Nagoya Math. J., **235** (2019), 201–226
DOI 10.1017/nmj.2018.8

ENRIQUES' CLASSIFICATION IN CHARACTERISTIC $p > 0$: THE P_{12}-THEOREM

FABRIZIO CATANESE AND BINRU LI

Dedicated to David Mumford on the occasion of his 80th birthday

Abstract. The main goal of this paper is to show that Castelnuovo–Enriques' P_{12}- theorem (a precise version of the rough classification of algebraic surfaces) also holds for algebraic surfaces S defined over an algebraically closed field k of positive characteristic ($\mathrm{char}(k) = p > 0$). The result relies on a main theorem describing the growth of the plurigenera for properly elliptic or properly quasielliptic surfaces (surfaces with Kodaira dimension equal to 1). We also discuss the limit cases, i.e., the families of surfaces which show that the result of the main theorem is sharp.

Introduction

The main technical result of the present article, expressed in modern language, is the following one:

Main Theorem. *Let S be a projective surface of Kodaira dimension 1 defined over an algebraically closed field k, and let K_S be a canonical divisor on S, so that $\Omega_S^2 \cong \mathcal{O}_S(K_S)$.*

Then the growth of the plurigenera $P_n(S) = \dim H^0(\mathcal{O}_S(nK_S)) = \dim H^0((\Omega_S^2)^{\otimes n})$ *satisfies:*

(1) $P_{12}(S) \geqslant 2$;
(2) *there exists* $n \leqslant 4$ *such that* $P_n(S) \geqslant 1$;
(3) *there exists* $n \leqslant 8$ *such that* $P_n(S) \geqslant 2$;
(4) $\forall n \geqslant 14$ $P_n(S) \geqslant 2$.

While (2)–(3) of the above theorem are new also in the classical case where k is a field of characteristic zero, (1) is due to Enriques [15] in characteristic 0 and (4) was shown by Katsura and Ueno [23] for elliptic surfaces in all

Received April 5, 2017. Revised February 5, 2018. Accepted February 5, 2018.
2010 Mathematics subject classification. 14J10, 14J27.
The present work took place in the framework of the ERC Advanced grant no. 340258, "TADMICAMT."

characteristics (but we reprove their result here as a part of the above more general statement). Needless to say, we use in the proof of our theorem many results, lemmas and propositions previously established by many authors, especially Bombieri–Mumford, Raynaud, and Katsura–Ueno [27], [5], [6], [31], [23].

Statement (1) is most important, which allows us to extend to the positive characteristic case the main classification theorem of Castelnuovo and Enriques. In modern language (see the next section for more details, and a more precise and informative statement), the classification theorem implies the following:

P_{12}-Theorem. *Let S be a projective surface defined over an algebraically closed field k.*

Then for the Kodaira dimension $\mathrm{Kod}(S)$ *we have:*

- (I) $\mathrm{Kod}(S) = -\infty \Longleftrightarrow P_{12}(S) = 0$;
- (II) $\mathrm{Kod}(S) = 0 \Longleftrightarrow P_{12}(S) = 1$;
- (III) $\mathrm{Kod}(S) = 1 \Longleftrightarrow P_{12}(S) \geqslant 2$ *and, for S minimal,* $K_S^2 = 0$;
- (IV) $\mathrm{Kod}(S) = 2 \Longleftrightarrow P_{12}(S) \geqslant 2$ *and, for S minimal,* $K_S^2 > 0$.

It should be observed that the estimates for the growth of the plurigenera are much weaker if one considers properly elliptic nonalgebraic surfaces, see [21], where the analogue of (4) of the main theorem for nonalgebraic surfaces is proved. Iitaka showed that, for $n \geqslant 86$, $H^0(\mathcal{O}_S(nK_S))$ yields the canonical elliptic fibration. One of the reasons why the estimate is much weaker depends on the failure of the Poincaré reducibility theorem, implying in the algebraic case that a certain monodromy group G is Abelian. Hence, for instance, if G is Abelian, it cannot be a Hurwitz group, i.e., G cannot have generators a, b, c of respective orders $(2, 3, 7)$ satisfying $abc = 1$.

Indeed (we omit here the simple proof), the analogue of statement (1) for nonalgebraic surfaces is that $P_{42} \geqslant 2$.

Concerning higher dimensional algebraic varieties, a natural question emerges:

Question 0.1. Given a projective manifold of X dimension N, is there a sharp number $d = d(N)$ such that:

(1) $\mathrm{Kod}(X) = -\infty \Longleftrightarrow P_d(X) = 0$;
(2) $\mathrm{Kod}(X) = 0 \Longleftrightarrow P_d(X) = 1$;
(3) $\mathrm{Kod}(X) \geqslant 1 \Longleftrightarrow P_d(X) \geqslant 2$?

 https://doi.org/10.1017/nmj.2018.8

Progress on a related question, about effectivity of the Iitaka fibration, was made, among others, by Fujino and Mori [18] and Birkar and Zhang [9].

§1. The classification theorem of Castelnuovo and Enriques

Let S be a nonsingular projective surface defined over an algebraically closed field k, and let K_S be a canonical divisor on S, so that $\Omega^2_S \cong \mathcal{O}_S(K_S)$. We assume that S is minimal: this means that there does not exist an irreducible exceptional curve C of the first kind, i.e., an integral curve C with $C^2 = K_S \cdot C = -1$. Let us recall the definition of the basic numerical invariants associated with S, which allow its birational classification.

For each integer $m \in \mathbb{N}$, we denote as usual, following Castelnuovo and Enriques, by

$$P_m(S) := h^0(S, mK_S),$$

the *mth plurigenus* of S.

In particular, the *geometric genus* is $p_g(S) := P_1(S)$, while the *arithmetic irregularity* is defined as $h(S) := h^1(\mathcal{O}_S)$, and the *arithmetic genus* is defined as

$$p_a(S) := p_g(S) - h(S) = \chi(\mathcal{O}_S) - 1.$$

To finish our comparison of classical and modern notation, recall that the *geometric irregularity* is defined as $q(S) := 1/2b_1(S)$, where $b_1(S)$ is the first l-adic Betti number of S, $b_1(S) := \dim_{\mathbb{Q}_l} H^1_{et}(S, \mathbb{Q}_l)$.

$q(S)$ is equal to the dimension of the Picard scheme $\mathrm{Pic}^0(S)$, and also (cf. [28, Chapter III. 13]) of the dual scheme $\mathrm{Pic}^0(\mathrm{Pic}^0(S))$, and of the Albanese variety $Alb(S) := \mathrm{Pic}^0(\mathrm{Pic}^0(S)_{\mathrm{red}})$.

The above numbers are all equal in characteristic zero: $q(S) = h(S) = h^0(\Omega^1_S)$, but not in characteristic $p > 0$, where one just has some inequalities.

Since $H^1(\mathcal{O}_S)$ is the Zariski tangent space to the Picard scheme at the origin [26, Lecture 24. 2°], one has the inequalities (cf. [5, pp. 34–35])

$$h(S) \geqslant q(S), \quad 2p_g(S) \geqslant \Delta := 2(h(S) - q(S)) = 2h(S) - b_1(S) \geqslant 0.$$

The inequality $h^0(\Omega^1_S) \geqslant q$ was shown by Igusa [19], and there are examples where the equality does not hold, cf. [20, p. 964, The Example], [25, p. 341, Corollary][1].

[1]The space of regular one forms on the Albanese variety A pulls back injectively to a subspace V of the space $H^0(\Omega^1_S)$, contained in the space of d-closed forms; it is an open

Moreover, the linear genus $p^{(1)}(S) := K_S^2 + 1$ is the arithmetic genus of any canonical divisor on the minimal surface. It is a birational invariant for every nonruled algebraic surface.

The classification of smooth projective integral curves C is given in terms of the genus $g(C) := h^0(\mathcal{O}_C(K_C))$:

(I) $g(C) = 0 \Longleftrightarrow C \cong \mathbb{P}^1$;

(II) $g(C) = 1 \Longleftrightarrow \mathcal{O}_C \cong \mathcal{O}_C(K_C) \Longleftrightarrow C$ *is an elliptic curve (it is isomorphic to a plane cubic curve)*;

(III) $g(C) \geqslant 2 \Longleftrightarrow C$ *is of general type, i.e.,* $H^0(C, \mathcal{O}_C(mK_C))$ *yields an embedding of* C *for all* $m \geqslant 3$.

Enriques and Castelnuovo ([15] and [10]) were able to classify surfaces essentially in terms of $P_{12}(S)$, as follows:

THEOREM 1.1. (P_{12}-theorem of Castelnuovo–Enriques) *Let* S *be a smooth projective surface defined over an algebraically closed field* k *of characteristic zero, and let* $p^{(1)}(S) := K_S^2 + 1$ *be the linear genus of a minimal model in the birational equivalence class of* S *where* $P_{12} > 0$. *Then*

(I) $P_{12}(S) = 0 \Longleftrightarrow S$ *is ruled* $\Longleftrightarrow S$ *is birational to a product* $C \times \mathbb{P}^1$, $g(C) = q(S) = h(S)$.

