2,3,4\}$ have $\psi(E_{i5}) = n-1$.

Assume that there is one of the strict transforms of the lines orthogonal to E_{34} which has $\psi = n-1$, wlog $\psi(E_{14}) = a_1 = n-1$. Then $\lambda_2 = 0$ implies $a_3 = a_5 = 0$ and $\psi(E_{25}) = n-1$. In particular, $\psi(E_{13}) = a_5 \neq n-1$. Moreover, $n + \psi(E_{45}) = \lambda_4 + \psi(E_{45}) = a_4 + a_5 + a_6 = a_4 + a_6 \le 2n-2$, hence $\psi(E_{45}) \neq n-1$.

Hence $\{E_{13}, E_{45}\} \subset M$.

Assume that also $\psi(E_{24}) = a_2 = n-1$. Then $a_4 = a_3 = 0$, since $\lambda_1 = 0$. But then, since $\lambda_4 = n$, $n \le a_4 + a_5 + a_6 = a_6$, a contradiction. Hence $\psi(E_{24}) \neq n-1$.

Assume that $\psi(E_{23}) = a_4 = n-1$. Then $a_2 = a_3 = 0$ and $a_6 = 2$ and this implies that $\psi(E_{35}) \neq n-1$. Therefore

$$\{E_{13}, E_{45}, E_{35}, E_{24}\} \subset M,$$

hence $\operatorname{rk} M \geq 4$.

If instead $\psi(E_{23}) \neq n-1$ (and $\psi(E_{14}) = n-1$), then

$$\{E_{13}, E_{45}, E_{23}, E_{24}\} \subset M,$$

and again $\operatorname{rk} M \geq 4$.

We can therefore assume that for all the strict transforms of the lines orthogonal to E_{34}: $\psi \neq n-1$, i.e.

$$\{E_{14}, E_{23}, E_{13}, E_{24}\} \subset M.$$

We need to show that either $\psi(E_{35}) \neq n-1$ or $\psi(E_{45}) \neq n-1$, then we are done.

Assume that $\psi(E_{35}) = \psi(E_{45}) = n-1$. Since $S = 2n$, this implies that $\psi(E_{15}) + \psi(E_{25}) = 2$. However $\psi(E_{15})+\psi(E_{25}) = a_2+a_3+a_4+a_1+a_3+a_5 = F(a_1,\dots,a_5)+a_3-a_5 \geq n+1$, a contradiction.

The last case in this subsection is

(7) $K_Y + \mathcal{L}_\psi \equiv -L + E_i + E_j + E_k$ for $i \neq j \neq k \in \{1,2,3,4\}$.

We have $F = 2n$, $S = 3n$ and wlog $\lambda_4 = n$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Then

$$K_Y + \mathcal{L}_\psi \equiv -L + E_1 + E_2 + E_3 \equiv -E_{i4} + E_{i5}$$

for all $i \in \{1,2,3\}$.

We set $A := E_{i5}$, $B := E_{i4}$ and we want to apply theorem 6.1, whence we have to verify that there is an i such that $\psi(E_{i4}) \neq n-1$ and that assumptions (2)-(4) of theorem 6.1 are satisfied.

We have for all $i \in \{1,2,3\}$ that $E_{i5}(E_{i5} - E_{i4}) = -1$, i.e., (2) holds. Since $\lambda_1 = \lambda_2 = \lambda_3 = 0$, it follows that $\psi(E_{j4}) = a_j \neq n-1$ for all $j \in \{1,2,3\}$, otherwise if e.g. $a_1 = n-1$, then $a_3 = a_5 = a_2 = a_6 = 0$, contradicting $F(a_1,\dots,a_5) = 2n$.

(3) is now satisfied since $E_{j4}E_{i5} = 1$ for $i \neq j$, hence

$$E_{i5}\Big(\Big(\sum_{\sigma:\psi(\sigma)\neq n-1}^{D_\sigma \neq E_{i5}} D_\sigma\Big) + E_{i5} - E_{i4}\Big) = \Big(\sum_{\sigma:\psi(\sigma)\neq n-1}^{D_\sigma \neq E_{i5}} D_\sigma\Big)E_{i5} - 1 \geq 2-1 = 1.$$

It remains to show (4), i.e. $M_i := \{D_\sigma : \psi(\sigma) \neq n-1, D_\sigma E_{i4} = 0\} \cup \{E_{i5}\}$ has rank 4 for some $i \in \{1,2,3\}$.

Observe that since $F = 2n$, at most two of the E_{ij}, $\{i,j\} \subset \{1,2,3\}$ have $\psi(E_{ij}) = n-1$. Assume wlog $\psi(E_{23}) = a_4 = \psi(E_{13}) = a_5 = n-1$. Since $\lambda_1 = \lambda_2 = 0$, this implies that $a_1 = a_3 = a_2 = 0$, whence $a_6 = 2$. But this contradicts $\lambda_4 = n$.

Assume now that for exactly one of the E_{ij}, $\{i,j\} \subset \{1,2,3\}$ have $\psi(E_{ij}) = n-1$, wlog $\psi(E_{23}) = a_4 = n-1$. Then, since $\lambda_1 = 0$, $a_2 = a_3 = 0$ and $\psi(E_{15}) = n-1$.

If $\psi(E_{i5}) \neq n-1$ for $i \neq 1$, then we are done since e.g. $\{E_{14}, E_{34}, E_{25}, E_{45}\} \subset M_2$.

