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$\mathfrak{S}_5$-equivariant syzygies for the Del Pezzo surface of degree 5

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Abstract
The Del Pezzo surface Y of degree 5 is the blow up of the plane in 4 general points, embedded in $\mathbb{P}^5$ by the system of cubics passing through these points. It is the simplest example of the Buchsbaum–Eisenbud theorem on arithmetically-Gorenstein subvarieties of codimension 3 being Pfaffian. Its automorphism group is the symmetric group $\mathfrak{S}_5$. We give canonical explicit $\mathfrak{S}_5$-invariant Pfaffian equations through a 6×6 antisymmetric matrix. We give concrete geometric descriptions of the irreducible representations of $\mathfrak{S}_5$. Finally, we give $\mathfrak{S}_5$-invariant equations for the embedding of Y inside $(\mathbb{P}^1)^5$, and show that they have the same Hilbert resolution as for the Del Pezzo of degree 4.

Keywords Del Pezzo surfaces · Pfaffian equations · Icosahedral symmetry · Representations of the symmetric group in 5 letters

Mathematics Subject Classification 13C14 · 13D02 · 14J25 · 14J26 · 14J45 · 14Q10 · 16E05 · 20B30

1 Introduction

Del Pezzo [12] (see also [9], page 313) showed that the smooth nondegenerate surfaces Y_n^2 of degree n in $\mathbb{P}^n$, except for the anticanonical embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^8$, are obtained as the blow-up of the projective plane in $(9-n)$ points of which no three are collinear, no six lie on a conic (here $n = 3, 4, 5, 6, 7, 8, 9$). After his classification these surfaces bear his name.

Del Pezzo surfaces are projectively unique as soon as $n \geq 5$, and we are interested (see [3], and the related papers [2] and [1]), about their defining equations, here particularly in the case $n = 5$.

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It is well known, at least since the work of Buchsbaum and Eisenbud [5] (see also [4]), that all surfaces Y in $\mathbb{P}^5$ which are arithmetically Cohen-Macaulay and subcanonical (i.e., the canonical sheaf $\omega_Y \cong \mathcal{O}_Y(r)$ for some integer r) are defined by the m-Pfaffians of an antisymmetric $(2m+1)\times(2m+1)$-matrix $A = -{}^tA$ of homogeneous forms (more precisely, the Pfaffians generate the ideal of polynomials vanishing on Y). The simplest nondegenerate case is the case $m=2$ of a generic 5×5 antisymmetric matrix A of linear forms: here $r=-1$ and we get the Del Pezzo surface Y_5^2 of degree 5 as Pfafffian locus of A, hence defined by the 5 quadratic equations which are the five 4×4-Pfaffians of A.

It suffices to choose a generic matrix, but an explicit matrix is better, as was for instance done in [13]. At any rate we are in this way far away from a normal form for the equations of Y_5^2, which might be desirable for many purposes. What do we mean by a normal form? To explain this, recall that Y_5^2 is indeed isomorphic to the moduli space of ordered quintuples of points in $\mathbb{P}^1$, and this isomorphism follows after showing that its group of automorphisms is $\mathfrak{S}_5$ (Y_5^2 is indeed in bijection with the set of projective equivalence classes of quintuples where no point of $\mathbb{P}^1$ occurs with multiplicity ≥ 3). Hence, we would like to have equations where this symmetry shows up, and which are ‘invariant’ for the symmetry group.

This requirement can be precisely specified as follows: we have the anticanonical (6-dimensional) vector space $V := H^0(\omega_Y^{-1}) = H^0(Y, \mathcal{O}_Y(-K_Y))$ and the 5-dimensional vector space W of quadratic forms vanishing on Y ($W \subset \operatorname{Sym}^2(V)$). Both vector spaces are representations of the symmetric group $\mathfrak{S}_5$ and the 5×5 antisymmetric matrix A of linear forms is seen, by the Buchsbaum-Eisenbud theory, to be an invariant tensor

$$A \in ((\Lambda^2 W)\otimes V)^{\mathfrak{S}_5}.$$

It was known through character theory ([10] with corrections done in [6, 7, 11]) that this invariant tensor is unique up to constants.

The main result of the present paper is to explicitly and canonically determine it. Now, it looks like there is no unique such representation of the tensor A, because in the end we have still to choose a basis for both vector spaces V and W. However, if we take the natural irreducible 4-dimensional representation U_4 of $\mathfrak{S}_5$, this is given by the invariant subspace $x_1 + x_2 + \cdots + x_5 = 0$ in the 5-dimensional vector space U_5' with coordinates $(x_1, \ldots, x_5)$ which are naturally permuted by $\mathfrak{S}_5$. So, there is a natural permutation representation U_5' yielding a natural basis for $U_5' \supset U_4$.

In our case, we first show that there is a natural basis s_{ij} ($1 \leq i \neq j \leq 3$) of V, showing that V is the regular representation of $\mathfrak{S}_3$ tensored with the sign character, and then we show that the space W is a 5-dimensional invariant subspace in a natural 6-dimensional subspace W' of $\operatorname{Sym}^2(V)$, related to a natural permutation representation of $\mathfrak{S}_5$[1], and with basis Q_{ij} again related to the regular representation of $\mathfrak{S}_3$ (the space W is then generated by the differences between two such quadratic forms Q_{ij}).

This leads to a normal form: we produce in Theorem 4.3 (which we reproduce here immediately below) in an elegant numerical way an explicit antisymmetric 6×6 matrix A' with entries in V (i.e., with entries linear forms) whose 15 4×4-Pfaffians are exactly twice the differences between the quadratic forms Q_{ij}.

[1] if ϵ is the sign character, indeed $W' \otimes \epsilon$ is the permutation representation corresponding to double combinatorial pentagons (the cosets in $\mathfrak{S}_5/\operatorname{Aff}(1, \mathbb{Z}/5)$).

Theorem 1.1 *Let Y be the del Pezzo surface of degree 5, embedded anticanonically in $\mathbb{P}^5$. Then the ideal of Y is generated by the 4×4 - Pfaffians of the $\mathfrak{S}_5$ - equivariant anti-symmetric 6×6-matrix $A' =$*

$$\begin{pmatrix}
0 & \begin{matrix} s_{21}+s_{23}- \\ -s_{31}-s_{32} \end{matrix} & \begin{matrix} s_{12}+s_{31}- \\ -s_{32}-s_{21} \end{matrix} & -s_{13}-s_{21} & s_{13}+s_{32} & \begin{matrix} s_{21}+s_{32}- \\ -s_{12}-s_{23} \end{matrix} \\
\begin{matrix} -s_{21}-s_{23}+ \\ +s_{31}+s_{32} \end{matrix} & 0 & s_{12}+s_{23} & \begin{matrix} s_{31}+s_{23}- \\ -s_{13}-s_{32} \end{matrix} & \begin{matrix} s_{13}+s_{21} \\ -s_{23}-s_{31} \end{matrix} & -s_{12}-s_{31} \\
\begin{matrix} -s_{12}-s_{31}+ \\ s_{32}+s_{21} \end{matrix} & -s_{12}-s_{23} & 0 & \begin{matrix} s_{12}+s_{13} \\ -s_{32}-s_{31} \end{matrix} & \begin{matrix} s_{12}-s_{21} \\ s_{31}-s_{13} \end{matrix} & s_{23}+s_{31} \\
s_{13}+s_{21} & \begin{matrix} -s_{31}-s_{23} \\ s_{13}+s_{32} \end{matrix} & \begin{matrix} -s_{12}-s_{13}+ \\ s_{32}+s_{31} \end{matrix} & 0 & -s_{21}-s_{32} & \begin{matrix} s_{23}+s_{12} \\ -s_{13}-s_{32} \end{matrix} \\
-s_{13}-s_{32} & \begin{matrix} -s_{13}-s_{21} \\ s_{23}+s_{31} \end{matrix} & \begin{matrix} -s_{12}+s_{21} \\ -s_{31}+s_{13} \end{matrix} & s_{21}+s_{32} & 0 & \begin{matrix} s_{12}+s_{13} \\ -s_{21}-s_{23} \end{matrix} \\
\begin{matrix} -s_{21}-s_{32} \\ s_{12}+s_{23} \end{matrix} & s_{12}+s_{31} & -s_{23}-s_{31} & \begin{matrix} -s_{23}-s_{12} \\ s_{13}+s_{23} \end{matrix} & \begin{matrix} -s_{12}-s_{13} \\ s_{21}+s_{23} \end{matrix} & 0
\end{pmatrix}$$

The action of the symmetric group $\mathfrak{S}_5$ on the entries of the matrix is given by the action on the vector space V, with basis s_{ij}, $i \neq j$, $1 \leq i,j \leq 3$, which is described in the proof of Theorem 2.6, tensored with the sign character ϵ (we shall denote $V' := V \otimes \epsilon$ and observe here that V, V' are isomorphic $\mathfrak{S}_5$ representations). The action on the matrix is determined by the action of $\mathfrak{S}_5$ on $W' \otimes W'$, in turn determined by the action of $\mathfrak{S}_5$ on the vector space W' with basis the quadrics Q_{ij}, $i \neq j$, $1 \leq i,j \leq 3$. The latter action (which is explicitly described in Remark 3.8) is a consequence of the former, since $Q_{ij} = s_{ij}\sigma_{ij}$, where the σ_{ij}'s are defined in Sect. 2 and are as follows: $\sigma_{ij} = \epsilon(s_{ik} - s_{ki} - s_{jk})$.

In practice, the action of two generators of $\mathfrak{S}_5$ is explicitly given as follows.

The transposition $\tau := (1,2)$ acts as

- $s_{ij} \mapsto s_{\tau(i)\tau(j)}$;
- $Q_{ij} \mapsto -Q_{\tau(i)\tau(j)}$.

The cycle $\varphi := (1,2,3,4,5)$ acts as

$$s_{12} \mapsto s_{31},\ s_{13} \mapsto \sigma_{12} = s_{13} - s_{31} - s_{23},\ s_{21} \mapsto s_{21},$$
$$s_{23} \mapsto -\sigma_{13} = s_{12} - s_{21} - s_{32},\ s_{31} \mapsto -\sigma_{32} = s_{13} - s_{31} + s_{21},\ s_{32} \mapsto -\sigma_{23} = s_{12} - s_{21} + s_{31},$$

and

$$Q_{12} \mapsto Q_{31},\ Q_{13} \mapsto Q_{12},\ Q_{21} \mapsto Q_{21},$$
$$Q_{23} \mapsto Q_{13},\ Q_{31} \mapsto Q_{32},\ Q_{32} \mapsto Q_{23}.$$

Our main theorem is also interesting because, even if we work sometimes using character theory and over a field of characteristic 0, in the end we produce equations which define the Del Pezzo surface of degree 5 over any field of characteristic different from 2 (in this case symmetric and antisymmetric tensors coincide, and the situation should be treated separately).

