SURFACES WITH $K^2 = p_g = 1$ AND THEIR PERIOD MAPPING.

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Introduction.

Recently a result of Kynef ([14]) drew attention on minimal surfaces S with $K^2 = p_g = 1$: he constructed a quotient of the Fermat sextic in $\mathbb{P}^3$ by a suitable action of $\mathbb{Z}/6$, with these invariants, such that the differential of the period mapping (see [7] , [9]) is not injective at it, thus answering negatively a problem posed by Griffiths in [8] .

One may remark however that the local Torelli theorem (injectivity of the infinitesimal period mapping) fails , for curves, exactly when one has an hyperelliptic curve ([7]), though the global Torelli theorem holds.

So one is motivated to study these surfaces and their period mapping.

They were first considered by Enriques in 1897, who proved their existence in [5] (see also [6] pag. 305) ; Bombieri ([1], pag. 201) proved rigorously that for these surfaces the tricanonical map is birational.

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Here we prove that the bicanonical map $\Phi = \Phi_{2K}$ is a morphism and that$^{++)}$ any such surface is a weighted complete intersection of type (6,6) in the weighted projective space $\mathbb{P}(1,2,2,3,3)$ (see [4] , [15] about the theory of weighted complete intersections).

We also show that these surfaces have equations in canonical form: this is a first step towards an explicit description of their moduli space, that we hope to accomplish in the future. Then we describe a geometric construction giving all the "special" surfaces,i.e. those for which Φ is a Galois covering (and it turns out that the Galois group is $\mathbb{Z}_{/2} + \mathbb{Z}_{/2}$).

Using this explicit description of our surfaces we prove that they are all diffeomorphic and simply connected, and that when K is ample the Kuranishi family is smooth of dimension 18 (as their local period space): our main result is that the differential of the period mapping is invertible outside an hypersurface, so that the period mapping is generally finite.

The 12 dimensional subfamily parametrizing "special" surfaces is strictly contained in the subvariety where the rank of the differential drops by 2 (the maximum possible amount) and we prove, by means of a more general result on deformations of cyclic coverings, that the restriction of the period mapping to this subfamily is locally 1-1 : this suggests that the period mapping might have no positive dimensional fibres, but we have not yet pursued such investigation.

One last remark is that our results on the failure of the local Torelli theorem for weighted complete intersections (w.c.i.) show that the

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$^{++)}$ The proof which appears here of this result is due to collaboration with Miles Reid.

restrictions put by S. Usui in his work [20] cannot all be eliminated.

I would like here to thank P. Griffiths for suggesting this research and I. Dolgachev for useful conversations.

Notations throughout the paper:

S is a minimal smooth surface with $p_g = K^2 = 1$

$x_\circ \in H^\circ(S, \mathcal{O}(K))$ the unique (up to constants) non zero section

$C = \mathrm{div}(x_\circ)$ the canonical curve

R the graded ring $\mathbb{C}[X_\circ, Y_1, Y_2, Z_3, Z_4]$, where $\deg X_\circ = 1$, $\deg Y_i = 2$, $\deg Z_i = 3$ $(i = 1,2, \quad j = 3,4)$

$W = \mathbb{C}[Y_1, Y_2, Z_3, Z_4]$ as a graded subring of R

R_m, W_m the graded parts of degree m of R, resp. W

$Q = Q(1,2,2,3,3) = \mathrm{proj}(R)$

$R(S) = \sum_{m=o}^{\infty} H^\circ(S, \mathcal{O}(mK))$ the canonical ring of S

$h^i(S,L) = \dim H^i(S,L)$ if L is a coherent sheaf on S

§ 1. STRUCTURE OF SURFACES S WITH $K^2 = P_g = 1$.

LEMMA 1. $P_m = h^{\circ}(S, \mathcal{O}(mK)) = \frac{1}{2}m(m-1) + 2$.

Proof. $P_m = \frac{1}{2}m(m-1)K^2 + \chi(\mathcal{O}_S)$ (see [1] pag. 185, or [11]) and by Theorems 11,15 of [1] $q=0$ and S has no torsion, so $\chi = 2$.

One can choose therefore y_1, y_2, z_3, z_4 such that x_o^2, y_1, y_2 are a basis of $H^{\circ}(S, \mathcal{O}(2K))$, and $x_o^3, x_o y_1, x_o y_2, z_3, z_4$ are a basis of $H^{\circ}(S, \mathcal{O}(3K))$.