(II) $P_{12} = 1 \Longleftrightarrow \mathcal{O}_S \cong \mathcal{O}_S(12K_S)$.

(III) $P_{12} \geqslant 2$ *and* $p^{(1)}(S) = 1 \Longleftrightarrow S$ *is properly elliptic, i.e.,* $H^0(S, \mathcal{O}_S (12K_S))$ *yields a fibration over a curve with general fibers elliptic curves*

(IV) $P_{12} \geqslant 2$ *and* $p^{(1)}(S) > 1 \Longleftrightarrow S$ *is of general type, i.e.,* $H^0(S, \mathcal{O}_S (mK_S))$ *yields a birational embedding of* S *for* m *large* ($m \geqslant 5$ *indeed suffices, as conjectured by Enriques in [17] and proven by Bombieri [3, Main Theorem]).*

Moreover, if S *is minimal, then in modern terminology:*

- Case (I) $S \cong \mathbb{P}^2$ *or* S *is a* $\mathbb{P}^1$*-bundle over a curve* C.
- Case (II) $p_g(S) = 1, q(S) = 2 \Longleftrightarrow \mathcal{O}_S \cong \mathcal{O}_S(K_S)$, $q(S) = 2 \Longleftrightarrow S$ *is an Abelian surface.*

question how to characterize V, for instance Illusie suggested V could be the intersection of the kernels of $d \circ C^m$, where C is the Cartier operator, and m is any positive integer; see [33], [13], [22].

- Case (II) $p_g(S) = 1, q(S) = 0 \Longleftrightarrow \mathcal{O}_S \cong \mathcal{O}_S(K_S)$, $q(S) = 0 \Longleftrightarrow S$ *is a K3 surface.*
- Case (II) $p_g(S) = 0, q(S) = 0 \Longleftrightarrow \mathcal{O}_S \ncong \mathcal{O}_S(K_S)$, $\mathcal{O}_S \cong \mathcal{O}_S(2K_S)$, $q(S) = 0 \Longleftrightarrow S$ *is an Enriques surface.*
- Case (II) $q(S) = 1 (\Rightarrow p_g(S) = 0) \Longleftrightarrow \mathcal{O}_S \ncong \mathcal{O}_S(K_S), \mathcal{O}_S \cong \mathcal{O}_S(mK_S)$, *for some* $m \in \{2, 3, 4, 6\}, q(S) = 1 \Longleftrightarrow S$ *is a hyperelliptic surface.*
- Case (III) $p_a(S) = -1 \Longleftrightarrow S \cong C \times E, g(E) = 1, g(C) = q(S) - 1.$

A modern account of the Castelnuovo–Enriques classification of surfaces was first given in [32], [24], then it appeared also in [4], [2] ([2] is the only text which mentions the P_{12}-theorem, in the historical note on p. 118), later also in [1] and [7].

REMARK 1.2. (i) Nowadays, cases (I)-(IV) are distinguished according to the Kodaira dimension, which is defined to be $-\infty$ if all the plurigenera vanish ($P_n = 0\ \forall n \geqslant 1$), otherwise it is defined as the maximal dimension of the image of some n-pluricanonical map (the map associated with $H^0(\mathcal{O}_X(nK_X))$).

(ii) The occurrence of the number 12 is rather miraculous: it first appears since, by the canonical divisor formula 1.6, in case (II) the equation

$$2 = \sum_j \left(1 - \frac{1}{m_j}\right)$$

admits only the following (positive) integer solutions:

$$(2, 2, 2, 2), \quad (3, 3, 3), \quad (2, 4, 4), \quad (2, 3, 6)$$

and then we get a set of integers m_j whose least common multiple is precisely 12.

Respectively, we have $2K_S \equiv 0, 3K_S \equiv 0, 4K_S \equiv 0, 6K_S \equiv 0$, where $D \equiv 0$ means that D is linearly equivalent to zero, i.e., $\mathcal{O}_S(D) \cong \mathcal{O}_S$. It follows that in case (II) we have $12K_S \equiv 0$, hence $P_{12} = 1$.

The second occurrence is more subtle, and is the heart of the theorem: in case (III) one has $P_{12} \geqslant 2$.

It is now customary (the name "key theorem" is due to [2]) to see the two major steps of surface classification as follows:

THEOREM 1.3. (Key theorem) *If S is minimal, then*
K_S *is* nef *(i.e., $K_S \cdot C \geqslant 0$ for all curves $C \subset S$) $\Longleftrightarrow S$ is nonruled.*

THEOREM 1.4. (Crucial theorem) *S is minimal, with $p_g(S)=0, q(S)=1 \Longleftrightarrow S$ is* isogenous *to a product, i.e., S is the quotient $(C_1 \times C_2)/G$ of a product of curves of genera*

$$g_1 := g(C_1) = 1, \quad g_2 := g(C_2) \geqslant 1,$$

by a free action of a finite group of product type (G acts faithfully on C_1, C_2 and we take the diagonal action $g(x,y) := (gx, gy)$), and such that moreover if we denote by $g'_j = g(C_j/G)$, then $g'_1 + g'_2 = 1$.

More precisely, let A be the Albanese variety of S, which is an elliptic curve and let

$$\alpha : S \to A$$

be the Albanese map.

Then either:

(1) *S is a hyperelliptic surface, $g_2 = 1$, G is a subgroup of translations of C_1, $A = C_1/G$, while $C_2/G \cong \mathbb{P}^1$.*
In this case all the fibers of the Albanese map are isomorphic to C_2, $P_{12}(S) = 1$ and S admits also an elliptic fibration $\psi : S \to C_2/G \cong \mathbb{P}^1$.
(2) *S is properly elliptic ($P_{12}(S) \geqslant 2$) and the genus $g = g_2$ of the Albanese fibers satisfies $g_2 \geqslant 2$: again G is a subgroup of translations of C_1, $A = C_1/G$, $C_2/G \cong \mathbb{P}^1$, all the fibers of the Albanese map are isomorphic to C_2.*
(3) *S is properly elliptic ($P_{12}(S) \geqslant 2$) and the genus $g = g_1$ of an Albanese fiber satisfies $g_1 = 1$: $A = C_2/G$, $C_1/G \cong \mathbb{P}^1$, and the fibers of the Albanese map*

$$\alpha : S = (C_1 \times C_2)/G \to A = C_2/G$$

are either isomorphic to the elliptic curve C_1 or are multiples of smooth elliptic curves isogenous to C_1.

REMARK 1.5. A crucial observation, used by Enriques in [15] for the P_{12}-theorem is that in the first two cases the group G is Abelian. The crucial ingredient is the canonical divisor (canonical bundle) formula, established by Enriques and Kodaira, and then extended to positive characteristic by Bombieri and Mumford.

THEOREM 1.6. [5, p. 27, Theorem 2] *Let $f : S \to C$ be a relatively minimal fibration such that the arithmetic genus of a fiber equals 1 (the general* *fiber is necessarily smooth elliptic in characteristic zero, but it can be rational with one cusp in characteristic 2 or 3: the latter is called the quasielliptic case).*

Let $\{q_1, \dots, q_r\} \subset C$ *be the set of points over which the fiber* $f^{-1}(q_i) = m_i F'_i$ *is a multiple fiber (i.e.,* $m_i \geqslant 2$ *and* F'_i *is not a multiple of any proper subdivisor), and consider the coherent sheaf* $R^1 f_*(\mathcal{O}_{\mathcal{S}})$ *on the smooth curve* C*, which decomposes as*

$$R^1 f_*(\mathcal{O}_S) = \mathcal{O}_C(L) \oplus T,$$

where $\mathcal{O}_C(L)$ *is an invertible sheaf and* T *is a torsion subsheaf with* $\operatorname{supp}(T) \subset \{q_1, \dots, q_r\}$*. The fibers over the points of* $\operatorname{supp}(T)$ *are called wild fibers, moreover* $T = 0$ *if* $\operatorname{char}(k) = 0$*.*

Then

$$K_S = f^*(\delta) + \sum_{i=1}^{r} a_i F'_i, \quad \delta := -L + K_C$$

where

(i) $0 \leqslant a_i < m_i$;
(ii) $a_i = m_i - 1$ *if* $m_i F'_i$ *is not wild (i.e.,* $q_i \notin \operatorname{supp}(T)$*);*
(iii) $d := \deg(\delta) = \deg(-L + K_C) = 2g(C) - 2 + \chi(\mathcal{O}_S) + length(T)$, *where* $g(C)$ *is the genus of* C*.*

Let us see how the above applies in characteristic zero and in the special subcase: $p_g = 0, q = 1$, the genus of the Albanese fibers equals 1, and there exists a multiple fiber.

