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The Bicanonical Map of Fake Projective Planes with an Automorphism

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We show, for several fake projective planes with a nontrivial group of automorphisms, that the bicanonical map is an embedding.

1 Introduction

Not always algebraic varieties are described via polynomial equations: sometimes they are constructed via **uniformization**: this means, as quotients of certain domains of a complex vector space, called bounded symmetric domains, via the action of discontinuous groups. Then general theorems (as [31]) imply the algebraicity of these quotient complex manifolds.

The problem concerning the algebro-geometrical properties of such varieties constructed via uniformization and especially the description of their projective embeddings (and the corresponding polynomial equations) lies at the crossroads of several allied fields: the theory of arithmetic groups and division algebras, complex algebraic and differential geometry, linear systems, use of group symmetries, and topological and homological tools in the study of quotient spaces.

Of particular importance are the so-called ball quotients, especially in dimension 2, since they yield the surfaces with the maximal possible canonical volume K^2 for a fixed value of the geometric genus p_g. For instance, the problem we deal with here

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is first of all concerned with the extremal case in the important theory of bicanonical maps of surfaces of general type (as established by Reider in [36]).

Second, Fake Projective Planes (here abbreviated as FPPs), originally discovered by Mumford in [34], are currently a subject of active research since their derived categories are quite different from the ones of standard projective spaces and lead to intriguing mysteries (see [5, 6, 13, 20, 27, 28]). For this purpose it is interesting to describe their geometry and their embeddings; in particular establishing, as we do here, that their bicanonical map is an embedding is an important tool towards writing their equations (as done for a pair of such FPPs by Borisov and the 2nd author in [7]).

Third, we show here how to effectively use topological arguments and symmetry for the study of linear systems on quotient varieties. Even if many arguments, taken by themselves, are easy to understand for the respective specialist on the topics used in their proof, we believe that it may be useful, for those who want to address similar problems, to see in play several different techniques used here to obtain several auxiliary results. Hence, we hope that our article may turn out to be useful for a wider readership.

After this general foreword, let us now pass to a more precise mathematical description of our results and their background.

A smooth compact complex surface with the same Betti numbers as the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ is either $\mathbb{P}^2_{\mathbb{C}}$ or is called a **fake projective plane**. Indeed, such a surface has $c_2 = 3$, $c_1^2 = K^2 = 9$, Picard number, and 2nd Betti number $= 1$; thus, its canonical class is either ample or anti-ample, and in the latter case it is isomorphic to $\mathbb{P}^2_{\mathbb{C}}$. In other words a FPP is a surface of general type with $p_g = 0$ and $c_1^2 = 3c_2 = 9$. Furthermore, its universal cover is the unit ball in $\mathbb{C}^2$ by [3] and [37], and its fundamental group is a co-compact arithmetic subgroup of $PU(2, 1)$ by [30].

Prasad and Yeung [35] enumerated commensurability classes of lattices that might contain the fundamental groups of FPPs. Their proof also shows that the automorphism group of an FPP has order 1, 3, 9, 7, or 21. Then Cartwright and Steger ([9, 10]) carried out a computer-based group theoretic enumeration to obtain a more precise result: there are exactly 50 distinct fundamental groups, each corresponding to a pair of FPPs, complex conjugate but not isomorphic to each other (by the result of [29]). They also computed the automorphism groups of all FPPs X. Four groups occur:

$$\mathrm{Aut}(X) \cong \{1\},\ C_3,\ C_3^2 \text{ or } G_{21} \cong C_7 : C_3,$$

where C_n is the cyclic group of order n and G_{21} is the unique non-abelian group of order 21 (semidirect product of C_7 with C_3, G_{21} is the group of affine transformations of

$\mathbb{Z}/7$ of the form $x \mapsto 2^i x + a, a \in \mathbb{Z}/7$). Among the 50 pairs, 33 admit a nontrivial group of automorphisms: three pairs have Aut $\cong G_{21}$, three pairs have Aut $\cong C_3^2$, and 27 pairs have Aut $\cong C_3$.

For each pair of FPPs the 1st homology group (abelianization of the fundamental group)

$$H_1(X, \mathbb{Z}) = Tor\left(H^2(X, \mathbb{Z})\right) = Tor(\mathrm{Pic}(X))$$

was also computed in [10].

By Reider's theorem [36] (see the next section) the bicanonical system of a ball quotient X is base point free; thus, it defines a morphism. If the ball quotient X has $\chi(X) \geq 2$, then $K_X^2 = 9\chi(X) \geq 10$, and since a ball quotient cannot contain a curve of geometric genus 0 or 1, the bicanonical map embeds X unless X contains a smooth genus 2 curve C with $C^2 = 0$, and $CK_X = 2$.

In the case $\chi(X) = 1$, for instance if we have an FPP, we are below the Reider condition $K_X^2 \geq 10$, and the question of the very ampleness of the bicanonical system is interesting.

An FPP X with automorphism group of order 21 cannot contain an effective curve with self-intersection 1, as was first proved in [27] (published in [28], see also [20]). Thus, by applying I. Reider's theorem, one sees that the bicanonical map of such an FPP is an embedding into $\mathbb{P}^9$ (see for instance [13]).

In addition to these three pairs of FPPs, for seven more pairs we confirm here the very ampleness of the bicanonical system.

Theorem 1.1. For the seven pairs of FPPs given in Table 1 the bicanonical map is an embedding into $\mathbb{P}^9$.

Table 1 Seven pairs of FPPs

X	Aut(X)	$H_1(X, \mathbb{Z})$	$\pi_1(X/C_3)$
$(a = 15, p = 2, \{3, 5\}, D_3)$	C_3	$C_3 \times C_7$	C_3
$(a = 15, p = 2, \{3, 5\}, 3_3)$	C_3	$C_2^2 \times C_3$	C_3
$(a = 15, p = 2, \{3, 5\}, (D3)_3)$	C_3	C_3	C_3
$(\mathcal{C}_2, p = 2, \{3\}, d_3D_3)$	C_3^2	C_7	$C_7, \{1\}, \{1\}, \{1\}$
$(\mathcal{C}_{10}, p = 2, \{17-\}, D_3)$	C_3	C_7	$\{1\}$
$(\mathcal{C}_{18}, p = 3, \emptyset, d_3D_3)$	C_3^2	$C_2^2 \times C_{13}$	$C_{13}, Q_8, \{1\}, \{1\}$
$(\mathcal{C}_2, p = 2, \emptyset, d_3D_3)$	C_3^2	$C_2 \times C_7$	C_{14}, S_3, C_2, C_2

Remark 1.2. (1) When $\mathrm{Aut}(X) \cong C_3^2$, there are four quotients corresponding to the four order 3 subgroups of $\mathrm{Aut}(X)$. Here Q_8 is the quaternion group and S_3 is the non-abelian group of order 6.

