Deformation of a generically finite map to a hypersurface embedding ☆

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A B S T R A C T

We give a structure theorem for projective manifolds W_0 with the property of admitting a 1-parameter deformation where W_t is a hypersurface in a projective smooth manifold Z_t.
Their structure is the one of special iterated univariate coverings which we call of normal type, which essentially means that the line bundles where the univariate coverings live are tensor powers of the normal bundle to the image X of W_0.
We give applications to the case where Z_t is projective space, respectively an Abelian variety.

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R É S U M É

Nous donnons un théorème de structure pour les variétés projectives W_0 avec la propriété d'admettre une déformation à 1 paramètre où W_t est une hypersurface dans une variété projective lisse Z_t. Leur structure est celle des revêtements univariées spéciales itérées que nous appelons de type normal : ce qui signifie essentiellement que le fibrées en droites où les revêtements univariées sont plongée sont des puissances tensoriels de le faisceau normal à l'image X de W_0. Nous donnons des applications au cas où Z_t est un espace projectif, respectivement une variété abélienne.

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1. Introduction

Many years ago Sernesi [7] showed that small deformations of complete intersections in projective space, of dimension $n \geq 2$ (the case of curves, $n = 1$ is of quite different nature, see e.g. [8]), are again complete intersections, unless the complete intersection defines a K3 surface (i.e., $n = 2$ and the canonical bundle is trivial). Hence, in particular, smooth hypersurfaces in projective space $\mathbb{P}^{n+1}$ form an open set in the Kuranishi space, respectively an open set in the moduli space when they are of general type, unless $n = 2$ and the degree equals 4. In considering the closure of this set in the moduli space, we have to deal with varieties W_0 of the same dimension, given together with a generically finite rational map $\phi_0 : W_0 \dashrightarrow \mathbb{P}^{n+1}$.

As shown by Horikawa in [5], already in the easiest nontrivial case $n = 2$, $deg(W_0) = 5$ the situation becomes rather complicated. But we show here that things are simpler in the case where ϕ_0 is a morphism.

A similar result to Sernesi's holds for hypersurfaces in an Abelian variety (Kodaira and Spencer's theorem 14.4 in [6]), and we can consider the closure of the locus of hypersurfaces X in Abelian varieties (for $n \geq 2$ the Abelian variety is just the Albanese variety of X) observing that in this case any limit W_0 has a generically finite Albanese map $\phi_0 : W_0 \to A_0$ (see for instance lemma 149 of [3]).

Also in this case we can ask the question of characterizing the morphisms ϕ_0 admitting a deformation which is a hypersurface embedding in some Abelian variety, deformation of the original one.

The main motivation for posing this question also in higher generality comes from the theory of topological methods to moduli theory, cf. [3]; and, more specifically, the theory of Inoue-type varieties, introduced in [1]. In the theory of Inoue-type varieties, one can describe their moduli spaces explicitly in the case where the morphism ϕ_0 has necessarily degree one onto its image. This is however a big restriction, and one would like to consider also the case where the morphism ϕ_0 has degree at least two. We strive therefore towards a theory of multiple Inoue-type varieties and, in order to do this, we restrict ourselves in this paper to the special case where ϕ_0 is a morphism which is generically finite onto its image, and where the canonical divisor of W_0 is ample.

To illustrate our main theorem, let us consider two simple examples, the first one where the image of W_0 is the smooth hypersurface $X := \{\sigma = 0\} \subset \mathbb{P}^{n+1}$, σ being a homogeneous polynomial of degree d. We let then W_0 be the complete intersection in the weighted projective space $\mathbb{P}(1, 1, \ldots, 1, d)$ defined by the equations

$$W_0 = \{(z_0, z_1, \ldots, z_{n+1}, w) | \sigma(z_0, z_1, \ldots, z_{n+1}) = 0,$$

$$P(z_0, z_1, \ldots, z_{n+1}, w) := w^m + \sum_{i=1}^{m} w^{m-i} a_i(z_0, z_1, \ldots, z_{n+1}) = 0\}. \quad \text{(I-1)}$$

We can easily deform the complete intersection by deforming the degree d equation adding a constant times the variable w, hence obtaining the following complete intersection:

$$P(z_0, z_1, \ldots, z_{n+1}, w) = 0, tw - \sigma(z_0, z_1, \ldots, z_{n+1}) = 0, t \in \mathbb{C}.$$

Clearly, for $t = 0$ we obtain the previous W_0, a degree m covering of the hypersurface $X = \{\sigma = 0\}$, whereas for $t \neq 0$ we can eliminate the variable w and obtain a hypersurface W_t in $\mathbb{P}^{n+1}$ with equation (of degree md)

$$P(z_0, z_1, \ldots, z_{n+1}, \sigma(z)/t) = 0.$$

Example 1.1 *(Iterated weighted deformations).* Now, one can iterate this process, and consider, in the weighted projective space

$$\mathbb{P}(1, 1, \ldots, 1, d, dm_1, \ldots, dm_k), \quad m_1|m_2| \ldots |m_k,$$

a complete intersection W of multidegrees $(d, dm_1, \ldots, dm_k, dm)$, where $m_k|m =: m_{k+1}$.

