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Fibred algebraic surfaces and commutators in the Symplectic group ☆

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We describe the minimal number of critical points and the minimal number s of singular fibres for a non isotrivial fibration of a surface S over a curve B of genus 1, exhibiting several examples and in particular constructing a fibration with $s = 1$ and irreducible singular fibre with 4 nodes.
Then we consider the associated factorizations in the mapping class group and in the symplectic group. We describe explicitly which products of transvections on homologically independent and disjoint circles are a commutator in the Symplectic group $Sp(2g, \mathbb{Z})$.

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Introduction

Our present work consists of two tightly related but different parts: the first is geometrical, and concerns fibrations $f : S \to B$ of a smooth complex algebraic surface over a smooth complex curve B, with special attention to the case where the base curve B has genus at most 1.

The second part is of algebraic nature, and determines which powers of products of certain standard transvections are a commutator in the symplectic group $Sp(2g, \mathbb{Z})$.

In the first section we begin describing the algebraic version of the so called Zeuthen-Segre formula, relating the topological Euler-Poincaré characteristic $e(S)$ with the number μ of singularities of the fibres of f (counted with multiplicity).

Then in Proposition 1.2 we consider the case where the genus b of the base curve B is 1, and describe the cases where μ is minimal ($\mu = 3$ or $= 4$). Both cases occur, the first case due to the existence of the Cartwright-Steger surface [5], the second due to Theorem 3.5 of section 3.

We proceed observing that if the base curve has genus $b \geq 2$, then there are non isotrivial fibrations without singular fibres; we also recall some basic lower bound for the number s of singular fibres of a moduli stable fibration when the base curve B has genus $b = 0, 1$. That $s = 1$ occurs for $b = 1$ was shown by Castorena Mendes-Lopes and Pirola in [6] (in their examples the singular fibre is reducible, and either with fibre genus $g = 9$, or with very high genus, here we show an example with $g = 10$), and we use a variant of their method in Theorem 3.5 to construct an example with $g = 9$ and irreducible and nodal singular fibre.

In section four we recall how to such a fibration corresponds a factorization in the Mapping class group $\mathcal{M}ap_g$, hence also in the symplectic group $Sp(2g, \mathbb{Z})$; and we recall several results, referring to [26] for results concerning symplectic fibrations.

The main point is that, if $f : S \to B$ is such that $b = 1$, and the fibre singularities are just nodes, we get that a product of Dehn twists is a commutator in $\mathcal{M}ap_g$, respectively a product of transvections is a commutator in $Sp(2g, \mathbb{Z})$.

In the next sections we treat rather exhaustively the purely algebraic question to determine which powers of the standard transvection, and of the product of transvections on homologically independent and disjoint circles, are a commutator in the Symplectic group $Sp(2g,\mathbb{Z})$.

While [26] states that the product of two Dehn twists cannot be a commutator in $\mathcal{M}ap_g$, we show that the corresponding product of transvections is a commutator in $Sp(2g,\mathbb{Z})$, for all $g \geq 2$.

1. Fibrations of compact complex surfaces over curves

Definition 1.1. Let $f : S \to B$ be a holomorphic map of a compact smooth (connected) complex surface S onto a smooth complex curve B of genus b.

By Sard's lemma, the fibre $F_y := f^{-1}(y)$ is smooth, except for a finite number of points $p_1, \dots p_s \in B$ (and then the fibres F_{p_j} are called the singular fibres).

(1) f is said to be a **fibration** if all smooth fibres are connected (equivalently, all fibres are connected). In this case we shall denote by g the genus of the fibres.

Consider a singular fibre $F_t = \sum n_i C_i$, where the C_i are distinct irreducible curves.

(2) Then the **divisorial singular locus of the fibre** is defined as the divisorial part of the critical scheme, $D_t := \sum (n_i - 1) C_i$, and the **Segre number of the fibre** is defined as

$$\mu_t := deg\mathcal{F} + D_t K_S - D_t^2,$$

where the sheaf $\mathcal{F}$ is concentrated in the singular points of the reduction $(F_t)_{red}$ of the fibre F_t, and is defined as the quotient of $\mathcal{O}_S$ by the ideal sheaf generated by the components of the vector $d\tau/s$, where $s = 0$ is the equation of D_t, and where τ is the pull-back of a local parameter at the point $t \in B$.

More concretely,

$$\tau = \Pi_j f_j^{n_j}, s = \tau/(\Pi_j f_j),$$

and the logarithmic derivative yields

$$d\tau = s[\sum_j n_j (df_j \Pi_{h\neq j} f_h)].$$

The following is the algebraic version of the Zeuthen-Segre formula, expressing how the topological Euler Poincaré characteristic $e(S)$ of S, equal to the second Chern class $c_2(S)$ of S, differs from the one of a fibre bundle (for a topological fibre bundle $e(S) = 4(g-1)(b-1)$) (see [7], [10], [9]).

Theorem 1.1 *(Modern Zeuthen-Segre formula).* Let $f : S \to B$ be a fibration of a smooth complex surface S onto a curve of genus b, and with fibres of genus g.

Then

$$e(S) = c_2(S) = 4(g-1)(b-1) + \mu,$$

where $\mu = \sum_{t \in B} \mu_t$, and μ_t is the Segre number defined above.

Moreover, $\mu_t \geq 0$, and indeed the Segre number μ_t is strictly positive, except if the fibre F_t is smooth or is a multiple of a smooth curve of genus $g = 1$.

The importance of the formula lies in the fact that the difference $\mu := e(S) - 4(b-1)(g-1)$ is always non negative.

It leads to an interpretation of μ as the total number of singular points of the fibres, counted with multiplicity, in the case where $g \neq 1$ (observe that if $g = 0$, then S is an iterated blow up of a $\mathbb{P}^1$-bundle over B, hence μ is equal to the number of blow ups, and also to the number of singular points on the fibres taken with their reduced structure).

Indeed, if the singularities of the fibres are isolated, then μ_t is the sum of the Milnor numbers of the singularities; in particular, it equals the number of singular points of the fibre if and only if all the singularities are **nodes**, i.e., critical points where there are local coordinates (x, y) such that locally $f = x^2 - y^2$ (equivalently, $f = xy$).

Most of the times, the formula is used in its non refined form: if $g > 1$, then either $\mu > 0$, or $\mu = 0$ and we have a differentiable fibre bundle.

The formula is well known using topology (see [2]), but the algebraic formula is very convenient for explicit calculations.

Let us look at the particular case where the base curve B has genus $b = 1$, hence $e(S) = \mu \geq 0$. If moreover $g \geq 2$, then we have the following proposition (using also some arguments from [8]):

Theorem 1.2. Let $f : S \to B$ be a fibration of a smooth complex surface S onto a curve of genus $b = 1$, and with fibres of genus $g \geq 2$.

Then either $e(S) = \mu = 0$, or $e(S) = \mu \geq 3$, equality holding if and only if S is a minimal surface S with $p_g(S) = q(S) = 1$ or $p_g(S) = q(S) = 2$, and with $K_S^2 = 9$. In particular S is then a ball quotient. Moreover, either

(I) all fibres are reduced, and the singular points of the fibres are either
  (I 1) 3 nodes, or
  (I 2) one tacnode ($f = y^2 - x^4$ in local coordinates), or
  (I 3) one node and one ordinary cusp ($f = y^2 - x^3$ in local coordinates), or
(II) we have one double fibre, twice a smooth curve of genus 2 (hence $g = 3$), plus one node.

If instead $e(S) = \mu = 4$ and S is minimal, then necessarily either

(1) $p_g(S) = q(S) = 1$ or
(2) $p_g(S) = q(S) = 2$, or
(3) $p_g(S) = q(S) = 3$, and then $g = 3$, the fibration has constant moduli and just two singular fibres, each twice a smooth curve of genus 2; or
(4) S is a product of two genus 2 curves in this case $p_g(S) = q(S) = 4$.

Proof. If S is not minimal, every (-1)-curve maps to a point, hence f factors as $p : S \to S'$, where S' is the minimal model, and $f' : S' \to B$. Since $e(S)$ equals $e(S')$ plus the number of blow ups, it suffices to prove the inequality in the case where S is minimal.