Then, for $n = 2$, since we have $\deg(\delta) = 0$, it follows that

$$2K_S = \sum_{i=1}^{r} (m_i - 2) F'_i + f^* \left(2\delta + \sum_{i=1}^{r} q_i \right).$$

The divisor $2\delta + \sum_{i=1}^{r} q_i$ is effective by the Riemann–Roch theorem on the elliptic curve A, so we have written $2K_S$ as the sum of two effective divisors.

Hence we obtain that $P_2 \geqslant 1$, and similarly one gets that $P_{12} \geqslant 6$.

§2. The P_{12}-theorem in positive characteristic

The extension of the Castelnuovo–Enriques classification of surfaces to the case of positive characteristic was achieved by Mumford and Bombieri (cf. [6, Section 3], [5, Theorem 1]).

In a remarkable series of three papers they got most of the following full result.

THEOREM 2.1. (P_{12}-theorem in positive characteristic) *Let S be a projective smooth surface defined over an algebraically closed field k of characteristic $p > 0$, and let $p^{(1)}(S) := K_S^2 + 1$ be the linear genus of a minimal model in the birational equivalence class of S where $P_{12} > 0$. Then*

(I) $P_{12}(S) = 0 \iff S$ *is ruled* $\iff S$ *is birational to a product* $C \times \mathbb{P}^1$, $g(C) = q(S) = h(S)$.

(II) $P_{12} = 1 \iff \mathcal{O}_S \cong \mathcal{O}_S(12K_S)$.

(III) $P_{12} \geqslant 2$ *and* $p^{(1)}(S) = 1 \iff S$ *is properly elliptic or properly quasielliptic, i.e.,* $H^0(S, \mathcal{O}_S(12K_S))$ *yields a fibration over a curve with general fibers either elliptic curves or rational curves with one cusp.*

(IV) $P_{12} \geqslant 2$ *and* $p^{(1)}(S) > 1 \iff S$ *is of general type, i.e.,* $H^0(S, \mathcal{O}_S(mK_S))$ *yields a birational embedding of* S *for* m *large (indeed,* $m \geqslant 5$ *suffices).*

Moreover, Bombieri and Mumford in [5] and [6] gave a full description of the surfaces in the classes (I) and (II) (with new nonclassical surfaces), but classes (II) and (III) were not distinguished by the behavior of the 12th plurigenus, but only by the Kodaira dimension, i.e., by the growth of $P_n(S)$ as $n \to \infty$.

The sharp statement ($\forall m \geqslant 5$) in case (IV), established by Bombieri [3, Main Theorem] in characteristic zero, was extended by Ekedahl's to the case of positive characteristic (cf. [14, Main Theorem], see also [11] and [12] for a somewhat simpler proof).

§3. Auxiliary results and proof of the P_{12}-theorem

Case (III) can be divided into two subcases: properly elliptic fibrations and properly quasielliptic fibrations.

Recall the definition of quasielliptic surfaces:

DEFINITION 3.1. A *quasielliptic surface* S is a nonsingular projective surface admitting a fibration $f: S \to C$ over a nonsingular projective curve C such that $f_*\mathcal{O}_S = \mathcal{O}_C$ and such that the general fibers of f are rational curves with one cusp.

If the fibration f is induced by $H^0(S, \mathcal{O}_S(nK_S))$ for some $n > 0$, we call S a *properly quasielliptic surface.*

REMARK 3.2. (1) By a result of Tate (cf. [34, Corollary 1]), quasielliptic fibrations only appear in characteristic 2 and 3.

(2) In case (III), where $P_n(S) := \dim H^0(S, \mathcal{O}_S(nK_S))$ grows linearly with n, S is necessarily properly elliptic or properly quasielliptic.

The case where S admits a properly elliptic fibration was treated by T. Katsura and K. Ueno who proved in [23, Theorem 5.2] that for any properly elliptic surface S, $\forall m \geqslant 14$, $P_m(S) \geqslant 2$ and showed the existence of an example where $P_{13} = 1$. They show that the situation is essentially the same as in characteristic zero. The fact that $P_{12}(S) \geqslant 2$ follows from our more general theorem, which uses several auxiliary results developed by Raynaud and Katsura–Ueno (they will be recalled in the sequel).

THEOREM 3.3. (Main Theorem) *Let $f\colon S \to C$ be a properly elliptic or quasielliptic fibration. Then:*

(1) $P_{12}(S) \geqslant 2$*;*
(2) *there exists $n \leqslant 4$ such that $P_n(S) \neq 0$;*
(3) *there exists $n \leqslant 8$ such that $P_n(S) \geqslant 2$;*
(4) $\forall n \geqslant 14$ $P_n(S) \geqslant 2$.

REMARK 3.4. Let us indicate the examples (see Remark 4.3) which show that in Theorem 3.3 the inequalities in our assumptions are best possible.

(2) and (4): in the notation of (2) of Theorem 1.4 we let $G = \mathbb{Z}/2 \oplus \mathbb{Z}/6$; in characteristic$\neq 2, 3$, G is isomorphic to a subgroup of any elliptic curve. In order to obtain a curve C_2 with an action of G such that $C_2/G \cong \mathbb{P}^1$ we consider a G-Galois covering C_2 of $\mathbb{P}^1$ branched in 3 points, and with local monodromies

$$(1, 0), (0, 1), (-1, -1).$$

In the characteristic zero case this exists by Riemann's existence theorem (since the sum of the three local monodromies equals zero). Indeed the resulting curve has affine equation $y^2 = x^6 - 1$, which is smooth in characteristic $\neq 2, 3$. Hence we get such a curve for any characteristic $\neq 2, 3$ (see [23, Example 4.6]).

The fibration $f\colon S \to C_2/G \cong \mathbb{P}^1$ is elliptic and has exactly three singular fibers, with multiplicities 2, 6, 6. It follows that

$$P_n(S) = \max\left\{0,\ -2n + 1 + \left[\frac{n}{2}\right] + 2 \cdot \left[\frac{5n}{6}\right]\right\},$$

where $[a]$ denotes the integral part of a.

It follows that $P_1 = P_2 = P_3 = 0$, $P_4 = P_5 = 1$, $P_6 = 2$, $P_{13} = 1$.

(2) and (3): in the notation of (2) of Theorem 1.4 we let $G = \mathbb{Z}/10$; in characteristic$\neq 2, 5$, G is isomorphic to a subgroup of any elliptic curve. In order to obtain a curve C_2 with an action of G such that $C_2/G \cong \mathbb{P}^1$ we consider a G-Galois covering C_2 of $\mathbb{P}^1$ branched in 3 points, and with local monodromies

$$(5), (4), (1).$$

This exists in characteristic zero by Riemann's existence theorem, since the sum of the three local monodromies is 10, which equals zero in G. Indeed, the resulting curve is defined by the affine equation $y^2 = x^5 - 1$ and we obtain therefore such a smooth curve for each characteristic $\neq 2, 5$.

The fibration $f : S \to C_2/G \cong \mathbb{P}^1$ is elliptic and has exactly three singular fibers, with multiplicities 2, 5, 10. It follows that

$$P_n(S) = \max\left\{0,\ -2n + 1 + \left[\frac{n}{2}\right] + \left[\frac{4n}{5}\right] + \left[\frac{9n}{10}\right]\right\}.$$

Follows that $P_1 = P_2 = P_3 = 0$, $P_4 = P_5 = P_6 = P_7 = 1$, $P_8 = P_9 = 2$, $P_{10} = 3$, $P_{11} = 1$, $P_{12} = P_{13} = 2$.

In the case of properly quasielliptic fibrations, we shall use some result of Raynaud, [31], and a corollary developed by Katsura and Ueno [23, Lemmas 2.3 and 2.4].

Given a multiple fiber mF' we denote by $\omega_n := \mathcal{O}_{nF'}(K_S + nF')$ the dualizing sheaf of nF'.

Observe that F' is an indecomposable divisor of elliptic type (see [27, p. 330][2] for the definition), hence (see [27, p. 332] [12, Theorem 3.3]) for any degree-zero divisor L on F', we have $h^0(\mathcal{O}_{F'}(L)) = h^1(\mathcal{O}_{F'}(L))$, and these dimensions are either $= 0$, or $= 1$, the latter case occurring if and only if $\mathcal{O}_{F'}(L) \cong \mathcal{O}_{F'}$.

Consider now the exact sequence

$$0 \to \mathcal{O}_{F'}(-(n-1)F') \to \mathcal{O}_{nF'} \to \mathcal{O}_{(n-1)F'} \to 0,$$

and apply the previous remark for $L = -(n-1)F'$ to deduce that

$$h^0(\mathcal{O}_{nF'}) = h^0(\mathcal{O}_{(n-1)F'}) \qquad \text{or} \qquad = h^0(\mathcal{O}_{(n-1)F'}) + 1.$$

[2] In Mumford's definition it is called "indecomposable of canonical type."

 https://doi.org/10.1017/nmj.2018.8

The second equality holds only if

$$(*) \qquad \mathcal{O}_{F'}((n-1)F') \cong \mathcal{O}_{F'}.$$

Conversely, if ($*$) holds, either $h^0(\mathcal{O}_{nF'}) = h^0(\mathcal{O}_{(n-1)F'})$ and $h^1(\mathcal{O}_{nF'}) = h^1(\mathcal{O}_{(n-1)F'})$, or both $h^0(\mathcal{O}_{nF'}) = h^0(\mathcal{O}_{(n-1)F'}) + 1$ and $h^1(\mathcal{O}_{nF'}) = h^1(\mathcal{O}_{(n-1)F'}) + 1$.