(2) By [10], the fundamental groups of these surfaces lift to $\mathrm{SU}(2,1)$; thus, the tautological line bundle of $\mathbb{P}^2$ restricted to the ball descends to give a cube root of the canonical bundle of X (see [32]). Note that the 1st three pairs in Table 1 have 3-torsion thus have several cube roots of K_X.

The 1st six pairs in Table 1 are dealt with via the following vanishing result.

Theorem 1.3. Let X be an FPP with a nontrivial C_3-action. Suppose that the quotient surface X/C_3 has $H_1(X/C_3, \mathbb{Z}) = 0$ or C_3. Then

$$H^0(X, L) = 0$$

for any ample line bundle L with $L^2 = 1$, or equivalently, X contains no effective curve D with $D^2 = 1$.

For the last pair in Table 1 we do not have a vanishing theorem. The surfaces possess either three curves D with $D^2 = 1$ or none. But even in the former case we prove the very ampleness of the bicanonical system (see Theorem 4.4).

In Section 5 we discuss three more pairs with a nontrivial C_3-action, for which we prove that the bicanonical map is an embedding outside three points, the fixed locus of the C_3-action.

2 Preliminaries

For the reader' s convenience, we recall the basic notation concerning linear systems and Reider's theorem [36] (by stating the expanded version given in Theorem 11.4 of [4]).

A linear system $|D|$ on a variety X is **very ample** if it yields an embedding of X: this means that it yields an injective map, which is a local embedding at each point. It is a well-known and very useful fact that this property amounts to the surjectivity of $H^0(\mathcal{O}_X(D)) \to H^0(\mathcal{O}_\zeta(D))$ for each length 2 subscheme ζ of X; such a subscheme consists either of two distinct reduced points P and Q or of a tangent vector at a point P. In the latter case we shall say, using the classical terminology, that ζ consists of two points P and Q, where Q is infinitely near to P. This terminology allows a shorter wording and notation (see for instance [12]).

Theorem 2.1. [36] Let L be nef divisor on a smooth projective surface X.

(1) Assume that $L^2 \geq 5$. If P is a base point of the linear system $|K_X + L|$, then P lies on an effective divisor D such that

(a) $DL = 0, \quad D^2 = -1$ or

(b) $DL = 1, \quad D^2 = 0$.

(2) Assume that $L^2 \geq 9$. If two different points P and Q, possibly infinitely near, are not base points of $|K_X + L|$ and fail to be separated by $|K_X + L|$, then they lie on an effective curve D, depending on P and Q, satisfying one of the following:

(a) $DL = 0, \quad D^2 = -2$ or -1;

(b) $DL = 1, \quad D^2 = -1$ or 0;

(c) $DL = 2, \quad D^2 = 0$;

(d) $L^2 = 9$ and L is numerically equivalent to $3D$.

A ball quotient cannot contain a curve of geometric genus 0 or 1. By Reider's theorem the bicanonical system of a ball quotient is base point free thus defines a morphism. Let X be an FPP and let

$$\Phi_{2,X} : X \to \mathbb{P}^9$$

be the bicanonical morphism.

Lemma 2.2. Let X be an FPP.

(1) If D is an effective curve on X with $D^2 = 1$, then D is an irreducible curve of arithmetic genus 3, $h^0(X, \mathcal{O}_X(K_X - D)) = 0$, and $h^0(X, \mathcal{O}_X(D)) = 1$. In particular X may contain at most finitely many curves D with $D^2 = 1$. Their number is bounded by $|H_1(X, \mathbb{Z})|$.

(2) If two different points P and Q on X (possibly infinitely near) are not separated by $\Phi_{2,X}$, then there is a curve D with $D^2 = 1$ containing P and Q such that $h^0(D, \mathcal{O}_D((K_X - D)|_D) = 1$ and $P + Q$ is the unique effective divisor in the linear system of $(K_X - D)|_D$. In particular, a curve D with $D^2 = 1$ may contain at most one pair of points (possibly infinitely near) that are not separated by $\Phi_{2,X}$. Such a curve D is uniquely determined by P and Q.

(3) The bicanonical map $\Phi_{2,X}$ yields an isomorphism with its image of the complement U of a finite set of points. The bicanonical image Σ is a surface with isolated singularities only and $\Phi_{2,X} : X \to \Sigma$ is the normalization map.

Proof. (1) The class of D is a generator of $Pic(X)$ *modulo* torsion; hence, it is irreducible: its arithmetic genus $p(D) = 3$ by adjunction, since $2p(D) - 2 = D^2 + K_X D = 1 + 3$. $h^0(X, K_X - D) = 0$ because D is effective and $p_g(X) = h^0(X, K_X) = 0$. For the last assertion, if $h^0(X, D) \geq 2$, then $h^0(X, 4D) \geq 5$. On the other hand, since $4D - K$ is ample, we have by Kodaira vanishing $h^1(X, 4D) = h^2(X, 4D) = 0$; hence, by Riemann–Roch, $h^0(X, 4D) = 3$: a contradiction.

(2) By Reider's theorem (Theorem 2.1), if the bicanonical system $|2K_X|$ does not separate two points P and Q (possibly infinitely near) then there exists a divisor D containing both P and Q and such that $K \equiv 3D$ *modulo* torsion ($\equiv$ denotes linear equivalence, according to the classical notation).