Write now $C = \operatorname{div}(x_o) = \Gamma + Z$, where $K \cdot \Gamma = 1$, $K \cdot Z = 0$.

LEMMA 2. If $D \in |2K|$ and $D \geq \Gamma$, $D = 2C$.

Proof. Write $D = D' + \Gamma$ and let D'' be the movable part of $|D'|$: $D'' \cdot K = 1$, so by the index theorem either D'' is homologous (hence linearly equivalent, as S has no torsion) to K, or $D''^2 \leq -1$, hence in both cases $h^{\circ}(S, \mathcal{O}(D'')) = 1$.

COROLLARY 3. $H^{\circ}(S, \mathcal{O}(4K)) = x_o \cdot H^{\circ}(S, \mathcal{O}(3K)) \oplus (\mathbb{C}y_1^2 \oplus \mathbb{C}y_1 y_2 \oplus \mathbb{C}y_2^2)$.

Proof. Because $P_4 = 8$ it is enough to prove that the two vector subspaces have no common line. Supposing the contrary, there would exist a section $s \in H^{\circ}(\mathcal{O}(3K))$, and costants $\lambda_1, \mu_1, \lambda_2, \mu_2$ such that $x_o \cdot s = (\lambda_1 y_1 + \mu_1 y_2)(\lambda_2 y_1 + \mu_2 y_2)$.

Taking the associated divisors $C+\mathrm{div}(s)=D_1+D_2$, where $D_i \in |2K|$, and one of the D_i, say D_1, is $\geq \Gamma$ therefore. By lemma 2 $D_1=2C$, hence $\lambda_1 y_1+\mu_1 y_2=cx_o^2$ for a suitable $c \in \mathbb{C}$, contradicting the independence of x_o^2, y_1, y_2.

THEOREM 1. $|2K|$ has no base points, so that $\Phi = \Phi_{2K}: S \longrightarrow \mathbb{P}^2$ is a morphism of degree 4.

Proof. If b were a base point of $|2K|$, then x_o, y_1, y_2 would vanish at b ; by coroll. 3 b would be a base point of $|4K|$, contradicting Theorem 2 of [1] .

Define an homomorphism $\alpha^*: R \longrightarrow R(S)$ by sending X_o to $x_o, \ldots . Z_4$ to z_4: by theorem 1 α^* induces a morphism $\alpha: S \rightarrow Q=Q(1,2,2,3,3)$.

Remark that Q is smooth outside the two $\mathbb{P}^1$'s $S_2=\{x_o=z_3=z_4=0\}$, $S_3=\{x_o=y_1=y_2=0\}$, and if $\mathbb{P}=\mathbb{P}(1,2,2,3,3) = Q-S_2-S_3$, on $\mathbb{P}$ $\mathcal{O}_Q(m)$ is an invertible sheaf for every integer m and $\forall\, a,b \in \mathbb{Z}$ one has an isomorphism $\mathcal{O}_Q(a) \otimes \mathcal{O}_Q(m)^{\otimes b} \longrightarrow \mathcal{O}_Q(a+bm)$ (compare [15] , exp. pages 619-624, and also cf. [4]).

PROPOSITION 4. $|3K|$ has no base points and $\alpha(S) \subset \mathbb{P}$.

Proof. If $\alpha(b) \in S_3$, then b is a base point of $|2K|$, if $\alpha(b) \in S_2$ b is a base point of $|3K|$: in view of theorem 1 we need to show that $|3K|$ has no base points.
If b is a base point of $|3K|$, we have that $b \in \Gamma - Z$: in fact $|3K|$ has no fixed part, but if $b \in E$, E irreducible with $K\cdot E=0$, any section of $\mathcal{O}(3K)$ vanishing on b vanishes

on E too.

Because of the exact sequence

$$0 \longrightarrow H^{\circ}(\mathcal{O}(2K-\Gamma)) \longrightarrow H^{\circ}(\mathcal{O}(2K)) \longrightarrow H^{\circ}(\mathcal{O}(2K)\otimes \mathcal{O}_{\Gamma}) \longrightarrow$$

of lemma 2, and the fact that $\Gamma \cdot 2K=2$, a general curve D of $|2K|$ passing through b is smooth at b , and one can suppose that the component of D to which b belongs is not rational (S being of general type).