Suppose that also $\psi(E_{45}) = n-1$. Then (since $\lambda_4 = n$):

$$2n-1 = a_4 + a_5 + a_6 = n-1+a_5+a_6 \implies a_5 + a_6 = n.$$

$F = 2n$ implies that $a_1 = 1$, in particular $\psi(E_{25})$ or $\psi(E_{35}) \neq n-1$. If $\psi(E_{35}) \neq n-1$, write $K_Y + \mathcal{L} = E_{25} - E_{24}$ and observe that

$$\{E_{14}, E_{34}, E_{35}, E_{12}\} \subset M_2.$$

Therefore we may assume that $\psi(E_{ij}) \neq n-1$ for all $\{i,j\} \subset \{1,2,3\}$.

If for some $i \leq 3$ we have $\psi(E_{i5}) \neq n-1$ we are done since, if (i,j,k) is a permutation of $(1,2,3)$,

$$\{E_{24}, E_{34}, E_{ij}, E_{ik}, E_{i5}\} \subset M_i.$$

If $\psi(E_{45}) \neq n-1$, then

$$\{E_{24}, E_{34}, E_{13}, E_{12}, E_{45}\} \subset M_1.$$

We conclude as follows: the case that $\psi(E_{i5}) = n-1$ for $i = 1,2,3,4$ contradicts $S = 3n$.

7.3. $F(a_1, \dots, a_5) = 3n$**.** Here the first case is:

(8) $K_Y + \mathcal{L}_\psi \equiv -E_{i5}$ for $i \in \{1,2,3,4\}$.

We have $F = 3n$, $S = n$. Wlog $i = 1$, then $\lambda_1 = 2n, \lambda_2 = \lambda_3 = \lambda_4 = n$. In particular, $\psi(E_{15}) \neq n-1$ by Remark 5.5. Since $A = 0$, B is irreducible, to apply Theorem 6.1 it suffices to verify that $M := \{D_\sigma : \psi(\sigma) \neq n-1, D_\sigma E_{15} = 0\}$ has rank 4.

Since $S = n$ we have that $\psi(E_{i5}) = n-1$ for at most one i. Assume wlog that $\psi(E_{25}) = n-1$. Since $\lambda_2 = n$ this implies

$$2n - 1 = a_1 + a_3 + a_5 \implies a_2 + a_4 + a_6 = n+1.$$

Since $\lambda_1 = 2n$, it follows that $a_2 + a_3 + a_4 \geq 2n$, hence $a_2 + a_4 \geq n+1 \Rightarrow a_2 + a_4 = n+1, a_6 = 0, a_3 = n-1$.

Hence at least one of a_1, a_5 is different from $n-1$. Then either $\{E_{35}, E_{45}, E_{12}, E_{13}\} \subset M$ or $\{E_{35}, E_{45}, E_{12}, E_{14}\} \subset M$ and we are done, unless $\psi(E_{i5}) \neq n-1$ for all i.

In this case $E_{35}, E_{45}, E_{25} \in M$ and it suffices to show that it cannot happen that $\psi(E_{14}) = \psi(E_{12}) = \psi(E_{13}) = n-1$, i.e. $a_1 = a_6 = a_5 = n-1$. But then (since $F = 3n$) $a_2 + a_3 + a_4 = 3$ contradicting $\lambda_1 = 2n$.

Remark 7.2. If $\lambda_i = 0$, then for the strict transforms of the three lines, say L_1, L_2, L_3, passing through p_i, we have $\psi(L_1) + \psi(L_2) + \psi(L_3) \leq n-1$, hence at least two of these lines have $\psi(\dots) \neq n-1$. Therefore $E_{i5}(\sum_{\psi(\sigma)\neq -1, D_\sigma \neq E_{i5}} D_\sigma) \geq 2$.

The next cases are:

(9) $K_Y + \mathcal{L}_\psi \equiv E_{j5} - E_{i5}$, for $i \neq j \in \{1,2,3,4\}$;
(10) $K_Y + \mathcal{L}_\psi \equiv 0$.

For case (9) we can assume wlog $j = 1, i = 2$, i.e. $K_Y + \mathcal{L}_\psi \equiv E_{15} - E_{25}$.

Observe again that $\psi(E_{25}) \neq n-1$, since $\lambda_2 = 2n$. Moreover, (2), (3) of Theorem 6.1 are satisfied, since $E_{15}(E_{15} - E_{25}) = -1$ and

$$E_{15}((\sum_{\sigma:\psi(\sigma)\neq n-1}^{D_\sigma\neq E_{15}} D_\sigma)+E_{15}-E_{25})=(\sum_{\sigma:\psi(\sigma)\neq n-1}^{D_\sigma\neq E_{15}} D_\sigma)E_{15}-1\geq 2-1=1.$$

The above inequality follows by remark 7.2.

We need to show: $M:=\{D_\sigma:\psi(\sigma)\neq n-1, D_\sigma E_{25}=0\}\cup\{E_{15}\}$ has rank 4. Observe that

$$M\subset\{D_\sigma:D_\sigma E_{25}=0\}=\{E_{24},E_{23},E_{12},E_{15},E_{35},E_{45}\}.$$

Assume that $\psi(E_{24})=a_2=n-1$. Then, since $\lambda_1=0$, we have $a_3=a_4=0$. But this contradicts $\lambda_2=2n$. The same argument for E_{23} shows that $\psi(E_{24}),\psi(E_{23})\neq n-1$.

Assume that $\psi(E_{34})=a_3=n-1$. Then again $\lambda_1=0$ implies that $a_2=a_4=0$. Since $\lambda_3=\lambda_4=n$ it follows that $\psi(E_{35}),\psi(E_{45})\neq n-1$ and then

$$\{E_{15},E_{23},E_{24},E_{35},E_{45}\}\subset M,$$

which implies that $\operatorname{rk} M=4$.

Therefore we can assume that $\psi(E_{34})\neq n-1$. Assume now that $\psi(E_{12})=a_6=n-1$. Then:

- $\psi(E_{35})=n-1 \implies a_1+a_2=n$;
- $\psi(E_{45})=n-1 \implies a_4+a_5=n$.