One sees immediately in fact that, again since $Pic(X)$ has rank equal to 1, and its torsion free part is generated by a divisor L with $L^2 = 1$, the alternatives (a), (b), and (c) in (2) of Theorem 2.1 are not possible ($D^2 \leq 0$ contradicts that D is numerically equivalent to a nontrivial multiple of L). Write then $K \equiv 3D + \tau$ for a torsion divisor class τ, and observe that

$$2K \equiv K + D + (2D + \tau).$$

By [16] and [12] in view of the exact sequence

$$0 \to \mathcal{I}_{P,Q}\omega_D(2D + \tau) \to \omega_D(2D + \tau) \to \mathbb{C}^2 \to 0$$

it must also hold that

$$H^1(D, \mathcal{I}_{P,Q}\omega_D(2D + \tau)) \cong \mathbb{C}.$$

Hence, there is an isomorphism $\mathcal{I}_{P,Q}(2D + \tau) \cong \mathcal{O}_D$; thus,

$$\mathcal{I}_{P,Q} \cong \mathcal{O}_D(-2D - \tau),$$

$\mathcal{I}_{P,Q}$ is invertible, and $P + Q$ is the unique divisor of a section $\in H^0(\mathcal{O}_D(2D + \tau)) \cong \mathbb{C}$.

If P and Q are contained in two different curves D_1 and D_2, then $D_1 D_2 \geq 2$, which is not possible since the curves D_1 and D_2 are numerically equivalent and have self-intersection 1. This proves the uniqueness of such a curve D.

(3) follows from (1) and (2). ■

In [24], all possible structures of the quotient surface of an FPP and its minimal resolution were classified.

Theorem 2.3. [24] Let X be an FPP with a group G acting on it. Then the fixed locus of any automorphism of G different from the identity consists of three isolated points. Moreover, the following hold.

(1) If $G = C_3$, then X/G is a $\mathbb{Q}$-homology projective plane with three singular points of type $1/3(1,2)$ and its minimal resolution is a minimal surface of general type with $p_g = 0$ and $K^2 = 3$.
(2) If $G = C_3^2$, then X/G is a $\mathbb{Q}$-homology projective plane with four singular points of type $1/3(1,2)$ and its minimal resolution is a minimal surface of general type with $p_g = 0$ and $K^2 = 1$.
(3) If $G = C_7$, then X/G is a $\mathbb{Q}$-homology projective plane with three singular points of type $1/7(1,5)$ and its minimal resolution is a $(2,3)$-, $(2,4)$-, or $(3,3)$-elliptic surface.
(4) If $G = 7:3$, then X/G is a $\mathbb{Q}$-homology projective plane with 4 singular points, where three of them are of type $1/3(1,2)$ and one of them is of type $1/7(1,5)$, and its minimal resolution is a $(2,3)$-, $(2,4)$-, or $(3,3)$-elliptic surface.

Here a $\mathbb{Q}$-homology projective plane is a normal projective surface with the same Betti numbers as $\mathbb{P}^2_{\mathbb{C}}$ (cf. [18, 19]). A normal projective surface with quotient singularities only is a $\mathbb{Q}$-homology projective plane if its 2nd Betti number is 1 (if the 1st Betti number were positive, then, since the Picard scheme is compact for a normal surface, looking at the Albanese map one sees that the Picard number is at least 2). An FPP is a nonsingular $\mathbb{Q}$-homology projective plane; hence, by the invariance of the class of the canonical divisor for automorphisms, every quotient is again a $\mathbb{Q}$-homology projective plane.

Lemma 2.4. On an FPP X there is no totally geodesic curve, smooth or singular.

Proof. The proof has the following two steps. We are indebted to Bruno Klingler, Inkang Kim, and a referee for their precious suggestions.

I) In general, if an arithmetic ball quotient X contains a totally geodesic (possibly singular) curve Y, then Y is a ball quotient, and we claim that Y is arithmetic.
This follows from the definition: if $\Gamma < G$ is an arithmetic lattice (it means that there exists a number field k such that Γ is commensurable with $G(\mathcal{O}_k) < G(k)$ where $\mathcal{O}_k$ is the ring of integers of k), one may assume, up to finite index, that Γ is contained in $G(\mathcal{O}_k)$. If the arithmetic ball quotient X corresponding to $\Gamma < G$ contains a totally geodesic space Y corresponding to $\Gamma' < G' < G$, then, since it is totally geodesic, there is an injection from Γ' to Γ (any element in Γ' yields a loop that is represented by a closed geodesic in X, which is contained in Y since Y is totally geodesic: hence, it is trivial in Γ' if

it is trivial in Γ). Here G is the isometry group of the complex n-ball and G' is the isometry group of the complex m-ball with $m < n$. Hence, $\Gamma' < \Gamma < G(\mathcal{O}_k)$ and consequently $\Gamma' < G(\mathcal{O}_k) \cap G' < G'(\mathcal{O}_k)$. Since $G'(\mathcal{O}_k)$ gives a finite volume ball quotient, and so does Γ', it follows that Γ' is a finite index subgroup of $G'(\mathcal{O}_k)$. Hence, Γ' is an arithmetic lattice in G'.

II) We use here a result of Möller and Toledo [33]. We represent the 2-ball as G/K, where G is the set of $\mathbb{R}$-valued points in a connected semisimple $\mathbb{Q}$-algebraic group $G_{\mathbb{Q}}$ and K is a maximal compact subgroup (then Γ is an irreducible arithmetic lattice in G).

Then Y is a Shimura curve, corresponding to a $\mathbb{Q}$-algebraic group $H_{\mathbb{Q}}$ with a homomorphism into $G_{\mathbb{Q}}$.

Moreover, there is a totally complex quadratic extension l of a totally real field k and a central division algebra $\mathcal{D}$ of degree $d = 1$ or $d = 3$ over l with an involution of the 2nd kind, such that $G_k = SU(h)$, where h is an Hermitian form on $\mathcal{D}^{3/d}$.

The case $d = 1$ does not occur for an FPP, as shown by Cartwright and Steger (see the Addendum of [35]), while in the case $d = 3$ there are no Shimura curves: Möller and Toledo show that the cubic division algebra $\mathcal{D}$ contains no proper non commutative subalgebras ([33, p. 901]). ■

When the central simple algebra $\mathcal{D}$ splits over l (as in the case of the Cartwright–Steger surface), it is a matrix algebra and the ball quotient always contains a totally geodesic curve, possibly singular.

Lemma 2.5. Let X be an FPP. If a curve D on X has $D^2 = 1$, then it is a smooth curve of genus 3.