It follows by prop. B of [2] that b is not a base point of $\omega_D = \mathcal{O}_D(3K)$: this is a contradiction by the exact sequence

$$0 \longrightarrow H^{\circ}(\mathcal{O}(K)) \longrightarrow H^{\circ}(\mathcal{O}(3K)) \longrightarrow H^{\circ}(\mathcal{O}_D(3K)) \longrightarrow 0 \quad .$$

Denote by I the ideal $\ker \alpha^*$: because $\dim R_6=19$, $P_6=\dim R(S)_6=17$, there exist two independent elements $f,g \in I_6$.

PROPOSITION 5. f,g are irreducible and $\alpha(S)=Y=\{f=g=0\}$.

Proof. If f is reducible, by coroll. 3 $f=X_o \cdot f'$, $f' \in R_5$, and $\alpha(S) \subset \{f'=g=0\}$. Denote now by $p:Q \longrightarrow \mathbb{P}^2$ the rational map given by (X_o^2, Y_1, Y_2) : clearly $\Phi = p \circ \alpha$. Now one gets a contradiction considering that

i) $p : \{f'=g=0\} \longrightarrow \mathbb{P}^2$ is of degree ≤ 2 , because $\{f'=0\}$ is irreducible and the variables Z_3, Z_4 appear at most quadratically in g , and linearly in f'(observe that $(-X_o, Y_1, Y_2, -Z_3, -Z_4) \cong (X_o, Y_1, Y_2, Z_3, Z_4)$).

ii) α is birational because the tricanonical map is such ([1], pag. 202).

iii) Φ is of degree four.

Finally $p : \{f=g=0\} \longrightarrow \mathbb{P}^2$ is of degree four by the same argument of i), hence $Y=\alpha(S)$ and is irreducible.

PROPOSITION 6. The subscheme of $\mathbb{P}$ weighted complete intersection of type (6,6), $Y=\{f=g=0\}$ is isomorphic to the canonical model of S. Therefore I is generated by f,g, and α^* induces an isomorphism $\alpha^*=R'=R/_I \longrightarrow R(S)$.

Proof. $\alpha : S \longrightarrow Y$ is a desingularization such that the pull back of the dualizing sheaf on Y is the canonical bundle K of S (as $\omega_Y \cong \mathcal{O}_Q(1)$ by [15] , prop. 3.3.): therefore Y has only rational double points as singularities and is the canonical model of S (cf. [1] , [16]).

THEOREM 2. The canonical models of minimal surfaces with $K^2=p_g=1$ correspond to weighted complete intersections Y of type (6,6) in $\mathbb{P}(1,2,2,3,3)$, with at most rational double points as singularities, and two surfaces are isomorphic iff their canonical models are projectively equivalent in $\mathbb{P}$.

Proof. If Y is as above, $\mathcal{O}_Y(1)$ is the canonical sheaf and by prop. 3.2. of [15] $\mathcal{O}_Y(1)^2=1$; again by prop. 3.3 of [15] R'_m is isomorphic to $H^0(Y,\mathcal{O}_Y(m))$, so our first assertion follows immediately.

Note that an isomorphism of two surfaces gives isomorphisms of the vector spaces $H^0(\mathcal{O}(mK))$, so the second statement is obvious after we describe the projective group of $\mathbb{P}$: it con sists of the invertible transformations of the following form

$X_o \longrightarrow dX_o$

$Y_i \longrightarrow d_{i1}Y_1 + d_{i2}Y_2 + d_{i0}X_o^2 \qquad i=1,2$

$Z_j \longrightarrow C_{j3}Z_3+C_{j4}Z_4+C_{j0}X_o^3 + C_{j1}X_oY_1 + C_{j2}X_oY_2$.

PROPOSITION 6. There exists a projective change of coordinates such that Y is defined by 2 equations in canonical form

$$f = Z_3^2+X_oZ_4(a_oX_o^2+a_1Y_1+a_2Y_2) + F_3(X_o^2,Y_1,Y_2)$$

$$g = Z_4^2 + X_oZ_3(b_oX_o^2+b_1Y_1+b_2Y_2) + G_3(X_o^2,Y_1,Y_2)$$

where F_3, G_3 are cubic forms.

<u>Proof</u>. Write $f = Q_1(Z_3,Z_4)+\dots\dots$(terms of deg ≤ 1 in the Z_j)

$g = Q_2(Z_3,Z_4)+\dots$.