We believe that our approach is also interesting from another point of view (as already announced in [6, 7]): the symmetries of Y_5^2 bear some similarity to those of the icosahedron, as it is well known ([8, 10]), and the irreducible representations of $\mathfrak{S}_5$ can be explicitly described via the geometry of Y_5^2. For instance, the representations V, W are irreducible, and together with the natural representation U_4 described above, they and their tensor product with the sign representation ϵ yield all irreducible representations of $\mathfrak{S}_5$ (V is the only one such that $V \cong V \otimes \epsilon$).

We also give explicit descriptions of the irreducible representations of $\mathfrak{S}_5$ via the geometry of Y_5^2. While it is easy to describe U_4 and $W \otimes \epsilon$ through natural permutation representations on $\mathfrak{S}_5/\mathfrak{S}_4$, respectively on $\mathfrak{S}_5/\operatorname{Aff}(1, \mathbb{Z}/5)$, we give another realization of the anticanonical vector space V,

$$V = H^0(\omega_{Y_5^2}^{-1}) = H^0(\mathcal{O}_{Y_5^2}(-K_{Y_5^2})) = H^0(\mathcal{O}_{Y_5^2}(1)),$$

through an explicit permutation representation of dimension 24, the one on the set of oriented combinatorial pentagons (the cosets in $\mathfrak{S}_5/(\mathbb{Z}/5)$). This is interesting, because we have then a bijection between the set of the 12 (unoriented, nondegenerate) combinatorial pentagons and the set of the 12 geometric nondegenerate pentagons contained in Y, and formed with quintuples out of the 10 lines contained in Y (these geometric pentagons are hyperplane sections of Y).

Finally, in a previous paper [3] we have written explicit equations of Del Pezzo surfaces in products $(\mathbb{P}^1)^h$, and here, for the Del Pezzo of degree 5, we give in Theorem 5.3 $\mathfrak{S}_5$ -invariant equations inside $(\mathbb{P}^1)^5$.

An interesting phenomenon is that the minimal (multigraded) Hilbert resolution for this embedding of the Del Pezzo of degree 5 is exactly the same as for the natural embedding in $(\mathbb{P}^1)^5$ of the Del Pezzo surfaces of degree 4, so that we have an example of a reducible Hilbert scheme.

2 Symmetries and the anticanonical system of the Del Pezzo surface of Degree 5

We recall some notation introduced in [7], as well as some intermediate results established there.

The Del Pezzo surface $Y := Y_5^2$ of degree 5 is the blow-up of the plane $\mathbb{P}^2$ in the 4 points $p_1, \dots, p_4$ of a projective basis, that is, we choose

$$p_1 = (1 : 0 : 0),\ p_2 = (0 : 1 : 0),\ p_3 = (0 : 0 : 1),\ p_4 = (1 : 1 : 1) = (-1 : -1 : -1).$$

Observe that p_i corresponds to a vector e_i, for $i = 1, 2, 3, 4$, where e_1, e_2, e_3 is a basis of a vector space U', and $e_1 + e_2 + e_3 + e_4 = 0$. In other words, $\mathbb{P}^2 = \mathbb{P}(U')$, where U' is the natural irreducible representation of $\mathfrak{S}_4$.

As already mentioned in the introduction, the Del Pezzo surface Y is indeed the moduli space of ordered quintuples of points in $\mathbb{P}^1$, and its automorphism group is isomorphic to $\mathfrak{S}_5$.

The obvious action of the symmetric group $\mathfrak{S}_4$ permuting the 4 points extends in fact to an action of the symmetric group $\mathfrak{S}_5$.

This can be seen as follows. The six lines in the plane joining pairs p_i, p_j can be labelled as $L_{i,j}$, with $i, j \in \{1, 2, 3, 4\}, i \neq j$.

Denote by $E_{i,5}$ the exceptional curve lying over the point p_i, and denote, for $i \neq j \in \{1, 2, 3, 4\}$, by $E_{h,k} = E_{k,h}$ the strict transform in Y of the line $L_{i,j}$, if $\{1, 2, 3, 4\} = \{i, j, h, k\}$. For each choice of 3 of the four points, $\{1, 2, 3, 4\} \setminus \{h\}$, consider the standard Cremona transformation σ_h based on these three points. To it we associate the

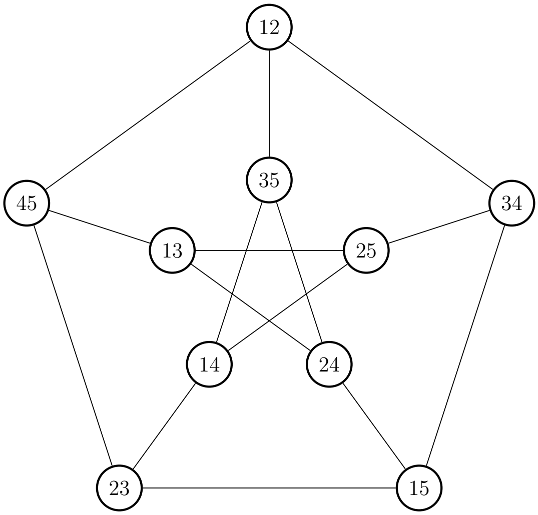

Fig. 1 The Petersen graph (the quotient of the dodecahedron via the antipodal map), which is the dual graph to the incidence correspondence of the lines on the quintic Del Pezzo surface

transposition $(h, 5) \in \mathfrak{S}_5$, and the upshot is that σ_h transforms the 10 (-1) curves $E_{i,j}$ via the action of $(h, 5)$ on pairs of elements in $\{1, 2, 3, 4, 5\}$.

There are five geometric objects permuted by $\mathfrak{S}_5$: namely, 5 fibrations $\varphi_i : Y \to \mathbb{P}^1$, induced, for $1 \le i \le 4$, by the projection with centre p_i, and, for $i = 5$, by the pencil of conics through the 4 points. Each fibration is a conic bundle, with exactly three singular fibres, corresponding to the possible partitions of type (2, 2) of the set $\{1, 2, 3, 4, 5\} \setminus \{i\}$.

The intersection pattern of the curves $E_{i,j}$, which generate the Picard group of Y is dictated by the simple rule (recall that $E_{i,j}^2 = -1, \ \forall i \neq j$)

$$E_{i,j} \cdot E_{h,k} = 1 \ \Leftrightarrow \{i,j\} \cap \{h,k\} = \emptyset, \ \ E_{i,j} \cdot E_{h,k} = 0 \Leftrightarrow \{i,j\} \cap \{h,k\} \neq \emptyset, \{i,j\}.$$

In this picture the three singular fibres of φ_1 are

$$E_{3,4} + E_{2,5}, \ E_{2,4} + E_{3,5}, \ E_{2,3} + E_{4,5}.$$

The relations among the $E_{i,j}$'s in the Picard group come from the linear equivalences $E_{3,4} + E_{2,5} \equiv E_{2,4} + E_{3,5} \equiv E_{2,3} + E_{4,5}$ and their $\mathfrak{S}_5$-orbits.

An important observation is that Y contains exactly ten lines (i.e., ten irreducible curves E with $E^2 = EK_Y = -1$), namely the lines $E_{i,j}$.

Their intersection pattern is described by the Petersen graph, whose vertices correspond to the curves $E_{i,j}$, and whose edges correspond to intersection points (cf. Fig. 1).

Remark 2.1 (1) The action of $\mathrm{Aut}\,(Y) \cong \mathfrak{S}_5$ can be described as the action on $\mathbb{P}^2$ by the following (birational) transformations:

- the above vectorial action of $\mathfrak{S}_4$ on U' (by which $\mathfrak{S}_4 \le \mathfrak{S}_5$ acts on $\mathbb{P}^2$ permuting p_1, p_2, p_3, p_4), i.e., in coordinates,
- for $\sigma \in \mathfrak{S}_3$

$$\sigma(x_1 : x_2 : x_3) = (x_{\sigma(1)} : x_{\sigma(2)} : x_{\sigma(3)}),$$

and for $\sigma = (3, 4)$:

$$\sigma(x_1 : x_2 : x_3) = (x_1 - x_3 : x_2 - x_3 : -x_3),$$

- the transposition (4, 5) is the standard Cremona transformation

$$(x_1 : x_2 : x_3) \mapsto \left(\frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}\right) = (x_2x_3 : x_1x_3 : x_1x_2).$$

(2) The equation of the six lines of the complete quadrangle in $\mathbb{P}^2$ (the lines $L_{i,j}$ joining p_i and p_j)

$$\Sigma := x_1x_2x_3(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$$

yields an eigenvector for the sign representation $\epsilon : \mathfrak{S}_5 \to \{\pm 1\}$.

In fact, the proper transform of div (Σ) is the sum of the 10 lines $E_{i,j}$ contained in Y, and it is a divisor in $|-2K_Y|$ invariant for any automorphism. Since there is a natural action of $\mathfrak{S}_5$ on $\mathcal{O}_Y(K_Y)$, we get a natural action of $\mathfrak{S}_5$ on $H^0(Y, \mathcal{O}_Y(-2K_Y))$ hence the proper transform of the equation Σ is a section of $H^0(Y, \mathcal{O}_Y(-2K_Y))$ which is an eigenvector for a character of $\mathfrak{S}_5$. There are only two characters, the sign and the trivial one: restricting to the subgroup $\mathfrak{S}_3$ we see that it must be the sign.

We consider now the anticanonical vector space

$$V := H^0(Y, \mathcal{O}_Y(-K_Y)).$$

We know that V is, via adjunction, naturally isomorphic to the vector space of cubics passing through the 4 points,

$$V' := \{F \in \mathbb{C}[x_1, x_2, x_3]_3 \mid F(p_1) = F(p_2) = F(p_3) = F(p_4) = 0\} \cong \mathbb{C}^6.$$

Since $\mathfrak{S}_5$ acts linearly on V we want to determine this action and translate this action on V'.

For this purpose we set, for $1 \leq i \neq j \leq 3$,

$$s_{ij} := x_ix_j(x_j - x_k),\ \{1, 2, 3\} = \{i, j, k\}.$$

Proposition 2.2

$$\{s_{ij} \mid 1 \leq i \neq j \leq 3\} \subset V'$$

is a basis of V'.