Then, necessarily, there exist constants $t_0, t_1, \ldots, t_k$ such that the equations of W have the following form, where the Q_j's are general weighted homogeneous polynomials of degree $= dm_j$ (in particular we assume them to be monic, so that the rational map to projective space is a morphism):

$$\begin{cases} \sigma(z) = w_0 t_0 \\ Q_1(w_0, z) = w_1 t_1 \\ \cdots \quad \cdots \\ Q_k(w_0, \ldots, w_{k-1}, z) = w_k t_k \\ Q_{k+1}(w_0, \ldots, w_k, z) = 0. \end{cases} \tag{I-2}$$

Again, if all the t_j's are $\neq 0$, we can eliminate the variables w_j, and we obtain a hypersurface $\{F(z) = 0\}$ in $\mathbb{P}^{n+1}$.

We claim that the above description generalizes, and the main idea of the following main theorem is that one can replace weighted projective space $\mathbb{P}(1, 1, \ldots, 1, d, dm_1, \ldots, dm_k)$, $m_1|m_2| \ldots |m_k$, by the total space of a direct sum of line bundles over some projective variety X, or $Z_0 \supset X$, or over a family $\mathcal{Z}$ of projective varieties.

The first assertion of the main theorem is that, in order that $\phi_0 : W_0 \to Z_0$ deforms to a hypersurface embedding, a necessary condition is that $\phi_0 : W_0 \to X := \phi_0(W_0)$ is a smooth iterated univariate covering of normal type (see the next section for this very restrictive condition).

The main theorem also gives sufficient conditions, given such a covering, for the existence of a deformation to a hypersurface embedding.

We then give the proof, and in the final section, we discuss the first applications to the case where the target manifold Z_0 is projective space or an Abelian variety.

We defer the applications to the theory of multiple Inoue type varieties to a future paper.

We work of course over the complex numbers, and in several situations we consider also more general compact complex manifolds than projective manifolds.

2. Statement of the main theorem

To give a clear statement of our results, we need to introduce the following terminology.

Definition 2.1.

i) Given a complex space (or a scheme) X, a **univariate covering** of X is a hypersurface Y, contained in a line bundle over X, and defined there as the zero set of a monic polynomial.

This means, $Y = \underline{Spec}(\mathcal{R})$, where $\mathcal{R}$ is the quotient algebra of the symmetric algebra over an invertible sheaf $\mathcal{L}$, $Sym(\mathcal{L}) = \oplus_{i \geq 0} \mathcal{L}^{\otimes i}$, by a monic (univariate) polynomial:

$$\mathcal{R} := Sym(\mathcal{L})/(P), P = w^m + a_1(x)w^{m-1} + a_2(x)w^{m-2} + \cdots + a_m(x).$$

Here $a_j \in H^0(X, \mathcal{L}^{\otimes j})$. The univariate covering is said to be **smooth** if both X and Y are smooth.

ii) An **iterated univariate covering** $W \to X$ is a composition of univariate coverings

$$f_{k+1} : W \to X_k, f_k : X_k \to X_{k-1}, \ldots, f_1 : X_1 \to X,$$

whose associated line bundles are denoted $\mathcal{L}_k, \mathcal{L}_{k-1}, \ldots, \mathcal{L}_1, \mathcal{L}_0$.

iii) In the case where $X \subset Z$ is a (smooth) hypersurface, we say that the iterated univariate covering is of **normal type** if

- all the line bundles $\mathcal{L}_j$ are pull back from X of a line bundle of the form $\mathcal{O}_X(m_j X)$, and moreover
- $m_1 | m_2 | \ldots | m_k$, and the degree of f_j equals $\frac{m_j}{m_{j-1}}$.
- we say that the iterated covering is **normally induced** if moreover all the coefficients $a_I(x)$ of the polynomials

$$Q_j(w_0, \ldots, w_{j-1}, x) = \sum_I a_I(x) w^I$$

describing the intermediate extensions are sections of a line bundle $\mathcal{O}_X(r(I)X)$ coming from $H^0(Z, \mathcal{O}_Z(r(I)X))$.

Remark 2.2. The property that the iterated univariate covering $W \to X$ is normally induced clearly means that it is the restriction to X of an iterated univariate covering of Z.

The property that the former is smooth does not necessarily imply that also the latter is smooth.

Definition 2.3. A 1-parameter deformation to hypersurface embedding consists of the following data:

(1) a one dimensional family of smooth projective varieties of dimension n (i.e., a smooth projective holomorphic map $p : \mathcal{W} \to T$ where T is a germ of a smooth holomorphic curve at a point $0 \in T$) mapping to another family $\pi : \mathcal{Z} \to T$ of smooth projective varieties of dimension $n + 1$ via a relative map $\Phi : \mathcal{W} \to \mathcal{Z}$ such that $\pi \circ \Phi = p$ (hence we have the following commutative diagram)

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\Phi} & \mathcal{Z} \\ & {}_{p}\searrow \quad \swarrow_{\pi} & \\ & T, & \end{array}$$
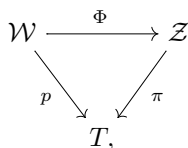
such that moreover

(2) for $t \neq 0$ in T, Φ_t is an embedding of $W_t := p^{-1}(t)$ in Z_t,
(3) the restriction of the map Φ on W_0 is a generically finite morphism of degree m, so that the image of $\Phi|_{W_0}$ is the cycle $\Sigma_0 := mX$ where X is a reduced hypersurface in Z_0, defined by an equation $X = \{\sigma = 0\}$.