I claim that the quadratic forms Q_1, Q_2 are not proportional: otherwise, by taking a linear combination of the 2 equations one would have $Q_2=0$, but then $p:Y \longrightarrow \mathbb{P}^2$ would have degree 2 and not 4.

By a transformation $Z_j \longrightarrow C_{j3}Z_3+C_{j4}Z_4$ one can suppose $Q_1=Z_3^2$, $Q_2=Z_4^2$: this is immediate if both Q_1, Q_2 have rank 1, while if, say, Q_1 has rank 2, one proceeds as follows.

First take coordinates such that $Q_1=Z_3\cdot Z_4$, then, subtracting to g a multiple of f one can get $Q_2=m_3Z_3^2+m_4Z_4^2$.

If m_3 and m_4 are $\neq 0$, one takes first new variables

$\sqrt{m_3}\, Z_3$, $\sqrt{m_4} Z_4$, so that for $f/\sqrt{m_3 m_4}$, g, Q_1, Q_2 have now the form $Z_3 Z_4$, $Z_3^2+Z_4^2$: then one takes variables Z_3', Z_4' with $Z_3=Z_3'-Z_4'$, $Z_4=Z_3'+Z_4'$ so $Q_1 = Z_3'^2-Z_4'^2$, $Q_2=2(Z_3'^2+Z_4'^2)$, and finally $\frac{2Q_1+Q_2}{4}$, $\frac{Q_2-2Q_1}{4}$ are in the desired form.

If, say m_4 is zero, one can suppose $Q_2=Z_3^2$: but we have a contradiction because then the point $(0,0,0,0,1)$ satisfies $f=g=0$, against the fact that $Y\subset \mathbb{P}$.

Finally, if now $f=Z_3^2+X_o Z_3(a_o' X_o^2+a_1' Y_1+a_2' Y_2)+\ldots.$ one kills the a_j' by completing the square, i.e. by taking $Z_3+\frac{1}{2}X_o(a_o' X_o^2+a_1' Y_1+a_2' Y_2)$ as new Z_3 coordinate, and analogously one then does for g acting on the Z_4 variable.

REMARK 7. If Y and Y' are defined by two canonical forms, they are isomorphic iff the canonical equations are equivalent under the projective subgroup

$$\begin{pmatrix} d & d_{10} & d_{20} & 0 & 0 \\ 0 & d_{11} & d_{21} & 0 & 0 \\ 0 & d_{12} & d_{22} & 0 & 0 \\ 0 & 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & 0 & c_{44} \end{pmatrix}$$

§ 2. A GEOMETRIC CONSTRUCTION OF THE "SPECIAL" SURFACES (Φ A GALOIS COVERING).

Consider $\mathbb{P}^2$ with coordinates (Y_0,Y_1,Y_2), denote by ℓ the line $\{Y_0=0\}$ and choose a reducible sextic curve F+G. In $\mathcal{O}_{\mathbb{P}^2}(3)$ take the double covering X of $\mathbb{P}^2$ branched along F+G, and let F+G have neither multiple components nor singular points of multiplicity ≥ 4, or of type (3,3): then (see e.g. [10] , pag. 47-50) X has only rational double points as singularities, and its minimal resolution $\tilde{X}$ is a K^3 surface (in fact if $p:\tilde{X} \longrightarrow \mathbb{P}^2$ is the double cover, $K_{\tilde{X}} = p^*(\mathcal{O}_{\mathbb{P}^2}(3)+K_{\mathbb{P}^2})=0$).

Suppose deg F = deg G = 3 and denote by $L=p^*(\ell)$, by $E_1,\ldots.E_p$ the rational curves with self-intersection -2 coming from the resolution, by F',G' the strict transforms of F,G.

One has $p^*(\mathcal{O}_{\mathbb{P}^2}(3)) \equiv 2F'+\sum r_iE_i$ (for some positive integers r_i)$\equiv$ $\equiv 2G'+\sum s_iE_i$. If we set $\mathcal{L} \doteq p^*(\mathcal{O}_{\mathbb{P}^2}(2))\otimes\mathcal{O}_{\tilde{X}}(-F'-\sum\left[\frac{r_i}{2}\right]E_i)$, and $J\subset\{1,\ldots p\}$ is the subset of indexes J for which r_j is odd, one can easily check that $L+\sum E_j\equiv 2\mathcal{L}$; therefore one takes a double cover $\tilde{S}$ of $\tilde{X}$ in $\mathcal{L}$, ramified over $L+\sum E_j$.