Since S is non ruled, we have (recall that $\chi(S) = 1 - q(S) + p_g(S)$) $K_S^2 \geq 0, \chi(S) \geq 0$.

By Noether's formula $12\chi(S) = K_S^2 + e(S)$, while the Bogomolov-Miyaoka-Yau inequality yields $K_S^2 \leq 9\chi(S)$, equality holding if and only if S is a ball quotient. Hence $e(S) \geq 3\chi(S)$; and if $\chi(S) = 0$, then necessarily also $K_S^2 = e(S) = 0$. Otherwise, $e(S) = \mu \geq 6$ unless $\chi(S) = 1$, which is equivalent to saying that $p_g(S) = q(S)$.

If $e(S) = 3$, then $\chi(S) = 1$, $K_S^2 = 9$ and we have a ball quotient. The map to B shows that $q(S) \geq b = 1$.

On the other hand, the classification of surfaces with $p_g = q$ shows that $p_g = q \leq 4$, equality holding if and only if S is a product $S = C_1 \times C_2$ of two genus 2 curves, in which case $K^2 = 8$ [4].

Moreover ([11], [20], see especially [14]) if $p_g = q = 3$ either $K^2 = 6$ or $K^2 = 8$; hence in the first case $e(S) = 6$, and in the second case $e(S) = 4$. In the latter case S is a quotient $(C \times D)/(\mathbb{Z}/2)$ where C, D are smooth curves of respective genera $2, 3$, and the group $\mathbb{Z}/2$ acts diagonally with $B := C/(\mathbb{Z}/2)$ of genus 1, and $E := D/(\mathbb{Z}/2)$ of genus 2. The only fibrations onto curves of strictly positive genus are the maps to B, respectively E. For the map $S \to B$ all the fibres are isomorphic to D, except two fibres which are the curve E counted with multiplicity 2.

Hence, if $e(S) = 3$, then necessarily $p_g = q = 1$ or $p_g = q = 2$. Moreover, since we have a ball quotient, K_S is ample and we claim that $D_t = 0$ for each non multiple fibre (this means that all n_i are equal to $n \geq 2$).

In fact, if F_t is not multiple, $D_t \cdot K_S = \sum_i (n_i - 1) K_S C_i$, while by Zariski's lemma $D_t^2 < 0$ if we do not have a multiple fibre. Since S is a ball quotient, it contains only curves of geometric genus ≥ 2, in particular of arithmetic genus ≥ 2: if C_i is not a submultiple of F_t, then $C_i^2 < 0$, hence $K_S \cdot C_i \geq 3$. This obviously contradicts $\mu \leq 3$.

If a fibre is multiple $F_t = nF'$, then

$$D_t K_S = (n-1) K_S F' = (n-1)(2g' - 2),$$

and this contribution is even, and ≤ 2 if and only if $n = 2$ and $g' = 2$. F' must be smooth, else its geometric genus would be ≤ 1, contradicting that S is a ball quotient. Hence there is only one multiple fibre, and a node on another fibre.

In fact, if a fibre is reduced, then μ_t is the sum of the Milnor numbers of the fibre singularities. Each point of multiplicity at least 3 has Milnor number at least 4, so all singularities are A_n singularities, i.e., double points, with local analytic equation $y^2 - x^{n+1}$. Their Milnor number is equal to n. $\square$

Remark 1.2.

(a) There exists a ball quotient with $q = p_g = 1$: it is the Cartwright-Steger surface [5]. For this Rito [22] asserts that there are exactly 3 singular fibres, each having a node as singularity.

(b) Ball quotients with $q = p_g = 2$ are conjectured not to exist (this would follow from the Cartwright-Steger classification if one could prove arithmeticity of their fundamental group).
The claimed proof of this fact in [29] is badly wrong.[1]

(c) a fibration with $e(S) = 4$ and with S a product of two genus 2 curves is constructed in a forthcoming section. For this the singular points of the fibres are exactly 4 nodes.

(d) Fibrations with $e(S) = 4$ and with $p_g = q = 2$ and fibre genus $4, 10$ can be obtained using a surface introduced by Polizzi, Rito and Roulleau in [21], as we now show.

Theorem 1.3. Let $p : S \to E \times E$ be the degree 4 map of the Polizzi-Rito-Roulleau surface to the Cartesian square of the Fermat elliptic curve. Set ϵ to be an automorphism of order three of the Fermat elliptic curve acting on a uniformizing parameter via multiplication with a primitive third root of 1, that we also call ϵ.

Then the maps

$$\phi_1,\ \phi_2 : E \times E \to E, \quad \phi_1(x, y) := x + y, \phi_2(x, y) := \epsilon y + x$$

produce fibrations $f_1, f_2 : S \to E$ with f_i defined as the Stein factorization of $\phi_i \circ p$. They have the following properties:

- f_2 has fibre genus $g = 4$, and two singular fibres, each consisting of a smooth genus 2 curve intersecting a smooth elliptic curve transversally in two points.
- f_1 has fibre genus $g = 10$, and only one singular fibre, having only nodes as singularities, consisting of a smooth genus 6 component intersecting two genus one components transversally each in respectively two points.

[1] According to two editors of Crelle, the article was indeed withdrawn, but published because of a technical error of the printer, which has not been publicly acknowledged by the editorial board of Crelle.

Proof. Recall that S is constructed in [21] through a degree two étale map $A \to E \times E$, corresponding to a character $\chi : \pi_1(E \times E) \to \mathbb{Z}/2$, and then S is a double cover of the blow up X of A in the two points which are the inverse image of the origin, ramified on the strict transforms of the four elliptic curves

$$\{x = 0\},\ \{y = 0\}, \{y = x\}, \{y = -\epsilon x\}$$

which meet pairwise transversally and exactly at the origin (observe that the authors use the primitive sixth root of 1, $\zeta := -\epsilon$, in their notation).

The fibres of ϕ_1 intersect these curves in respectively $1, 1, 4, 3$ points, those of ϕ_2 intersect these curves in respectively $1, 1, 1, 3$ points.

Hence the general fibre of f_1 has genus g satisfying $2g - 2 = 4 \cdot \frac{9}{2} \Rightarrow g = 10$, while the same calculation would seem to show that the general fibre of f_2 has genus g satisfying $2g - 2 = 4 \cdot \frac{6}{2} \Rightarrow g = 7$: however the fibres of $\phi_2 \circ p$ are, as we shall now show, not connected, and consist of two connected components of genus $g = 4$.

The only singular fibre of f_1 is the one over 0, which contains the inverse images of the two exceptional divisors D_1 and D_2 and of the fibre of ϕ_1 over 0.

Blowing up the origin in $E \times E$ we see that the exceptional divisor D intersects each of the four curves in one point, while the fibre of ϕ_1 intersects the last two curves in respectively $3, 2$ points.

Hence the singular fibre of f_1 contains two genus one components which are disjoint and intersect the rest of the fibre transversally respectively in two points. The rest of the fibre is an irreducible smooth component of genus equal to 6, because, as we shall now show, the inverse image in A of the fibre of ϕ_1 is a connected elliptic curve, and then we take a double covering branched in 10 points.

While the authors take a basis $\zeta e_1, \zeta e_2, e_1, e_2$ of the lattice $\pi_1(E \times E)$, we take the more natural basis $e_1, \epsilon e_1, e_2, \epsilon e_2$. In the second basis, and using an additive notation, the character χ takes values $(0, 1, 1, 1)$ on the four basis vectors.

It follows then easily that the restriction of the double covering is nontrivial on the five curves

$$\{x = 0\},\ \{y = 0\}, \{y = x\}, \{y = -\epsilon x\}, \{y = -x\},$$

(and their translates). In particular, the fibres of $\phi_1 \circ p$ are connected.

While, for the fibres of ϕ_2, like $\{\epsilon y + x = 0\}$, the values of the character χ on the two periods $(e_1 - e_2 - \epsilon e_2), (\epsilon e_1 - e_2)$ are equal to $0 + 1 + 1 = 0, 1 - 1 = 0$.

Hence the inverse image of fibre of ϕ_2 splits into two components and we get a fibration f_2 whose fibres are of genus $g = 4$, $f_2 : S \to E'$, where E' is an étale cover of degree 2 of E.