Proof. For a curve C on a ball quotient Z

$$3(2g(C') - 2) \geq 2K_Z C,$$

where C' is the normalization of C, with equality iff C is totally geodesic [38]. In our case, since $D^2 = 1$, we have $K_X D = 3$ and $p_a(D) = 3$. By Lemma 2.4 D is not totally geodesic, so the above inequality implies that $g(D') \geq 3$. ■

Remark 2.6. A referee informed us that another proof of Lemma 2.5 was obtained in [13] (see proof of Proposition 4.2), using the theory of the Toledo invariant [14].

Lemma 2.7. Let C be a smooth curve on a smooth complex surface X with $C^2 > 0$. Then the natural restriction map

$$Tor\ Pic(X) \to Pic(C)$$

is injective.

Proof. Let τ be a nontrivial torsion line bundle on X. Then it defines an unramified cover $X' \to X$ of finite degree, say $d > 1$. If $\tau|C$ is trivial, then C splits into a disjoint union of curves $C_1, \dots, C_d$ in X' with $C_i^2 = C^2 > 0$, contradicting the Hodge index theorem. ■

3 Proof of Theorem 1.3

In this section we prove Theorem 1.3.

First, we state a general result on the 1st homology group of a quotient space $Y = X/G$.

Recall here (cf. 6.7 of [11]) that, for a $\mathbb{Z}[G]$-module M, the group of coinvariants M_G is the quotient of M by the submodule generated by $Im(g-1)$, for $g \in G$. In particular, M_G is the quotient of M *modulo* the relations $g_i(x) \equiv x$, for a system of generators g_i of G.

The functor $M \mapsto M_G$ is the same as tensor product with the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$, that is, $M_G = M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$. Recall that tensor product is right exact, and that the left derived functors are the homology groups $H_i(G, M)$. In particular,

$$H_1(G, \mathbb{Z}) = G^{ab},\ H_i(G, \mathbb{Z}[G]) = 0,\ \forall i \geq 1, H_0(G, \mathbb{Z}[G]) = \mathbb{Z}.$$

Proposition 3.1. Assume that X is a good topological space (arcwise connected and semilocally 1-connected) and assume that the group G is a properly discontinuous group of homeomorphisms of X. Let $Y = X/G$ be the quotient space. Then

(I) If G is generated by the stabilizer subgroups G_x, then $H_1(X/G, \mathbb{Z})$ is a quotient of the group of coinvariants $H_1(X, \mathbb{Z})_G$,

$$H_1(X, \mathbb{Z})_G \twoheadrightarrow H_1(X/G, \mathbb{Z}).$$

(II) More generally, if $K(X)$ is the normal subgroup generated by the stabilizer subgroups G_x, then $H_1(X/G, \mathbb{Z})$ is an extension of a quotient of the group of coinvariants $H_1(X,\mathbb{Z})_G$ by the abelianization of $G/K(X)$, that is, the following sequence is exact:

$$H_1(X,\mathbb{Z})_G \to H_1(X/G,\mathbb{Z}) \to (G/K(X))^{ab} \to 0.$$

(III) If X is homotopically equivalent to a simplicial complex on which G acts simplicially, and with only isolated fixed points, then,
the kernel of the homomorphism $H_1(X,\mathbb{Z})_G \to H_1(X/G,\mathbb{Z})$ is generated by the image of a group $H_1(G,Z_0)$ sitting in an exact sequence:

$$H_2(G,\mathbb{Z}) \to H_1(G,Z_0) \to H_1(G,C_0) \to H_1(G,\mathbb{Z}) = G^{ab},$$

where $H_1(G,C_0)$ is the direct sum of groups of the form

$$H_1(G,\mathbb{Z}[G/G']) \cong (G')^{ab},$$

where G' is a subgroup of G (here $\mathbb{Z}[G/G']$ is just a module over the group ring $\mathbb{Z}[G]$).

(IV) In particular, if G is a finite abelian group, G is generated by stabilizers, and $H_1(X,\mathbb{Z})_G$ is a torsion group of order relatively prime to $|G|$, then $H_1(X,\mathbb{Z})_G = H_1(X/G,\mathbb{Z})$.

Proof. Let $p:\tilde{X}\to X$ be the universal cover, $\pi := \pi_1(X)$, so that $X = \tilde{X}/\pi$.

The group G (cf. 6.1 of [11]) admits an exact sequence

$$1 \to \pi \to \Gamma \to G \to 1,$$

where Γ acts properly discontinuously on $\tilde{X}$ and $Y = X/G = \tilde{X}/\Gamma$.

By the theorem of Armstrong [1, 2] we have that $\pi_1(Y) = \Gamma/K$, where K is the subgroup generated by stabilizers Γ_z, for $z \in \tilde{X}$. As π acts freely, Γ_z maps isomorphically to the stabilizer G_x, if $x = p(z)$. Indeed, for each $z \in p^{-1}(x)$ there is a splitting of G_x, and changing z only changes Γ_z up to conjugation by π.

In particular, K maps on to the normal subgroup $K(X)$ of G generated by the stabilizers G_x, and we have an exact sequence

$$1 \to \pi/(\pi\cap K) \cong (\pi K)/K \to \pi_1(Y) = \Gamma/K \to G/K(X) = \Gamma/(\pi K) \to 1.$$

Set

$$H := H_1(X,\mathbb{Z}) = \pi^{ab},\ H' := H_1(Y,\mathbb{Z}) = \pi_1(Y)^{ab} = (\Gamma/K)^{ab} = \Gamma/(K[\Gamma,\Gamma]);$$

hence, an exact sequence

$$1 \to (\pi K[\Gamma,\Gamma])/(K[\Gamma,\Gamma]) \to H' \to (G/K(X))^{ab} = \Gamma/(\pi K[\Gamma,\Gamma]) \to 1.$$

The left-hand side equals $\pi/(\pi \cap (K[\Gamma,\Gamma]))$ and is clearly a quotient of $H = \pi/[\pi,\pi]$. Moreover, for each $\gamma \in \Gamma, \phi \in \pi$, we have that $\gamma\phi\gamma^{-1}$ and ϕ has the same image in H'. So, for each $g \in G$, G acts trivially on the image of H inside the kernel of the surjection $H' \to (G/K(X))^{ab}$. So, we get an exact sequence

$$H_G \to H' \to (G/K(X))^{ab} \to 0,$$

and (I) and (II) are proven.