If both equalities occur, $F=3n\Rightarrow a_3=1$, which contradicts $\lambda_2=2n$. Therefore either $\psi(E_{45})\neq n-1$ or $\psi(E_{35})\neq n-1$. We can assume wlog $\psi(E_{45})\neq n-1$. Then:

$$\{E_{15},E_{23},E_{24},E_{45}\}\subset M,$$

which implies that $\operatorname{rk} M=4$.

Therefore we can assume that $\psi(E_{12})=a_6\neq n-1$ and we are done since $\{E_{15},E_{23},E_{24},E_{12}\}\subset M$.

For case (10) we are done by Theorem 6.1, since by Proposition 5.12 it holds $\operatorname{rk}\{D_\sigma:\psi(\sigma)\neq -1\}=5$.

In this subsection we are left with the following two cases:

(11) $K_Y+\mathcal{L}_\psi\equiv E_{j5}+E_{i5}-E_{k5}$, for $i\neq j\neq k\in\{1,2,3,4\}$;
(12) $K_Y+\mathcal{L}_\psi\equiv E_{i5}$ for $i\in\{1,2,3,4\}$.

In case (11) we can assume wlog that $K_Y+\mathcal{L}_\psi\equiv E_{15}+E_{25}-E_{35}$. We have $F=S=3n$, $\lambda_4=n$, $\lambda_1=\lambda_2=0$ and $\lambda_3=2n$.

Observe that again $\psi(E_{35})\neq n-1$ since $\lambda_3=2n$. Moreover, (2), (3) of Theorem 6.1 are satisfied, since $E_{i5}(E_{15}+E_{25}-E_{35})=-1$ for $i=1,2$ and (by Remark 7.2)

$$E_{i5}((\sum_{\sigma:\psi(\sigma)\neq n-1}^{D_\sigma\neq E_{15},E_{25}} D_\sigma)+E_{15}+E_{25}-E_{35})=(\sum_{\sigma:\psi(\sigma)\neq n-1}^{D_\sigma\neq E_{15},E_{25}} D_\sigma)E_{i5}-1\geq 2-1=1.$$

We need to show that $M := \{D_\sigma : \psi(\sigma) \neq n-1, D_\sigma E_{35} = 0\} \cup \{E_{15}, E_{25}\}$ has rank 4. Observe that

$$M \subset \{D_\sigma : D_\sigma E_{35} = 0\} \cup \{E_{15}, E_{25}\} = \{E_{15}, E_{25}, E_{45}, E_{34}, E_{23}, E_{13}\},$$

and $\psi(E_{35}) \neq n-1$.

Since $\lambda_1 = \lambda_2 = 0$ we see that $a_3 = \psi(E_{34}) \neq n-1$ (else $a_2 = a_4 = a_1 = a_5 = 0$, contradicting $F = 3n$); moreover $a_4 = \psi(E_{23}) \neq n-1$ and $a_5 = \psi(E_{13}) \neq n-1$. In fact $a_4 = n-1$ implies $a_2 = a_3 = 0$, contradicting $\lambda_3 = 2n$, similarly $a_5 = n-1$ contradicts $\lambda_3 = 2n$.

Hence $\{E_{15}, E_{25}, E_{34}, E_{23}, E_{13}\} \subset M$ and we are done.

Consider the next case (12), where we can apply again remark 7.2 to infer that property (3) is satisfied; if $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} \cup \{E_{i5}\} = 5$, we can apply Theorem 6.1. If instead $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} \cup \{E_{i5}\} < 5$, then wlog we can assume that $i = 4$ and it is easy to see that $\psi = (n-1, n-1, n-1, 1, 1)$. Since this case occurs for $n = 5$, we are done.

7.4. $F(a_1, \ldots, a_5) = 4n$. The first case is:

(13) $K_Y + \mathcal{L}_\psi \equiv L - E_1 - E_2 - E_3 - E_4 \equiv E_{34} - E_{35} - E_{45}$.

Here the divisors appearing in A and B are disjoint, conditions (1)-(2) of Theorem 6.1 are fulfilled. Moreover, since $\lambda_i = 2n$ for all i, we have $\psi(E_{i5}) \neq n-1$ for all $i \in \{1, 2, 3, 4\}$, hence also condition (3) is fulfilled.

We have, for $i = 1, 2, 3$, $K_Y + \mathcal{L}_\psi \equiv E_{i4} - E_{i5} - E_{45} = E_{34} - E_{35} - E_{45}$.

Then

$$\{E_{ij} : E_{ij}.E_{35} = E_{ij}.E_{45} = 0\} = \{E_{15}, E_{25}, E_{34}\}.$$

Note that this set has rank 3 and is equal to $\{D_\sigma : \psi(\sigma) \neq n-1, D_\sigma E_{35} = D_\sigma E_{45} = 0\} \cup \{E_{34}\}$.

Therefore we are done if we can show that $R = 0$ (cf. Theorem 6.1, (4)) for some choice of $i \in \{1, 2, 3\}$. The only case that does not work is the case where $\psi(E_{ij}) \neq n-1$ for all $i, j \in \{1, 2, 3\}$. To handle this case we consider:

- $\mathcal{F}' := \Omega^1_Y(\log E_{ij} : (i,j) \neq (1,2),\ \psi(E_{ij}) \neq n-1)(E_{34} - E_{35} - E_{45})$,
- $\mathcal{F} := \Omega^1_Y(\log E_{ij}, \psi(E_{ij}) \neq n-1)(E_{34} - E_{35} - E_{45})$.