The singular fibres lie over the origin in E, hence we get two singular fibres for the two points of E' lying over the origin in E. Thus the number of singular fibres s of f_2 equals 2, and each singular fibre is the inverse image of the union of an elliptic curve and an exceptional curve ($\cong \mathbb{P}^1$). Since the double covering of the elliptic curve is branched on 2 points, while the double cover of the exceptional curve is branched in 4 points, we obtain the desired assertion. □

2. Number of singular fibres of a fibration

Once we fix b, g and we consider fibrations $f : S \to B$ with fibres F of genus g, and genus b of the base curve B, the Zeuthen-Segre formula which we have discussed in the previous section gives a relation between the topological Euler characteristic $e(S)$ and the number of singular fibres of f, counted with multiplicity.

In particular, if there is only a finite number c of critical points of f, it gives an upper bound for the number c.

This upper bound must obviously depend on $e(S)$, as shows the case where $b = 1$: in fact, in this case there exist unramified coverings $B' \to B$ of arbitrary degree m, and the fibre product

$$f' : S' := S \times_B B' \to B'$$

has both numbers $c' = m \cdot c$ and $e(S') = m \cdot e(S)$.

The Zeuthen-Segre formula says also that if there are no singular fibres, for $g \geq 2$, then necessarily $e(S) = 4(b-1)(g-1)$. There are two ways in which this situation can occur (see [10] for more details), since then S is relatively minimal and one can apply Arakelov's theorem asserting that

$$K_S^2 \geq 8(b-1)(g-1),$$

equality holding iff all the smooth fibres are isomorphic.

As a consequence, there are two cases when $e(S) = 4(b-1)(g-1)$:

Étale bundles: $K_S^2 = 8(b-1)(g-1)$, and there is a Galois unramified covering $B' \to B$ such that

$$S' := S \times_B B' \cong B' \times F,$$

Kodaira fibrations: $K_S^2 > 8(b-1)(g-1)$, and not all fibres are biholomorphic.

The only restrictions for Kodaira fibrations are that $b \geq 2, g \geq 3$, and for all such values of b, g we have Kodaira fibrations.

Assume now that $b \leq 1$, and assume that not all smooth fibres are biholomorphic. Let then

$$B^* := \{t \in B | F_t \text{ is smooth } \}.$$

Then the universal cover of B^* admits a non constant holomorphic map into the Siegel space $\mathcal{H}_g$, which is biholomorphic to a bounded domain.

The conclusion is that, for $b = 1$, there must be at least one singular fibre, whereas for $b = 0$ the number of singular fibres must be at least 3.

With the stronger hypothesis that the fibration is moduli stable, i.e., all singular fibres have only nodes as singularities and do not possess a smooth rational curve intersecting the other components in two points or less, one gets a better estimate [3], [28], [30]:

Theorem 2.4. Let $f : S \to B$ be a moduli stable fibration with $g \geq 1$. Then the number s of singular fibres is at least:

(1) $s \geq 4$ for $b = 0, g \geq 1$,
(2) $s \geq 5$ for $b = 0, g \geq 2$,
(3) $s \geq 6$ for $b = 0, g \geq 3$,
(4) $s \geq 2$ for $b = 1, g = 2$.

For $b = 1$ Ishida [15] constructed a Catanese-Ciliberto surface with $g = 3$, $K_S^2 = 3$, $p_g = q = 1$ having only one singular fibre: but in this case the singular fibre is not a stable curve, it is isomorphic to the union of 4 lines in the plane passing through the same point (note that here the Milnor number is 9, and that for a plane quartic curve the number of singular points is at most 6, so that there is no stable curve with $g = 3$ and with 9 nodes).

Parshin [19] claimed that for a moduli stable fibration with $b = 1$ one should have $s \geq 2$, but the claim was contradicted by [6] who constructed an example with $s = 1$ and with reducible singular fibre.

In the next section, using a variation of the method of [6], we construct an example where there is only one singular fibre, irreducible and with 4 nodes (the number of nodes should be the smallest one, see Remark 1.2).

This example will play a role also in the later sections.

3. A fibration over an elliptic curve with only one singular fibre, irreducible and nodal

Theorem 3.5. There exists fibrations $f : S \to B$, where B is a smooth curve of genus $b = 1$, and the fibres of f are smooth curves of genus $g = 9$, with the exception of a unique singular fibre, which is an irreducible nodal curve with 4 nodes. Moreover, S is the product $C_1 \times C_2$ of two smooth genus 2 curves.

Proof. We achieve the result in three steps.

Step 1: we construct, for $i = 1, 2$, a degree 4 covering

$$f_i : C_i \to B,$$

branched only over $O \in B$, and with $f_i^{-1}(O)$ consisting of two (necessarily simple) ramification points.

Step 2: taking O to be the neutral element of the group law on the genus 1 curve B, we set, as in [6],

$$S := C_1 \times C_2, \text{ and } f(x_1, x_2) := f_1(x_1) - f_2(x_2).$$

Hence $x := (x_1, x_2)$ is a critical point for f if and only if both x_1 is a critical point for f_1 and x_2 is a critical point for f_2.

By the choice made in step 1, we have 4 such critical points, and for each of them $f(x) = O$. Hence f has only one singular fibre $F_O := f^{-1}(O)$, which possesses exactly 4 singular points.

By simple ramification, there is a local coordinate t around O, and there are local coordinates z_i around x_i such that in these coordinates $f_i(z_i) = z_i^2$. Therefore, at a critical point x, there are local coordinates (z_1, z_2) such that

$$f(z_1, z_2) = z_1^2 - z_2^2,$$

and we have a nodal singularity of the fibre $F_O = \{f(z_1, z_2) = 0\}$.

Step 3: we shall show, based on the explicit construction in step 1, that the singular fibre $F_O = f^{-1}(O)$, which is the fibre product $C_1 \times_B C_2$, is irreducible.

We observe moreover that the fibre F_y over a point $y \in B$ is the fibre product of C_1 and C_2 via the respective maps $f_1 - y$ and f_2: hence there are exactly $2 \cdot 2 \cdot 4 = 16$ simple ramification points for the map $F_y \to B$, hence $g(F_y) = 9$.

Construction of step 1: We construct the two respective coverings $f_i : C_i \to B$ using Riemann existence's theorem, and we let B be any elliptic curve, with a fixed point O which we take as neutral element for the group law.

Since the local monodromy at the point O is a double transposition, the monodromy $\mu_i : \pi_1(B \setminus \{O\})$ factors through the orbifold fundamental group

$$\Gamma := \pi_1^{orb}(B; 2O) := \langle \alpha, \beta, \gamma | [\alpha, \beta] = \gamma, \gamma^2 = 1 \rangle,$$

where γ represents a simple loop around the point O.

We define then two homomorphisms,

$$\mu_1 : \Gamma \to \mathfrak{S}_4, \mu_1(\alpha) := (1, 2, 3, 4), \ \mu_1(\beta) := (1, 2)(3, 4),$$
$$\mu_2 : \Gamma \to \mathfrak{S}_4, \mu_2(\alpha) := (1, 2, 3), \ \mu_2(\beta) := (1, 4)(2, 3).$$

In both constructions $\mu_i(\beta)$ is an element of the Klein group $\mathcal{K} \cong (\mathbb{Z}/2)^2$, consisting of the three double transpositions and of the identity, as well as $\mu_i(\alpha\beta\alpha^{-1}) = \mu_i(\gamma\beta)$. We have respectively

$$\mu_1(\alpha\beta\alpha^{-1}) = (2,3)(4,1), \quad \mu_2(\alpha\beta\alpha^{-1}) = (2,4)(1,3),$$

so that $\mu_1(\alpha\beta\alpha^{-1}) \neq \mu_1(\beta)$ and $\mu_i(\gamma)$ is the third nontrivial element in $\mathcal{K}$, a double transposition as desired.

The conclusion is that $\mu_i(\Gamma)$ contains the normal subgroup $\mathcal{K}$ and is generated by $\mathcal{K}$ and $\mu_i(\alpha)$.

Hence $\mu_1(\Gamma)$ is the dihedral group D_4, while $\mu_2(\Gamma)$ is the alternating group $\mathfrak{A}_4$.