Proof All 6 cubic forms vanish on $p_1, \dots, p_4$.

s_{12}, s_{21} are the only ones not divisible by x_3, and they are independent modulo (x_3), since their divisors on the line $x_3 = 0$ are, respectively $2p_1 + p_2$, $p_1 + 2p_2$.

It suffices to show that the other 4 are independent after division by x_3. Now, $\frac{s_{23}}{x_3}, \frac{s_{32}}{x_3}$ yield x_2x_3, respectively x_2^2 modulo (x_1), hence it suffices to show that $\frac{s_{13}}{x_3x_1}, \frac{s_{31}}{x_3x_1}$ are independent: but these are respectively $(x_3 - x_2), (x_1 - x_2)$. □

We set

$$\sigma_{ij} := \frac{\Sigma}{s_{ij}} = \frac{\Sigma}{x_ix_j(x_j - x_k)} = \epsilon x_k(x_i - x_j)(x_k - x_i),$$

where $\epsilon := \epsilon(\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix})$ is the sign of the permutation.

Then we have:

$$\sigma_{ij} = \epsilon(s_{ik} - s_{ki} - s_{jk}).$$

Remark 2.3 (1) We observe as a parenthetical remark that the σ_{ij}'s span a 4-dimensional sub vector space of V. In fact,

$$\alpha(\sigma_{12} + \sigma_{13}) + \beta(\sigma_{21} + \sigma_{23}) + \gamma(\sigma_{31} + \sigma_{32}) = 0,$$

for all $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha + \beta + \gamma = 0$.

(2) Observe that for $\mathfrak{S}_3 \le \mathfrak{S}_5$, $\mathfrak{S}_3$ acts on homogeneous polynomials by permuting x_1, x_2, x_3, and that V' is the regular representation of $\mathfrak{S}_3$.

Consider now the Euler sequence on $\mathbb{P}^2 := \mathbb{P}(U'), U' \cong \mathbb{C}^3$:

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to U' \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to \Theta_{\mathbb{P}^2} \to 0.$$

From this follows

$$\mathcal{O}_{\mathbb{P}^2}(-K_{\mathbb{P}^2}) \cong \wedge^3 U' \otimes \mathcal{O}_{\mathbb{P}^2}(3).$$

Therefore the isomorphism $\mathbb{C}[x_1, x_2, x_3]_3 \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-K_{\mathbb{P}^2}))$ is given by the map:

$$P \mapsto P(dx_1 \wedge dx_2 \wedge dx_3)^{-1}.$$

Hence we identify V' with V via the map: $V' \to V,\ s_{ij} \mapsto s_{ij}(dx_1 \wedge dx_2 \wedge dx_3)^{-1}$. In this way we can read out on V' the given action of $\mathfrak{S}_5$ on V. We obtain immediately:

Lemma 2.4 *$\mathfrak{S}_3 \le \mathfrak{S}_5$ acts on V by*

$$\tau(s_{ij}) = \epsilon(\tau) s_{\tau(i)\tau(j)}.$$

Hence the representation V restricted to $\mathfrak{S}_3 \le \mathfrak{S}_5$ is the regular representation tensored by the sign character ϵ, while V restricted to $\mathfrak{S}_4 \le \mathfrak{S}_5$ is isomorphic to the subspace of $S^3((U')^\vee) \otimes \epsilon$ corresponding to the polynomials vanishing in $p_1, \dots, p_4$.

Proof Direct computation for $\mathfrak{S}_3$, for $\mathfrak{S}_4$ observe that the tranposition (3, 4) acts sending

$$dx_1 \wedge dx_2 \wedge dx_3 \mapsto d(x_1 - x_3) \wedge d(x_2 - x_3) \wedge d(-x_3) = -dx_1 \wedge dx_2 \wedge dx_3. \tag{2.1}$$

□

Remark 2.5 Since $s_{ij}\sigma_{ij} = \Sigma$ and Σ is an eigenvector for the sign representation, the previous lemma implies immediately that for $\tau \in \mathfrak{S}_3 \le \mathfrak{S}_5$ we have:

$$\tau(\sigma_{ij}) = \sigma_{\tau(i)\tau(j)}.$$

We prove the following

Theorem 2.6 (1) *The character vector χ_V of the $\mathfrak{S}_5$-representation $V = H^0(Y, \mathcal{O}_Y(-K_Y))$ is equal to* $(6, 0, -2, 0, 0, 1, 0)$. *Therefore $H^0(Y, \mathcal{O}_Y(-K_Y))$ is the unique six dimensional irreducible representation of $\mathfrak{S}_5$.*

(2) *The character of* $\bigwedge^6 V$ *is the sign character* $\epsilon = \chi_2$.

For the convenience of the reader and for the purpose of fixing the notation we recall in Table 1 the character table of $\mathfrak{S}_5$. Here, cf. [14], pages 199–202, the natural permutation representation on $\mathbb{Z}/5$ yields $\chi_1 + \chi_4$, while $\Lambda^2(\chi_4) = \chi_7$, and $S^2(\chi_4) = \chi_1 + \chi_4 + \chi_5$. Finally, $\chi_3 = \chi_4 \otimes \chi_2$, $\chi_6 = \chi_5 \otimes \chi_2$, and obviously $\chi_7 = \chi_7 \otimes \chi_2$.

Remark 2.7 Indeed, as we shall also see later, $V = W_7$, $W = W_5$ and $U_4 = W_4$.

Proof (1) We have already observed that for $\tau \in \mathfrak{S}_3$ we have $\tau(s_{ij}) = \epsilon(\tau) s_{\tau(i)\tau(j)}$.

Recall that the transposition $\tau := (3, 4)$ acts as

$$\tau \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 \\ x_2 - x_3 \\ -x_3 \end{pmatrix}.$$

We use formula 2.1, obtaining that (3, 4) acts by:

- $s_{12} = x_1 x_2 (x_2 - x_3) \mapsto -(x_1 - x_3)(x_2 - x_3) x_2 = \sigma_{31}$;
- $s_{13} = x_1 x_3 (x_3 - x_2) \mapsto -(x_1 - x_3)(-x_3)(-x_2) = s_{23}$;
- $s_{21} = x_2 x_1 (x_1 - x_3) \mapsto -(x_2 - x_3)(x_1 - x_3) x_1 = -\sigma_{32}$;
- $s_{23} \mapsto s_{13}$;
- $s_{31} = x_1 x_3 (x_1 - x_2) \mapsto -(x_1 - x_3)(-x_3)(x_1 - x_2) = -\sigma_{12}$;
- $s_{32} = x_2 x_3 (x_2 - x_1) \mapsto -(x_2 - x_3)(-x_3)(x_2 - x_1) = \sigma_{21}$.

Next we consider the action of $\tau = (4, 5)$. This time we do not have an action on the space of polynomials, hence we work on the affine chart $\{x_3 = 1\}$.

Then: $\tau \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{x_1} \\ \frac{1}{x_2} \end{pmatrix}$ and we have that (4, 5) acts by:

$$\tau((dx_1 \wedge dx_2)^{-1}) = \left(d\frac{1}{x_1} \wedge d\frac{1}{x_2} \right)^{-1} = \left(\frac{1}{x_1^2} dx_1 \wedge \frac{1}{x_2^2} dx_2 \right)^{-1} = x_1^2 x_2^2 (dx_1 \wedge dx_2)^{-1}.$$

Table 1 Character table of $\mathfrak{S}_5$

Conj. Class	1	(1, 2)	(1, 2)(3, 4)	(1, 2, 3)	(1, 2, 3, 4)	(1, 2, 3, 4, 5)	(1, 2, 3)(4, 5)
$\chi_1 = \chi_{W_1}$	1	1	1	1	1	1	1
$\chi_2 = \chi_{W_2} =: \epsilon$	1	−1	1	1	−1	1	−1
$\chi_3 = \chi_{W_3}$	4	−2	0	1	0	−1	1
$\chi_4 = \chi_{W_4}$	4	2	0	1	0	−1	−1
$\chi_5 = \chi_{W_5}$	5	1	1	−1	−1	0	1
$\chi_6 = \chi_{W_6}$	5	−1	1	−1	1	0	−1
$\chi_7 = \chi_{W_7}$	6	0	−2	0	0	1	0

Therefore (again here $\{i,j,k\} = \{1,2,3\}$)

$$\begin{aligned} s_{ij}(dx_1 \wedge dx_2)^{-1} =& x_i x_j (x_j - x_k)(dx_1 \wedge dx_2)^{-1} \mapsto \frac{1}{x_i}\frac{1}{x_j}\frac{x_k - x_j}{x_j x_k} x_1^2 x_2^2 (dx_1 \wedge dx_2)^{-1} \\ =& \frac{x_1^2 x_2^2 x_3^2}{x_i x_j^2 x_k}(x_k - x_j)(dx_1 \wedge dx_2)^{-1} = x_i x_k (x_k - x_j)(dx_1 \wedge dx_2)^{-1} \\ =& s_{ik}(dx_1 \wedge dx_2)^{-1}. \end{aligned} \tag{2.2}$$

Now we are ready to calculate the trace of τ for

$$\tau \in \{(1,2), (1,2)(3,4), (1,2,3), (1,2,3,4), (1,2,3,4,5), (1,2,3)(4,5)\}.$$

$\underline{\tau = (1,2)}$:

$$s_{12} \leftrightarrow -s_{21},\ s_{13} \leftrightarrow -s_{23},\ s_{31} \leftrightarrow -s_{32},$$

hence $\chi_V(\tau) = \operatorname{trace}(\tau) = 0$.

$\underline{\tau = (1,2)(3,4)}$:

$$\begin{aligned} &s_{12} \mapsto \sigma_{32} = s_{21} + s_{13} - s_{31},\ s_{13} \mapsto -s_{13},\ s_{21} \mapsto -\sigma_{31} = s_{12} + s_{23} - s_{32}, \\ &s_{23} \mapsto -s_{23},\ s_{31} \mapsto -\sigma_{21} = s_{23} - s_{13} - s_{32},\ s_{32} \mapsto \sigma_{12} = s_{13} - s_{31} - s_{23}, \end{aligned}$$

hence $\chi_V(\tau) = \operatorname{trace}(\tau) = -2$.

If we use the character table of $\mathfrak{S}_5$ we can now immediately conclude, since V has dimension 6 and the trace of (1, 2)(3, 4) is nonnegative for all other representations, except that $\chi_7((1,2)(3,4)) = -2$.