We have the exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{O}_{E_{12}}(E_{34} - E_{35} - E_{45}) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \to 0.$$

Therefore it suffices to show that $H^1(Y, \mathcal{F}') = 0$, which is true since the assumption (4) of Theorem 6.1 is satisfied for $\mathcal{F}'$.

Consider case

(14) $K_Y + \mathcal{L}_\psi \equiv L - E_i - E_j - E_k$ for $i \neq j \neq k \in \{1, 2, 3, 4\}$.

Wlog we can assume $(i, j, k) = (1, 2, 3)$, hence

$$K_Y + \mathcal{L}_\psi \equiv E_{i4} - E_{i5},\ \ i \in \{1, 2, 3\}.$$

Since $\lambda_i = 2n$ for $i = 1, 2, 3$, we have $\psi(E_{i5}) \neq n-1$ for $i = 1, 2, 3$. We easily see that it suffices to verify condition (4) of Theorem 6.1. But since $\lambda_4 = n$ we have that $a_4 + a_5 + a_6 < 2n$ whence we can assume wlog $a_4 \neq n-1$ and choose $i = 3$ in the decomposition $A - B$. Therefore

$$E_{25}, E_{15}, E_{23}, E_{34} \in \{D_\sigma : \psi(\sigma) \neq n-1, D_\sigma E_{35} = 0\} \cup \{E_{34}\},$$

and we are done.

In case

(15) $K_Y + \mathcal{L}_\psi \equiv L - E_i - E_j$, for $i \neq j \in \{1, 2, 3, 4\}$

wlog $K_Y + \mathcal{L}_\psi \equiv E_{34} = A$. Conditions (1), (2) of Theorem 6.1 are obviously true ($B = 0$), condition (3) is satisfied since $\psi(E_{15}), \psi(E_{15}) \neq n-1$ (as $\lambda_1, \lambda_2 = 2n$).

We are done if $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} = 5$. If instead $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} < 5$, by the proof of proposition 5.12 we have that $\psi = (n-1, n-1, 3, n-1, n-1)$; since this case occurs for $n = 5$ we are done.

The last case of this subsection is

(16) $K_Y + \mathcal{L}_\psi \equiv L - E_i \equiv X_i$ for $i \in \{1, 2, 3, 4\}$.

Here $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} = 5$ by proposition 5.12 since X_i is not irreducible. We now verify the assumptions of Theorem 6.1. Wlog $i = 1$ and $K_Y + \mathcal{L}_\psi \equiv E_{34} + E_{25}$. (1) and (2) are clear and (3) is verified for E_{34} since $\psi(E_{15}) \neq n-1$ and $E_{34}E_{15} = 1$. If condition (3) is not verified for E_{25}, this means that $\psi = n-1$ for all lines through the point P_2. We can however vary the decomposition $X_1 \equiv E_{24} + E_{35} \equiv E_{23} + E_{45}$, and if for each (3) does not hold for E_{35}, respectively E_{45}, the value of ψ equals $n-1$ for all the lines, contradicting $F = 4n$ (as $n \geq 4$).

7.5. $F(a_1, \ldots, a_5) = 5n$. Here we have

(17) $K_Y + \mathcal{L}_\psi \equiv 2L - E_1 - E_2 - E_3 - E_4 \equiv X_5$.

We know that $\mathrm{rk}\{D_\sigma : \psi(\sigma) \neq -1\} = 5$, and $K_Y + \mathcal{L}_\psi \equiv E_{14} + E_{23}$. Also here we verify the assumptions of Theorem 6.1: (1) and (2) are clear and (3) is verified since $\psi(E_{i5}) \neq -1$ for all i.

This concludes the proof of the Main Theorem 5.1.

8. Proof of Theorem 5.1 for $n = 4, 6$

The result follows from the following:

Proposition 8.1. *Let S be the Kummer covering of Y branched in D of exponent n, where $n \in \{3, 4, 6\}$. Then*

(1) $H^1(S, \Theta_S) = 0$ for $n = 4, 6$.
(2) $H^1(S, \Theta_S) \neq 0$ for $n = 3$.

Proof. For assertion 2) consider $\psi = (2, 2, 2, 1, 1)$. Then $K_Y + \mathcal{L}_\psi \equiv 0$. Moreover,

$$\{D_\sigma : \psi(\sigma) \neq -1\} = \{E_{23}, E_{12}, E_{13}, E_{45}\}.$$

Then

$$\pi_*(\Omega^1_S \otimes \Omega^2_S)^\psi = \Omega^1_Y(\log E_{23}, \log E_{13}, \log E_{12}, \log E_{45}) =: \mathcal{F},$$

and $\chi(\mathcal{F}) = -1$.

1) For $n = 6$ we have to show that (cf. Remark 5.13, 2))

$$H^1(Y, \Omega^1_Y(\log E_{15}, \log E_{25}, \log E_{35}, \log E_{45})(E_{14} + E_{23})) = 0.$$

But this follows from Theorem 6.1.

For $n = 4$, we need to show that (cf. Remark 5.13, 3))

- $H^1(Y, \Omega^1_Y(\log E_{14}, \log E_{12}, \log E_{15}, \log E_{25}, \log E_{45})(E_{35} - E_{15})) = 0;$
- $H^1(Y, \Omega^1_Y(\log E_{14}, \log E_{12}, \log E_{15}, \log E_{25}, \log E_{45})(E_{34} - E_{45})) = 0;$
- $H^1(Y, \Omega^1_Y(\log E_{23}, \log E_{34}, \log E_{24})(E_{15})) = 0.$

The vanishing of the first two cohomology groups follows from Theorem 6.1.