We let then f_i to be the degree 4 covering associated to the monodromy $\mu_i : \Gamma \to \mathfrak{S}_4$.

Proof of the assertion of Step 3: The normalization of the singular fibre of f, which is the fibre product of C_1 and C_2 over B, is a degree 16 covering of B associated to the product monodromy

$$\mu_1 \times \mu_2 : \Gamma \to \mathfrak{S}_4 \times \mathfrak{S}_4 \subset \mathfrak{S}_{16}.$$

The irreducibility of this fibre product amounts therefore to the transitivity of the monodromy $\mu(\Gamma) := \mu_1 \times \mu_2(\Gamma)$ on the product set $\{1,2,3,4\} \times \{1,2,3,4\}$.

Indeed, the $\mu(\alpha)$-orbit has cardinality 12 and contains $\Sigma := \{1,2,3,4\} \times \{1,2,3\}$. Moreover, each element not in Σ is of the form $(a,4)$ and $\mu(\beta)$ sends $(a,4)$ to an element $(y,1)$ which lies in Σ, whence there is a unique orbit, the monodromy is transitive, and the unique singular fibre F_O of f is irreducible. $\square$

Remark 3.1. Since we took B to be any elliptic curve, we see that our construction leads to a one-parameter family of such fibrations. And since a deformation of a product of curves is again a product, we see that any deformation of f which has exactly one singular fibre must be as in our construction.

4. Fibrations and factorizations in the mapping class group

Let as usual now $f : S \to B$ be a fibration of an algebraic surface onto a curve B of genus b, such that the fibres F_t of f have genus g.

As before, we let B^* be the complement of the s critical values $p_1, \dots, p_s$ of f. We denote the s singular fibres by $f^{-1}(p_i) =: F_i$, and set $S^* := f^{-1}(B^*) = S \setminus (F_1 \cup \dots F_s)$.

Then $f^* : S^* \to B^*$ is a differentiable fibre bundle, and its monodromy defines homomorphisms

$$\pi_1(B^*, t_0) \to \mathcal{M}ap_g \to Sp(2g, \mathbb{Z}),$$

where the second homomorphism corresponds geometrically to the bundle $\mathcal{J}^*$ of Jacobian varieties with fibres $J_t := Jac(F_t) = Pic^0(F_t)$.

Here $\mathcal{M}ap_g$ is the Mapping class group $\mathcal{D}iff^+(F_0)/\mathcal{D}iff^0(F_0)$, introduced by Dehn in [12], and we let $\nu : \pi_1(B^*, t_0) \to \mathcal{M}ap_g$ be the geometric monodromy (see also [9]).

Fixing a geometric basis, the fundamental group $\pi_1(B^*, t_0)$ is isomorphic to the group

$$\pi_b(s) := \langle \alpha_1, \beta_1, \dots \alpha_b, \beta_b, \gamma_1, \dots, \gamma_s | \Pi_1^s \gamma_i \Pi_1^b [\alpha_j, \beta_j] = 1 \rangle,$$

and the image $\delta_i := \nu(\gamma_i)$ is a conjugate of the local monodromy around p_i.

In the case where the only fibre singularity is a node, then δ_i is a Dehn twist around the vanishing cycle, a circle c_i, whose image in the Symplectic group is the Picard-Lefschetz transvection associated to the homology class c of c_i:

$$T_c, \ T_c(v) := v + (c, v)c,$$

(here (c, v) denotes the intersection pairing on the base fibre F_0, a smooth curve of genus g).

It is customary to view the monodromy as a factorization

$$\Pi_1^s \delta_i \Pi_1^b [\alpha'_j, \beta'_j] = 1$$

in the Mapping class group (just let $\alpha'_j := \nu(\alpha_j), \beta'_j := \nu(\beta_j)$).

Completing work of Moishezon [18] and Kas [16], Matsumoto showed the following result (theorems 2.6 and 2.4 of [17]):

Theorem 4.6. Given a factorization $\Pi_1^s \delta_i \Pi_1^b [\alpha'_j, \beta'_j] = 1$ in the mapping class group $\mathcal{M}ap_g$, for $g \geq 1$, there is a differentiable Lefschetz fibration $f : M \to B$, whose monodromy corresponds to such a factorization, if and only if the δ_i are negative Dehn twists about an essential simple closed curve.

Moreover, two such fibrations are equivalent, for $g \geq 2$, if and only if the corresponding factorizations are equivalent, via change of a geometric basis in $\pi_1(B^*, b_0)$ and via simultaneous conjugation of all the factors $\delta_i, \alpha'_j, \beta'_j$ by a fixed element $a \in \mathcal{M}ap_g$.

In the above theorem a simple closed curve c is said to be essential if it is not the boundary of a disk. There are two cases: if its homology class in $H_1(F_0, \mathbb{Z})$ is non trivial (hence the complementary set is connected) then c is said to be nonseparating, or of type I. Else, the complementary set is disconnected, the curve is said to be separating, or of type II, and pinching the curve to a point one gets the union of two curves of respective genera $h \geq 1$, $(g - h) \geq 1$, meeting in a point.

Remark 4.1. Matsumoto takes the more restrictive definition in which M is oriented, and that at the critical points there are complex coordinates z_1, z_2 such that not only F is locally given by $z_1 z_2$, but also the complex orientation coincides with the global orientation. One says then that the Lefschetz fibration is orientable.

Kas does not make this requirement, so there is no requirement imposed on the Dehn twists δ_i occurring in the factorization.

An important question is whether a factorization comes from a holomorphic fibration: the case of fibre genus $g = 2$ was treated by Siebert and Tian [25].

A similar question can be posed, requiring M to be a symplectic 4-manifold, and that there is a local symplectomorphism yielding the local complex coordinates (z_1, z_2) (we take here the standard symplectic structure on the target $\mathbb{C}^2$). This question was however answered by Gompf [13], see also [1], who showed that any orientable Lefschetz fibration comes from a symplectic Lefschetz fibration.

Matsumoto showed, for $g = 2$ orientable Lefschetz fibrations, that the number m of singular fibres of type I, and the number n of singular fibres of type II satisfy the congruence

$$m + 2n \equiv 0 \in \mathbb{Z}/10.$$

Indeed, the Abelianization of $\mathcal{M}ap_2$ is isomorphic to $\mathbb{Z}/10$.

We refer to [26] for more information about the minimal number of singular fibres for an orientable Lefschetz fibration over a curve of genus b, the cases $b = 0, 1$ being the open cases. Stipsicz and Yun state that for $b = 1$ the number s of singular fibres is at least 3. The bound would be sharp in genus $g = 19$ because of the Cartwright-Steger surface. Our example in Theorem 1.3 shows that already in genus $g \geq 4$ we have a product of 4 Dehn twists which is a commutator.

In the case $b = 1$, the existence of such a factorization is equivalent to the assertion that a product of s Dehn twists is a commutator in the Mapping class group.

In view of this, in the next section we focus on a related question, when is the product of certain transvections a commutator in the $Sp(2g, \mathbb{Z})$.

5. Commutators in the Symplectic group $Sp(2g, \mathbb{Z})$, $g \leq 2$

5.1. The case $g = 1$

As a warm up, let us begin with the case $g = 1$, where $Sp(2, \mathbb{Z}) = SL(2, \mathbb{Z})$.

In this case the group surjects to the group $\mathbb{P}SL(2, \mathbb{Z})$ of integral Möbius transformation. It is known, see for instance [23], that $\mathbb{P}SL(2, \mathbb{Z})$ is the free product $(\mathbb{Z}/2) * (\mathbb{Z}/3)$, where the first generator comes from the matrix A, the second generator comes from the matrix B,

$$A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; B := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

We consider the standard transvection $T = T_{e_1}$, with matrix

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

giving rise to the projectivity $z \mapsto z + 1$.

Proposition 5.7.

(i) One has $T^{-1} = AB$, hence the image of T^{-1} in the Abelianization $(\mathbb{Z}/2) \times (\mathbb{Z}/3) \cong (\mathbb{Z}/6)$ of $\mathbb{P}SL(2, \mathbb{Z})$ is equal to $(1, 1) \equiv 1 \in \mathbb{Z}/6$, and no power T^m, for m not divisible by 6, is a product of commutators.
(ii) T^{2m} is a commutator in $GL(2, \mathbb{Z})$ $\forall m$.
(iii) T^m is a product of commutators in $GL(2, \mathbb{Z})$ only if m is even.
(iv) No power T^m, $m \neq 0$, is a commutator in $SL(2, \mathbb{Z})$.