The E_j's become exceptional of the first kind in $\tilde{S}$, and after blowing them down I get a surface $S \xrightarrow{\Phi} \mathbb{P}^2$, for which $\Phi^*(\mathcal{O}_{\mathbb{P}^2}(1))\equiv 2K_S$; clearly then $K^2=1$, and $p_g=1$ because $H^{\circ}(\tilde{S},K_{\tilde{S}}) \cong H^{\circ}(\tilde{X},\mathcal{L}) + H^{\circ}(\tilde{X},\mathcal{O}_{\tilde{X}})$.

REMARK 8. If one chooses F of degree 5, G of degree 1, $H^{\circ}(\tilde{X},\mathcal{L})$ becomes 1-dimensional, so that one gets thus a surface S with $K^2=1$, $p_g=2$.

DEFINITION 9. A surface S is called "special" if it is obtained in the above described way.

PROPOSITION 10. A special S is simply connected and is a Galois covering of $\mathbb{P}^2$ with group $\mathbb{Z}/_2 + \mathbb{Z}/_2$. The canonical model of S in $\mathbb{P}(1,2,2,3,3)$ has equations

$$\begin{cases} z_3^2 = F(x_o^2,y_1,y_2) \\ z_4^2 = G(x_o^2,y_1,y_2) \end{cases}$$

F,G being the equations of the 2 cubics.

<u>Proof</u>. For any S denote by R the ramification divisor of Φ, and by $B=\Phi(R)$. $\pi_1(S)$ is a quotient of $\pi_1(S-R)$, a subgroup of index 4 in $\pi_1(\mathbb{P}^2-B)$.
If S is special $B=\ell+F+G$; if then F,G are smooth, the mutual intersections of ℓ,F,G are transversal, $\pi_1(\mathbb{P}^2-B)$ is abelian ([21]).
But S has no torsion, $q=0$ ([1] theorems 11 and 15), so $\pi_1(S)=0$. Take now $\zeta_i=\left(Y_j/Y_i\right)^3\zeta_j$ as a fibre coordinate in $\mathcal{O}_{\mathbb{P}^2}(3)$ (X is defined by $\zeta_i^2 = F\cdot G\cdot Y_i^{-6}$), and $\xi_i = (Y_j/Y_i)^2 \frac{\zeta_{j3}}{\zeta_{i3}}\xi_j$ as a fibre coordinate in $\mathcal{L}$, where

ζ_{i3} is a local coordinate for F', ζ_{i4} for G'. (So that $\zeta_i = \zeta_{i3} \cdot \zeta_{i4}$).

If X_o is the section corresponding to (ξ_i), Z_h to (ζ_{ih}), after easy manipulations one obtains the desired equations.

THEOREM 3. The special surfaces are exactly those for which Φ is a Galois covering, and they form a 12-dimensional family.

Proof. If $\Phi : S \to \mathbb{P}^2$ is Galois, let G be the Galois group (a priori it can be $\mathbb{Z}/_4$ or $\mathbb{Z}/_2 + \mathbb{Z}/_2$, but our proof will imply that the first case does not occur). Consider that if $C = \Gamma + Z$ is, as usual, the canonical curve, Φ makes Γ a double cover of a line; hence the image of G in $\mathrm{Aut}(\Gamma)$ has order two, so exists an involution $\sigma \in G$ (i.e. σ^2=Identity) leaving Γ pointwise fixed, and σ is biregular, S being of general type.

We will first give a proof in the case when K is ample (so $\Gamma = C$). The proof of the following lemma is elementary and well known.

LEMMA 11. Let V be an n-dimensional manifold, σ a biregular involution, F the set of fixed points of σ.

Then $F = \bigcup_{r=0}^{n-1} F_r$, where each F_r is a closed submanifold (possibly empty or disconnected) of dimension r, and if $P \in F_r$ one can choose local coordinates $(z_1, \ldots z_r, z_{r+1}, \ldots, z_n)$ such that $\sigma(z_1, \ldots z_n) = (z_1, \ldots z_r, -z_{r+1}, \ldots -z_n)$.