For the second case observe that for $n = 5$ and $\psi = (4, 1, 1, 1, 4)$ we have

$$\pi_*(\Omega^1_S \otimes \Omega^2_S)^\psi = \Omega^1_Y(\log E_{23}, \log E_{34}, \log E_{24}, \log E_{15})(E_{15})) =: \mathcal{F}.$$

We kow that $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$ and since

$$\Omega^1_Y(\log E_{23}, \log E_{34}, \log E_{24})(E_{15}) \subset \mathcal{F},$$

it follows that

$$h^0(Y, \Omega^1_Y(\log E_{23}, \log E_{34}, \log E_{24})(E_{15})) = 0 = \chi(\Omega^1_Y(\log E_{23}, \log E_{34}, \log E_{24})(E_{15})).$$

This proves the claim.

□

9. Iterated Campedelli-Burniat type configurations

Recall that the configuration of the complete quadrangle consists of four points which we can describe as the vertices of an equilateral triangle, $e_1 := (1, 0, 0), e_2 := (0, 1, 0), e_3 := (0, 0, 1)$, plus the barycentre $e_4 := (1, 1, 1)$; the six lines joining them pairwise can be described as the sides $L_i := \{x_i = 0\}, i = 1, 2, 3$, of the triangle plus the three medians $\Lambda_i = \{x_j = x_k\}, \{i, j, k\} = \{1, 2, 3\}$.

The configuration of the complete quadrangle yields a configuration of six lines and seven points, once we add the three middle points of the sides

$$e'_i \in L_i, \ x_i = 0, \ x_j = x_k = 1.$$

These in turn form an equilateral triangle with barycentre e_4, and we can define the contraction linear map as the unique projectivity satisfying

$$A : \mathbb{P}^2 \to \mathbb{P}^2, \ A(e_i) = e'_i, \ A(e_4) = e_4.$$

A acts on the dual space of linear forms by

$$A(x_i) = -x_i + x_j + x_k,$$

hence, as one can easily verify,

$$A(\Lambda_i) = \Lambda_i, \ \forall i = 1, 2, 3.$$

Definition 9.1. We define inductively

- $e_i(n) := A^n(e_i)$,
- $L_i(n) := A^n(L_i)$,
- $\mathcal{C}_{CB}(0) := (\bigcup_1^3 L_i) \cup (\bigcup_1^3 \Lambda_i)$,
- $\mathcal{C}_{CB}(n) := (\bigcup_{0 \le m \le n, i=1,2,3} A^m L_i) \cup (\bigcup_1^3 \Lambda_i) = A(\mathcal{C}_{CB}(n-1)) \cup \mathcal{C}_{CB}(0)$,
- $a_n := (1 - (-2)^{n+1}) = a_{n-1} + 3(-2)^n$,
- $3b_n := |1 - (-2)^n| = |a_{n-1}| = (-1)^{n+1} a_{n-1}$,
- $c_n := b_n + (-1)^n$.

We shall call $\mathcal{C}_{CB}(n)$ the *nth iterated Campedelli-Burniat configuration.*

We can write explicit formulae as follows, keeping in mind the $\mathfrak{S}_3$- invariance of the configurations.

Proposition 9.2.

$$L_i(n) = \{a_n x_i + a_{n-1}(x_j + x_k) = 0\}.$$

$$e_1(n) = (c_n, b_n, b_n).$$

Proof. For $n = 0$ the formulae are true, and inductively

$$A(a_n x_i + a_{n-1}(x_j + x_k)) = a_n(-x_i + x_j + x_k) + a_{n-1} 2x_i =$$

$$= (-a_n + 2a_{n-1})x_i + a_n(x_j + x_k) = (a_n + 3(-2)^{n+1})x_i + a_n(x_j + x_k).$$

Also,

$$A(e_1(n) = A(c_n, b_n, b_n) = (2b_n, c_n + b_n, c_n + b_n),$$

and it suffices to see that $c_{n+1} = 2b_n, \ c_n + b_n = b_{n+1}$.

Indeed,

$$3(c_n+b_n) = 2|1-(-2)^n|+3(-1)^n = (-1)^{n-1}(2-2(-2)^n-3) = (-1)^{n-1}(-1+(-2)^{n+1}) =$$

$$= (-1)^n(1 - (-2)^{n+1}) = 3b_{n+1},$$

then clearly

$$2b_n = c_n + b_n - (-1)^n = b_{n+1} + (-1)^{n+1} = c_{n+1}.$$

$\square$

To finish the description of the configuration, we determine the intersection points of the lines of the configuration $\mathcal{C}_{CB}(n)$.

- $\Lambda_i \cap \Lambda_j = \{e_4\}, \ i \neq j$
- $\Lambda_1 \cap L_1(n) = \{(-2a_{n-1}, a_n, a_n)\} = \{(2b_n, b_{n+1}, b_{n+1})\} = \{e_1(n)\}$

- $\Lambda_2 \cap L_1(n) = \{(a_{n-1}, -(a_n + a_{n-1}), , a_{n-1})\} =$
  $= \{((-1)^{n+1}3b_n, -2(1-(-2)^{n-1}), (-1)^{n+1}3b_n) = (3b_n, (-1)^n 2(1-(-2)^{n-1}), 3b_n) =$
  $(3b_n, 2 \cdot 3 \cdot b_{n-1}, 3b_n) = (b_n, 2b_{n-1}, b_n)$,
- $\Lambda_1 \cap L_1 = \{(0, 1, -1)\}$,
- for $i \neq 1$ $\Lambda_1 \cap L_i = \{(-a_{n-1}, , 0, a_n)\} = \{(b_n, 0, , b_{n+1})\}$.