Proof. An immediate calculation shows that $T^{-1} = AB$, hence assertion (i) follows.

(ii) follows by taking

$$C := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

hence

$$CT^m C^{-1}(T^m)^{-1} = T^{-2m}.$$

(iv) assuming that

$$T^m = X^{-1}Y XY^{-1},\ m \in \mathbb{Z}, m \neq 0,$$

equivalently

$$Z := X^{-1}Y X = T^m Y.$$

Setting

$$Y := \begin{pmatrix} r & s \\ t & u \end{pmatrix},$$

we get

$$Z := \begin{pmatrix} r + mt & s + mu \\ t & u \end{pmatrix}.$$

Since Y and Z are conjugate, they have the same trace, hence:

$$r + mt + u = r + u \Rightarrow t = 0.$$

Since Y has determinant $= 1$, we obtain

$$ru = 1 \Rightarrow r = u = \pm 1,$$

and possibly replacing Y with $-Y$ we obtain

$$r = u = 1 \Rightarrow Y = T^s.$$

Whence,

$$T^s X = X T^{m+s},$$

hence $X(e_1)$ is an eigenvector for T^s; since $s \neq 0$, X is also upper triangular, hence a power of T and we reach a contradiction.

(iii) we observe that reduction modulo 2 yields a projection $GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/2\mathbb{Z}) \simeq \mathfrak{S}_3$ and that T is sent by this projection to a transposition, hence an odd permutation. □

The fact that the condition of being a commutator changes drastically, if one allows orientation reversing transformations, occurs also in higher genus. For instance Szepietowski [27] proved:

Theorem 5.8. Let c be an essential closed circle in a compact complex curve X of genus $g \geq 3$: then any power of the Dehn twist δ_c is a commutator in the extended mapping class group

$$\mathcal{M}ap_g^e = \mathcal{D}iff(X)/\mathcal{D}iff^0(X).$$

5.2. The case $g = 2$

We consider the lattice $\mathbb{Z}^4$ with its canonical basis e_1, e_2, e_3, e_4, and define the symplectic form $(\cdot|\cdot)$ on $\mathbb{Z}^4$ by setting

$$(e_1|e_2) = 1 = (e_3|e_4)$$

and

$$(e_i|e_j) = 0, \qquad \text{for } \{i,j\} \notin \{\{1,2\},\{3,4\}\}.$$

The matrix of this symplectic form is then

$$J_2 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We denote by $Sp(4, \mathbb{Z})$ the corresponding symplectic group, i.e. the group of 4×4 matrices X with integral coefficients satisfying

$${}^tX \cdot J_2 \cdot X = J_2.$$

Let now $T \in \mathrm{Sp}_4(\mathbb{Z})$ be the matrix

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We prove in this subsection the following result

Theorem 5.1. *Let $m \in \mathbb{Z}$ be an integer; the power T^m of T is a commutator in the group $Sp(4, \mathbb{Z})$ if and only if m is even. In particular, T itself is not a commutator.*

One direction of the equivalence will follow from reduction modulo 2: we shall prove the following stronger result

Theorem 5.2. *The reduction modulo 2 of the matrix T does not belong to the group generated by the commutators in the group $Sp(4, \mathbb{F}_2)$.*

The reverse implication in Theorem 5.1 shall result from a simple explicit construction given below. Note that as a consequence of this latter implication, every power of T is a commutator in $Sp_4(4, \mathbb{F}_p)$ for every odd prime p.

Here is the explicit realization of even powers T^{2m} as commutators: set

$$X = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & 0 & 0 & m \\ 0 & 1 & 0 & 0 \\ 0 & m & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.9}$$

Then one verifies that X, Y are indeed symplectic matrices and $XYX^{-1}Y^{-1} = T^{2m}$.

We shall give two proofs of Theorem 5.2. The first proof, which occupies the rest of this section, makes use of the isomorphism between the group $Sp(4, \mathbb{F}_2)$ and the symmetric group $\mathfrak{S}_6$.

This isomorphism has a nice and classical geometric interpretation, which we now briefly describe, in the spirit of the first part of our work.

Recall that every algebraic curve C of genus $g = 2$ has a canonical map which is a double covering of the projective line branched in six points, so that there is an involution on C, called the hyperelliptic involution, whose six fixed points $P_1, \dots, P_6$ are called the Weierstrass points (they are the critical points for the canonical map).

Hence every curve $\mathcal{C}$ of genus 2 admits an affine model of equation

$$y^2 = (x - \alpha_1) \cdots (x - \alpha_6)$$

for pairwise distinct complex numbers $\alpha_1, \dots, \alpha_6$ (thus $P_i = (\alpha_i, 0)$).

Given any fibration $S^* \to B^*$ in curves of genus 2 and a point $b_0 \in B^*$, the action of the fundamental group $\pi_1(B^*, b_0)$ on the fibre of b_0 gives, as described above, a morphism $\pi_1(B^*, b_0) \to \mathcal{M}ap_2$; this morphism also induces a permutation of the six Weierstrass points, hence a representation $\pi_1(B^*, b_0) \to \mathfrak{S}_6$.

On the other hand, the first homology group $H_1(C, \mathbb{Z}/2)$ is isomorphic to the subgroup $Pic^0(C)[2]$ of the 2-torsion points in the Jacobian variety $Jac(C) \cong Pic^0(C)$.

This subgroup is isomorphic to $(\mathbb{Z}/2)^4$. Since $2P_i \equiv 2P_j \equiv K_C$ (here K_C is the (degree two) canonical divisor of C), and since $div(y) \equiv \sum_i P_i - 3K_C$, it has a basis given by the differences $P_1 - P_2, P_2 - P_3, P_3 - P_4, P_4 - P_5$ (indeed, $\sum_i P_i = \ div(y) \equiv 3K_C \Rightarrow (P_1 - P_2) + (P_3 - P_4) + (P_5 - P_6) \equiv 0$).

The morphism $\mathcal{M}ap_2 \to Sp(4, \mathbb{Z})$ can be composed with reduction modulo 2, thus giving a homomorphism $\mathcal{M}ap_2 \to Sp(4, \mathbb{Z}/2\mathbb{Z})$, where the symplectic form modulo 2 is called the Weil pairing on the group $Pic^0(C)[2]$, and corresponds to cup product in cohomology.

To see that the two groups $Sp(4, \mathbb{Z}/2\mathbb{Z})$ and $\mathfrak{S}_6$ are indeed isomorphic, we observe that the half twist on a simple arc joining α_i and α_j, which yields a transposition exchanging the two points P_i, P_j, lifts to a Dehn twist $\delta_{i,j}$ which maps to a transvection $T_{i,j}$ on the class corresponding to $P_i - P_j$.

Hence we have defined a homomorphism $\mathfrak{S}_6 \hookrightarrow \mathrm{GL}(H_1(C, \mathbb{Z}/2\mathbb{Z}))$, which is an embedding because a permutation fixes all the basis vectors if and only if it is the identity. Moreover, the two groups have the same cardinality, hence we have an isomorphism.

We describe now more formally the isomorphism $\mathfrak{S}_6 \simeq Sp(4, \mathbb{F}_2)$ following §10.1.12 of Serre's book [24].

Let $H \subset \mathbb{F}_2^6$ be the hyperplane of equation $\sum_{i=1}^6 x_i = 0$. Consider the alternating bilinear form $H \times H \to \mathbb{F}_2$ sending $(x, y) \mapsto \sum_i x_i y_i$. The vector $(1, \dots, 1)$ is orthogonal to the whole space, and the induced bilinear form on the four dimensional vector space $V := H / < (1, \dots, 1) >$ turns out to be non degenerate. The group $\mathfrak{S}_6$ acts naturally on $\mathbb{F}_2^6$ leaving H invariant; also it conserves the bilinear form and fixes the point $(1, \dots, 1)$, so it acts (faithfully) on V as a group of symplectic automorphisms. Hence we obtain an embedding $\mathfrak{S}_6 \hookrightarrow Sp(V) = Sp(4, \mathbb{F}_2)$. To prove that this embedding is in fact an isomorphism, we compare the orders of the two groups.