For completeness we calculate explicitly the action of the other representatives of all the conjugacy classes (thereby giving a selfcontained proof that V is irreducible).

$\underline{\tau = (1,2,3)}$:

$$\begin{aligned} &s_{12} \mapsto s_{23},\ s_{13} \mapsto s_{21},\ s_{21} \mapsto s_{32}, \\ &s_{23} \mapsto s_{31},\ s_{31} \mapsto s_{12},\ s_{32} \mapsto s_{13}, \end{aligned}$$

hence $\chi_V(\tau) = \operatorname{trace}(\tau) = 0$.

$\underline{\tau = (1,2,3,4)}$:

$$\begin{aligned} &s_{12} \mapsto \sigma_{12} = s_{13} - s_{31} - s_{23},\ s_{13} \mapsto s_{31},\ s_{21} \mapsto -\sigma_{13} = s_{12} - s_{21} - s_{32}, \\ &s_{23} \mapsto s_{21},\ s_{31} \mapsto -\sigma_{23} = s_{12} - s_{21} + s_{31},\ s_{32} \mapsto \sigma_{32} = s_{13} - s_{31} + s_{21}, \end{aligned}$$

hence $\chi_V(\tau) = \operatorname{trace}(\tau) = 0$.

$\underline{\tau = (1,2,3,4,5)}$:

$$\begin{aligned} &s_{12} \mapsto s_{31},\ s_{13} \mapsto \sigma_{12} = s_{13} - s_{31} - s_{23},\ s_{21} \mapsto s_{21}, \\ &s_{23} \mapsto -\sigma_{13} = s_{12} - s_{21} - s_{32},\ s_{31} \mapsto -\sigma_{32} = s_{13} - s_{31} + s_{21},\ s_{32} \mapsto -\sigma_{23} = s_{12} - s_{21} + s_{31}, \end{aligned}$$

hence $\chi_V(\tau) = \operatorname{trace}(\tau) = 1$.

$\underline{\tau = (1,2,3)(4,5)}$:

$$\begin{aligned} &s_{12} \mapsto s_{21},\ s_{13} \mapsto s_{23},\ s_{21} \mapsto s_{31}, \\ &s_{23} \mapsto s_{32},\ s_{31} \mapsto s_{13},\ s_{32} \mapsto s_{12}, \end{aligned}$$

hence $\chi_V(\tau) = \operatorname{trace}(\tau) = 0$.

The above explicit calculations show that the character vector for the representation V is

$$\chi_V = (6, 0, -2, 0, 0, 1, 0).$$

hence $\langle \chi_V, \chi_V \rangle = 1$ and V is an irreducible representation.

(2) A basis of $\bigwedge^6 V$ consists of the element $\omega := s_{12} \wedge s_{13} \wedge s_{21} \wedge s_{23} \wedge s_{31} \wedge s_{32}$. Then $\tau = (1, 2)$ maps ω to

$$s_{21} \wedge s_{23} \wedge s_{12} \wedge s_{13} \wedge s_{32} \wedge s_{31} = -\omega,$$

whence the claim follows. □

3 Quadratic equations for the Del Pezzo surface of degree 5 and geometrical descriptions of $\mathfrak{S}_5$-representations

We observe that by Riemann-Roch and Kodaira vanishing we have:

$$h^0(\mathcal{O}_Y(-2K_Y)) = \chi(\mathcal{O}_Y(-2K_Y)) = \frac{1}{2}(-2K_Y)(-3K_Y) + 1 = 16.$$

The above formula can also be checked by direct calculation of the space of homogeneous polynomials $F(x_1, x_2, x_3)$ of degree 6 vanishing of multiplicity 2 at the points $p_1, \dots, p_4$. This direct calculation also shows that the natural morphism of $\mathfrak{S}_5$-representations:

$$\Phi : \operatorname{Sym}^2(V) \cong \mathbb{C}^{21} \to H^0(Y, \mathcal{O}_Y(2))(= H^0(Y, \mathcal{O}_Y(-2K_Y))) \cong \mathbb{C}^{16}.$$

is surjective.

We set $W := \ker(\Phi)(\cong \mathbb{C}^5) = H^0(Y, \mathcal{I}_Y(2))$.

Observe that, setting $Q_{ij} := s_{ij}\sigma_{ij}, 1 \leq i \neq j \leq 3$, we have that $\Phi(Q_{ij}) = \Sigma \in H^0(Y, \mathcal{O}_Y(2))$. It is easy to verify that the set $\{Q_{ij} \mid 1 \leq i \neq j \leq 3\}$ is $\mathbb{C}$-linearly independent and therefore

$$\begin{aligned} &q_{12} := Q_{12} - Q_{32},\ q_{13} := Q_{13} - Q_{32},\ q_{21} := Q_{21} - Q_{32}, \\ &q_{23} := Q_{23} - Q_{32},\ q_{31} := Q_{31} - Q_{32} \end{aligned}$$

are linearly independent elements of W, hence form a basis.

If we denote by W' the vector subspace of $\operatorname{Sym}^2(V)$ spanned by the Q_{ij}, $1 \leq i \neq j \leq 3$, then we have the following inclusions, where $\mathcal{I}$ is the sheaf of ideals of the functions vanishing on Y:

$$H^0(Y, \mathcal{I}_Y(2)) = W \subset W' = \langle Q_{ij} \mid\ 1 \leq i \neq j \leq 3 \rangle \subset \operatorname{Sym}^2(V).$$

3.1 Geometrical and combinatorial pentagons

We want to give now some geometrical background to the choice of the sections s_{ij}, σ_{ij}, and the quadratic forms Q_{ij}, which makes our calculations less mysterious.

Definition 3.1 Let $\mathcal{S}$ be a set. Then

(1) an ordered *combinatorial* n-gon on $\mathcal{S}$ is a map $p : \mathbb{Z}/n \to \mathcal{S}$;
(2) the i-th *side* L_i is the restriction of p to $\{i, i+1\}$;

(3) an ordered combinatorial n-gon on $\mathcal{S}$ is said to be *nondegenerate* if p is injective;
(4) an *oriented combinatorial n-gon* on $\mathcal{S}$ is an equivalence class of ordered combinatorial n-gons on $\mathcal{S}$ for the action of $\mathbb{Z}/n$ given by composition on the source $p(i) \sim p(i+a)$;
(5) a(n unoriented) *combinatorial n-gon* on $\mathcal{S}$ is an equivalence class of oriented combinatorial n-gons on $\mathcal{S}$, where $p(i) \sim p(-i)$ (that is, an equivalence class of ordered n-gons for the action of the dihedral group D_n, $p(i) \sim p(\pm i + a)$);
(6) the *neighbouring n-gon* of $p(i)$ is the unoriented n-gon $p(2i)$; it is nondegenerate, for n odd, iff p is nondegenerate, moreover in this case it has no sides in common with the n-gon $p(i)$ once $n \geq 5$;
(7) a *double combinatorial pentagon* ($n = 5$) is the (unordered) pair of two neighbouring nondegenerate (unoriented) combinatorial pentagons.
(8) If $\mathcal{S}$ is a linear space, then a *geometrical* n-gon is the union of n distinct lines L_i associated to a combinatorial pentagon $p(i)$ in such a way that L_i is the line joining $p(i)$ with $p(i+1)$.

Remark 3.2 If the set $\mathcal{S}$ has n elements, and we consider only nondegenerate n-gons, then the objects in item (1) are the elements of the set $\mathfrak{S}_n$, those in item (4) are the cosets $\mathfrak{S}_n/(\mathbb{Z}/n)$, for item (5) we get the cosets $\mathfrak{S}_n/D_n$, and finally for item (7) we have the cosets $\mathfrak{S}_5/\operatorname{Aff}(1, \mathbb{Z}/5)$.

Proposition 3.3 *Let Y be a Del Pezzo surface of degree 5.*

(a) *The geometrical pentagons contained in Y are exactly 12. They are given by the divisors of zeros of the 12 sections s_{ij}, σ_{ij}.*
(b) *There is a bijection between the set of such geometrical pentagons and the set of combinatorial nondegenerate pentagons on* $\{1, 2, 3, 4, 5\}$. *This bijection associates to the combinatorial pentagon* $[i \mapsto p(i)]$ *the geometrical pentagon*

$$E_{p(1)p(2)} + E_{p(2)p(3)} + E_{p(3)p(4)} + E_{p(4)p(5)} + E_{p(5)p(1)}.$$

(c) *There is a bijection between the set of the 6 quadrics* $\{Q_{ij}\}$ *and the set of double combinatorial pentagons on* $\{1, 2, 3, 4, 5\}$.
(d) *Moreover, the subset of V*

$$\mathcal{M} := \{\pm s_{ij}, \pm \sigma_{ij}\}$$

is an orbit for the action of $\mathfrak{S}_5$ on V, and the stabilizer of s_{21} is the cyclic subgroup generated by $(1, 2, 3, 4, 5)$. *In particular $\mathfrak{S}_4$ acts simply transitively on $\mathcal{M}$.*

Figures 2 and 3 illustrate items (6), (7), (b) and (c) above.

Proof We know that the only lines contained in Y are the 10 lines E_{ij}, and it is straightforward to verify that their union is the divisor $\operatorname{div}(\Sigma)$.

Let $L_1 + L_2 \cdots + L_5$ be a geometrical pentagon; then L_i intersects L_{i+1}, hence we get 5 distinct pairs A_i of elements in $\{1, 2, 3, 4, 5\}$ such that A_i and A_{i+1} are disjoint, $\forall i \in \mathbb{Z}/5$.