Defining now $\hat{e}_1 := (0, 1, -1)$ and analogously $\hat{e}_i$, we have

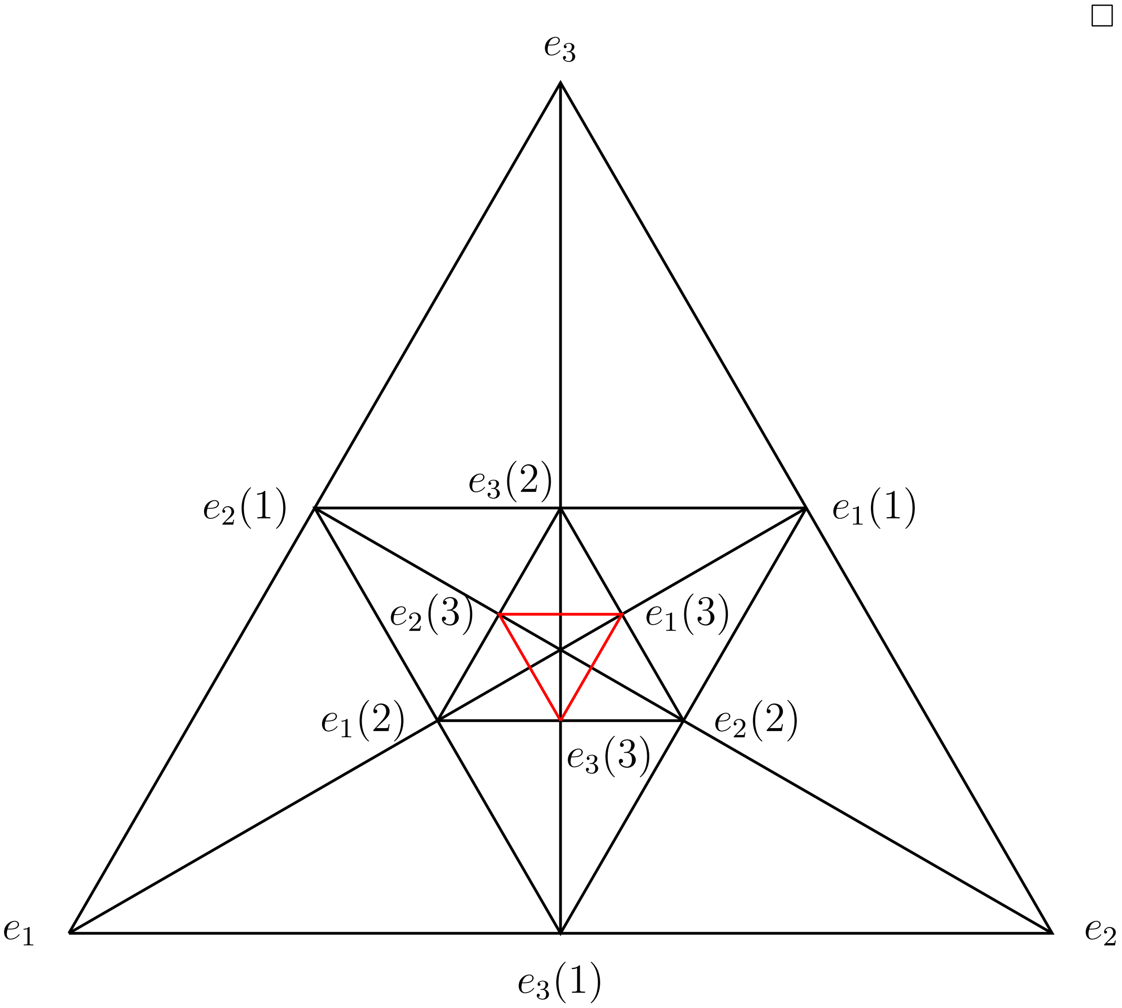
Proposition 9.3. *The configuration $\mathcal{C}_{CB}(n)$ consists of $3(n+2)$ lines. The points $e_i, i = 1, 2, 3, 4$ are triple points, while the points $e_i(m)$, $m \leq n$ are quadruple points, the points $e_i(n+1)$ are double points; while through the points $\hat{e}_i$ pass $(n+1)$ lines of the configuration, finally the points $L_i(m) \cap L_j$, $m \leq n, i \neq j$ and their transforms under A^i, $i + m \leq n$ are double points.*

Hence, if t_m is the number of vertices of the configuration with valency equal to m, we have

$$t_2 = 3n(n-1) + 3, t_3 = 4, t_4 = 3n, t_{n+1} = 3.$$

Proof. It suffices to use the previous formulae, observing that the only point which is fixed by $\mathfrak{S}_3$ is the point e_4.

Observe also that $t_2 + 3t_3 + 6t_4 + 3\frac{1}{2}n(n+1) = \frac{1}{2}(3(n+2)(3(n+2)-1))$. $\square$



References

[BBF12] Nathan Barker, Nigel Boston, and Ben Fairbairn, *A note on Beauville p-groups*, Exp. Math. **21** (2012), no. 3, 298–306, DOI 10.1080/10586458.2012.669267. MR2988581 ↑15

[BHH87] Gottfried Barthel, Friedrich Hirzebruch, and Thomas Höfer, *Geradenkonfigurationen und Algebraische Flächen*, Aspects of Mathematics, D4, Friedr. Vieweg & Sohn, Braunschweig, 1987 (German). MR912097 ↑6

[BC08] Ingrid C. Bauer and Fabrizio Catanese, *A volume maximizing canonical surface in 3-space*, Comment. Math. Helv. **83** (2008), no. 2, 387–406, DOI 10.4171/CMH/129. MR2390050 ↑1, 23

[BCG08] I. C. Bauer, F. Catanese, and F. Grunewald, *The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves*, Pure Appl. Math. Q. **4** (2008), no. 2, Special Issue: In honor of Fedor Bogomolov., 547–586, DOI 10.4310/PAMQ.2008.v4.n2.a10. MR2400886 ↑9