For (III), observe that H is the 1st homology group of the complex of simplicial chains in X

$$C_2 \to C_1 \to Z_0,$$

where we take as Z_0 the group of degree zero 0-chains.

We hence have several exact sequences, where Z_i is the subgroup of i-cycles, and B_i is the group of i-boundaries:

$$0 \to Z_1 \to C_1 \to Z_0 \to 0,$$

$$0 \to B_1 \to Z_1 \to H \to 0,$$

$$0 \to Z_2 \to C_2 \to B_1 \to 0$$

Applying the functor of coinvariants we get exact sequences

$$B_{1,G} \to Z_{1,G} \to H_G \to 0,$$

$$C_{2,G} \to B_{1,G} \to 0,$$

$$H_1(G, Z_0) \to Z_{1,G} \to C_{1,G} \to Z_{0,G} \to 0.$$

Denote now by H'' the homology of the complex

$$C_{2,G} \to C_{1,G} \to Z_{0,G}.$$

By what we have observed above, H'' is a quotient of $Z_{1,G}$ by the subgroup generated by the image of $B_{1,G}$ and by the image of $H_1(G, Z_0)$, hence a quotient of H_G by the image of $H_1(G, Z_0)$.

Now, by our hypothesis, for $i \geq 1$, $C_{i,G} = C'_i$, the group of simplicial i-chains on $Y = X/G$. This is true since G acts freely on i-chains, for $i \geq 1$.

In addition, the 1st homology group H' of X/G is the homology of the complex

$$C'_2 \to C'_1 \to Z'_0.$$

Moreover, $Z_0 \to Z'_0$ factors through $Z_{0,G}$; hence, H'/H'' equals the quotient of two kernels

$$ker\left(C_{1,G} \to Z'_0\right)/ker\left(C_{1,G} \to Z_{0,G}\right),$$

and we shall now again see that it is isomorphic to $(G/K(X))^{ab}$. Indeed, if a 1-cycle on Y maps to zero in Z'_0, then it lifts to a 1-cycle on X with boundary of the form $x - g(x)$. Adding zero, a path from one vertex to another minus the same path, we can take x to lie on any fiber over a vertex of Y. In particular, if $g \in G_x$, we get $x - g(x) = 0$; hence, similarly, adding a path from x to z, minus its transform via g, we get $z - g(z) = x - g(x) = 0$ for each other vertex z. Finally, since $z = g(z) \Rightarrow hz = gh(z) = hg(z)$ (use the fact that we work in the group of coinvariants for the 2nd equality), we obtain that our quotient H'/H'' equals $(G/K(X))^{ab}$.

Finally, the module C_0 is a direct sum $M \oplus M'$, where M is a free module corresponding to vertices of X on which G acts freely, and M' is the direct sum of modules corresponding to orbits of vertices with a nontrivial stabilizer G', hence modules of the form $\mathbb{Z}[G/G']$.

For the former summand we have $H_1(G,M) = 0$ and for the latter the Hochshild–Lyndon–Serre spectral sequence, if G' is a normal subgroup, yields

$$H_1\left(G, \mathbb{Z}\left[G/G'\right]\right) = \left(G'\right)^{ab}.$$

If G' is not normal, the same assertion follows from Shapiro's lemma (see [8, 6.2, p. 73]), since the $\mathbb{Z}[G]$-module $\mathbb{Z}[G/G']$ is just the representation of G induced from the trivial representation $\mathbb{Z}$ of G'. From the exact sequence

$$0 \to Z_0 \to C_0 \to \mathbb{Z} \to 0$$

the exact group homology sequence yields an exact sequence

$$H_2(G,\mathbb{Z}) \to H_1(G,Z_0) \to H_1(G,C_0) \to H_1(G,\mathbb{Z}) = G^{ab} \to Z_{0,G} \to C_{0,G} \to \mathbb{Z}.$$

(IV) follows easily once we observe that the finite group $H_2(G,\mathbb{Z}) \cong H^2(G,\mathbb{C}^*)$ has exponent dividing $|G|$ (cf. Theorem 6.14 of [21]). ■

Example 3.2. Let X be a hyperelliptic curve of genus g and $G = \mathbb{Z}/2$ acting via the hyperelliptic involution, so that $X/G = \mathbb{P}^1$. Here $H_G = (\mathbb{Z}/2)^{2g}$, while $H_1(X/G, \mathbb{Z}) = H_1(\mathbb{P}^1, \mathbb{Z}) = 0$.

This example shows that the kernel of the homomorphism $H_1(X, \mathbb{Z})_G \to H_1(X/G, \mathbb{Z})$ might be everything (observe that the description of this kernel in III of Proposition 3.1 is not completely explicit!).

To get an example where the locus of fixed points has complex codimension 2, consider more generally the product X of two hyperelliptic curves of genera g_1 and g_2, where $G = \mathbb{Z}/2$ acts diagonally via the two hyperelliptic involutions. Again the quotient is simply connected, $H_G = (\mathbb{Z}/2)^{2g_1+2g_2}$.

If $G = \langle g \rangle$ is cyclic, then $g^i - 1 = (g-1)(g^{i-1} + \ldots + 1)$; hence, $Im(g^i - 1) \subset Im(g-1)$ for all i and, as already mentioned,

$$M_G = M_g := M/Im(g-1).$$

Lemma 3.3. (I) If a cyclic group G of order m acts on an abelian group H, and if m is coprime to the order $|h|$ of every element $h \in H$, then

$$H_G \cong H^G.$$

(II) If a finite cyclic group G acts on X with only isolated fixed points, G is generated by the stabilizer subgroups, and $H_1(X, \mathbb{Z})$ is finite and has order coprime to $|G|$, then

$$H_1(X/G, \mathbb{Z}) = H_1(X, \mathbb{Z})_G = H_1(X, \mathbb{Z})^G.$$

Proof. (I) Let $g \in G$ be a generator: g has order m. Consider the trace homomorphism $\mathrm{Tr}(g) = 1 + g + g^2 + \ldots + g^{m-1} : H \to H$. Since

$$(g-1)\mathrm{Tr}(g) = \mathrm{Tr}(g)(g-1) = g^m - 1 = 0$$

we have

$$Im(\mathrm{Tr}(g)) \subset \mathrm{Ker}(g-1) = H^G, \ \ Im(g-1) \subset \mathrm{Ker}(\mathrm{Tr}(g)).$$

We will show that both are equalities under the assumption.