In case K is ample $C \subset F_1$, but we have equality because if $D \subset F_1$, $C \cdot D \geq 1$, contradicting the smoothness of F_1.
Therefore $S/_\sigma = X$ is a, possibly singular, K3 surface and $\Phi : S \to \mathbb{P}^2$ factors as $S \xrightarrow{\pi} X \xrightarrow{p} \mathbb{P}^2$. Because the ramification divisor R of Φ is $\equiv 7K$, by Porteous' formula or an explicit computation that we will see in prop. 12, X is a double cover of $\mathbb{P}^2$ ramified over a sextic curve B': now in $\tilde{X}$ the divisor $\pi(C) + \sum E_j$, the E_j's being the rational (-2) curves coming from the resolution of the isolated fixed points of σ, is divisible by 2 in $\mathrm{Pic}(\tilde{X})$ iff B' is reducible in components of odd degree, and by remark 8 one concludes that B' consists of two cubics.
When K is not ample this proof becomes more cumbersome, so we use a different idea in the general case, using the representation of σ on the vector spaces $H^0(\mathcal{O}(mK))$, hence on $\mathbb{P}(1,2,2,3,3)$.
Clearly $\sigma(x_o) = \pm x_o$, σ is the identity on $H^0(\mathcal{O}(2K))$, and one can choose z_3, z_4 so that they are eigenvectors for σ: therefore one can assume that σ acts on $\mathbb{P}$ by one of the two trasformations $(X_o, Y_1, Y_2, Z_3, Z_4) \to (-X_o, Y_1, Y_2, Z_3, Z_4)$ or $(-X_o, Y_1, Y_2, -Z_3, Z_4)$.
In the second case the fixed locus of σ on S is contained in $\{x_o = z_3 = 0\} \cup \{x_o = z_4 = 0\}$, i.e. a finite set of points ($|3K|$ has no base points): but the whole curve Γ is pointwise fixed, so this case cannot occur.
$I_6 \subset R_6$ being σ invariant, one can assume that f, g are eigenvectors for σ themselves: but the monomials in R_6

are eigenvectors of eigenvalue (-1) if X_o appears with an odd power, and $(+1)$ if X_o appears with even power. f,g not being divisible by X_o, the corresponding eigenvalues are both $+1$, and f,g are sum of monomials where X_o appears only with even power: proceeding as in proposition 6 one can find coordinates where f,g have the form

$$\begin{cases} Z_3^2 - F_3(X_o^2,Y_1,Y_2)=0 \\ Z_4^2 - G_3(X_o^2,Y_1,Y_2)=0 \end{cases}$$

The last statement follows from remark 7.

PROPOSITION 12. If B is the branch locus of $\Phi : S \longrightarrow \mathbb{P}^2$, and the canonical model of S has equations as in prop. 6

$$f=Z_3^2+X_oZ_4\,\alpha_1(X_o^2,Y_1,Y_2)+F_3(X_o^2,Y_1,Y_2)$$
$$g=Z_4^2+X_oZ_3\,\beta_1(X_o^2,Y_1,Y_2)+G_3(X_o^2,Y_1,Y_2)$$

(α_1, β_1 being linear forms)

the equation of B is $Y_o^2(F^2G^2+\alpha_1^2\,Y_o\,G^3+F^3\beta_1^2Y_o-\frac{27}{256}\alpha_1^4\,\beta_1^4\cdot Y_o^4+$ $+\frac{9}{8}Y_o^2\,\beta^2\,\alpha^2FG)$. If S is not special $\pi_1(\mathbb{P}^2-B)$ is not abelian.

<u>Proof</u>. We first write the equation of R, given by the vanishing of the jacobian of f,g, $Y_o=X_o^2,Y_1Y_2$, easily computed to be $2X_o\left(\frac{\partial f}{\partial Z_3}\cdot\frac{\partial g}{\partial Z_4}-\frac{\partial f}{\partial Z_4}\cdot\frac{\partial g}{\partial Z_3}\right)=2X_o(4Z_3\cdot Z_4-X_o^2\,\alpha_1\,\beta_1)$.