Put in concrete terms, one can take a local coordinate t for T at 0, and write, locally around $\{t = 0\}$ the equation of the image $\Sigma := \Phi(\mathcal{W})$ in $\mathcal{Z}$ via the Taylor series development in t, in terms of local coordinates $z = (z_1, \ldots, z_n, z_{n+1})$ on Z_0,

$$\Sigma(z, t) : \sigma(z)^m + t\sigma_1(z) + t^2\sigma_2(z) + \cdots + t^{m-1}\sigma_{m-1}(z) + \ldots = 0.$$

$\mathcal{W}$ is a resolution of Σ and the next theorems indicates exactly the sequence of blow-ups needed in order to obtain the resolution $\mathcal{W}$.

Theorem 2.4. *(A) Suppose we have a 1-parameter deformation to hypersurface embedding*

$$
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\ \Phi\ } & \mathcal{Z} \\
& {}_{p}\searrow \quad \swarrow{}_{\pi} & \\
& T, &
\end{array}
$$
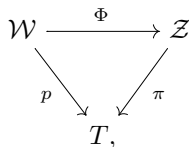
and assume that K_{W_0} is ample.

Then we have:

(A1) X is smooth,

(A2) There are line bundles $\mathcal{L}_0, \ldots, \mathcal{L}_k$ on $\mathcal{Z}$, such that $\mathcal{L}_j|_{Z_0} = \mathcal{O}_{Z_0}(m_j X)$ for $j = 0, \ldots, k$, with $1 = m_0 | m_1 | m_2 \ldots | m_k | m_{k+1} := m$ (here m is the degree of the morphism $\Phi_0 : W_0 \to X$), and such that W_0 is a complete intersection in $\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_k|_{Z_0}$, with Φ_0 a normally induced iterated smooth univariate covering.

(A3) $\mathcal{W}$ is obtained from $\Sigma := \Phi(\mathcal{W})$ by a finite sequence of blow-ups.

Moreover the local equations of $\mathcal{W}$ are of the following standard form

$$
\begin{cases}
\sigma(z) = w_0 t \\
Q_1(w_0, z) = w_1 t \\
\cdots \quad \cdots \\
Q_k(w_0, \ldots, w_{k-1}, z) = w_k t \\
Q_{k+1}(w_0, \ldots, w_k, z, t) = 0.
\end{cases}
\tag{II-1}
$$

(B1) Conversely, take any smooth iterated univariate covering of normal type

$$
\phi_0 : W_0 \to X \subset Z_0
$$

and take any 1-parameter family $\mathcal{Z}$ of deformations of Z_0.

Then the line bundle $\mathcal{O}_{Z_0}(X)$ extends to a line bundle $\mathcal{L}_0$ on the whole family $\mathcal{Z}$. And W_0 deforms to a hypersurface embedding if, for all $i \geq 2$, every section in $H^0(Z_0, \mathcal{O}_{Z_0}(iX))$ and every section in $H^0(X, \mathcal{O}_X(iX))$ comes from a section in $H^0(\mathcal{Z}, \mathcal{L}_0^{\otimes i})$.

(B2) This holds in particular, when the family $\mathcal{Z}$ is trivial, $\mathcal{Z} \cong Z_0 \times T$, if the necessary condition of being normally induced is fulfilled.

Remark 2.5. (b1) More precisely, in (B1) above, there is a family $\mathcal{W}$ such that $\mathcal{W}$ is a complete intersection in $\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_k$ ($\mathcal{L}_i = \mathcal{L}_0^{\otimes m_i}$), and $\mathcal{W}$ is given as above; moreover, for $t \neq 0$ in T, the morphism Φ_t, induced on W_t by the bundle projection on Z_t, is an embedding.

(b2) Sufficient condition in (B2) is the surjectivity of $H^0(Z_0, \mathcal{O}_{Z_0}(iX)) \to H^0(X, \mathcal{O}_X(iX))$ for $i \geq 2$; this is implied by $H^1(Z_0, \mathcal{O}_{Z_0}(iX)) = 0, \forall i \geq 1$.

Remark 2.6. The line bundle $\mathcal{O}_{Z_0}(X)$ extends to a line bundle $\mathcal{L}_0$ on the whole family $\mathcal{Z}$, because of the Lefschetz (1,1) theorem, since $\mathcal{O}_{Z_0}(mX)$ does.

Observe moreover that there is a (non-canonical) isomorphism

$$
\mathrm{Pic}^0(Z_0) \cong \mathrm{Pic}^0(Z_t),
$$

whereas in general there is no isomorphism of $\mathrm{Pic}(Z_0)$ with $\mathrm{Pic}(Z_t)$.

Remark 2.7. Consider the family of submanifolds of weighted projective spaces given in Example 1.1, with equations (I-2), and consider the 1-parameter deformation where we set $t_j = t^{n_j}$.