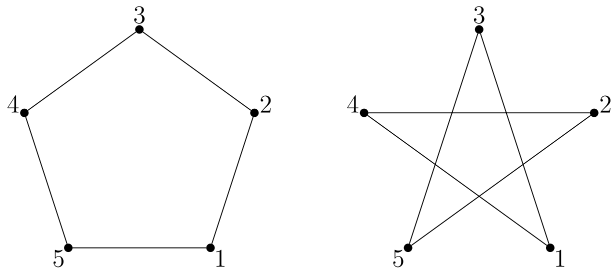

Fig. 2 A double combinatorial pentagon

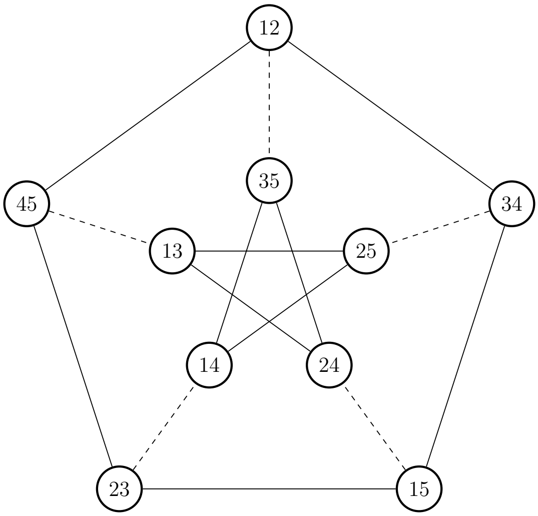

Fig. 3 The pair of disjoint geometric pentagons in the Petersen graph associated to the above double combinatorial pentagon

Hence A_{i+1} is disjoint from the union of A_i and A_{i+2}, in particular A_i and A_{i+2} have exactly one element in common.

We consider the[2] 5-gon B_i, where $B_i := A_{2i}$. We denote by

$$p(i+1) := B_i \cap B_{i+1} = A_{2i} \cap A_{2i+2}.$$

Hence $B_i = \{p(i), p(i+1)\}$, and we get a nondegenerate combinatorial pentagon.

Observe that we can recover the A_i's from the B_j's, simply letting $A_i := B_{2i}$.

By item (5) above the number of nondegenerate combinatorial pentagons on $(1, 2, 3, 4, 5)$ is $120/10 = 12$.

Direct inspection shows then that

$$\operatorname{div}(s_{12}) = E_{14} + E_{42} + E_{23} + E_{35} + E_{51}.$$

Similar formulae can be computed directly for $\operatorname{div}(s_{ij})$, we may however observe that $\{\operatorname{div}(s_{ij})\}$ is the $\mathfrak{S}_3$-orbit of $\operatorname{div}(s_{12})$, hence $\operatorname{div}(s_{ij})$ is a geometric pentagon.

[2] it is the dually neighbouring pentagon, where the dual n-gon is associated to the sequence of sides $L_1, \dots, L_n$.

Since div (Σ) is the union of the 10 lines of Y, follows that div (σ_{ij}) is associated to the neighbouring combinatorial pentagon of the one of div (s_{ij}).

Hence the quadrics Q_{ij} correspond bijectively to double pentagons (these are 6).

The calculations performed above in the proof of Theorem 2.6 show that if $\tau := (1,2,3)(4,5)$, then $\langle\tau\rangle$ permutes simply transitively the set $\{s_{ij}\}$, hence it sends the set $\{\sigma_{ij}\}$ to the set $\{-\sigma_{ij}\}$. While $\mathfrak{S}_3$ permutes the set $\mathcal{M}' := \{\pm s_{ij}\}$, hence also the set $\{\pm\sigma_{ij}\}$. Finally, (3, 4) permutes the set $\mathcal{M}$. Since the above elements generate $\mathfrak{S}_5$, $\mathcal{M}$ is $\mathfrak{S}_5$-invariant. The orbit of s_{21} inside $\mathcal{M}'$ for the subgroup generated by $\mathfrak{S}_3$ and $\tau := (1,2,3)(4,5)$ has at least 7 elements, hence it equals $\mathcal{M}'$. Since (3, 4) sends s_{21} to $-\sigma_{32}$, follows that $\mathcal{M}$ is a single $\mathfrak{S}_5$-orbit. Hence the stabilizer of s_{21} has cardinality 5: but we know that it contains (1, 2, 3, 4, 5).

Indeed,

$$\operatorname{div}(s_{21}) = E_{25} + E_{53} + E_{31} + E_{14} + E_{42} = \sum_{i=1}^{5} E_{i,i+3}.$$

□

We can summarize part of our discussion in the following corollary.[3]

Corollary 3.4 *There is a natural bijection between the set $\mathcal{M}$ and the set of oriented nondegenerate combinatorial pentagons on $\{1,2,3,4,5\}$, in such a way that $-m$ is the oppositely oriented pentagon of m. $\mathcal{M}/\pm 1$ is then in bijection with the set $\mathcal{P}$ of pentagons, where $\pm\sigma_{ij}$ corresponds to the neighbouring pentagon of $\pm s_{ij}$.*

Defining

$$\sigma(p)(i) := p(2i),$$

we get an order 4 transformation σ on $\mathcal{M}$,such that $\sigma^2(m) = -m\ \forall m \in \mathcal{M}$, and inducing an involution on $\mathcal{M}/\pm 1$, which exchanges $\pm s_{ij}$ with $\pm\sigma_{ij}$, hence such that $(\mathcal{M}/\pm 1)/\sigma$ is in bijection with the set $\mathcal{DP}$ of double pentagons.

Proof The bijection follows immediately from the fact that we have two transitive actions, and the stabilizer of s_{21} is the cyclic subgroup generated by (1, 2, 3, 4, 5), which is also the stabilizer of the standard pentagon corresponding to the identity map.

We define then $\sigma(m)$ by the property that it associates to an oriented pentagon $p(i)$ the neighbouring oriented pentagon $p(2i)$; from this definition follows that $\sigma^2(m) = -m$. Moreover, since $-m$ is the pentagon $p(-i)$, and $p(2(-i)) = p(-(2i))$ we obtain $\sigma(-m) = -\sigma(m)$. Hence $\sigma(m)m = \sigma(-m)(-m)$ inside $S^2(V)$.[4] □

Recall now that we have bijections:

[3] a referee suggested a lengthy alternative proof of this corollary using the isomorphism with the ring of invariants of five points on the line; here to an oriented pentagon corresponds a product of Plücker coordinates of a 2×5 matrix, as done in the paper by Howard, Millson, Snowden, Vakil : 'The ideal of relations for the ring of invariants of n points on the line', JEMS 14 (2012), 1, 1–60.

[4] Indeed $\sigma(s_{ij}) = \sigma_{ij}, \sigma(\sigma_{ij}) = -s_{ij}$. Observe moreover that, by what we observed in remark 1.3, σ is not induced by a linear map of V!

- $\mathcal{M} \cong \mathfrak{S}_5/(\mathbb{Z}/5) = \mathfrak{S}_5/\langle(1,2,3,4,5)\rangle$ (oriented pentagons)
- $\mathcal{P} \cong \mathcal{M}/\pm 1 \cong \mathfrak{S}_5/D_5 = \mathfrak{S}_5/\langle(1,2,3,4,5),(1,4)(2,3)\rangle$ (pentagons)
- $\mathcal{DP} \cong (\mathcal{M}/\pm 1)/\sigma \cong \mathfrak{S}_5/\operatorname{Aff}(1,\mathbb{Z}/5) = \mathfrak{S}_5/\langle(1,2,3,4,5),(1,2,4,3)\rangle$ (double pentagons)

Remark 3.5 Observe that σ corresponds to multiplication on the right by (1, 2, 4, 3) on the cosets gH, where H is any of the above three subgroups.

Next we show that all irreducible representations of $\mathfrak{S}_5$ of dimension different from 4[5] are contained in the permutation representation associated to the set $\mathcal{M}$, i.e., to the set of oriented pentagons.

Theorem 3.6 *Let M be the permutation representation associated to $\mathcal{M}$, D the permutation representation associated to $\mathcal{DP}$, P the permutation representation associated to $\mathcal{P}$.*

Then

- $W' = W \oplus \epsilon = D \otimes \epsilon$, and
- $M = V \oplus (V \otimes \epsilon) \oplus D \oplus (D \otimes \epsilon) = V \oplus V \oplus [(W \otimes \epsilon) \oplus \chi_1] \oplus [W \oplus \epsilon]$.

Proof First of all, σ induces an action of $\mathbb{Z}/4$ on M, hence M splits according to the 4 eigenvalues $\pm\sqrt{-1}, \pm 1$.

The eigenspace for $+1$ is generated by the vectors (of M!)

$$m + \sigma(m) + (-m) + (-\sigma(m)),$$

hence it clearly corresponds to the representation D on the double pentagons.

The eigenspace D' for -1 is generated by the vectors

$$m - (\sigma(m)) + (-m) - (-\sigma(m)),$$

and together with the previous yields as direct sum the $+1$-eigenspace for $-1 = \sigma^2$, which is clearly P, the representation associated to the pentagons. Observe that D' contains the sign representation ϵ.

The eigenspace V' for $\sqrt{-1}$ is generated by the vectors $m - \sqrt{-1}(\sigma(m)) - (-m) + \sqrt{-1}(-\sigma(m))$. The Galois group of $\lambda^2 = -1$ (complex conjugation) yields an isomorphism of V' with the eigenspace V'' for $-\sqrt{-1}$.

Step 1 $V' \cong V$.

This follows immediately from Schur's Lemma: indeed M surjects onto V, hence V appears as a summand of M. But D and D' are reducible, hence P maps trivially to V, and V' is isomorphic to V. Therefore we have

$$M = D \oplus D' \oplus V^{\oplus 2},$$

and we know that $V \cong V \otimes \epsilon$.

[5] These are easily gotten by the standard permutation representation on 5 elements!

Step 2 We calculate the character of M.

In order to achieve this, we use the following obvious

Lemma 3.7 *Consider a transitive permutation representation of a finite group G on G/H. Then its character χ_H is given by:*

$$\chi_H(\gamma) = |\{gH|\gamma gH = gH\}| = \frac{1}{|H|}|\{g|\gamma \in gHg^{-1}\}|.$$

In the case of the oriented pentagons, H is spanned by (1, 2, 3, 4, 5) and $\chi_H(\gamma) = 0$ unless γ has order dividing 5, in which case $\chi_H(Id) = 24$, while $\chi_H((1,2,3,4,5)) = 4$, since the normalizer of H is the affine group $\mathrm{Aff}(1, \mathbb{Z}/5)$.

Thus in this case the character vector equals

$$\chi_M = (24, 0, 0, 0, 0, 4, 0).$$

Step 3 $\chi_D = \chi_6 + \chi_1$, $D' = D \otimes \epsilon$.

Subtracting twice the character of V, we obtain $\chi_P = (12, 0, 4, 0, 0, 2, 0)$. Alternatively, and also in order to calculate the character of D, we again apply the above lemma.

If we consider pentagons and double pentagons (i.e. $H = D_5$ resp. $H = \mathrm{Aff}(1, \mathbb{Z}/5)$), observe that in both cases gHg^{-1} contains no elements of order 3 and in the case of pentagons no transpositions and no 4-cycles. Hence the character is zero, in the case of pentagons, unless we have 5-cycles, double transpositions, or the identity.