If $h \in \mathrm{Ker}(g-1) = H^G$, then, choosing an integer a such that $am \equiv 1 \bmod |h|$, we see that $\mathrm{Tr}(g)(ah) = mah = h$; hence, $h \in Im(\mathrm{Tr}(g))$.

If $h \in \mathrm{Ker}(\mathrm{Tr}(g))$, then, again by choosing a positive integer a such that $am \equiv 1$ mod $|h|$, one has

$$\begin{aligned} h &= -g(h) - g^2(h) - \dots - g^{am-1}(h) \\ &= (g-1)(g(h) + 2g^2(h) + \dots + (am-1)g^{am-1}(h)) - (am-1)h \\ &= (g-1)(g(h) + 2g^2(h) + \dots + (am-1)g^{am-1}(h)) \in Im(g-1). \end{aligned}$$

(II) This follows from (I) and Proposition 3.1. ■

Corollary 3.4. Let X be an FPP with $\mathrm{Aut}(X) \cong C_7 : C_3$. Then

$$H_1(X,\mathbb{Z})^{C_7} \cong H_1(X,\mathbb{Z})_{C_7} \cong H_1(X/C_7,\mathbb{Z}) = \pi_1(X/C_7) \cong 0 \text{ or } C_2.$$

More precisely, the C_7 action on $H_1(X,\mathbb{Z})$ fixes no 2-torsion element in the case of $H_1(X,\mathbb{Z}) \cong C_2^3,\ C_2^6$ and only one in the case of $H_1(X,\mathbb{Z}) \cong C_2^4$.

Proof. Recall that, by [10], the three pairs of FPPs with $\mathrm{Aut}(X) \cong G_{21}$ have torsion groups

$$H_1(X,\mathbb{Z}) \cong C_2^3,\ C_2^4,\ C_2^6,$$

respectively. By Proposition 3.1 and Lemma 3.3

$$H_1(X,\mathbb{Z})^{C_7} \cong H_1(X,\mathbb{Z})_{C_7} \cong H_1(X/C_7,\mathbb{Z}).$$

By Theorem 2.3, since an (a,b)-elliptic surface has fundamental group isomorphic to the cyclic group of order $gcd(a,b)$ [15], we see that

$$\pi_1(X/C_7) = \pi_1(X/C_7)^{ab} = H_1(X/C_7,\mathbb{Z})$$

is of order at most 3, hence either 0 or C_2 (this coincides with the computation of $\pi_1(X/C_7)$ in [10].) Since the polynomial $x^7 - 1$ in $C_2[x]$ is the product of thee prime factors $(x+1)(x^3+x^2+1)(x^3+x+1)$ we see that any linear action of C_7 on a vector space C_2^n is a direct sum of subspaces of cardinality 2 or 8. Thus, the C_7 action on $H_1(X,\mathbb{Z})$ fixes no 2-torsion element in the case of $H_1(X,\mathbb{Z}) \cong C_2^3,\ C_2^6$ and one in the case of $H_1(X,\mathbb{Z}) \cong C_2^4$. ■

Remark 3.5. (1) This is no strange. In fact, $\mathrm{Aut}(C_2^3) \cong GL(3,2) \cong PSL(2,7)$, a simple group of order 168 containing a subgroup $\cong G_{21}$.

(2) Corollary 3.4 is a crucial point missing in the proof of the paper by S.-K. Yeung [A surface of maximal canonical degree, Math. Ann. 368 (2017), 1171-1189], where the proof of the base point freeness of $|K_M|$ is based on his wrong claim that the C_7 action on $H_1(X,\mathbb{Z}) = C_2^4$ is trivial. Thus, the main result of the paper is not proven at all.

Lemma 3.6. Let F be a finite abelian group of order not divisible by 9. Suppose that F admits an order 3 automorphism σ such that the group of coinvariants $F_\sigma \cong C_3$ or 0. Then for every $t \in F$

$$t + \sigma(t) + \sigma^2(t) = 0 \text{ in } F.$$

Proof. In the case

$$0 = F_\sigma = F/(\sigma - 1)F$$

$1 - \sigma$ is invertible; hence,

$$0 = \sigma^3 - 1 = (\sigma - 1)\left(\sigma^2 + \sigma + 1\right) \Rightarrow \left(\sigma^2 + \sigma + 1\right) = 0.$$

Note that the action of σ on F is the product of its actions on the p-primary summands of F.

In the 2nd case $F_\sigma \cong C_3$, the previous argument applies for p-primary summands of F for $p \neq 3$.

For $p = 3$ clearly $\sigma = 1$; hence, $(\sigma^2 + \sigma + 1) = 3 = 0$. ■

Now we prove Theorem 1.3.

Suppose that X has no 3-torsion (in $H_1(X,\mathbb{Z})$, hence also in $Pic(X)$). Then K_X has a unique cube root L_0. Then L_0 is fixed by every automorphism. Let L be an ample line bundle with $L^2 = 1$. Then $L = L_0 + t$ for some torsion line bundle t, and

$$\sigma^*(L_0 + t) = L_0 + \sigma^*(t), \;\; \sigma^{2*}(L_0 + t) = L_0 + \sigma^{2*}(t).$$

By the above lemma, $t + \sigma^*(t) + \sigma^{2*}(t) = 0$. Thus,

$$(L_0 + t) + \sigma^*(L_0 + t) + \sigma^{2*}(L_0 + t) = 3L_0 + (t + \sigma^*(t) + \sigma^{2*}(t)) = 3L_0 = K_X.$$

If $L_0 + t$ is effective, then all the three summands are effective, so is K_X, contradicting $p_g(X) = 0$. Thus, $H^0(X, L_0 + t) = 0$.