This in any case shows that the function $h^0(\mathcal{O}_{nF'})$ is monotonously nondecreasing. One says that n is a *jumping value* if $n \geqslant 1$ and $h^0(\mathcal{O}_{nF'}) = h^0(\mathcal{O}_{(n-1)F'}) + 1$. Considering all the $n \geqslant 1$, we can then define the first jumping value, the second, and so on (they are then clearly $\geqslant 2$).

Recall now:

PROPOSITION 3.5. [5, Proposition 4] *Since $(\mathcal{O}_{F'_i}(F'_i))$ is a torsion element in the Picard group of F'_i, we consider its torsion order:*

$$\nu_i := \mathrm{order}(\mathcal{O}_{F'_i}(F'_i)).$$

We have then

(1) *ν_i divides both m_i and $a_i + 1$;*
(2) *letting $p = \mathrm{char}(k)$, there exists an integer $e_i \geqslant 0$ such that $m_i = \nu_i \cdot p^{e_i}$;*
(3) *$h^0(F'_i, \mathcal{O}_{(\nu_i+1)F'_i}) = 2$, $h^0(F_i, \mathcal{O}_{\nu_i F'_i}) = 1$, so that $\nu_i + 1$ is a jumping value;*
(4) *$h^0(F'_i, \mathcal{O}_{rF'_i})$ is a nondecreasing function of r.*

Proof. In the proof we suppress the subscripts.
(4) follows from the above discussion.
(3) consider the exact sequence:

$$0 \to \mathcal{O}_{F'} \cong \mathcal{O}_{(\nu+1)F'}(-\nu F') \to \mathcal{O}_{(\nu+1)F'} \to \mathcal{O}_{\nu F'} \to 0.$$

Passing to the exact cohomology sequence we get

$$0 \to k \cong H^0(\mathcal{O}_{(\nu+1)F'}(-\nu F')) \to H^0(\mathcal{O}_{(\nu+1)F'}) \to H^0(\mathcal{O}_{\nu F'}) \to \cdots .$$

By our previous argument, we see that $H^0(\mathcal{O}_{\nu F'}) \simeq k$. Since $H^0(\mathcal{O}_{(\nu+1)F'}(-\nu F'))$ is a space of functions which have value zero on F', and the constants in $H^0(\mathcal{O}_{(\nu+1)F})$ are mapped to constants in $H^0(\mathcal{O}_{\nu F'})$, we see that $h^0(\mathcal{O}_{(\nu+1)F'}) = 2$.

(1) ν divides m since mF' is trivial in a neighborhood of F'.

Note that $\mathcal{O}_{F'}((a+1)F) \simeq \omega_{F'}$, $\omega_{F'}$ is nef and has degree zero and $h^0(\omega_{F'}) = h^1(\mathcal{O}_{F'}) = h^0(\mathcal{O}_{F'}) = 1$. Since F' is indecomposable of elliptic type, from the previous discussion, we have $\omega_{F'} \simeq \mathcal{O}_{F'}$, hence $\nu|(a+1)$.

(2) Set $o_n := \mathrm{Ord}(\mathcal{O}_{nF'}(F'))$. Since $\mathcal{O}_{nF'}(mF')$ is trivial for any $n \geqslant 1$, we have $o_n|m$. Using Lemma 3.8 (ii) we see that o_{n-1} divides o_n and $o_n = p^{e_n}\nu$ for some nonnegative integer e_n. Noting that m is the order of F' in the formal neighborhood of F' (cf. [30, Proposition 6.1.11 (3)]) it follows that $m|o_n$ for large n (cf. [30, Lemme 6.4.4]), hence $m = o_n$ for some large n, therefore $m = p^e\nu$ for some $e \geqslant 0$. ∎

Using Proposition 3.5, we get the following corollary:

Corollary 3.6. [5, p. 30, Corollary] *If $h^1(S, \mathcal{O}_S) \leqslant 1$, we have either*

$$a_i = m_i - 1$$

or

$$a_i = m_i - 1 - \nu_i.$$

More precise results are the following two lemmas of Raynaud (cf. [31], [8, Section 2]).

Lemma 3.7. [31, Corollaire 3.7.6], [8, Lemma 2.1.8] *Let $f: S \to C$ be an elliptic or quasielliptic fibration with $f^{-1}(q) = mF'$ a multiple fiber over $q \in C$. Then for any integer $n \geqslant 2$:*

(i) *The dualizing sheaf $\omega_n := \mathcal{O}_{nF'}(K_S + nF')$ of nF' is nontrivial iff $h^0(\omega_n) = h^0(\omega_{n-1})$.*
(ii) *ω_n is trivial iff $h^0(\omega_n) = h^0(\omega_{n-1}) + 1$.*

Lemma 3.8. [31, Lemma 3.7.7] *Notation being as in Lemma 3.7, observe that the invertible sheaves $\mathcal{O}_{nF'}(F')$ are torsion elements in the Picard group of nF'. There are only two possibilities for their torsion orders. Setting $o_n := \mathrm{Ord}(\mathcal{O}_{nF'}(F'))$ (hence $o_1 = \nu$), we have*

(i) *$o_n = o_{n-1}$;*
(ii) *$o_n = p\ o_{n-1}$.*

Moreover, case (ii) occurs only if ω_n is trivial.

Proof. Setting $\mathfrak{N} := \mathcal{O}_{F'}(-(n-1)F')$, we consider the following two exact sequences:

$$0 \to \mathfrak{N} \to \mathcal{O}_{nF'} \to \mathcal{O}_{(n-1)F'} \to 0. \tag{2}$$

$$0 \to 1 + \mathfrak{N} \to \mathcal{O}^*_{nF'} \to \mathcal{O}^*_{(n-1)F'} \to 0. \tag{3}$$

Since $\mathfrak{N}^2 = 0$, the map $x \mapsto 1 + x$ defines an isomorphism of abelian sheaves:

$$\beta \colon \mathfrak{N} \simeq 1 + \mathfrak{N}.$$

Taking the induced long exact sequences of (2) and (3) and observing that $H^2(F', \mathfrak{N}) \simeq H^2(F', 1 + \mathfrak{N}) = 0$, we get

$$H^0(\mathcal{O}_{(n-1)F'}) \xrightarrow{\partial} H^1(\mathfrak{N}) \to H^1(\mathcal{O}_{nF'}) \xrightarrow{\alpha} H^1(\mathcal{O}_{(n-1)F'}) \to 0 \tag{4}$$

and

$$H^0(\mathcal{O}^*_{(n-1)F'}) \xrightarrow{\partial^*} H^1(1 + \mathfrak{N}) \to \mathrm{Pic}(nF') \xrightarrow{\alpha^*} \mathrm{Pic}((n-1)F') \to 0. \tag{5}$$

By a result of Oort (cf. [29, §6]), we have that $H^1(\beta)(\mathrm{Im}(\partial)) = \mathrm{Im}(\partial^*)$.

Since $H^1(\mathfrak{N})$ is a $\mathbb{Z}/p\mathbb{Z}$-vector space, we see that any element in $\ker(\alpha^*)$ has pth power equal to 1, hence we have $o_n = o_{n-1}$ or $o_n = po_{n-1}$.

If $o_n = po_{n-1}$, then $\ker(\alpha^*) \neq \{1\}$ and hence $\ker(\alpha) \neq \{0\}$. Since $h^1(nF', \mathcal{O}_{nF'}) = h^0(\omega_n)$, by Lemma 3.7 we have that $h^0(\omega_n) = h^0(\omega_{n-1}) + 1$ and ω_n is trivial. □

Assume that we have a multiple fiber over the point q_j, and denote by t_j the length of the skyscraper sheaf T at q_j.

Then, by the base change theorem we have

$$t_j + 1 = \mathrm{rk}_{q_j} \mathcal{R}^1 f_*(\mathcal{O}_S) = h^1(\mathcal{O}_{m_j F'_j}) = h^0(\mathcal{O}_{m_j F'_j}).$$

The two lemmas by Raynaud imply the following useful corollary.