A 5-cycle fixes 2 pentagons, and exactly one double pentagon. A double transposition fixes 4 pentagons, and 2 double pentagons. A 4-cycle fixes 2 double pentagons: for instance, if $i \mapsto 2i$ fixes $\{(0, a_1, a_2, a_3, a_4), (0, a_2, a_4, a_1, a_3)\}$, then it must preserve the block decomposition $\{1, 2, 3, 4\} = \{a_1, a_4\} \cup \{a_2, a_3\}$ corresponding to the two sets of neighbours of 0; since it does not fix any pentagon, it must exchange these two sets, and then it must be either $a_2 = 2a_1$ or $a_3 = 2a_1$. Up to choosing an appropriate representative, we may assume $a_1 = 1$; then it must be $a_4 = -1$, and the only two choices are $a_2 = 2$ or $a_2 = -2$.

We conclude from these calculations that

$$\begin{aligned}\chi_P =&(12, 0, 4, 0, 0, 2, 0),\\ \chi_D =&(6, 0, 2, 0, 2, 1, 0) = \chi_1 + \chi_6\end{aligned}$$

We recover in this way that

$$\chi_M - \chi_P = (12, 0, -4, 0, 0, 2, 0) = 2\chi_V = 2\chi_7.$$

and moreover we obtain:

$$\chi_P - \chi_D = (6, 0, 2, 0, -2, 1, 0) = \chi_{D\otimes\epsilon}.$$

This implies that $D' = D \otimes \epsilon$.

Step 4 We shall show that $D \cong W' \otimes \epsilon$ and $W' \cong W \oplus \epsilon$.

In fact, $W' = \langle Q_{ij} \rangle$ surjects onto the one dimensional representation ϵ corresponding to $\mathbb{C}\Sigma \subset H^0(\mathcal{O}_Y(2))$. In particular, $W' = W \oplus \epsilon$.

Observe now that, for each $\tau \in \mathfrak{S}_3$,

$$\tau(Q_{ij}) = \epsilon(\tau) Q_{\tau(i)\tau(j)},$$

while, defining

$$d_{ij} := \big((s_{ij}) + (-s_{ij}) + (\sigma_{ij}) + (-\sigma_{ij})\big),$$

we have $\tau(d_{ij}) = d_{\tau(i)\tau(j)}$.

Hence W' and $D \otimes \epsilon$ have the same character on $\mathfrak{S}_3$. It follows that also W' is the direct sum of an irreducible five dimensional representation with a one dimensional one, and either $W \cong D$ or $W \cong D \otimes \epsilon$. The first possibility is excluded since W' contains ϵ. □

Remark 3.8 We have indeed precise formulae for the action on the space W'.

By Lemma 2.4 and Remark 2.5 we know that for $\tau \in \mathfrak{S}_3$ we have that $\tau(Q_{ij}) = \operatorname{sgn}(\tau) Q_{\tau(i)\tau(j)}$.

In particular:

$\underline{\tau = (1,2)}$:

$$Q_{12} \leftrightarrow -Q_{21},\ Q_{13} \leftrightarrow -Q_{23},\ Q_{31} \leftrightarrow -Q_{32}.$$

From the proof of Theorem 2.6 we see that other elements of $\mathfrak{S}_5$ act as follows:

$\underline{\tau = (3,4)}$:

$$Q_{12} \leftrightarrow -Q_{31},\ Q_{13} \leftrightarrow -Q_{23},\ Q_{21} \leftrightarrow -Q_{32},$$

$\underline{\tau = (4,5)}$:

$$Q_{12} \leftrightarrow -Q_{13},\ Q_{21} \leftrightarrow -Q_{23},\ Q_{31} \leftrightarrow -Q_{32},$$

$\underline{\tau = (1,2)(3,4)}$:

$$Q_{12} \leftrightarrow Q_{32},\ Q_{13} \mapsto Q_{13},\ Q_{21} \leftrightarrow Q_{31},\ Q_{23} \mapsto Q_{23}.$$

$\underline{\tau = (1,2,3,4)}$:

$$Q_{12} \mapsto -Q_{12},\ Q_{13} \mapsto -Q_{31},\ Q_{21} \mapsto -Q_{13},$$
$$Q_{23} \mapsto -Q_{21},\ Q_{31} \mapsto -Q_{23},\ Q_{32} \mapsto -Q_{32}.$$

$\underline{\tau = (1,2,3,4,5)}$:

$$Q_{12} \mapsto Q_{31},\ Q_{13} \mapsto Q_{12},\ Q_{21} \mapsto Q_{21},$$
$$Q_{23} \mapsto Q_{13},\ Q_{31} \mapsto Q_{32},\ Q_{32} \mapsto Q_{23}.$$

$\underline{\tau = (1,2,3)(4,5)}$:

$$Q_{12} \mapsto -Q_{21},\ Q_{13} \mapsto -Q_{23},\ Q_{21} \mapsto -Q_{31},$$
$$Q_{23} \mapsto -Q_{32},\ Q_{31} \mapsto -Q_{13},\ Q_{32} \mapsto -Q_{12}.$$

For future use we also show:

Lemma 3.9 *The determinant of W is the trivial representation, equivalently, $\bigwedge^6 W' = \epsilon$.*

Proof Since $W' \cong W \oplus \epsilon$, we have that $\bigwedge^6 W' = \epsilon$ if and only if $\bigwedge^5 W = \chi_1$.

A basis of $\bigwedge^6 W'$ consists of the element $\omega := Q_{12} \wedge Q_{13} \wedge Q_{21} \wedge Q_{23} \wedge Q_{31} \wedge Q_{32}$. Then $\tau = (1, 2)$ maps ω to

$$Q_{21} \wedge Q_{23} \wedge Q_{12} \wedge Q_{13} \wedge Q_{32} \wedge Q_{31} = -\omega,$$

whence the claim follows. $\square$

4 $\mathfrak{S}_5$-equivariant resolution of the sheaf of regular functions on the Del Pezzo surface $Y \subset \mathbb{P}^5$ of degree 5

By Theorem 2.6, 2) we know that $\chi_{\bigwedge^6 V} = \chi_2 = \epsilon$.

Consider the Euler sequence on $\mathbb{P}^5 := \mathbb{P}(V)$, $V = H^0(Y, \mathcal{O}_Y(-K_Y))$:

$$0 \to \Omega^1_{\mathbb{P}^5} \to V \otimes \mathcal{O}_{\mathbb{P}^5}(-1) \to \mathcal{O}_{\mathbb{P}^5} \to 0.$$

Then

$$\bigwedge^6 (V \otimes \mathcal{O}_{\mathbb{P}^5}(-1)) \cong \bigwedge^5 \Omega^1_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}^5},$$

and this implies that

$$\omega_{\mathbb{P}^5} \cong (\det V) \otimes \mathcal{O}_{\mathbb{P}^5}(-6) \cong \mathcal{O}_{\mathbb{P}^5}(-6) \otimes \epsilon,$$

where ϵ is the sign representation.

We take a Hilbert resolution of $\mathcal{O}_Y$ ($P := \mathbb{P}^5$):

$$0 \to U \otimes \mathcal{O}_P(-5) \to W'' \otimes \mathcal{O}_P(-3) \to W \otimes \mathcal{O}_P(-2) \to \mathcal{O}_P \to \mathcal{O}_Y \to 0, \qquad (4.1)$$

where W, W'', U are $\mathfrak{S}_5$-representations (a posteriori they shall be irreducible of respective dimensions 5, 5, 1, at this stage we only know that $\dim(W) = 5$).

Applying $\mathcal{H}om(\cdot, \omega_P)$ to the above resolution, we obtain a resolution of

$$\begin{aligned} &\mathcal{E}xt^3(\mathcal{O}_Y, \omega_P) \cong \omega_Y \cong \mathcal{O}_Y(-1) : \\ &0 \to \omega_P \to W^\vee \otimes \omega_P(2) \to (W'')^\vee \otimes \omega_P(3) \to U^\vee \otimes \omega_P(5) \to \mathcal{O}_Y(-1) \to 0. \end{aligned} \qquad (4.2)$$

This equals

$$\begin{aligned} &0 \to \epsilon \otimes \mathcal{O}_P(-6) \to \epsilon \otimes W^\vee \otimes \mathcal{O}_P(-4) \to (W'')^\vee \otimes \epsilon \otimes \mathcal{O}_P(-3) \\ &\quad \to \epsilon \otimes U^\vee \otimes \mathcal{O}_P(-1) \to \mathcal{O}_Y(-1) \to 0, \end{aligned} \qquad (4.3)$$

which after twisting by $\mathcal{O}_P(1)$ becomes:

$$\begin{aligned} &0 \to \epsilon \otimes \mathcal{O}_P(-5) \to \epsilon \otimes W^\vee \otimes \mathcal{O}_P(-3) \to (W'')^\vee \otimes \epsilon \otimes \mathcal{O}_P(-2) \\ &\quad \to \epsilon \otimes U^\vee \otimes \mathcal{O}_P \to \mathcal{O}_Y \to 0. \end{aligned} \qquad (4.4)$$

By the uniqueness of a minimal graded free resolution up to isomorphism, we get that (4.1) is isomorphic to (4.4), hence:

$$U \cong \epsilon, \quad W'' \cong W^\vee \otimes \epsilon.$$

Therefore we have proven the following:

Proposition 4.1 *The self dual Hilbert resolution of $\mathcal{O}_Y$ is given by:*

$$0 \to \epsilon \otimes \mathcal{O}_P(-5) \xrightarrow{B^\vee} \epsilon \otimes W^\vee \otimes \mathcal{O}_P(-3) \xrightarrow{A} W \otimes \mathcal{O}_P(-2) \xrightarrow{B} \mathcal{O}_P \to \mathcal{O}_Y \to 0, \tag{4.5}$$

where $A \in W \otimes (W \otimes \epsilon) \otimes V$, $B \in W^\vee \otimes S^2(V) = \mathrm{Hom}(W, S^2(V))$, *and* A *is anti symmetric.*

Observe that $W \otimes (W \otimes \epsilon) \otimes V \cong W \otimes W \otimes V$ as representations, since $V \cong V \otimes \epsilon$. Therefore we look for $A \in (\wedge^2 W \otimes V)^{\mathfrak{S}_5}$, and we claim that A is uniquely determined up to scalars.