Suppose that X has nontrivial 3-torsion, so that K_X has a cube root L_0, but this is not unique (see Remark under Table 1). In this case it is possible that no such divisor class L_0 is fixed by the order 3 automorphism σ. If $\sigma^*(L_0) = L_0$, then the previous argument shows that $H^0(X, L_0 + t) = 0$. If $\sigma^*(L_0) = L_0 + t_3$ for some 3-torsion element t_3, then for any torsion line bundle t

$$\sigma^*(L_0 + t) = L_0 + t_3 + \sigma^*(t), \;\; \sigma^{2*}(L_0 + t) = L_0 + 2t_3 + \sigma^{2*}(t).$$

Here $\sigma^*(t_3) = t_3$, because the 3-adic part of $Pic(X)$ for an FPP X has rank at most 1. By the above lemma, $t + \sigma^*(t) + \sigma^{2*}(t) = 0$. Thus,

$$(L_0 + t) + \sigma^*(L_0 + t) + \sigma^{2*}(L_0 + t) = 3L_0 + 3t_3 = K_X.$$

Since K_X is not effective, none of the three summands is effective.

4 FPPs with Aut$(X) = C_3^2$ and $H_1(X, \mathbb{Z}) = C_{14}$

This is one of the 3 pairs of FPPs with Aut$(X) = C_3^2$. The other two pairs are also listed in Table 1. Note that the unique 2-torsion element is fixed by every automorphism.

If Aut$(X) = C_3^2$ acts trivially on $H_1(X, \mathbb{Z})$, then $H_1(X, \mathbb{Z})_{\mathrm{Aut}(X)} = C_{14}$. This group has order coprime to 3, thus by Proposition 3.1 $H_1(X/\mathrm{Aut}(X), \mathbb{Z}) = C_{14}$. But $X/\mathrm{Aut}(X)$ has four A_2-singularities and its minimal resolution is a numerical Godeaux surface (a minimal surface of general type with $p_g = 0$ and $K^2 = 1$). This is impossible, as a numerical Godeaux surface has a torsion group of order ≤ 5. Thus, Aut$(X) = C_3^2$ does not act trivially on $H_1(X, \mathbb{Z})$.

Note that

$$Aut(H_1(X, \mathbb{Z})) \cong (C_2 \times C_7)^* \cong C_6.$$

Thus, we have

$$\mathrm{Ker}(Aut(X) \to Aut(H_1(X, \mathbb{Z})) \cong C_3.$$

If $\sigma \in$ Aut$(X) = C_3^2$ acts trivially on $H_1(X, \mathbb{Z})$, then by Proposition 3.1 and Lemma 3.3

$$H_1(X/\langle\sigma\rangle, \mathbb{Z}) = H_1(X, \mathbb{Z})_\sigma = H_1(X, \mathbb{Z})^\sigma = C_{14}.$$

If $\sigma \in$ Aut$(X) = C_3^2$ does not act trivially on $H_1(X, \mathbb{Z})$, then it fixes the 2-torsion element and permutes the six 7-torsion elements and the six 14-torsion elements; hence,

$$H_1(X/\langle\sigma\rangle, \mathbb{Z}) = H_1(X, \mathbb{Z})_\sigma = H_1(X, \mathbb{Z})^\sigma = C_2.$$

This coincides with the computation of Cartwright and Steger [10]:

$$\pi_1(X/C_3) = C_{14}, \;\; S_3, \;\; C_2, \;\; C_2$$

for four order 3 subgroups of Aut$(X) = C_3^2$, respectively (see Table 1).

Since X has no 3-torsion, it has a unique cubic root of K_X. Let $L_0 \in \mathrm{Pic}(X)$ be the unique cubic root of K_X.

First, we recall the following vanishing result from ([28, Theorem 0.2 and its proof]).

Theorem 4.1. [28] Let X be an FPP with $\mathrm{Aut}(X) \cong C_3^2$. Then $H^0(X, 2L_0 + t) = 0$ for any $Aut(X)$-invariant torsion line bundle t. In particular, $H^0(X, 2L_0) = 0$.

Remark 4.2. Among the three pairs of FPPs with $\mathrm{Aut}(X) \cong C_3^2$, only the pair with $H_1(X, \mathbb{Z}) = C_{14}$ has an $Aut(X)$-invariant nontrivial torsion line bundle, which corresponds to the unique 2-torsion in $H_1(X, \mathbb{Z})$. It follows that for this pair of FPPs, $\mathcal{O}_X, -(L_0 + t_2), -2L_0$ also form an exceptional collection.

Lemma 4.3. For an FPP X with $\mathrm{Aut}(X) = C_3^2$ and $H_1(X, \mathbb{Z}) = C_{14}$,

$$H^0(X, L_0 + t) = 0$$

for any torsion element $t \in Pic(X)$, except possibly for three 14-torsion elements that are rotated by an order 3 automorphism.

Proof. Suppose that $t = t_7$ is a 7-torsion element. We know that there is an automorphism $\sigma \in Aut(X)$ such that $\sigma^*(t) = 2t$ (by replacing it by σ^2 if $\sigma^*(t) = 4t$). Thus,

$$\sigma^*(L_0 + t) = L_0 + 2t, \;\; \sigma^{2*}(L_0 + t) = L_0 + 4t.$$

Since $p_g(X) = H^0(X, K_X) = 0$ and

$$(L_0 + t) + (L_0 + 2t) + (L_0 + 4t) = 3L_0 = K_X,$$

we have $H^0(X, L_0 + t) = 0$.

Suppose that $t = t_2$ is the unique 2-torsion element. It is fixed by every automorphism. Thus,

$$(L_0 + t_2) + \sigma^*(L_0 + t_2) + \sigma^{2*}(L_0 + t_2) = 3(L_0 + t_2) = K_X + t_2$$

and one cannot use the previous argument. But the vanishing $H^0(X, L_0 + t_2) = 0$ follows from Theorem 4.1, since $2(L_0 + t_2) = 2L_0$ is $\mathrm{Aut}(X)$-invariant.