Corollary 3.9. [31, Lemma 3.7.9], [8, Lemma 2.1.11], [23, Lemmas 2.3–2.4]

(1) *Let $n_j^{(i)}$ be the ith jumping value of a wild fiber mF'_j (recall that $n_j^{(i)} \geqslant 2$). Setting $\nu_j := \mathrm{Ord}(\mathcal{O}_{F'_j}(F'_j))$, we have*

$$n_j^{(1)} = \nu_j + 1,$$

and

$$n_j^{(2)} = \begin{cases} 2\nu_j + 1 & \text{if } \operatorname{Ord}(\mathcal{O}_{(\nu_j+1)F'_j}) = \nu_j, \\ (p+1)\nu_j + 1 & \text{if } \operatorname{Ord}(\mathcal{O}_{(\nu_j+1)F'_j}) = p\nu_j. \end{cases}$$

(2) *If* $h^0(\mathcal{O}_{mF'_j}) = 2 \Leftrightarrow t_j = 1$, *then* $a_j = m_j - 1$ *or* $a_j = m_j - 1 - \nu_j$.

(3) *If* $h^0(\mathcal{O}_{mF'_j}) = 3 \Leftrightarrow t_j = 2$, *then* $a_j = m_j - 1$, $a_j = m_j - 1 - \nu_j$, $a_j = m_j - 1 - 2\nu_j$, *or* $a_j = m_j - 1 - (p+1)\nu_j$.

Finally, Katsura and Ueno proved for elliptic fibrations in characteristic p the analogue of a result which in characteristic zero follows from the description of the fundamental group of the complement of a finite set of points on $\mathbb{P}^1$.

Definition 3.10. [23, Definition 3.1] Let $f: S \to \mathbb{P}^1$ be an elliptic fibration with $\chi(S, \mathcal{O}_S) = 0$, let $m_i F'_i$, $i = 1, \ldots, k$, be the multiple fibers, and let as usual ν_i be the order of $\mathcal{O}_{F'_i}(F'_i)$.

Then S is said to be *of type* $(m_1, \ldots, m_r | \nu_1, \ldots, \nu_r)$.

Definition 3.11. [23, Definition 3.2] Given $1 \leqslant i \leqslant r$, we say that two sequences $(m_1, \ldots, m_r | \nu_1, \ldots, \nu_r)$ satisfy condition U_i, if there exist integers $n_1, \ldots, n_r$ (depending on i) such that

- $n_i \equiv 1 \bmod \nu_i$ and
- $\sum_{j=1}^r n_j / m_j \in \mathbb{Z}$.

Theorem 3.12. [23, Theorem 3.3] *In the situation of Definition 3.10, the sequences* $(m_1, \ldots, m_r | \nu_1, \ldots, \nu_r)$ *satisfy condition* U_i $\forall i = 1, 2, \ldots, r$.

§4. Proof of the main theorem 3.3

Let $f: S \to C$ be a relatively minimal properly elliptic or properly quasielliptic fibration. Set here $g := g(C)$ and set $t := \mathit{length}(T)$, where T is the torsion sheaf appearing in the canonical bundle formula.

The first important observation is that in the canonical bundle formula the term $\chi(\mathcal{O}_S)$ is $\geqslant 0$, by Mumford's extension of Castelnuovo's theorem [27, p. 330].

The case $\chi(\mathcal{O}_S) + t \geqslant 1$ and $g \geqslant 1$ follows from the inequality

$$P_n(S) = h^0(\mathcal{O}_S(nK_S)) \geqslant h^0(\mathcal{O}_C(n\delta)) \geqslant g + (n-1) \geqslant n.$$

If $g \geqslant 2$, and $\chi(\mathcal{O}_S) = t = 0$, then we are done, since $P_n \geqslant (2n-1)(g-1)$.

If instead $g = 1$ and $\chi(\mathcal{O}_S) = t = 0$, then there are no wild fibers, hence

$$nK_S = \sum_j n(m_j - 1)F'_j = D + m_j \left\{ \frac{n(m_j - 1)}{m_j} \right\} F'_j,$$

where $D := \sum_j [n(m_j - 1)/m_j] F_j$.

Since the canonical divisor is nef and not numerically trivial, we have $\sum_j (1 - 1/m_j) > 0$, therefore D is an effective divisor (with integral coefficients).

Hence

$$(**) \quad P_n(S) \geqslant \sum_j \left[\frac{n(m_j - 1)}{m_j} \right] \geqslant \left[\frac{n}{2} \right].$$

We may therefore assume that $g = 0$.

If $g = 0$, and $\chi(\mathcal{O}_S) + t \geqslant 3$, then $P_n(S) \geqslant n + 1$.

If $g = t = 0$, and $\chi(\mathcal{O}_S) = 2$, then again there are no wild fibers and the same argument as in $(**)$ yields

$$P_n(S) \geqslant 1 + \sum_j \left[\frac{n(m_j - 1)}{m_j} \right] \geqslant 1 + \left[\frac{n}{2} \right].$$

We are left with the following possibilities:

Case (1) $\chi(\mathcal{O}_S) = 1$, $t = 1$ and $g = 0$;
Case (2) $\chi(\mathcal{O}_S) = 0$, $t = 2$ and $g = 0$;
Case (3) $\chi(\mathcal{O}_S) = 1$, $t = 0$ and $g = 0$;
Case (4) $\chi(\mathcal{O}_S) = 0$, $t = 1$ and $g = 0$;
Case (5) $\chi(\mathcal{O}_S) = t = 0$ and $g = 0$.

The next lemma shows that, except possibly in cases (1) and (3), we have only to consider the properly elliptic case.

LEMMA 4.1. *There exists no quasielliptic fibration $f : S \to \mathbb{P}^1$ with $\chi(\mathcal{O}_S) = 0$.*

Proof. Assume we have such a fibration.

Let $\alpha : S \to A$ be the Albanese map of S and assume that $q := \dim(A) \geqslant 1$. Since a general fiber of f is a cuspidal rational curve, whose image in A must be a single point, we see that α factors through f. Hence the image of α is a point: since the image generates A, A is a point, and $q = 0$, a contradiction.

We conclude that $q = 0$, hence $p_g \geqslant h$ and $\chi(\mathcal{O}_S) \geqslant 1$, a contradiction. $\square$

Let us now proceed with the proof.

LEMMA 4.2. *We write*

$$K_S \equiv dF + \sum_i a_i F_i',$$

where F is a fiber of f. Then we have $p_g(S) = \max(0, d+1)$.

Proof. Indeed, if $p_g \geqslant 1$, we can write $|K_S| = |M| + \Phi$, where Φ is the fixed part, and where the movable part is of the form $(p_g - 1)F$.

Hence K_S is linearly equivalent to an effective divisor D of the form $(p_g - 1)F + \sum_i b_i F_i'$, with $0 \leqslant b_i < m_i$.

If $d \geqslant 0$, then $p_g = d + 1$ and the fixed part $\Phi = \sum_i a_i F_i'$.

Otherwise, if $d < 0$, and we assume $p_g \geqslant 1$ we have a linear equivalence of effective divisors: $(|d| + p_g - 1)F + \sum_i b_i F_i' \equiv \sum_i a_i F_i'$, which shows that $|d| + p_g - 1 = 0$, a contradiction. ∎

Hence in our cases we have respectively:

Case (1) $\chi(\mathcal{O}_S) = 1,\ t = 1, h = 1, p_g = 1$ and $g = 0$;
Case (2) $\chi(\mathcal{O}_S) = 0, t = 2, h = 2, p_g = 1$ and $g = 0$;
Case (3) $\chi(\mathcal{O}_S) = 1, t = 0, h = 1, p_g = 0$ and $g = 0$;
Case (4) $\chi(\mathcal{O}_S) = 0, t = 1, h = 1, p_g = 0$ and $g = 0$;
Case (5) $\chi(\mathcal{O}_S) = t = 0, h = 1, p_g = 0$ and $g = 0$.

Observe therefore that Corollary 3.6 applies in all cases except (2).

Case (1): $K_S \equiv \sum_i a_i F_i'$, and if there exists a multiple fiber for which $a_j = m_j - 1$, we are done, since $P_n \geqslant [n/2] + 1$.

Otherwise, there is exactly one multiple fiber, wild, with $t_j = 1$, and by Corollary 3.6 and Proposition 3.5 $a := a_j$ satisfies

$$a = m - 1 - \nu = \nu(p^e - 1) - 1 > 0.$$

If $\nu = 1$, we obtain $a/m = m - 2/m \geqslant 1/3$. If $\nu \geqslant 2$, then we get

$$\frac{a}{m} \geqslant \frac{p^e - 1 - \frac{1}{2}}{p^e} = \frac{2p^e - 3}{2p^e} \geqslant \frac{1}{4}.$$

Thus the inequality

$$P_n \geqslant \begin{cases} \left[\dfrac{n}{3}\right] + 1 & \text{if } \nu = 1, \\[2ex] \left[\dfrac{n}{4}\right] + 1 & \text{if } \nu \geqslant 2, \end{cases}$$

holds. ∎

Case (2): Again $K_S \equiv \sum_i a_i F_i'$, and if there exists a multiple fiber for which $a_j = m_j - 1$, we are done, since then $P_n \geqslant [n/2] + 1$.