Lemma 4.2 *The natural inclusion*

$$\wedge^2 W \otimes V \subset \wedge^2 W' \otimes V$$

induces an isomorphism

$$(\wedge^2 W \otimes V)^{\mathfrak{S}_5} \cong (\wedge^2 W' \otimes V)^{\mathfrak{S}_5} \cong \mathbb{C}.$$

Proof Observe that $\wedge^2 W' \cong \wedge^2(W \oplus W_2) \cong \wedge^2 W \oplus (W \otimes W_2) \cong \wedge^2 W \oplus W_6$, where $W_6 \cong W \otimes \epsilon$ and $\wedge^2 W \cong V \oplus W_3$ (as it is easily checked by the character formula

$$\chi_{\wedge^2 W}(g) = \frac{1}{2}(\chi_W(g)^2 - \chi_W(g^2)).)$$

By Schur's lemma it follows that

$$(\wedge^2 W' \otimes V)^{\mathfrak{S}_5} \cong ((\wedge^2 W \oplus W_6) \otimes V)^{\mathfrak{S}_5} \cong (\wedge^2 W \otimes V)^{\mathfrak{S}_5} \cong$$
$$\cong ((V \oplus W_3) \otimes V)^{\mathfrak{S}_5} \cong (V \otimes V)^{\mathfrak{S}_5} \cong \mathbb{C}.$$

□

We want to find the (up to a constant) unique $\mathfrak{S}_5$-equivariant map A, such that if A corresponds to a tensor $\alpha \in (\bigwedge^2 W \otimes V)^{\mathfrak{S}_5}$, then $\alpha \wedge \alpha = B$.

We have that $\alpha \wedge \alpha \in \bigwedge^4 W \otimes S^2 V$ and, since $\bigwedge^5 W \cong \mathbb{C}$ is the trivial representation, hence the pairing

$$\bigwedge^4 W \times W \to \bigwedge^5 W \cong \mathbb{C}$$

identifies $\bigwedge^4 W$ with $W^\vee$, therefore

$$\alpha \wedge \alpha = B \in W^\vee \otimes S^2(V).$$

Since we want to find the (up to a constant) unique element $A \in (\bigwedge^2 W' \otimes V)^{\mathfrak{S}_5}$, we write for $1 \leq i \neq j \leq 3, 1 \leq h \neq k \leq 3, 1 \leq m \neq n \leq 3$,:

$$A := \sum_{ij,hk,mn} a_{ij,hk,mn} (Q_{ij} \wedge Q_{hk}) \otimes s_{mn}, \; A_{ij,hk} := \sum_{mn} a_{ij,hk,mn} s_{mn},$$

and use the lexicographical order for (ij), resp. (hk), resp (mn).

Moreover, we use that $A_{ij,hk}$ is skew-symmetric, i.e. $A_{ij,hk} = -A_{hk,ij}$.
We are going to prove the following:

Theorem 4.3 *Let Y be the del Pezzo surface of degree 5, embedded anticanonically in $\mathbb{P}^5$. Then the ideal of Y is generated by the 4×4 - Pfaffians of the $\mathfrak{S}_5$ - equivariant anti-symmetric 6×6-matrix $A =$*

$$\begin{pmatrix}
0 & s_{21}+s_{23}-s_{31}-s_{32} & s_{12}+s_{31}-s_{32}-s_{21} & -s_{13}-s_{21} & s_{13}+s_{32} & s_{21}+s_{32}-s_{12}-s_{23} \\
-s_{21}-s_{23}+s_{31}+s_{32} & 0 & s_{12}+s_{23} & s_{31}+s_{23}-s_{13}-s_{32} & s_{13}+s_{21}-s_{23}-s_{31} & -s_{12}-s_{31} \\
-s_{12}-s_{31}+s_{32}+s_{21} & -s_{12}-s_{23} & 0 & s_{12}+s_{13}-s_{32}-s_{31} & s_{12}-s_{21}+s_{31}-s_{13} & s_{23}+s_{31} \\
s_{13}+s_{21} & -s_{31}-s_{23}+s_{13}+s_{32} & -s_{12}-s_{13}+s_{32}+s_{31} & 0 & -s_{21}-s_{32} & s_{23}+s_{12}-s_{13}-s_{32} \\
-s_{13}-s_{32} & -s_{13}-s_{21}+s_{23}+s_{31} & -s_{12}+s_{21}-s_{31}+s_{13} & s_{21}+s_{32} & 0 & s_{12}+s_{13}-s_{21}-s_{23} \\
-s_{21}-s_{32}+s_{12}+s_{23} & s_{12}+s_{31} & -s_{23}-s_{31} & -s_{23}-s_{12}+s_{13}+s_{23} & -s_{12}-s_{13}+s_{21}+s_{23} & 0
\end{pmatrix}$$

We first prove the following:

Proposition 4.4 *For $\{i, j, k\} = \{1, 2, 3\}$ we have:*

(1) $A_{ij,ik} = s_{ji} + s_{jk} - s_{ki} - s_{kj}$;
(2) $A_{ik,kj} = -s_{ij} - s_{ki}$;
(3) $A_{ij,ki} = s_{ik} + s_{kj}$;
(4) $A_{ik,ki} = s_{ik} + s_{ji} - s_{jk} - s_{ki}$;
(5) $A_{ij,kj} = s_{ji} - s_{ij} + s_{kj} - s_{jk}$.

Proof Observe that $\tau = (4, 5)$ sends $Q_{12} \wedge Q_{13}$ to $-Q_{12} \wedge Q_{13}$. Therefore $A_{12,13} = -\tau A_{12,13}$, since A is $\mathfrak{S}_5$ - invariant. Write

$$A_{12,13} = \alpha_{12}s_{12} + \alpha_{13}s_{13} + \alpha_{21}s_{21} + \alpha_{23}s_{23} + \alpha_{31}s_{31} + \alpha_{32}s_{32},$$

then we have

$$\begin{aligned} A_{12,13} &= \alpha_{12}s_{12} + \alpha_{13}s_{13} + \alpha_{21}s_{21} + \alpha_{23}s_{23} + \alpha_{31}s_{31} + \alpha_{32}s_{32} \\ &= -\tau A_{12,13} = \alpha_{12}s_{13} + \alpha_{13}s_{12} + \alpha_{21}s_{23} + \alpha_{23}s_{21} + \alpha_{31}s_{32} + \alpha_{32}s_{31}. \end{aligned} \tag{4.6}$$

This implies

$$\alpha_{12} = \alpha_{13}, \ \alpha_{21} = \alpha_{23}, \ \alpha_{31} = \alpha_{32}.$$

On the other hand, if we apply $\tau = (2, 3)$, we get:

$$\begin{aligned} A_{12,13} &= \alpha_{12}s_{12} + \alpha_{13}s_{13} + \alpha_{21}s_{21} + \alpha_{23}s_{23} + \alpha_{31}s_{31} + \alpha_{32}s_{32} \\ &= -\tau A_{12,13} = -\alpha_{12}s_{13} - \alpha_{13}s_{12} - \alpha_{21}s_{31} - \alpha_{23}s_{32} - \alpha_{31}s_{21} - \alpha_{32}s_{23}, \end{aligned} \tag{4.7}$$

hence

$$\alpha_{12} = \alpha_{13} = 0, \ \alpha_{21} = \alpha_{23} = -\alpha_{31} = -\alpha_{32}.$$

W.l.o.g. we set $\alpha_{21} = 1$ and we have

$$A_{12,13} = s_{21} + s_{23} - s_{31} - s_{32}. \tag{4.8}$$

Assume that $\{i,j,k\} = \{1,2,3\}$ and let $\sigma := \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$. Apply σ to equation (4.8), then we see that

$$A_{ij,ik} = s_{ji} + s_{jk} - s_{ki} - s_{kj}.$$

In particular we have: $A_{21,23} = s_{12} + s_{13} - s_{31} - s_{32}$. If we apply $\tau = (3,4)$, we obtain:

$$\begin{aligned} A_{13,32} =& - \tau A_{21,23} = -\tau(s_{12} + s_{13} - s_{31} - s_{32}) = \sigma_{31} + s_{23} - \sigma_{21} + \sigma_{12} \\ =& \, s_{32} - s_{23} - s_{12} + s_{23} - s_{32} + s_{23} - s_{13} + s_{13} - s_{31} - s_{23} = -s_{12} - s_{31}. \end{aligned} \tag{4.9}$$

Applying σ we get:

$$A_{ik,kj} = -s_{ij} - s_{ki}.$$

For $\tau = (4,5)$ we have $A_{12,31} = \tau A_{13,32} = s_{13} + s_{32}$, and applying σ we get:

$$A_{ij,ki} = s_{ik} + s_{kj}.$$

We apply $\tau = (3,4)$ to the equation $A_{12,23} = -s_{13} - s_{21}$ and obtain

$$A_{13,31} = -\tau A_{12,23} = \tau(s_{13} + s_{21}) = -s_{23} + \sigma_{32} = -s_{23} + s_{13} - s_{31} + s_{21}.$$

The general equation (4) follows again applying $\sigma := \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$.

Finally we apply $\tau = (4,5)$: $A_{12,32} = \tau A_{13,31} = s_{21} - s_{12} + s_{32} - s_{23}$ and applying $\sigma := \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$ yields

$$A_{ij,kj} = s_{ji} - s_{ij} + s_{kj} - s_{jk}.$$

□

Proof of Theorem 4.3 The matrix follows immediately from Proposition 4.4. A straightforward calculation shows that the 15 (4×4) - Pfaffians of the matrix A are:

$$\begin{aligned} P_1 =& 2Q_{12} - 2Q_{13}, \ P_2 = 2Q_{12} - 2Q_{32}, \ P_3 = 2Q_{21} - 2Q_{12}, \ P_4 = 2Q_{13} - 2Q_{21}, \\ P_5 =& 2Q_{23} - 2Q_{13}, \ P_6 = 2Q_{32} - 2Q_{23}, \ P_7 = 2Q_{21} - 2Q_{32}, \ P_8 = 2Q_{31} - 2Q_{32}, \\ P_9 =& 2Q_{31} - 2Q_{21}, \ P_{10} = 2Q_{13} - 2Q_{31}, \ P_{11} = 2Q_{31} - 2Q_{12}, \ P_{12} = 2Q_{23} - 2Q_{12}, \\ P_{13} =& 2Q_{21} - 2Q_{23}, \ P_{14} = 2Q_{32} - 2Q_{13}, \ P_{15} = 2Q_{12} - 2Q_{23}. \end{aligned}$$

This proves the theorem. □

5 $\mathfrak{S}_5$-invariant equations of $Y \subset (\mathbb{P}^1)^5$

As already mentioned in the first section there are five geometric objects permuted by the automorphism group $\mathfrak{S}_5$ of the del Pezzo surface Y of degree five: namely, 5 fibrations $\varphi_i : Y \to \mathbb{P}^1$, induced, for $1 \le i \le 4$, by the projection with centre p_i, and, for $i = 5$, by the pencil of conics through the 4 points (viewing the Del Pezzo surface as the moduli space of five ordered points on the projective line, these fibrations are just the maps to the moduli space of four ordered points on the projective line obtained forgetting the j-th of the five points). Each fibration is a conic bundle, with exactly three singular fibres, correponding to the possible partitions of type (2, 2) of the set $\{1, 2, 3, 4, 5\} \setminus \{i\}$.