Then the equations considered in the proof of the theorem are many more than equations (I-2), since for instance from the equation $\sigma(z) = t^{n_0} w_0$ we recover w_0 not directly but only after an iterated procedure: we inductively set $\sigma(z) = t^a v_{a-1}$, so finally we get $w_0 = v_{n_0-1}$.

3. Auxiliary results and proof of the main theorem

The following observation plays an important role in the proof.

Lemma 3.1. *Suppose we have a one dimensional smooth family $p : \mathcal{W} \to T$ of smooth projective varieties of dimension n mapping to another flat family $q : \mathcal{Y} \to T$ of projective varieties of the same dimension via a relative map $\Psi : \mathcal{W} \to \mathcal{Y}$ over a smooth holomorphic curve T such that $q \circ \Psi = p$, i.e. we have the commutative diagram*

$$\mathcal{W} \xrightarrow{\quad \Psi \quad} \mathcal{Y}$$
$$p \searrow \quad \swarrow q$$
$$T.$$
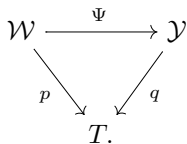
Assume that

(1) $\mathcal{Y}$ is normal and Gorenstein,
(2) Ψ is birational,
(3) for $t \neq 0$ in T, Ψ induces an isomorphism,
(4) K_{W_0} is ample.

Then we have that Ψ is an isomorphism, in particular $W_0 \cong Y_0$.

Proof. We have $K_{\mathcal{W}} = \Psi^*(K_{\mathcal{Y}}) + \mathcal{B}$. Since we assume that Ψ induces an isomorphism for $t \neq 0$ in T, the support of the Cartier divisor $\mathcal{B}$ is contained in W_0, which is irreducible.

Now Y_0 has dimension n and the morphism $\Psi_0 : W_0 \to Y_0$ is generically finite, hence we conclude that $\mathcal{B} = 0$. In particular, $K_{\mathcal{W}} = \Psi^*(K_{\mathcal{Y}})$; restricting to the special fiber, we obtain $K_{W_0} = \Psi_0^*(K_{Y_0})$. Since by assumption K_{W_0} is ample, Ψ_0 is finite, hence also Ψ is finite, hence an isomorphism in view of the normality of $\mathcal{Y}$. $\square$

Remark 3.2. Without the assumption that K_{W_0} is ample one can only assert that Y_0 is normal with at most canonical singularities.

Lemma 3.3. *In the hypotheses of Theorem 2.4, we have that $\sigma^{m-i} \,|\, \sigma_i$ for $i = 1, \ldots, m-1$.*

Proof. Since the map $W_0 \to X$ is a generically finite map of degree m, given a general point p of X, the inverse image of p consists of m points $q_1, \ldots, q_m$, and at each q_i the rank of the derivative of the morphism $W_0 \to Z_0$ is equal to n. Hence we get local coordinates $(w_1, \ldots, w_n, t)$ for $\mathcal{W}$ at q_i and local coordinates $(z_1, \ldots, z_n, z_{n+1}, t)$ for $\mathcal{Z}$ at p (not depending on i) such that Φ is given by a function f_i

$$f_i(w_1, \ldots, w_n, t) = (w_1, \ldots, w_n, \varphi_i(w_1, \ldots, w_n, t), t)$$

such that

$$f_i(w_1, \ldots, w_n, 0) = (w_1, \ldots, w_n, 0, 0).$$

Here $\varphi_i(w_1, \ldots, w_n, t)$ is a holomorphic function.

Hence at $p \in X$, there are variables $(z_1, \ldots, z_n, z_{n+1}, t)$ such that $\sigma = z_{n+1}$, and Σ consists of m smooth branches with equation

$$\sigma - \varphi_i(z_1, \ldots, z_n, t) = 0, \sigma = z_{n+1}$$

for $i = 1, \ldots, m$.

Hence, setting $z := (z_1, \ldots, z_n)$ the local equation of Σ is

$$\prod_{i=1}^{m}(\sigma - \varphi_i(z, t)) = \sigma^m - a_1(z, t)\sigma^{m-1} + \ldots + a_m(z, t) \tag{III-1}$$

and $t^i | a_i$, since $t | \varphi_i$. $\square$

Remark 3.4. Since we assume that Φ_t is an embedding when $t \neq 0$, for a fixed value of $z := (z_1, \ldots, z_n)$ equation (III-1) yields a plane curve with m smooth branches. Equivalently, we may view equation (III-1) as giving a plane curve over a non algebraically closed field (the fraction field of the ring of power series in z).

Lemma 3.5. *Assume the same hypotheses of part (A) of Theorem 2.4 with $n = 1$. Let $p \in W_0$ be any point. Then there is a neighborhood U of p in W_0 such that $\phi_0(U)$ is a smooth curve of Z_0 at $\phi_0(p)$.*

Proof. We have a factorization $\phi_0 = \nu \circ \psi_0$ near the point p

$$W_0 \xrightarrow{\psi_0} X^{\mathrm{nor}} \xrightarrow{\nu} X$$

where $\nu : X^{\mathrm{nor}} \to X$ is the normalization.