Suppose that $H^0(X, L_0+t_2+t_7) \neq 0$ for some 7-torsion element t_7. By Theorem 4.1 we know that there is an automorphism $\sigma \in Aut(X)$ such that $\sigma^*(t_7) = 2t_7$. Thus

$$\sigma^*(L_0 + t_2 + t_7) = L_0 + t_2 + 2t_7, \;\; \sigma^{2*}(L_0 + t_2 + t_7) = L_0 + t_2 + 4t_7$$

and these two line bundles are effective. We know that $H^0(X, 2L_0) = 0$. Since

$$(L_0 + t_2 + at_7) + (L_0 + t_2 + (7 - a)t_7) = 2L_0,$$

we have

$$H^0(X, L_0 + t_2 + 6t_7) = H^0(X, L_0 + t_2 + 5t_7) = H^0(X, L_0 + t_2 + 3t_7) = 0.$$

■

From now on, assume that $H^0(X, L_0 + t_2 + t_7) \neq 0$ for some 7-torsion element t_7. Then

$$H^0(X, L_0 + t_2 + t_7) \cong \mathbb{C}.$$

Let D_1 be the unique effective curve in the linear system, that is,

$$D_1 \equiv L_0 + t_2 + t_7.$$

Define

$$D_2 = \sigma^* D_1 \equiv L_0 + t_2 + 2t_7, \;\; D_3 = \sigma^* D_2 \equiv L_0 + t_2 + 4t_7.$$

There is another automorphism $\nu \in Aut(X)$ acting trivially on $H_1(X, \mathbb{Z}) = C_{14}$. Then

$$\nu^*(M) = M$$

for any line bundle M. In particular,

$$\nu^*(D_i) = D_i, \;\; i = 1, 2, 3.$$

By Lemma 2.5, each D_i is a smooth curve of genus 3. Note that the intersection number

$$D_i D_j = 1, \;\; i, j = 1, 2, 3.$$

Hence, D_i and D_j intersect transversally in a point x_{ij}. Then D_1, D_2, and D_3 form a triangle with vertices x_{ij}. (If the three curves intersect at a point x, then both σ and ν fix x, impossible by Theorem 2.3.) We know by Theorem 2.3 that the fixed locus of ν consists of three isolated points, so we infer that

$$Fix(\nu) = \{x_{12}, x_{23}, x_{31}\}.$$

Theorem 4.4. Let X be an FPP with $\text{Aut}(X) = C_3^2$ and $H_1(X, \mathbb{Z}) = C_{14}$. The bicanonical map $\Phi_{2,X}$ of X is an embedding.

Proof. If two different points P and Q, with Q possibly infinitely near to P, are not separated by the bicanonical system, then they must belong to one of the three curves D_i, say D_1. We know that $P + Q$ is the unique divisor of $H^0(D_1, \mathcal{O}_{D_1}((K_X - D_1)|D_1))$. Since

ν preserves the line bundle $K_X - D_1$ and the curve D_1, it preserves the divisor $P + Q$. Since $\nu|D_1$ is of order 3, both P and Q are fixed points of $\nu|D_1$. If Q is not infinitely near to P, then

$$P + Q = x_{12} + x_{13}.$$

Thus, in our previous notation, we should have

$$\mathcal{O}_{D_1}(D_2 + D_3) \cong \mathcal{O}_{D_1}(x_{12} + x_{13}) \cong \mathcal{O}_{D_1}(K_X - D_1);$$

hence

$$\mathcal{O}_{D_1}(t_2) \cong \mathcal{O}_{D_1}(K_X - D_1 - D_2 - D_3) \cong \mathcal{O}_{D_1}.$$

This contradicts however Lemma 2.7.

If Q is infinitely near to P, then $P + Q|D_1 = 2x_{12}$ or $P + Q|D_1 = 2x_{13}$. In the former case we must have

$$\mathcal{O}_{D_1}(2D_2) \cong \mathcal{O}_{D_1}(2x_{12}) \cong \mathcal{O}_{D_1}(K_X - D_1);$$

hence

$$\mathcal{O}_{D_1}(t_2 - 5t_7) \cong \mathcal{O}_{D_1}(K_X - D_1 - 2D_2) \cong \mathcal{O}_{D_1},$$

contradicting again Lemma 2.7. The argument in the latter case is identical. ∎

5 FPPs with $H_1(X, \mathbb{Z}) = C_6$

Among the 50 pairs of FPPs, exactly three pairs have $H_1(X, \mathbb{Z}) = C_6$, as listed in Table 2. Moreover, they all have $\mathrm{Aut}(X) = C_3$. Since $\mathrm{Aut}(H_1(X, \mathbb{Z})) \cong C_2$ in this case, the automorphism group acts trivially on $H_1(X, \mathbb{Z}) = C_6$. Consider the set

$$\mathcal{L} = \{L \in \mathrm{Pic}(X) \mid L \text{ ample with } L^2 = 1\}.$$

This set has the same cardinality as $H_1(X, \mathbb{Z}) = C_6$. First, note that $\mathrm{Aut}(X)$ may act nontrivially on the set $\mathcal{L}$ even though $\mathrm{Aut}(X)$ acts trivially on $H_1(X, \mathbb{Z}) = C_6$, which is the set of differences of two elements of $\mathcal{L}$.

Table 2 FPPs with $H_1(X, \mathbb{Z}) = C_6$

X	Aut(X)	$H_1(X, \mathbb{Z})$	$\pi_1(X/C_3)$
(a=15, p=2, {3}, $(D3)_3$)	C_3	C_6	C_6
($\mathcal{C}_{18}$, p=3, {2}, $(dD)_3$)	C_3	C_6	C_6
($\mathcal{C}_{18}$, p=3, {2}, $(d^2D)_3$)	C_3	C_6	C_6

If $\mathrm{Aut}(X) = C_3$ acts nontrivially on the set $\mathcal{L}$, then the previous argument shows that no element of $\mathcal{L}$ can be effective; hence, we have the very ampleness of the bicanonical system.

If $\mathrm{Aut}(X)$ acts trivially on the set $\mathcal{L}$, then a similar argument shows that at most two elements of $\mathcal{L}$ can be effective. In this case, if X has only one curve D with $D^2 = 1$, then $\mathrm{Aut}(X) = C_3$ fixes 2 points on D and the bicanonical map embeds away from the two points, and if X has two curves D_1 and D_2 with $D_1^2 = D_2^2 = D_1 D_2 = 1$, then $\mathrm{Aut}(X) = C_3$ fixes three points on $D_1 \cup D_2$ and the bicanonical map embeds away from the three points.

Remark 5.1. The last two pairs in Table 2 have fundamental group that does not lift to $\mathrm{SU}(2,1)$ by [10], so their canonical classes are not divisible by 3.

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