[BC12] Ingrid Bauer and Fabrizio Catanese, *Inoue type manifolds and Inoue surfaces: a connected component of the moduli space of surfaces with $K^2 = 7$, $p_g = 0$*, Geometry and arithmetic, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 23–56, DOI 10.4171/119-1/2, (to appear in print) (English, with English and Italian summaries). MR2987651 ↑14

[Bea83] Arnaud Beauville, *Some remarks on Kähler manifolds with $c_1 = 0$*, Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39, Birkhäuser Boston, Boston, MA, 1983, pp. 1–26, DOI 10.1007/BF02592068, (to appear in print). MR728605 ↑15

[Bor84] Ciprian Borcea, *Moduli for Kodaira surfaces*, Compositio Math. **52** (1984), no. 3, 373–380. MR756728 ↑

[BW74] D. M. Burns Jr. and Jonathan M. Wahl, *Local contributions to global deformations of surfaces*, Invent. Math. **26** (1974), 67–88. MR0349675 ↑4, 9

[CV60] Eugenio Calabi and Edoardo Vesentini, *On compact, locally symmetric Kähler manifolds*, Ann. of Math. (2) **71** (1960), 472–507. MR0111058 ↑33

[Cat83] F. Catanese, *Moduli of surfaces of general type*, Algebraic geometry—open problems (Ravello, 1982), Lecture Notes in Math., vol. 997, Springer, Berlin-New York, 1983, pp. 90–112. MR714742 ↑7

[Cat84] ———, *On the moduli spaces of surfaces of general type*, J. Differential Geom. **19** (1984), no. 2, 483–515. MR755236 (86h:14031) ↑

[Cat88] ———, *Moduli of algebraic surfaces*, Theory of moduli (Montecatini Terme, 1985), Lecture Notes in Math., vol. 1337, Springer, Berlin, 1988, pp. 1–83, DOI 10.1007/BFb0082806, (to appear in print). MR963062 ↑9

[Cat00] Fabrizio Catanese, *Fibred surfaces, varieties isogenous to a product and related moduli spaces*, Amer. J. Math. **122** (2000), no. 1, 1–44. MR1737256 ↑4

[Cat15] F. Catanese, *Topological methods in moduli theory*, Bull. Math. Sci. **5** (2015), no. 3, 287–449, DOI 10.1007/s13373-015-0070-1. MR3404712 ↑14

[CD14] Fabrizio Catanese and Michael Dettweiler, *The direct image of the relative dualizing sheaf needs not be semiample*, C. R. Math. Acad. Sci. Paris **352** (2014), no. 3, 241–244, DOI 10.1016/j.crma.2013.12.015 (English, with English and French summaries). MR3167573 ↑1

[CD13] ———, *Vector bundles on curves coming from variation of Hodge structures* (2013), 26 pp., available at arXiv:1311.3232[math.AG]. ↑1

[CD16] ———, *Vector bundles on curves coming from variation of Hodge structures*, Internat. J. Math. **27** (2016), no. 7, 1640001, 25, DOI 10.1142/S0129167X16400012. MR3521586 ↑1

[CR09] Fabrizio Catanese and Sönke Rollenske, *Double Kodaira fibrations*, J. Reine Angew. Math. **628** (2009), 205–233, DOI 10.1515/CRELLE.2009.024. MR2503241 ↑4, 13

[CFG15] Elisabetta Colombo, Paola Frediani, and Alessandro Ghigi, *On totally geodesic submanifolds in the Jacobian locus*, Internat. J. Math. **26** (2015), no. 1, 1550005, 21, DOI 10.1142/S0129167X15500056. MR3313651 ↑2

[Dab82] Krzysztof Dabrowski, *Moduli spaces for Hopf surfaces*, Math. Ann. **259** (1982), no. 2, 201–225, DOI 10.1007/BF01457309. MR656662 ↑11

[EV92] Hélène Esnault and Eckart Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser Verlag, Basel, 1992. MR1193913 ↑30

[FGvGvL13] Dino Festi, Alice Garbagnati, Bert van Geemen, and Ronald van Luijk, *The Cayley-Oguiso automorphism of positive entropy on a K3 surface*, J. Mod. Dyn. **7** (2013), no. 1, 75–97, DOI 10.3934/jmd.2013.7.75. MR3071466 ↑19

[FG65] Wolfgang Fischer and Hans Grauert, *Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1965** (1965), 89–94 (German). MR0184258 ↑7, 8

[FGP15] Paola Frediani, Alessandro Ghigi, and Matteo Penegini, *Shimura varieties in the Torelli locus via Galois coverings*, Int. Math. Res. Not. IMRN **20** (2015), 10595–10623, DOI 10.1093/imrn/rnu272. MR3455876 ↑2

[Gie77] D. Gieseker, *Global moduli for surfaces of general type*, Invent. Math. **43** (1977), no. 3, 233–282. MR0498596 ↑10

[Khi85] F. Khirtsebrukh, *Algebraic surfaces with extremal Chern numbers (based on a dissertation by T. Höfer, Bonn, 1984)*, Uspekhi Mat. Nauk **40** (1985), no. 4(244), 121–129 (Russian). Translated from the English by I. A. Skornyakov; International conference on current problems in algebra and analysis (Moscow-Leningrad, 1984). MR807793 ↑6

[Hir83] F. Hirzebruch, *Arrangements of lines and algebraic surfaces*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser, Boston, Mass., 1983, pp. 113–140. MR717609 ↑4

[Hwa95] Jun-Muk Hwang, *Nondeformability of the complex hyperquadric*, Invent. Math. **120** (1995), no. 2, 317–338, DOI 10.1007/BF01241131. MR1329044 ↑3