There are respective local coordinates w around $p \in W_0$, u around $\psi_0(p)$, x, y around $\phi_0(p)$ such that:

– $\psi_0(w) = w^h = u$.
– $x(u), y(u)$ are a Puiseux parametrization of the branch associated to $\phi_0(p)$. If the branch is nonsingular, without loss of generality $y \equiv 0$ and there is nothing to prove.
– Otherwise the branch is singular, and we have a Puiseux parametrization of the form

$$\begin{cases} x = u^d \\ y = u^e + g(u)u^{e+1}, \text{ where } e > d. \end{cases}$$

Hence the branch is the zero set of a pseudo-polynomial in x,

$$x^d + a_1(y)x^{d-1} + \cdots + a_d(y).$$

Now, we can write, locally identifying $\mathcal{Z}$ to $\mathbb{C}^2 \times T$, $\Phi(w, t)$ as

$$\Phi(w, t) = (\phi(w, t), t)$$

and $\phi(w, t)$ as

$$\begin{cases} x = w^{dh} + t\phi_1(w,t) \\ y = w^{eh}(1 + g(w^h)w^h) + t\phi_2(w,t) \end{cases}$$

The link of the branch $\phi_0(p)$ is, by Zariski's theorem [9], an iterated non-trivial toral knot, running h times. However, L_0 is isotopic to L_t, which is gotten by the image of the circle $|w| = \epsilon$ under $\phi(w,t)$.

Observe that the map $\phi(w,t)$, by purity of branch locus, ramifies on a curve R, which is not contained in $\{t = 0\}$: since W_t is embedded for $t \neq 0$, it follows that R is exceptional and that the curves W_t are a family of curves through the origin $x = y = 0 \in \mathbb{C}^2$. Moreover, again since W_t embeds, the reduced curve R_{red} is a smooth curve which projects isomorphically to the t-axis, so we may assume without loss of generality that $R_{red} = \{w = 0\}$.

The link L_t is contained in the submanifold W_t of a four dimensional ball B around the origin; W_t is a smooth holomorphic curve through the point $\phi(0,t) = 0$, hence L_t yields an unknotted circle $S^1 \subset S^3$. We have derived a contradiction from assuming $d > 1$, and that the branch is singular. $\quad \square$

Lemma 3.6. *Assume the same hypotheses of Theorem 2.4. Let*

$$W_0^f = \{p \in W_0 \mid \phi_0 \text{ is finite at } p\}.$$

Then if $p \neq p' \in W_0^f$ and $\phi_0(p) = \phi_0(p') = y_0$, then the germs of analytic subsets $Y_0 := \phi_0(U_p)$ and $Y_0' := \phi_0(U_{p'})$ coincide.

Proof. Cut now $\mathbb{C}^{n+1}$ with a general linear $\mathbb{C}^2$ passing through y_0, so that we get curves $\mathbb{C}^2 \cap Y_0 = C$, $\mathbb{C}^2 \cap Y_0' = C'$, passing through y_0. Now $(C \cdot C')_{y_0} = d \geq 1$, and indeed d is a topological invariant, it is the linking number of $C \cap S^3$ and $C' \cap S^3$.

For $|t| << 1$, we can deform the curves C, C' which are in the image of W_0, to curves $C(t), C'(t)$ in the image of W_t: these have the property that in a neighborhood of $y_0 \in \mathbb{C}^2$ they intersect in d points, counted with multiplicities. Hence, if $y(t) \in C(t) \cap C'(t)$, then there are $p(t) \neq p'(t) \in W_t$ with $\phi_0(p(t)) = \phi_0(p'(t))$. This is a contradiction. $\quad \square$

Proposition 3.7. *Assume the same hypotheses of Theorem 2.4. Then X is normal.*

Proof. X is a hypersurface in a smooth manifold, hence it is normal if and only if it is smooth outside of codimension 2 in X.

The image of the points in $W_0 \setminus W_0^f$ is a Zariski closed subset of codimension ≥ 2 in X.

Hence it suffices to consider points $q \in X$ which are image points only of points $p \in W_0^f$.

By Lemma 3.6 the germ of X at q equals the image of the germ of W_0 at p.

Let $\mathcal{C}_k = \{p \in W_0 \mid \mathrm{rank}(D\phi_0)_p = k\}$. Then $\phi_0(\mathcal{C}_k)$ has dimension $\leq n-2$ for $k \leq n-2$, and we conclude that X is smooth outside of codimension 2 unless $\dim \phi_0(\mathcal{C}_{n-1}) = n-1$.

It also suffices to consider the general point p of $\mathcal{C} := \mathcal{C}_{n-1}$, where $\mathcal{C}$ is smooth of dimension $n-1$ and $\mathrm{rank}(D(\phi_0|_{\mathcal{C}})) = n-1$.

Let $(\hat{d} - 1)$ be the multiplicity of $\mathcal{C}$ in the locus given by the $n \times n$ minors of the derivative matrix.

There are local coordinates (v,w) in a neighborhood of $p \in W_0$, with $v = (v_1, \ldots, v_{n-1})$, such that $\mathcal{C} = \{w = 0\}$ and such that

$$\phi_0(v, w) = (v, x(v, w), y(v, w)).$$

The locus given by the $n \times n$ minors of the derivative matrix is then just the locus $\frac{\partial x}{\partial w} = 0, \frac{\partial y}{\partial w} = 0$.