The order of $Sp(4, \mathbb{F}_2)$ can be computed as follows: the set of non-degenerate planes in $\mathbb{F}_2^4$ has cardinality $15 \cdot 8/6 = 20$, since one can choose a non-zero vector v_1 in 15 ways and a second vector $v_2 \in \mathbb{F}_2^4 \setminus v_1^\perp$ in $16 - 8 = 8$ ways. Hence there are $15 \cdot 8$ possibilities for the ordered base (v_1, v_2) and each plane admits six order bases, hence the cardinality of the set of non-degenerate planes is 20. The group $Sp(4,\mathbb{F}_2)$ acts transitively on the set of non-degenerate planes and the stabilizer of any such plane is isomorphic to $\mathrm{SL}_2(\mathbb{F}_2)\times \mathrm{SL}_2(\mathbb{F}_2)\simeq \mathfrak{S}_3\times\mathfrak{S}_3$, so has order 36. It follows that

$$|Sp(4,\mathbb{F}_2)| = 20\times 36 = 720 = 6!$$

We then obtain the sought isomorphism $\mathfrak{S}_6 \simeq Sp(4,\mathbb{F}_2)$.

We want to prove that the matrix T corresponds, via this isomorphism, to an odd permutation in $\mathfrak{S}_6$, hence it does not belong to the derived subgroup of $Sp(4,\mathbb{F}_2)$.

Now, T has order two, and every even permutation of order two in $\mathfrak{S}_6$ is conjugate to the permutation $(1,2)\circ(3,4)$. To prove the theorem, it then suffices to show that this permutation gives rise to a matrix in $Sp(4,\mathbb{F}_2)$ which is not conjugate to T.

The quotient space $V = H/ < (1,\dots,1) >$ is represented by the vectors $(x_1,\dots,x_6)$ with vanishing last coordinate x_6 and vanishing sum of the coordinates. A basis is provided by $v_1 = (1,0,0,0,1,0), v_2 = (0,1,0,0,1,0), v_3 = (0,0,1,0,1,0)$ and $v_4 = (0,0,0,1,1,0)$.

The permutation $(1,2)\circ(3,4)$ sends

$$\begin{matrix} v_1 & \leftrightarrow & v_2 \\ v_3 & \leftrightarrow & v_4 \end{matrix}$$

hence corresponds to the matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which is not conjugate, not even in $\mathrm{SL}_4(\mathbb{F}_2)$, to the matrix T (compare the ranks of $T+I$ and $S+I$). This ends the proof of Theorem 5.2.

Actually, it turns out that the matrix T corresponds to a permutation of $\mathfrak{S}_6$ conjugate to $(1,2)\circ(3,4)\circ(5,6)$.

6. Alternative proof for $g = 2$

Our second proof, which will only be sketched here, is more involved but has the advantage of admitting some extensions in higher dimensions.

Notation. For a vector space V and vectors $v_1,\dots,v_k\in V$, we denote by $< v_1,\dots,v_k >$ the sub-vector space generated by $v_1,\dots,v_k$. When Λ is a lattice (or a $\mathbb{Z}$-module), and $v_1,\dots,v_k$ are k elements of Λ, we denote by the same symbol $< v_1,\dots,v_k >$ the $\mathbb{Z}$-module generated by $v_1,\dots,v_k$, when no confusion can arise.

For a vector $v\in\mathbb{Z}^4$ (or more generally in a module with symplectic form $(\cdot|\cdot)$), the symbol $v^\perp$ denotes its orthogonal with respect to the given symplectic form.

We start with the following Proposition, which in fact holds in arbitrary characteristic not dividing the integer m appearing in the statement, and is one of the main tool in the proof of the Theorem.

Proposition 6.1. *Let $m \neq 0$ be an integer and suppose that*

$$T^m = XYX^{-1}Y^{-1} \tag{6.10}$$

for two matrices $X, Y \in Sp(4, \mathbb{Z})$. Then either e_1 is an eigenvector for both X and Y, or the orbit of e_1 under the subgroup generated by X and Y is contained in a two dimensional sub-lattice of $\mathbb{Z}^4$, contained in $e_1^\perp$ and invariant under X and Y.

Remarks. (1) The above proposition could be extended in higher dimensions: for the analogue in dimension $2g$, the result would be that the orbit of e_1 under the group generated by X and Y would be contained in an invariant subgroup $\Lambda \subset \mathbb{Z}^{2g}$, satisfying $\Lambda \subset \Lambda^\perp \subset e_1^\perp$. (2) We have stated the proposition over the integers, but we could have worked over any field (of characteristic not dividing m); in that case we would speak of sub-vector spaces instead of sub-lattices.

Proof. Let us put $\Delta : T - I$, where $I = I_4$ is the identity matrix. Note that $\Delta^2 = 0$ and that $\Delta v = 0$ for each $v \in e_1^\perp$. Also, $T^n = I + n\Delta$, for all $n \in \mathbb{Z}$.

We note the useful equality

$$e_1^\perp =< e_1, e_3, e_4 >= \ker \Delta$$

(where we identify the matrix Δ with the multiplication-by-Δ endomorphism of $\mathbb{Z}^4$). We shall also keep in mind that $\Delta \cdot \mathbb{Z}^4 =< e_1 >$.

We can rewrite equation (6.10) in the form

$$XY - YX = m\Delta YX, \tag{6.11}$$

as well as

$$X^{-1}Y^{-1} - Y^{-1}X^{-1} = mY^{-1}X^{-1}\Delta. \tag{6.12}$$

It immediately follows from the first of the two identities that $\mathrm{Tr}\Delta YX = 0$, which means precisely that the coefficient on the first column – second row of YX vanishes. This property can be stated as

$$YXe_1 \in e_1^\perp.$$

Also, interchanging X, Y turns their commutator into its inverse $T^{-m} = I - m\Delta$ and repeating the argument we also obtain

$$YXe_1 \in e_1^\perp.$$

Again using equation (6.10) one gets

$$XYX^{-1} - Y = m\Delta Y$$

and, noting that interchanging X, Y turns their commutator T^m into T^{-m},

$$YXY^{-1} - X = -m\Delta X.$$

From these relations we obtain as above that $\mathrm{Tr}\Delta X = \mathrm{Tr}\Delta Y = 0$, i.e. $Xe_1 \in e_1^\perp$, $Ye_1 \in e_1^\perp$. Summarizing we have

$$Xe_1, Ye_1, XYe_1, YXe_1 \in e_1^\perp.$$

We now notice that the commutator $XYX^{-1}Y^{-1}$ does not change if we replace X by XZ, where Z commutes with Y, or Y by YZ, where Z commutes with X; hence taking for Z any power of Y in the first case and any power of X in the second case, we also obtain that for each $n \in \mathbb{Z}$,

$$XY^n e_1,\ YX^n e_1 \in e_1^\perp. \tag{6.13}$$

The identity (6.12) implies, considering that $\ker \Delta = e_1^\perp$, that

$$X^{-1}Y^{-1}v = Y^{-1}X^{-1}v$$

for each $v \in e_1^\perp$.

Also, observing that all monomials in X, Y are symplectic matrices, and that for every symplectic matrix F the relation $(Fe_1|e_1) = 0$ implies $(F^{-1}e_1|e_1) = 0$, we also obtain that for all $n \in \mathbb{Z}$:

$$X^n Y^{-1} e_1,\ Y^n X^{-1} e_1 \in e_1^\perp. \tag{6.14}$$

We now pause to prove the following

Claim. *The orbit of e_1 under X (resp. under Y) is contained in a proper sub-vector space of $\mathbb{Q}^4$.*

Proof of the Claim. This follows from the relations $YX^n e_1 \in e_1^\perp$, included in (6.13) and valid for all $n \in \mathbb{Z}$, which imply that the orbit of e_1 under X is included in the hyperplane $Y^{-1}(e_1^\perp) = (Y^{-1}e_1)^\perp$. Of course, the relations $XY^n e_1 \in e_1^\perp$ imply the same conclusion for the orbit under Y. This proves our claim.