We have proven in [3] the following result

Theorem 5.1 *Y embeds into $(\mathbb{P}^1)^4$ via $\varphi_1 \times \cdots \times \varphi_4$ and in $(\mathbb{P}^1)^5$ via $\varphi_1 \times \cdots \times \varphi_5$.*

Moreover, we showed[6]

Theorem 5.2 *Let $\Sigma \subset (\mathbb{P}^1)^4 =: Q$, with coordinates*

$$(v_1 : v_2), (w_1 : w_2), (z_1 : z_2), (t'_1 : t'_2),$$

be the image of the Del Pezzo surface Y via $\varphi_3 \times \varphi_1 \times \varphi_2 \times \varphi_4$. Then the equations of Σ are given by the four 3×3-minors of the following Hilbert-Burch matrix:

$$A := \begin{pmatrix} t'_2 & -t'_1 & t'_1 + t'_2 \\ v_1 & v_2 & 0 \\ w_2 & 0 & w_1 \\ 0 & -z_1 & z_2 \end{pmatrix}. \tag{5.1}$$

In particular, we have a Hilbert-Burch resolution:

$$0 \to (\mathcal{O}_Q(-\sum_{i=1}^{4} H_i))^{\oplus 3} \to \bigoplus_{j=1}^{4}(\mathcal{O}_Q(-\sum_{i=1}^{4} H_i + H_j)) \to \mathcal{O}_Q \to \mathcal{O}_\Sigma \to 0, \tag{5.2}$$

where H_i is the pullback to Q of a point in $\mathbb{P}^1$ under the i-th projection.

Observe first that each pencil $\varphi_i : Y \to \mathbb{P}^1$ can be rewritten as

$$\varphi_i : Y \to \Lambda_i \subset \mathbb{P}^2,$$

where $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5$ are the lines in $\mathbb{P}^2$ defined by equations which are consequences of the following equalities:

[6] In the previous paper the coordinates which are here denoted (t'_1, t'_2) were denoted (t_1, t_2); whereas here we reserve the notation (t_1, t_2) for $(t'_1, -t'_2)$ in order to show the $\mathfrak{S}_5$-symmetry,

$$
\begin{aligned}
y_1 &:= x_3 - x_2,\\
y_2 &:= x_1 - x_3,\\
y_3 &:= x_2 - x_1,\\
y_1 + y_2 + y_3 &= 0,\\
x_1y_1 + x_2y_2 + x_3y_3 &= 0.
\end{aligned}
$$

We set

$$
\begin{aligned}
\Lambda_1 &: = \{w_1 - w_2 + w_3 = 0\},\\
\Lambda_2 &: = \{z_1 - z_2 + z_3 = 0\},\\
\Lambda_3 &: = \{v_1 - v_2 + v_3 = 0\},\\
\Lambda_4 &: = \{t_1 - t_2 + t_3 = 0\},\\
\Lambda_5 &: = \{s_1 - s_2 + s_3 = 0\},
\end{aligned}
$$

and then the map

$$\varphi_3 \times \varphi_1 \times \varphi_2 \times \varphi_4 \times \varphi_5 \, : \, Y \to \Lambda_3 \times \Lambda_1 \times \Lambda_2 \times \Lambda_4 \times \Lambda_5$$

is expressed by:

$$
\begin{aligned}
(v_1, v_2, v_3) &: = (x_1, x_2, y_3), (w_1, w_2, w_3) := (x_2, x_3, y_1),\\
(z_1, z_2, z_3) &: = (x_3, x_1, y_2), (t_1, t_2, t_3) := (y_1, -y_2, y_3),\\
(s_1, s_2, s_3) &: = (x_1y_1 : -x_2y_2 : x_3y_3).
\end{aligned}
$$

The equation of the image of the Del Pezzo surface Y in $(\mathbb{P}^1)^3 = \Lambda_3 \times \Lambda_1 \times \Lambda_2$ under the map map $\varphi_3 \times \varphi_1 \times \varphi_2$ is then

$$v_1w_1z_1 - v_2w_2z_2 = 0.$$

We give now the action of $\mathfrak{S}_5$ on the pencils φ_i in order to determine the 10 ($\mathfrak{S}_5$-invariant) equations of the image of Y under the map $\varphi_3 \times \varphi_1 \times \varphi_2 \times \varphi_4 \times \varphi_5$, which correspond to the 10 possible projections of $(\mathbb{P}^1)^5$ to $(\mathbb{P}^1)^3$.

In fact, $\mathfrak{S}_5$ acts by permuting the indices of φ_i, but a permutation $\tau \in \mathfrak{S}_5$ maps φ_i to $\lambda_\tau \circ \varphi_{\tau(i)}$, where λ_τ is a projectivity of $\mathbb{P}^1$.

A straightforward computation (using the formulae for the action of $\mathfrak{S}_5$ on $\mathbb{P}^2$ by birational maps given in Sect. 2) gives the following table.

τ	$(1, 2)$	$(2, 3)$	$(3, 4)$	$(4, 5)$
$\tau(v_1 : v_2 : v_3)$	$(v_2 : v_1 : -v_3)$	$(z_2 : z_1 : -z_3)$	$(t_2 : t_1 : -t_3)$	$(v_2 : v_1 : -v_3)$
$\tau(w_1 : w_2 : w_3)$	$(z_2 : z_1 : -z_3)$	$(w_2 : w_1 : -w_3)$	$(w_3 : w_2 : w_1)$	$(w_2 : w_1 : -w_3)$
$\tau(z_1 : z_2 : z_3)$	$(w_2 : w_1 : -w_3)$	$(v_2 : v_1 : -v_3)$	$(-z_1 : z_3 : z_2)$	$(z_2 : z_1 : -z_3)$
$\tau(t_1 : t_2 : t_3)$	$(t_2 : t_1 : -t_3)$	$(-t_1 : t_3 : t_2)$	$(v_2 : v_1 : -v_3)$	$(s_1 : s_2 : s_3)$
$\tau(s_1 : s_2 : s_3)$	$(s_2 : s_1 : -s_3)$	$(-s_1 : s_3 : s_2)$	$(s_2 : s_1 : -s_3)$	$(t_1 : t_2 : t_3)$

Theorem 5.3 *Let* $\Sigma \subset (\mathbb{P}^1)^5 = \Lambda_3 \times \Lambda_1 \times \Lambda_2 \times \Lambda_4 \times \Lambda_5 \subset (\mathbb{P}^2)^5$, *with coordinates*

$$(v_1 : v_2 : v_3), (w_1 : w_2 : w_3), (z_1 : z_2 : z_3), (t_1 : t_2 : t_3), (s_1 : s_2 : s_3)$$

be the image of the Del Pezzo surface Y under the map $\varphi_3 \times \varphi_1 \times \varphi_2 \times \varphi_4 \times \varphi_5$. Then the equations of Σ are the following:

$$
\begin{aligned}
(1)&v_1w_1z_1 - v_2w_2z_2 = 0,\\
(2)&v_3w_1t_1 - v_2w_3t_3 = 0,\\
(3)&v_1z_3t_3 + v_3z_2t_2 = 0,\\
(4)&w_3z_1t_2 + w_2z_3t_1 = 0,\\
(5)&t_1v_1s_2 - t_2v_2s_1 = 0,\\
(6)&t_1z_2s_3 - t_3z_1s_1 = 0,\\
(7)&t_3w_2s_2 - t_2w_1s_3 = 0,\\
(8)&v_3w_2s_1 - v_1w_3s_3 = 0,\\
(9)&v_2z_3s_3 + v_3z_1s_2 = 0,\\
(10)&w_3z_2s_2 + w_1z_3s_1 = 0.
\end{aligned}
$$

Proof The equations (2)–(10) are obtained from the first one using the above table described in the following diagram:

$$1) \xrightarrow{(3,4)} 4) \xrightarrow{(4,5)} 10)$$

$$4) \xrightarrow{(2,3)} 2) \xrightarrow{(1,2)} 3) \xrightarrow{(4,5)} 9) \xrightarrow{(3,4)} 6) \xrightarrow{(2,3)} 5),$$

$$10) \xrightarrow{(2,3)} 8) \xrightarrow{(3,4)} 7),$$

□

Remark 5.4 We have seen that the equations of

$$Y' := (\varphi_1 \times \ldots \times \varphi_5)(Y) \subset (\mathbb{P}^1)^5$$

are the ten equations obtained by the ten coordinate projections $P := (\mathbb{P}^1)^5 \to (\mathbb{P}^1)^3$ and therefore we have an exact sequence

$$
\begin{aligned}
0 \to (\mathcal{O}_P(-\sum_{i=1}^{5} H_i))^{\oplus 6} \to (\bigoplus_{j=1}^{5} \mathcal{O}_P(-\sum_{i=1}^{5} H_i + H_j))^{\oplus 3} \to \\
\to \bigoplus_{h<k}(\mathcal{O}_P(-\sum_{i=1}^{5} H_i + H_k + H_h)) \to \mathcal{O}_P \to \mathcal{O}_{Y'} \to 0,
\end{aligned}
\tag{5.3}
$$

where the first syzygies are the pull-backs of the syzygies obtained for each projection $(\mathbb{P}^1)^5 \to (\mathbb{P}^1)^4$.

Observe that the shape of this resolution is the same as the Eagon-Northcott complex for a Del Pezzo surface S_4 of degree 4, cf [3].

But if this resolution were associated to a 5×3 matrix of the same type, then we would get a Del Pezzo surface of degree 4 and not of degree 5.

Hence (since K^2 is invariant for smooth deformations) we have a Hilbert scheme which is reducible (since the open set corresponding to smooth surfaces is disconnected).

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