Without loss of generality, at the general point of $\mathcal{C}$ we can assume that

$$x(v, w) = \sum_{i=\hat{d}}^{\infty} a_i(v) w^i,$$

and, since $a_{\hat{d}}(x) \not\equiv 0$, we may assume that $a_{\hat{d}}(0) \neq 0$. Hence we choose coordinates x, y with

$$\begin{cases} x = w^{\hat{d}}, \\ y = w^e + \cdots \text{ with } e > \hat{d} \text{ and } \hat{d} \nmid e. \end{cases}$$

Then, if $y \not\equiv 0$, we get, for any v in a neighborhood of 0, a singular curve branch. The same argument as for the case $n = 1$ applied to $x(0, w), y(0, w)$ gives a contradiction. So we have established the proof. $\quad\square$

Lemma 3.8. *Let C be a germ of a plane curve singularity consisting of m smooth branches with non vertical tangents, i.e. the local equation of C is $F(y, t) = 0$, with*

$$\begin{aligned} F(y, t) &:= \\ &= \prod_{i=1}^{m}(y - \varphi_i(t)) \\ &= y^m - \sigma_1(\varphi_1, \ldots, \varphi_m)y^{m-1} + \sigma_2(\varphi_1, \ldots, \varphi_m)y^{m-2} + \cdots + (-1)^m \varphi_1 \cdots \varphi_m. \end{aligned}$$

Here, σ_i is the i-th elementary symmetric function, and $t | \varphi_i(t)$, since $(0, 0)$ is the singular point, hence t^i divides $\sigma_i(\varphi_1, \ldots, \varphi_m)$.

Then the singularity can be resolved by iterated blow-ups of the form:

$$\begin{cases} y = w_0 t \\ D_1(w_0) = w_1 t \\ D_2(w_1) = w_2 t \\ \cdots \quad \cdots \\ D_k(w_{k-1}) = w_k t \end{cases}$$

where the D_j's are monic polynomials.

Proof. $F(y, t)$ is a pseudo-polynomial. Write

$$F(y, t) = y^m + t\, b_1(t)y^{m-1} + \cdots + t^m b_m(t).$$

The first blow-up yields

$$P(w_0, t) = w_0^m + b_1(t)w_0^{m-1} + \cdots + b_m(t).$$

Let $P(w_0, t) = P_0(w_0) + tP_1(w_0) + t^2 P_2(w_0) + \cdots$ where $P_0(w_0)$ is a monic polynomial of degree m and $\deg_{w_0} P_j(w_0) \leq m - 1$ for $j \geq 1$. Looking at the two partial derivatives for $t = 0$, the proper transform C_0 is smooth if and only if

$$P_0(w_0) = 0, \ \frac{\partial}{\partial w_0} P_0(w_0) = 0, \ P_1(w_0) = 0$$

have no common roots.

If not, let $D_1(w_0) = $ gcd of the above three polynomials, so that $P_0(w_0) = D_1^r \cdot G$ for some $2 \leq r \leq m$ such that $r \cdot deg(D_1) \leq m$.

To continue, observe that D_1 is again monic, so there is no tangent $t = 0$, and it suffices to blow-up $t = D_1 = 0$ setting $D_1(w_0) = w_1 t$.

We get equations

$$D_1(w_0)^r \cdot G(w_0) + tD_1(w_0)\tilde{P}(w_0) + t^2 \cdots$$
$$= t^r [w_1^r \cdot G(w_0) + w_1^{r-1} \cdots],$$

where the divisibility of the second term by w_1^{r-1}, and similarly for the next terms follows by applying once more Lemma 3.3.

We continue this process until all branches are separated. $\square$

Proof of Theorem 2.4. Recall the Taylor series development in t of the equation in $\mathcal{Z}$ of the image $\Sigma :=$ $\Phi(\mathcal{W})$:

$$\Sigma(z, t) := \sigma(z)^m + t\sigma_1(z) + t^2\sigma_2(z) + \cdots + t^{m-1}\sigma_{m-1}(z) + \cdots.$$

Choose a general point p in X. In order to show that the process terminates we shall at a later moment consider a germ of plane curve C passing through p, consisting of m smooth branches with non vertical tangents, and obtained as a linear section of Σ (i.e., C is obtained by setting $(z_1, \ldots, z_n) = $ constant in appropriate local coordinates).

Observe that, by Lemma 3.3, $\sigma^i | \sigma_{m-i}$ for $i = 1, \ldots, m-1$. So we can rewrite the equation of Σ as follows:

$$\Sigma(z, t) := \sigma(z)^m + a_1(z, t)\sigma(z)^{m-1} + a_2(z, t)\sigma(z)^{m-2} + \cdots + a_m(z, t) = 0 \qquad \text{(III-2)}$$

where $t^i | a_i(z, t)$.