We now prove that such sub-spaces must be one or two-dimensional:

Claim. *The vector space generated by the orbit of e_1 under X (resp. under Y) cannot be three-dimensional.*

Proof of the Claim. Suppose by contradiction that such a vector space has dimension 3. Then it admits the base $(X^{-1}e_1, e_1, Xe_1)$. Since the three vectors $X^{-1}e_1, e_1, Xe_1$ all belong to $e_1^\perp$, as we have seen in (6.13), (6.14), this vector space must coincide with $e_1^\perp$, which then is an invariant subspace for X. But if a symplectic operator leaves invariant a subspace, it also leaves invariant its orthogonal, so in this case the line $< e_1 >$ would be X-invariant, contrary to our assumption that $e_1, Xe_1, X^{-1}e_1$ are linearly independent.

Then only three cases must be considered for the proof of the proposition:

(1) The two orbits of e_1 (under X and Y) are contained in a line; this line is then $< e_1 >$ and in this case the assertion of the proposition is plainly verified.
(2) The vector e_1 is an eigenvector for X and its orbit under Y is contained in a plane W; in this case we must show that W is contained in $e_1^\perp$ and that it is X-invariant. Of course, the symmetric situation, when e_1 is an eigenvector only of Y, is treated in exactly the same way.
(3) The two orbits generate planes W_X, W_Y. In this case we must show that $W_X = W_Y$ and that this common plane is contained in $e_1^\perp$.

Let us consider now the second case: $Xe_1 = \pm e_1$ and the orbit of e_1 under Y generates a plane $W =< e_1, Ye_1 >=< e_1, Y^{-1}e_1 >$. Since $X^{-1}Y^{-1}$ and $Y^{-1}X^{-1}$ coincide in e_1 and $Xe_1 = \pm e_1$, we have $X^{-1}Y^{-1}e_1 = \pm Y^{-1}e_1$, so both $e_1, Y^{-1}e_1$ are eigenvectors for X, so W is X-invariant. Since $Y^{-1}e_1 \in e_1^\perp$, the inclusion $W \subset e_1^\perp$ holds, and the verification of the proposition in this case is complete.

In the last case to examine, let

$$W_X =< e_1, Xe_1 >=< e_1, X^{-1}e_1 >, \qquad W_Y =< e_1, Ye_1 >=< e_1, Y^{-1}e_1 >$$

and again note that $W_X \subset e_1^\perp, W_Y \subset e_1^\perp$. If $W_X \neq W_Y$, then the subspace generated by W_X and W_Y would coincide with the hyperplane $e_1^\perp$ and would be generated by e_1, Xe_1, Ye_1. Now, since $XYe_1 \in e_1^\perp$, we would obtain that $e_1^\perp$ is X-invariant, so again e_1 would be an eigenvector for X, contrary to our assumptions. So we cannot have $W_X \neq W_Y$ and the proposition is proved in this last case too. □

Thanks to Proposition 6.1, we can divide the proof of Theorem 5.2 into two cases, according to the orbit of e_1 under X, Y being a line or a plane.

`First case:` e_1 `is an eigenvector for both` X `and` Y. We reduce, possibly after changing X, Y by their opposite, to matrices X, Y of the form

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \tag{6.15}$$

Put $\Delta = T - I$, where $I = I_4$ is the identity matrix. Recall that Δ is nilpotent, satisfying $\Delta^2 = 0$ which implies that for every $m \in \mathbb{Z}$,

$$T^m = I + m\Delta$$

The following relations, whose proof is left to the reader, hold:

$$X\Delta = Y\Delta = \Delta$$

$$\Delta X = \pm\Delta,\ \Delta Y = \pm\Delta.$$

From the above one easily deduces:

$$XY - YX = \pm m\Delta. \tag{6.16}$$

Our aim now is proving that the relation (6.16) cannot hold for any odd integer m. This will follow from an argument modulo 2, leading to the next proposition (where Δ will denote the reduction of the previous matrix Δ modulo 2):

Proposition 6.2. *The equation*

$$XY - YX = \Delta \tag{6.17}$$

admits no solution in matrices $X, Y \in Sp(4, \mathbb{F}_2)$ satisfying (6.15).

Let us suppose to have a solution (X, Y) to $XY - YX = \Delta$ in $Sp(4, \mathbb{F}_2)$ of the form (6.15). From the above relations we deduce that Δ, so also T, commute with X and Y. Replacing if necessary X by TX, which does not change the commutator, we can suppose that the coefficient on the first line – second column on X vanishes. We can suppose the same for Y, so X, Y will both be of the form

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}. \tag{6.18}$$

The following lemma ensures that we can basically choose the form of the second column too.

Lemma 6.3. *Let k be any field and (X, Y) be a solution to the equation (6.17) with $X, Y \in Sp(4, k)$ of the above form (6.18). Then $Xe_2 \neq e_2$ and $Ye_2 \neq e_2$. Also, $Xe_2 \neq Ye_2$.*

Proof. Suppose by contradiction that $Xe_2 = e_2$ (the argument is symmetrical if $Ye_2 = e_2$). Then, since the plane $< e_1, e_2 >$ is invariant by multiplication by X, the same must be true of its orthogonal, which is $< e_3, e_4 >$. Now, write $Ye_2 = e_2 + v$, where $v \in < e_3, e_4 >$ (this is certainly possible since Y is of the form (6.18)). The relation (6.17) applied to the vector e_2 gives

$$e_1 = \Delta e_2 = XYe_2 - YXe_2 = X(e_2 + v) - Ye_2 = e_2 + Xv - e_2 - v = Xv - v,$$

which is impossible since v and Xv belong to the plane $< e_3, e_4 >$. This proves the first two inequalities. Suppose now $Xe_2 = Ye_2$. Then from (6.17) applied to the vector e_2 we obtain, writing $Xe_2 = e_2 + v = Ye_2$, with $v \in < e_3, e_4 >$,

$$XYe_2 - YXe_2 = Xv - Yv = e_1.$$

But from $Xe_2 = e_2 + v = Ye_2$ and $(e_2|v) = 0$ we obtain

$$(e_2 + v|Xv) = (e_2 + v|Yv) = 0,$$

so $(e_2 + v|Xv - Yv) = 0$ which contradicts $Xv - Yv = e_1$ (since $(e_1|v) = 0$ and $(e_1|e_2) = 1$). $\square$

Let us now go back to characteristic 2. Thanks to the above lemma and the form (6.18) for X we can write $Xe_2 = e_2 + w$ for some non-zero vector $w \in < e_3, e_4 >$. Also, again by the above lemma, $Ye_2 = e_2 + w'$ for some vector $w' \neq w$ in the plane $< e_3, e_4 >$. Since w, w' are distinct non zero vector in the plane $< e_3, e_4 >$, necessarily $(w|w') = 1$, so we can suppose without loss of generality that $w = e_3$ and $w' = e_4$. Then, remembering that X, Y are symplectic, we deduce that they take the form

$$X = \begin{pmatrix} 1 & 0 & a & c \\ 0 & 1 & 0 & 0 \\ 0 & 1 & b & d \\ 0 & 0 & a & c \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & 0 & e & g \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e & g \\ 0 & 1 & f & h \end{pmatrix} \tag{6.19}$$

for some scalars $a, b, c \in \mathbb{F}_2$ with $ad \neq bc$ and $eh \neq fg$. But then, applying once again the relation (6.17) to the vector e_2 we obtain $Xe_4 - e_4 = Ye_3 - e_3 + e_1$, i.e.

$$\begin{pmatrix} c \\ 0 \\ d \\ c-1 \end{pmatrix} = \begin{pmatrix} e+1 \\ 0 \\ e-1 \\ f \end{pmatrix}.$$

Using the fact that either $e = 0$ or $e - 1 = e + 1 = 0$, at least one of the two determinants $ad - bc$ and $eh - fg$ must vanish. This contradiction achieves the proof in the first case.

`Second case: a plane containing` e_1 `and contained in` $e_1^{\perp}$ `is invariant under` X `and` Y`.` Without loss of generality, we can suppose that this plane is $< e_1, e_3 >$. It is more convenient to write the matrices with respect to the ordered basis (e_1, e_3, e_2, e_4). With respect to this new ordered basis, the symplectic form is expressed by the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where $I = I_2$ denotes the 2×2 identity matrix and 0 the 2×2 null-matrix.