Otherwise there are only wild fibers, either one with $t_1 = 2$, or two with $t_1 = t_2 = 1$. In the latter case by Corollary 3.9 we have $a_j = m_j - 1 - \nu_j$, and we argue as in Case (1).

In the former case, we are left (set $m := m_1$, $a := a_1$, and $\nu := \nu_1$) with the case $a = m - 1 - 2\nu$ or $a = m - 1 - (p+1)\nu$. It is clear that the first case will give a better estimate than the second. Thus, we have only to consider the second case.

Here

$$\frac{a}{m} = \frac{p^e - p - 1 - \frac{1}{\nu}}{p^e},$$

which is a monotonously increasing function of e, ν, and p.

We must have $e \geqslant 2$. For $e = 2$ and $\nu = 1$, we must have $p \geqslant 3$.

In conclusion, for $\nu = 1$, $a/m \geqslant \min(4/9, 4/8) = 4/9 \Rightarrow P_n \geqslant [4n/9] + 1$.

Instead, for $\nu \geqslant 2$, the minimum is given in the case $p = 2$, $e = 2$ and $\nu = 2$, and we obtain $a/m \geqslant 1/8$.

In this case, we get $P_n = [n/8] + 1$, which would be a limit case.[3] □

Case (3): Here $K_S \equiv -F + \sum_i^r a_i F_i'$, where F is a fiber of f. Since K_S is $\mathbb{Q}$-linearly equivalent to an effective divisor, for any ample divisor H on S, we have

$$K_S.H = \left(-1 + \sum_{i=1}^{r} \frac{a_i}{m_i} \right) F.H > 0,$$

which is equivalent to

$$(\circledast) \qquad -1 + \sum_{i=1}^{r} \frac{a_i}{m_i} > 0.$$

It follows that $r \geqslant 2$. The condition $t = 0$ implies that there is no wild fiber, hence $a_i = m_i - 1$ for all i.

If $r \geqslant 3$, one sees easily that $P_n \geqslant [n/2] + 1$ and we are done.

If $r = 2$, we have at least one $m_i \geqslant 3$, therefore we get

$$P_n \geqslant \left[\frac{n}{2} \right] + \left[\frac{2n}{3} \right] - n + 1.$$

We have $P_n \geqslant 1$ for $n \geqslant 3$ and $P_n \geqslant 2$ for $n \geqslant 8$. □

[3] However the actual existence of this case with only one multiple fiber, and the above numerical characters, is unclear to us.

Case (4): Here $K_S \equiv -F + \sum_i^r a_i F_i'$, where F is a fiber of f. For the same reason as in Case (3), we have $r \geqslant 2$. Since $t = 1$, there exists exactly one wild fiber, say $m_1 F_1'$, with $t_1 = 1$: by Corollary 3.6 and Proposition 3.5 $a_1 = m_1 - 1$, or $a_1 = m_1 - 1 - \nu_1$. Hence we can rewrite K_S as follows:

$$K_S \equiv -F + a_1 F_1' + \sum_{i=2}^{r} (m_i - 1) F_i',$$

so that

$$P_n = \max\left(0, 1 - n + \left[\frac{n a_1}{m_1}\right] + \sum_{i=2}^{r} \left[\frac{n(m_i - 1)}{m_i}\right]\right).$$

If $r \geqslant 4$, or if $r = 3$ and $a_1 = m_1 - 1$, we have $P_n \geqslant 1 - n + 3[n/2]$, and writing $n = 2k + s$, $s \in \{0, 1\}$, we get $P_n \geqslant 1 - 2k - s + 3k = 1 + k - s$, which is $\geqslant 1$ for $n \geqslant 2$, and $\geqslant 2$ for $n \geqslant 4$.

In the case where $r = 3$ and $a_1 = m_1 - 1 - \nu_1$, consider first the case $a_1 = 0$. Then ($\circledast$) implies that m_2 or $m_3 \geqslant 3$, and we get

$$P_n \geqslant 1 - n + \left[\frac{2n}{3}\right] + \left[\frac{n}{2}\right].$$

Writing $n = 2k + s$ with $s \in \{0, 1\}$, we get

$$P_n \geqslant 1 - 2k - s + k + \left[\frac{k + 2s}{3}\right] + k = 1 + \left[\frac{k + 2s}{3}\right] - s,$$

which is

$$\begin{cases} 1 + \left[\dfrac{k}{3}\right] & \text{if } s = 0, \\ \left[\dfrac{k+2}{3}\right] & \text{if } s = 1. \end{cases}$$

Hence we get $P_n \geqslant 1$ for $n \geqslant 2$, $P_6 \geqslant 2$ and $P_n \geqslant 2$ for $n \geqslant 8$.

If $a_1 > 0$, we are done if m_2 or m_3 is $\geqslant 3$. The remaining case is $m_2 = m_3 = 2$, and Condition U_1 implies that there exists an integer l such that $(l\nu_1 + 1)/(p^{e_1}\nu_1) \in \mathbb{Z}$, which implies $\nu_1 = 1$. Therefore we conclude that

$$\frac{a_1}{m_1} = \frac{m_1 - 1 - \nu_1}{m_1} \geqslant \frac{m_1 - 2}{m_1},$$

whence $m_1 \geqslant 3$ and $a_1/m_1 \geqslant 1/3$. Hence we have

$$P_n \geqslant 1 - n + \left[\frac{n}{3}\right] + 2\left[\frac{n}{2}\right].$$

We get

$$P_n \geqslant \begin{cases} 1 + \left[\dfrac{2k}{3}\right] & n = 2k \\ \left[\dfrac{2k+1}{3}\right] & n = 2k+1. \end{cases}$$

Hence we have $P_2 \geqslant 1$, $P_4 \geqslant 2$, and $P_n \geqslant 2$ for $n \geqslant 6$.

We are left with the case $r = 2$.

Assume first that $a_1 = m_1 - 1$: the situation is then identical to the case $r = 3$ and $a_1 = 0$, and we are done.

We may therefore assume that $a_1 = m_1 - 1 - \nu_1 > 0$, and Inequality (⁂) becomes now

$$(⁂⁂) \qquad 1 - \frac{1 + \frac{1}{\nu_1}}{p^{e_1}} - \frac{1}{m_2} > 0,$$

and we have

$$P_n \geqslant 1 - n + \left[\frac{n(p^{e_1} - 1 - \frac{1}{\nu_1})}{p^{e_1}}\right] + \left[\frac{n(m_2 - 1)}{m_2}\right].$$

Conditions U_1 and U_2 imply that $\nu_1 | m_2$ and $m_2 | m_1 = p^{e_1}\nu_1$, hence $m_2 = \nu_1 p^{\epsilon}$ and $\epsilon \leqslant e_1$.

If $\nu_1 = 1$, then an immediate consequence is that $m_2 \geqslant p$. Moreover, combining with (⁂⁂), we get $p^{e_1} \geqslant 5$ or $p^{e_1} = p^{\epsilon} = 4$; however, in the latter case, we have

$$(***) \quad P_n \geqslant f_n := 1 - n + \left[\frac{n}{2}\right] + \left[\frac{3n}{4}\right] = f_s + k, \;\; n = 4k + s, \;\; 0 \leqslant s \leqslant 3,$$

$$f_s = 1, 0, 1, 1, \;\; s = 0, 1, 2, 3.$$

Let us consider the former case:

- If $p \geqslant 5$, then $m_2 \geqslant 5$, hence $P_n \geqslant f_n := 1 - n + [3n/5] + [4n/5]$. Writing $n = 5k + s$ with $0 \leqslant s \leqslant 4$, we get

$$P_n \geqslant f_n = 2k + f_s, \quad f_s = 1, 0, 1, 1, 2, \quad s = 0, 1, 2, 3, 4.$$

  Therefore we have $P_n \geqslant 1$ for $n \geqslant 2$, and $P_n \geqslant 2$ for $n \geqslant 4$.

- If $p = 3$, then $e_1 \geqslant 2$ and $m_2 \geqslant 3$. It follows that $P_n \geqslant f_n := 1 - n + [7n/9] + [2n/3]$. Writing $n = 3k + s$ with $0 \leqslant s \leqslant 2$, we get

$$\begin{aligned} P_n &\geqslant 1 - 3k - s + 2k + \left[\frac{3k + 7s}{9}\right] + 2k + \left[\frac{2s}{3}\right] \\ &= 1 + k + \left[\frac{3k + 7s}{9}\right] + \left[\frac{2s}{3}\right] - s. \end{aligned}$$

  Hence $P_n \geqslant 1 + k$ except for the case $k = 0$ and $s = 1$, which implies that $P_n \geqslant 1$ for $n \geqslant 2$, and $P_n \geqslant 2$ for $n \geqslant 3$.
- If finally $p = 2$, observe that $e_1 \geqslant 3$ and $m_2 \geqslant 2$, hence we have $P_n \geqslant f_n := 1 - n + [3n/4] + [n/2]$. This case which was already treated in $(***)$.

In the following, we assume $\nu_1 \geqslant 2$.