Please observe that the above equation is not just a local equation, but that it is a global equation for a section of a line bundle on $\mathcal{Z}$ (see Remark 2.6)

$$\mathcal{O}_{\mathcal{Z}}(\Sigma) = \mathcal{L}_0^{\otimes m}.$$

Now, by setting $\sigma = tw_0$ and $a_i(z, t) = t^i b_i(z, t)$ in equation (III-2), we obtain the equation

$$P_{w_0}(w_0, z, t) := w_0^m + b_1(z, t)w_0^{m-1} + \cdots + b_m(z, t) = 0.$$

This is a hypersurface and its singular locus is contained in $t = 0$ by our assumption. Write

$$b_j(z, t) = b_{j,0}(z) + tb_{j,1}(z) + \cdots,$$

so that the $b_{j,0}(z)$'s are sections on Z_0 of a line bundle of the form $\mathcal{O}_{Z_0}(iX)$ (this observation shall be repeated in the sequel, leading to the proof that we get a normally induced covering of X).

Hence we can write

$$P(w_0, z) := w_0^m + b_{1,0}(z)w_0^{m-1} + \cdots + b_{m,0}(z).$$

Hence the equation P_{w_0} has the development

$$P_{w_0}(w_0, z, t) = P(w_0, z) + t\sum_{j=1}^{m} b_{j,1}w_0^{m-j} + t^2 \cdots.$$

Set $\hat{P}(w_0, z) := \sum_{j=1}^{m} b_{j,1} w_0^{m-j}$; then the gradient of P_{w_0} for $t = 0$ equals

$$(\frac{\partial P}{\partial w_0}, \hat{P}, \frac{\partial P}{\partial z_1}, \frac{\partial P}{\partial z_2}, \dots).$$

The singular locus is contained in $t = 0$ and is there given by

$$\begin{cases} P(w_0, z) = 0 \\ \frac{\partial}{\partial w_0} P(w_0, z) = 0 \\ \hat{P}(w_0, z) = 0 \\ \frac{\partial}{\partial z_i} P(w_0, z) = 0 \ (\text{for } i = 1, \dots, n). \end{cases}$$

The hypersurface defined by $P_{w_0}(w_0, z, t) = 0$ is normal unless it is singular in codimension 1. Assume that the hypersurface defined by $P_{w_0}(w_0, z, t) = 0$ is not normal, so that it is singular in codimension 1. Observe that the square free part of $P(w_0, z)$ is irreducible because its zero set is the image of an irreducible variety W_0, hence for $t = 0$ the vanishing of $P(w_0, z)$ should imply the vanishing of the other polynomials.

Since $P(w_0, z) = 0 \Rightarrow \partial P(w_0, z)/\partial w_0 = 0$, hence $P(w_0, z) = Q(w_0, z)^r$, for some $2 \le r \le m$.

And $Q(w_0, z)$ is again irreducible and $Q(w_0, z) | \hat{P}(w_0, z)$. Now, since $r \ge 2$,

$$\partial P(w_0, z)/\partial z_i = rQ(w_0, z)^{r-1} \partial Q(w_0, z)/\partial z_i,$$

so the last conditions are automatically fulfilled.

If we write the equation $P_{w_0}(w_0, z, t)$ in terms of $Q(w_0, z)$, then

$$Q(w_0, z)^r + tQ(w_0, z)\tilde{P}(w_0, z, t) + t^2 \cdots .$$

Then again by Lemma 3.3, $Q(w_0, z)^{r-2} | \tilde{P}(w, z, t)$, and so on.

Observe that $Q(w_0, z) = 0$ gives a covering W_0' of X of degree m/r.

In view of Lemma 3.8, we set $Q(w_0, z) = w_1 t$. Then we get an iterated covering as in Lemma 3.8, where $P = D_1^r = Q^r$. We consider now a germ of plane curve C through the general point $p \in X$ consisting of m smooth branches with non vertical tangents by taking linear sections. Since the resolution of C is obtained by a finite sequence of blow-ups, we get an iterated covering as in Lemma 3.8, which is normal and a complete intersection.

Lemma 3.1 implies that we have then obtained $\mathcal{W}$.

Since W_0 is smooth, let us set $t = 0$ in the above equations (II-1), and observe that the matrix of derivatives has triangular form. So, smoothness of W_0 implies the smoothness of X and of all the intermediate coverings.

Therefore we have shown (A1), (A2) and (A3).

Let us show now the converse, (B1) and (B2).

Assume we are given an iterated smooth univariate covering W_0 of X which is normally induced, defined by equations:

$$\begin{cases} \sigma(z) = 0 \\ Q_1(w_0, z) = 0 \\ \cdots \quad \cdots \\ Q_k(w_0, \dots, w_{k-1}, z) = 0 \\ Q_{k+1}(w_0, \dots, w_k, z) = 0. \end{cases} \tag{III-3}$$

Claim I): if $H^0(Z_0, \mathcal{O}_{Z_0}(iX))$ surjects onto $H^0(X, \mathcal{O}_X(iX))$ for each $i \geq 2$, then every iterated univariate covering W_0 of X is normally induced, i.e., it extends to an iterated univariate covering of Z_0.

Proof. First of all we can put the equations of the iterated covering W_0 of X in Tschirnhausen form. Here the polynomial equation of a univariate covering is said to be in Tschirnhausen form if $a_1(x) \equiv 0$, and every covering can be put in Tschirnhausen form after an automorphism replacing w with $w - \frac{1}{m}a_1(x)$.