The conjugate matrices, still denoted by X, Y, will take the form

$$X = \begin{pmatrix} A & AR \\ 0 & {}^tA^{-1} \end{pmatrix}, \qquad Y = \begin{pmatrix} B & BS \\ 0 & {}^tB^{-1} \end{pmatrix}$$

for two matrices $A, B \in \mathrm{GL}_2(\mathbb{Z})$ and symmetric matrices R, S (with integral coefficients). The matrix corresponding to T in this new basis is

$$T' := \begin{pmatrix} I & E \\ 0 & I \end{pmatrix}$$

where $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Now the condition $XYX^{-1}Y^{-1} = T'$ is equivalent to $XY - YX = (T' - I)YX$, which amounts to the two conditions

$$AB = BA, \qquad ABS + AR({}^tB^{-1}) - BAR - BS({}^tA^{-1}) = E({}^tB^{-1})({}^tA^{-1}). \tag{6.20}$$

Now we prove that:

The above equation has no solution (A, B, R, S) with $A, B \in \mathrm{GL}_2(\mathbb{F}_2)$ and R, S symmetric.

To prove this claim, rewrite the second equality, after using the commutativity of A, B, as

$$AB\left(S + B^{-1}R({}^tB^{-1}) - R - A^{-1}S({}^tA^{-1})\right) = E({}^tB^{-1})({}^tA^{-1}).$$

Observe that the right-hand side has rank one. We then conclude via the following lemma, which implies that the symmetric matrix inside the parenthesis cannot have rank one:

Lemma 6.4. *Let $T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be a 2×2 symmetric matrix with coefficients in $\mathbb{F}_2$. Let $X \in \mathrm{GL}_2(\mathbb{F}_2)$ be an invertible matrix. Write $X \cdot T \cdot {}^tX$ as $\begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix}$. Then*

$$a + b + c = a' + b' + c'.$$

In particular, every linear combination of symmetric matrices of the form $T - X \cdot T \cdot {}^tX$ has rank zero or two.

Proof. Recall that a two-dimensional vector space over $\mathbb{F}_2$ contains exactly three non-zero vectors v_1, v_2, v_3, and that their sum vanishes. To a symmetric matrix $T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ corresponds a symmetric bilinear form $(\cdot|\cdot)$ on $\mathbb{F}_2^2$. The quantity $a+b+c$ equals the sum

$$(v_1|v_1) + (v_2|v_2) + (v_1|v_2) = (v_1|v_2) + (v_2|v_3) + (v_3|v_1)$$

which is invariant under permutations of v_1, v_2, v_3, i.e. under transformations $T \mapsto X\,T\,{}^tX$. $\square$

Remark. Note that the explicit matrices (5.9) correspond to the following solution of the equation (6.20) with E replaced by mE:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & m/2 \\ m/2 & 0 \end{pmatrix}.$$

7. Commutators in the Symplectic group $Sp(2g, \mathbb{Z})$, $g \geq 3$

We now show:

Theorem 7.21. In every dimension $2g$ with $g \geq 3$, for every $m \geq 0$ there exist symplectic matrices $X, Y \in Sp(2g, \mathbb{Z})$ whose commutator equals

$$T^m = \begin{pmatrix} I + m\Delta & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & I & 0 \\ 0 & 0 & \dots & 0 & I \end{pmatrix}$$

Here, as before, Δ is the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Proof. Clearly, it suffices to prove this statement in the case $g = 3$, i.e. for $Sp(6, \mathbb{Z})$.
Here is a concrete example:

$$X = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & 0 & 0 & m & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & m & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & m & 1 \end{pmatrix}. \quad \square$$

We note that both X and Y are unipotent. In general, we can prove that for every symplectic solution (X, Y) of $XYX^{-1}Y^{-1} = T$, both X and Y must have an eigenvalue equal to ± 1.

Remark 7.1. We want now to show that without the hypothesis that the matrices be symplectic, we have examples even in dimension ≤ 4. For instance, the following pair of (unipotent) matrices in $\mathrm{SL}_3(\mathbb{Z})$

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m & 1 \end{pmatrix}$$

provides a solution to the equation

$$XYX^{-1}Y^{-1} = \begin{pmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = T^m.$$

7.1. The case $g \geq 3$, more general

For the sake of simplicity, we introduce the following notation:

Definition 7.2. Using the standard inclusion of $Sp(2g', \mathbb{Z}) \subset Sp(2g, \mathbb{Z})$ for $g \geq g'$, we define T_1 as the image of the matrix

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

in every $Sp(2g, \mathbb{Z})$.

We define, for $g \geq 2$, T_2 as the image of the matrix

$$T_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and we define similarly $T_k \in Sp(2g, \mathbb{Z})$, for $g \geq k$.

We have the following

Theorem 7.22. T_2 is always a commutator ($g \geq 2$); also T_3 is always a commutator ($g \geq 3$).

Proof. Here is an explicit solution for T_2: $XYX^{-1}Y^{-1} = T_2$ where

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For T_3 we have: $T_3 = XYX^{-1}Y^{-1}$ with

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \square$$

The idea for constructing X, Y comes from the following remark: for every non-zero complex number λ, setting

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & (\lambda^2 - 1)^{-1} \\ 0 & 1 \end{pmatrix}$$

we have

$$ABA^{-1}B^{-1} = T_1.$$

We then look for a number field $\mathbb{K}$ containing a unit (of its ring of integers) λ such that $\lambda^2 - 1$ is also a unit. Letting $n = [\mathbb{K} : \mathbb{Q}]$ be its degree, we can view $\mathbb{K}^2$ as a vector space of dimension $2n$ over $\mathbb{Q}$. The two matrices A, B defined above induce automorphisms of this vector space, and in a suitable basis define two matrices $\tilde{A}, \tilde{B} \in \mathrm{SL}_{2n}(\mathbb{Z})$ satisfying $[A, B] = T_n$. The problem is defining a symplectic form $\mathbb{K}^2 \to \mathbb{Q}$ inducing the standard one on $\mathbb{Q}^{2n}$, after identification $\mathbb{K}^2 \simeq \mathbb{Q}^{2n}$.

It turns out that for $n = 2$ there is only one choice for the number field $\mathbb{K}$, namely the field $\mathbb{Q}(\lambda)$ where λ is the 'golden ratio' satisfying $\lambda^2 = 1 + \lambda$. Identifying $\mathbb{K}^2 \simeq \mathbb{Q}^4$ via the basis $\binom{1}{0}, \binom{0}{1}, \binom{\lambda}{0}, \binom{0}{\lambda}$ we obtain from A, B the matrices X, Y of the theorem.

For $n = 3$, again we have only one choice for the cubic number field, namely the field $\mathbb{Q}(\lambda)$ where

$$\lambda^3 = 2\lambda^2 + \lambda - 1.$$

The basis to be used to identify $\mathbb{K}^2$ with $\mathbb{Q}^6$ is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda \end{pmatrix}, \begin{pmatrix} 1 + \lambda - \lambda^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 + \lambda - \lambda^2 \end{pmatrix}.$$

Again, the matrices X, Y are then obtained from the action of A, B on $\mathbb{K}^2 \simeq \mathbb{Q}^6$.

We can now obtain a more general result using our previous results and the following:

Remark 7.3.

(1) Assume that $A_i \in Sp(2g_i, \mathbb{Z})$ is a commutator, for $i = 1, 2$.
Then $A_1 \oplus A_2 \in Sp(2(g_1 + g_2), \mathbb{Z})$ is also commutator.

(2) In particular, this holds for A_2 equal to the identity matrix.

Summarising, we obtained the following

Theorem 7.23.

(1) T_1^m is never a commutator for $g = 1$.
(2) T_1^m is a commutator if and only if m is even, for $g = 2$.
(3) T_1^m is always a commutator for $g \geq 3$.
(4) T_k is a commutator for all $g \geq k \geq 2$.
(5) T_k^m is a commutator for all $g \geq 2k$, when m is even.
(6) T_k^m is a commutator for all $g \geq 3k$, when m is odd.

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