- If $p^{e_1} \geqslant 4$, we have that $P_n \geqslant 1 - n + [5n/8] + [n/2]$. Writing $n = 2k + s$ with $s \in \{0, 1\}$, we get

$$\begin{aligned} P_n &\geqslant 1 - 2k - s + k + \left[\frac{2k + 5s}{8}\right] + k \\ &= 1 + \left[\frac{2k + 5s}{8}\right] - s. \end{aligned}$$

  It follows that $P_n \geqslant 1 + [k/4]$ for $s = 0$ and $P_n \geqslant [(2k + 5)/8]$ for $s = 1$. In the worst case where $p^{e_1} = 4$ and $\nu_1 = m_2 = 2$ (this case does not actually occur by Condition U_1), we have that $P_1 = P_3 = 0$, $P_2 = P_4 = P_5 = P_6 = P_7 = 1$, $P_8 = 2$ and $P_n \geqslant 2$ for $n \geqslant 12$.
- If $p^{e_1} = 3$, we cannot have $m_2 = \nu_1 = 2$, since this would contradict inequality $(\ast\ast)$. Hence we have either $m_2,\ \nu_1 \geqslant 3$ or $\nu_1 = 2$, $m_2 = 6$. We obtain

$$(\ast 1) \qquad P_n \geqslant 1 - n + \left[\frac{5n}{9}\right] + \left[\frac{2n}{3}\right]$$

  respectively

$$(\ast 2) \qquad P_n \geqslant 1 - n + \left[\frac{n}{2}\right] + \left[\frac{5n}{6}\right].$$

  For $(\ast 1)$, writing $n = 3k + s$ with $0 \leqslant s \leqslant 2$, we get

$$P_n \geqslant 1 + \left[\frac{6k + 5s}{9}\right] + \left[\frac{2s}{3}\right] - s,$$

which implies that $P_n \geqslant 1 + [2k/3]$ for $s = 0$, $P_n \geqslant [(6k+5)/9]$ for $s = 1$ and $P_n \geqslant 1 + [(6k+1)/9]$ for $s = 2$. Hence $P_n \geqslant 1$ for $n \geqslant 2$, $P_6 \geqslant 2$, and $P_n \geqslant 2$ for $n \geqslant 8$.

For $(*2)$, writing $n = 2k + s$ with $s \in \{0, 1\}$, we get

$$P_n \geqslant 1 + \left[\frac{4k+5s}{6}\right] - s,$$

it follows that $P_n \geqslant 1 + [2k/3]$ for $s = 0$ and $P_n \geqslant [(4k+5)/6]$ for $s = 1$. We see that $P_n \geqslant 1$ for $n \geqslant 2$, and $P_n \geqslant 2$ for $n \geqslant 4$.

- If $p^{e_1} = 2$, we have either $m_2 = \nu_1,\ \nu_1 \geqslant 4$ or $m_2 = 2\nu_1,\ \nu_1 \geqslant 3$. It follows that

$$P_n \geqslant 1 - n + \left[\frac{3n}{8}\right] + \left[\frac{3n}{4}\right] \tag{$*3$}$$

respectively

$$P_n \geqslant 1 - n + \left[\frac{n}{3}\right] + \left[\frac{5n}{6}\right]. \tag{$*4$}$$

For $(*3)$, writing $n = 4k + s$ with $0 \leqslant s \leqslant 3$, we get

$$P_n \geqslant 1 + \left[\frac{4k+3s}{8}\right] + \left[\frac{3s}{4}\right] - s,$$

which equals $1 + [k/2]$ for $s = 0$, $[(4k+3)/8]$ for $s = 1$, $[(4k+6)/8]$ for $s = 2$, and $1 + [(4k+1)/8]$ for $s = 3$. In the worst numerical case $\nu_1 = m_2 = 4$ (this case does not actually occur by Condition U_1), we have $P_3 = P_4 = P_6 = P_7 = 1$, $P_2 = P_5 = 0$, $P_8 = 2$, $P_{12} = 2$, $P_{13} = 1$, and $P_n \geqslant 2$ for $n \geqslant 14$.

For $(*4)$, writing $n = 3k + s$ with $0 \leqslant s \leqslant 2$, we get

$$P_n \geqslant 1 + \left[\frac{3k+5s}{6}\right] - s,$$

hence $P_n \geqslant 1 + [k/2]$ for $s = 0$, $\geqslant [(3k+5)/6]$ for $s = 1$, and $\geqslant [(3k+4)/6]$ for $s = 2$. We conclude that $P_n \geqslant 1$ for $n \geqslant 3$, $P_6 \geqslant 2$, and $P_n \geqslant 2$ for $n \geqslant 9$. □

Case (5): Here $K_S \equiv -2F + \sum_{i=1}^{r}(m_i - 1)F_i'$, since $t = 0$ implies that there is no wild fiber.

In view of Theorem 3.12 this situation is exactly as in the classical case (of characteristic 0). However, our main theorem is new also in the classical case, so we proceed to treat case (4).

We may assume that

$$m_1 \leqslant m_2 \leqslant \cdots \leqslant m_r,$$

and we recall that

$$(\star) \qquad P_n = \max\left(0, 1 - 2n + \sum_j \left[\frac{n(m_j - 1)}{m_j}\right]\right).$$

For $r \geqslant 5$ we have $P_n \geqslant 1 - 2n + 5[n/2]$, and writing $n = 2k + s, s \in \{0, 1\}$, we get $P_n \geqslant 1 - 4k - 2s + 5k = 1 + k - 2s$, which is at least 1 for $n \geqslant 4$, and $\geqslant 2$ for $n \geqslant 6$.

Assume $r = 4$ and observe once more that the right hand side of ($\star$) is an increasing function of the multiplicities m_j, hence the worst case is $(2, 2, 3, 3)$. However, the worst case would be $(2, 2, 2, 3)$, this case does not occur, since Condition U_4 is not fulfilled.

Hence we obtain

$$P_n \geqslant 1 - 2n + 2\left[\frac{n}{2}\right] + 2\left[\frac{2n}{3}\right] = 1 + \left(2\left[\frac{n}{2}\right] - n\right) + \left(2\left[\frac{2n}{3}\right] - n\right).$$

For even numbers $n = 2k$, we get $P_n = 1 + 2[k/3]$, which is $\geqslant 1$, and $\geqslant 3$ whenever $n \geqslant 6$. For odd numbers $n = 2k + 1$ we get

$$P_n = 2\left[\frac{4k+2}{3}\right] - 2k - 1 = 2\left[\frac{k+2}{3}\right] - 1,$$

which is $\geqslant 1$ for $n \geqslant 3$, $\geqslant 3$ whenever $n \geqslant 9$.

In the case $r = 3$, note that Conditions U_1, U_2, U_3 are equivalent to the condition that m_k divides $\mathrm{LCM}(m_i, m_j)$ for each choice of $\{i, j, k\} = \{1, 2, 3\}$.

Assume that $m_1 \geqslant 4$: by monotonicity, the worst case is $(4, 4, 4)$, where, setting $n = 4k + s, 0 \leqslant s \leqslant 3$,

$$P_n \geqslant 3\left[\frac{3n}{4}\right] - 2n + 1 = 3\left(3k + \left[\frac{3s}{4}\right]\right) - 8k - 2s + 1 = 1 + k + 3\left[\frac{3s}{4}\right] - 2s.$$

We get

- $k+1$ for $s=0,3$;
- k for $s=2$;
- $k-1$ for $s=1$.

Hence $P_3=1$, $P_4=2$, $P_n \geqslant 2$ for $n \geqslant 10$.

Assume that $m_1=3$. Then 3 divides $\mathrm{LCM}(m_2,m_3)$. Hence either 3 divides both m_2 and m_3 or 3 does not divide m_2 or m_3.

Keeping in mind the positivity of K_S, equivalent here to $\sum_j (1/m_j) < 1$, each case leads to the worst possible case, i.e., one maximizing $\sum_j (1/m_j) < 1$.

(1) $(3,3a,3b)$, $a|b$, $b|a \Rightarrow a=b \Rightarrow (3,3a,3a)$: the worst case $(3,6,6)$;
(2) $(3,c,3b)$ c not divisible by 3, $3b|3c$, $c|b \Rightarrow b=c \Rightarrow (3,c,3c)$: worst case $(3,4,12)$;
(3) $(3,3a,c)$ c not divisible by 3, $3a|3c$, $c|a \Rightarrow a=c \Rightarrow (3,3a,a)$: same as in the previous case.

Recall the plurigenus formula, here it gives respectively

$$(3,6,6): \quad P_n = \max(0, F(n)), \quad F(n) := 1 - 2n + \left[\frac{2n}{3}\right] + 2\left[\frac{5n}{6}\right].$$

$$(3,4,12): \quad P_n = \max(0, F(n)), \quad F(n) := 1 - 2n + \left[\frac{2n}{3}\right] + \left[\frac{3n}{4}\right] + \left[\frac{11n}{12}\right].$$

In the former case, writing $n = 6k+s$, $0 \leqslant s \leqslant 5$, we get

$$F(n) = 2k + F(s), \quad \text{and} \quad F(s) = 1, -1, 0, 1, 1, 2 (s = 0, 1, \ldots 5)$$

hence $P_3 \geqslant 1$, $P_5 \geqslant 2$, and $P_n \geqslant 2$ for $n \geqslant 8$.