[HM98] Jun-Muk Hwang and Ngaiming Mok, *Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation*, Invent. Math. **131** (1998), no. 2, 393–418, DOI 10.1007/s002220050209. MR1608587 ↑3

[Ino74] Masahisa Inoue, *On surfaces of Class* VII_0, Invent. Math. **24** (1974), 269–310. MR0342734 ↑3, 9, 10, 11

[JY85] Jürgen Jost and Shing-Tung Yau, *A strong rigidity theorem for a certain class of compact complex analytic surfaces*, Math. Ann. **271** (1985), no. 1, 143–152, DOI 10.1007/BF01455803. MR779612 ↑3

[Kat75] Masahide Kato, *Topology of Hopf surfaces*, J. Math. Soc. Japan **27** (1975), 222–238. MR0402128 ↑11

[Kod63] K. Kodaira, *On stability of compact submanifolds of complex manifolds*, Amer. J. Math. **85** (1963), 79–94. MR0153033 ↑10

[Kod64] ———, *On the structure of compact complex analytic surfaces. I*, Amer. J. Math. **86** (1964), 751–798. MR0187255 ↑11

[Kod68] ———, *On the structure of compact complex analytic surfaces. III*, Amer. J. Math. **90** (1968), 55–83. MR0228019 ↑11

[Kur62] M. Kuranishi, *On the locally complete families of complex analytic structures*, Ann. of Math. (2) **75** (1962), 536–577. MR0141139 ↑7

[Kur65] ———, *New proof for the existence of locally complete families of complex structures*, Proc. Conf. Complex Analysis (Minneapolis, 1964), Springer, Berlin, 1965, pp. 142–154. MR0176496 ↑7

[Mee11] Laurent Meersseman, *Foliated structure of the Kuranishi space and isomorphisms of deformation families of compact complex manifolds*, Ann. Sci. Éc. Norm. Supér.

(4) **44** (2011), no. 3, 495–525 (English, with English and French summaries). MR2839457 ↑8

[Mok88] Ngaiming Mok, *Strong rigidity of irreducible quotients of polydiscs of finite volume*, Math. Ann. **282** (1988), no. 4, 555–577, DOI 10.1007/BF01462883. MR970219 ↑3

[Moo10] Ben Moonen, *Special subvarieties arising from families of cyclic covers of the projective line*, Doc. Math. **15** (2010), 793–819. MR2735989 ↑2

[Mos73] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973. Annals of Mathematics Studies, No. 78. MR0385004 ↑3

[MS80] G. D. Mostow and Yum Tong Siu, *A compact Kähler surface of negative curvature not covered by the ball*, Ann. of Math. (2) **112** (1980), no. 2, 321–360, DOI 10.2307/1971149. MR592294 ↑4, 5, 13

[Pan11] Dmitri Panov, *Complex surfaces with CAT(0) metrics*, Geom. Funct. Anal. **21** (2011), no. 5, 1218–1238, DOI 10.1007/s00039-011-0133-8. MR2846388 ↑5

[Par91] Rita Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. **417** (1991), 191–213, DOI 10.1515/crll.1991.417.191. MR1103912 (92g:14012) ↑5, 24

[Rou11] Xavier Roulleau, *The Fano surface of the Fermat cubic threefold, the del Pezzo surface of degree 5 and a ball quotient*, Proc. Amer. Math. Soc. **139** (2011), no. 10, 3405–3412, DOI 10.1090/S0002-9939-2011-10847-5. MR2813372 ↑19

[Sei92] Wolfgang K. Seiler, *Deformations of ruled surfaces*, J. Reine Angew. Math. **426** (1992), 203–219, DOI 10.1515/crll.1992.426.203. MR1155754 ↑11

[Siu80] Yum Tong Siu, *The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds*, Ann. of Math. (2) **112** (1980), no. 1, 73–111, DOI 10.2307/1971321. MR584075 ↑3, 5

[Siu89] ______, *Nondeformability of the complex projective space*, J. Reine Angew. Math. **399** (1989), 208–219, DOI 10.1515/crll.1989.399.208. MR1004139 ↑3

[Siu91] ______, *Global nondeformability of the complex projective space*, Prospects in complex geometry (Katata and Kyoto, 1989), Lecture Notes in Math., vol. 1468, Springer, Berlin, 1991, pp. 254–280, DOI 10.1007/BFb0086197, (to appear in print). MR1123546 ↑3

[Tel94] Andrei Dumitru Teleman, *Projectively flat surfaces and Bogomolov's theorem on class* VII_0 *surfaces*, Internat. J. Math. **5** (1994), no. 2, 253–264, DOI 10.1142/S0129167X94000152. MR1266285 ↑10, 11

[Vak06] Ravi Vakil, *Murphy's law in algebraic geometry: badly-behaved deformation spaces*, Invent. Math. **164** (2006), no. 3, 569–590, DOI 10.1007/s00222-005-0481-9. MR2227692 ↑5

[Wav69] John J. Wavrik, *Obstructions to the existence of a space of moduli*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 403–414. MR0254882 ↑7

[Zhe99] Fangyang Zheng, *Hirzebruch-Kato surfaces, Deligne-Mostow's construction, and new examples of negatively curved compact Kähler surfaces*, Comm. Anal. Geom. **7** (1999), no. 4, 755–786, DOI 10.4310/CAG.1999.v7.n4.a4. MR1714937 ↑5

Mathematisches Institut, Universität Bayreuth, 95440 Bayreuth, Germany.

E-mail address: `Ingrid.Bauer@uni-bayreuth.de`

Mathematisches Institut, Universität Bayreuth, 95440 Bayreuth, Germany.

E-mail address: `Fabrizio.Catanese@uni-bayreuth.de`