Second, the coefficients $a_{j,I}$ of the polynomials Q_j are now given by sections of line bundles of the form $\mathcal{O}_X(n(I,j)X)$, where $n(I,j) \geq 2$. By assumption, they extend to sections of $\mathcal{O}_{Z_0}(n(I,j)X)$.

That this holds if $H^1(Z_0, \mathcal{O}_{Z_0}(iX)) = 0, \forall i \geq 1$, follows immediately from the long exact cohomology sequence associated to the exact sequence

$$0 \to \mathcal{O}_{Z_0}(iX) \to \mathcal{O}_{Z_0}((i+1)X) \to \mathcal{O}_X((i+1)X) \to 0,$$

for $i \geq 1$. $\square$

Let's pass to the proof of (B1).

By our assumption, $\sigma(z)$ extends to a section $\sigma(z,t)$ of the line bundle $\mathcal{L}_0$ on the whole family $\mathcal{Z}$. Similarly the sections $a_{j,I}$ extend to sections $a_{j,I}(z,t)$ over $\mathcal{Z}$, hence we can also extend the polynomials $Q_j(w_0, \ldots, w_{j-1}, z)$ to polynomials $Q_j(w_0, \ldots, w_{j-1}, z, t)$.

Then we define the iterated univariate covering $\mathcal{W}$ of $\mathcal{Z}$ via the following equations:

$$\begin{cases} \sigma(z,t) = w_0 t \\ Q_1(w_0, z, t) = w_1 t \\ \cdots \quad \cdots \\ Q_k(w_0, \ldots, w_{k-1}, z, t) = w_k t \\ Q_{k+1}(w_0, \ldots, w_k, z, t) = 0. \end{cases} \tag{III-4}$$

To finish the proof of (B2), observe that if the family $\mathcal{Z}$ is trivial, $\mathcal{Z} = Z_0 \times T$, then obviously

$$H^0(Z_0, \mathcal{O}_{Z_0}(iX)) \subset H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(iX)),$$

hence there is no problem to extend the iterated univariate covering to one of $\mathcal{Z}$. $\square$

4. Applications

The first applications that we shall give are, more or less, direct corollaries of the previous general results.

4.1. Hypersurfaces in projective space

Corollary 4.1. *The smooth manifolds W_0 with ample canonical divisor which admit a 1-parameter deformation to a hypersurface embedding*

$$p : \mathcal{W} \to T, \quad \Phi : \mathcal{W} \to \mathbb{P}^{n+1} \times T$$

(here, for $t \neq 0$, W_t is a smooth hypersurface in $\mathbb{P}^{n+1}$) are exactly the iterated weighted deformations of Example 1.1 with $md > n+2$, and the family $\mathcal{W}$ is a pull back from the family in (I-2).

The class of such manifolds W_0 is open in the Kuranishi space for $n \geq 2$.

Proof. Either W_0 is a hypersurface, and there is nothing to prove (set $m = 1$), or the degree of $\phi_0 : W_0 \to X$ is $m \geq 2$.

Then Theorem 2.4 applies, and it is easy to see that we get a manifold in the family (I-2). The converse is direct (set $t = t_j \ \forall j$ in the family (I-2)).

The proof of the second statement follows imitating quite closely Sernesi's argument in [7] for the case of complete intersections in weighted projective spaces.[1] $\quad \square$

4.2. Hypersurfaces in an Abelian variety

Corollary 4.2. *The smooth manifolds with ample canonical divisor W_0 which admit a 1-parameter deformation $p : \mathcal{W} \to T$ where, for $t \neq 0$, W_t is a smooth hypersurface in an Abelian variety A_t, are, for $n \geq 2$, exactly the iterated smooth univariate coverings $W_0 \to X$ of normal type, where X is an ample divisor in an Abelian variety A_0.*

Proof. A_t is, for $t \neq 0$, the Albanese variety of W_t, since $H^i(\mathcal{O}_{A_t}(-W_t)) = 0$ for $i = 1, 2$, hence the exact cohomology sequence associated to the exact sequence

$$0 \to \mathcal{O}_{A_t}(-W_t) \to \mathcal{O}_{A_t} \to \mathcal{O}_{W_t} \to 0$$

yields $h^1(\mathcal{O}_{A_t}) = h^1(\mathcal{O}_{W_t})$.

We obtain therefore a morphism

$$\Phi : \mathcal{W} \to \mathcal{A}, \ A_t = Alb(W_t)$$

that induces a 1-parameter deformation to hypersurface embedding in Abelian varieties by Lemma 149 of [3].

Our main theorem applies, in particular (B2) holds since we have $H^1(\mathcal{O}_{A_0}(iX)) = 0$ for all $i \geq 1$. $\quad \square$

4.3. Multiple Inoue-type varieties

We can now extend the definition of Inoue type variety (see [1] and [3]) in the following way:

Definition 4.3. A primary multiple Inoue-type variety W_0 of smoothing type is a normally induced smooth iterated univariate covering $W_0 \to X$, where X is a smooth ample subvariety of a projective classifying space Z_0.

A multiple Inoue-type variety Y_0 of smoothing type is a quotient W_0/G, for the free action of a finite group, of a primary multiple Inoue-type variety W_0 of smoothing type.

We do not give further applications here, hopefully in a future paper.

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