In the latter case, writing $n = 12k+s$, $0 \leqslant s \leqslant 11$, we get

$$F(n) = 4k + F(s), \; F(0) = 1 \Rightarrow F(s) = -s + \left[\frac{2s}{3}\right] + \left[\frac{3s}{4}\right] \quad \text{for } s \geqslant 1,$$
$$F(s) = 1, -1, 0, 1, 1, 1, 2, 2, 3, 3, 3, 4, \quad 0 \leqslant s \leqslant 11,$$

hence $P_n \geqslant 1$ for $n \geqslant 3$, $P_n \geqslant 2$ for $n \geqslant 6$.

Assume finally $m_1 = 2$. Then one of m_2 and m_3 is even. If $m_j = c$ is odd and $m_i = 2b$, then $c|b$ and $2b|2c \Rightarrow b = c$, hence we get $(2, b, 2b)$ and the worst case is $(2, 5, 10)$, which was already considered in Remark 3.4.

Similarly, in case $(2, 2a, 2b)$ again $a = b$, hence we get a triple $(2, 2a, 2a)$ and the worst case is the case $(2, 6, 6)$, which was already considered in Remark 3.4.

REMARK 4.3. Our analysis allows us also to see (cf. Remark 3.4) which the possible cases are where the estimates are sharp in the main theorem.

- (2): $P_1 = P_2 = P_3 = 0$ in Case (4) for triples $(2, b, 2b)$, $b \geqslant 5, 2 \nmid b$, or $(2, 2a, 2a)$, $a \geqslant 3$.
- (3): $P_n \leqslant 1$ for $n \leqslant 7$ in Case (4) for the triple $(2, 5, 10)$ and possibly in Case (2) with one wild fiber and $p = \nu = e = 2$.
- (4): $P_{13} = 1$ in Case (4) for the triple $(2, 6, 6)$.

Acknowledgments. Both authors gratefully acknowledge the support of the ERC Advanced Grant no. 340258, "TADMICAMT." The first author would like to thank Ciro Ciliberto, Mirella Manaresi and Sandro Verra for organizing a Conference in memory of Federigo Enriques 70 years after his death: their invitation provided stimulus to extend Enriques' P_{12}-theorem to positive characteristic; and Igor Dolgachev for his question about item (3) of the main theorem. The second author would like to thank Michel Raynaud for helpful communications.

REFERENCES

[1] L. Bădescu, *Suprafeţe Algebrice*, Editura Academiei Pepublicii Socialiste România, Bucharest, 1981.

[2] A. Beauville, *Surfaces Algébriques Complexes*, Astérisque, No. 54, Société Mathématique de France, 1978.

[3] E. Bombieri, *Canonical models of surfaces of general type*, Publ. Math. Inst. Hautes Études Sci. **42** (1973), 171–219.

[4] E. Bombieri and D. Husemoller, "*Classification and embeddings of surfaces*", in *Algebraic Geometry (Proc. Symp. Pure Math., Vol. 29, Humboldt State University, Arcata, California, 1974)*, American Mathematical Society, Providence, RI, 1975, 329–420.

[5] E. Bombieri and D. Mumford, "*Enriques' classification of surfaces in chap. p. II*", *Complex Analysis and Algebraic Geometry, Iwanami Shoten, Tokyo*, 23–42. 1977.

[6] E. Bombieri and D. Mumford, *Enriques' classification of surfaces in char. p. III*, Invent. Math. **35** (1976), 197–232.

[7] W. Barth, C. Peters and A. Van de Ven, *Compact Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 4, Springer, Berlin, 1984.

[8] A. Bertapelle and J. Tong, *On torsors under elliptic curves and Serre's pro-algebraic structures*, Math. Z. **277**(1–2) (2014), 91–147.

[9] C. Birkar and D. Q. Zhang, *Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs*, Publ. Math. Inst. Hautes Études Sci. **123** (2016), 283–331.

[10] G. Castelnuovo and F. Enriques, "*Die algebraische Flächen vom Gesichtspunkte der birationalen transformationen aus*", in *Enz. d. Math. Wiss. III C 6 b, (Band III 2, Heft 6)*, 1915, 674–768.
[11] F. Catanese and M. Franciosi, "*Divisors of small genus on algebraic surfaces and projective embeddings*", in *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, Israel Math. Conf. Proc. **9**, Bar-Ilan University, Ramat Gan, 1996, 109–140.
[12] F. Catanese, M. Franciosi, K. Hulek and M. Reid, *Embeddings of curves and surfaces*, Nagoya Math. J. **154** (1999), 185–220.
[13] P. Deligne and L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89**(2) (1987), 247–270.
[14] T. Ekedahl, *Canonical models of surfaces of general type in positive characteristic*, Publ. Math. Inst. Hautes Études Sci. **67** (1988), 97–144.
[15] F. Enriques, *Sulla classificazione delle superficie algebriche e particolarmente sulle superficie di genere Lineare $p^{(1)} = 1$*, Rend. Acc. Lincei, s. 5^a **23** (1914), 206–214.
[16] F. Enriques, "*Lezioni sulla teoria delle superficie algebriche*", *Raccolte da L. Campedelli. I*, Padova, CEDAM, 1932.
[17] F. Enriques, *Le Superficie Algebriche*, Nicola Zanichelli, Bologna, 1949.
[18] O. Fujino and S. Mori., *A canonical bundle formula*, J. Differ. Geom. **56** (2000), 167–188.
[19] J. Igusa, *A fundamental inequality in the theory of Picard varieties*, Proc. Natl. Acad. Sci. USA **41** (1955), 317–320.
[20] J. Igusa, *On some problems in abstract algebraic geometry*, Proc. Natl. Acad. Sci. USA **41** (1955), 964–967.
[21] S. Iitaka., *Deformations of compact complex surfaces. II*, J. Math. Soc. Japan **22** (1970), 247–261.
[22] L. Illusie and M. Raynaud, *Les suites spectrales associées au complexe de De Rham–Witt*, Publ. Math. Inst. Hautes Études Sci. **57** (1983), 73–212.
[23] T. Katsura and K. Ueno, *On elliptic surfaces in characteristic p*, Math. Ann. **272**(3) (1985), 291–330.
[24] K. Kodaira, *On the structure of complex analytic surfaces. IV*, Am. J. Math. **90** (1968), 1048–1066.
[25] D. Mumford, *Pathologies of modular algebraic surfaces*, Am. J. Math. **83** (1961), 339–342.
[26] D. Mumford, "*Lectures on curves on an algebraic surface*", in *With a Section by G. M. Bergman*, Annals of Mathematics Studies **59**, Princeton University Press, NJ, 1966.
[27] D. Mumford, "*Enriques' classification of surfaces in char p. I*", in *Global Analysis (Papers in Honor of K. Kodaira)*, University Tokyo Press, Tokyo, 1969, 325–339.
[28] D. Mumford, "*Abelian varieties*", in *Tata Institute of Fundamental Research Studies in Mathematics, No. 5 Published for the Tata Institute of Fundamental Research, Bombay*, Oxford University Press, 1970.
[29] F. Oort., *Sur le schéma de Picard*, Bull. Soc. Math. Fr. **90** (1962), 1–14.
[30] M. Raynaud, *Spécialisation du foncteur de Picard*, Publ. Math. Inst. Hautes Études Sci. **38** (1970), 27–76.
[31] M. Raynaud, "*Surfaces elliptiques et quasi-elliptiques*". Manuscript (1976).
[32] I. R. Šafarevič, B. G. Averbuh, Ju. R. Vaĭnberg, A. B. Žižčenko, Ju. I. Manin, B. G. Moĭšezon, G. N. Tjurina and A. N. Tjurin, *Algebraic surfaces*, Tr. Mat. Inst. Steklova **75** (1965).

[33] C. S. Seshadri, *L'opération de Cartier. Applications*, Variétés de Picard. Sém. C. Chevalley 3 (1958/59) **6** (1960), 1–26.

[34] J. Tate, *Genus change in inseparable extensions of function fields*, Proc. Am. Math. Soc. **3** (1952), 400–406.

Fabrizio Catanese
Lehrstuhl Mathematik VIII
Mathematisches Institut der Universität Bayreuth
NW II, Universitätsstr. 30
95447 Bayreuth
Germany
Fabrizio.Catanese@uni-bayreuth.de

Binru Li
Lehrstuhl Mathematik VIII
Mathematisches Institut der Universität Bayreuth
NW II, Universitätsstr. 30
95447 Bayreuth
Germany
Binru.Li@uni-bayreuth.de