It is clear that $\Phi_*(X_o=0)$ gives twice the line $Y_o=0$, while, to compute the branch locus when $Y_o\neq 0$ we proceed as follows: given X_o,Y_1,Y_2, f and g can be considered as two conic

equations in the plane of coordinates (Z_3, Z_4) and one must write when two conics intersect in less than 4 points.
Now the pencil of conics $\lambda f+\mu g$ has ≤ 3 base points iff there are not 3 distinct degenerate conics in the pencil, i.e. when the discriminant of the cubic equation in (λ,μ), given by

$$\det \begin{vmatrix} \lambda & 0 & \mu X_o \beta_1/2 \\ 0 & \mu & \lambda X_o \alpha_1/2 \\ \frac{\mu\beta_1 X_o}{2} & \frac{\lambda X_o \alpha_1}{2} & \lambda F+\mu G \end{vmatrix} = 0$$

vanishes: using the expression for the discriminant of a cubic equation one obtains the above written equation for B.
For the second statement, consider that the group of covering transformations of $\Phi : S-R \to \mathbb{P}^2-B$ is given by $N/_{\pi_1(S-R)}$, where N is the normalizer of $\pi_1(S-R) \hookrightarrow \pi_1(\mathbb{P}^2-B)$, so that if $\pi_1(S-B)$ is abelian Φ is a Galois covering (because a covering transformation can be extended to a biregular automorphism of S).
One can indeed check that the general B has 6 tacnodes on the line $Y_o=0$.

PROPOSITION 13. All the minimal surfaces with $K^2=p_g=1$ are diffeomorphic. In particular they are all simply connected.

<u>Proof</u>. By proposition 6 there exists a family with connected smooth base containing all the canonical models. By the results of [18] follows that all the nonsingular models are deformation of each other, hence they are all diffeomor-

phic.

We have already proven in prop. 10 that a general special S is simply connected.

REMARK 14. One can easily compute that the surface constructed by Kynef in [14] is a "special" surface corresponding to the following choice of the two cubics: $F=2Y_1^3-(Y_2^3-Y_0^3)$.

$$G=2Y_1^3+(Y_2^3-Y_0^3)$$

F and G have double contact on 3 points lying in the line $Y_1=0$, and these contribute 3 points of type A_5 on the singular K3 surface X, 3 points of type A_2 on the canonical model of S, which however are disjoint from the canonical curve. The (-2) rational curves, the smooth curve C, the two elliptic curves with self-intersection -1, each covering twice via Φ the line $Y_1=0$, form an interesting configuration on the surface S.

REMARK 15. If S is special, the Galois group being $\mathbb{Z}/_2 + \mathbb{Z}/_2$, it is easy to see that there are two more geometric constructions for S: take the double cover of $\mathbb{P}^2$ branched on $\ell + F$, then the double cover branched on the inverse image of G plus some rational (-2) curves (and the same with G in the place of F).

§ 3. THE INFINITESIMAL PERIOD MAPPING.

Consider the Kuranishi family of deformations of S: its tangent space at the point representing S is naturally identified with $H^1(S,T_S)$, T_S denoting the tangent sheaf of S. By [7], the differential of the period mapping $\mu: H^1(S,T_S) \longrightarrow \mathrm{Hom}(H^{\circ}(\Omega^2_S), H^1(\Omega^1_S))$ is obtained via the bilinear mapping $H^1(S,T_S)\times H^{\circ}(\Omega^2_S) \longrightarrow H^1(T_S \otimes \Omega^2_S)$, and the natural isomorphism $\Omega^1_S \cong T_S \otimes \Omega^2_S$.

The injectivity of μ("Local Torelli" problem) assumes an easy form when $p_g=1$: it means that if x_o is the usual non zero section of $H^{\circ}(\Omega^2_S) = H^{\circ}(\mathcal{O}_S(K))$, $H^1(T_S)\xrightarrow{\text{mult by } x_o} H^1(\Omega^1_S)$ multiplication by x_o is injective.

S being of general type $H^{\circ}(T_S)=0$, the morphism μ fits into the exact sequence of cohomology

$$\underset{0}{\overset{\Vert}{H^{\circ}(\Omega^1_S)}} \longrightarrow H^{\circ}(\Omega^1_S \otimes \mathcal{O}_C) \longrightarrow H^1(T_S) \xrightarrow{\ \mu\ } H^1(\Omega^1_S) \longrightarrow$$

and here μ is injective iff $h^{\circ}(\Omega^1_S \otimes \mathcal{O}_C)=0$.

In the rest of the paragraph we will assume that the canonical model of S is smooth, hence isomorphic to S.

PROPOSITION 16. If S is special $\ker \mu$ is 2 dimensional.

Proof. There is an involution σ on S leaving C pointwise fixed and C is smooth, so $\Omega^1_S \otimes \mathcal{O}_C$ splits into the (+1) and (−1) eigenspaces for σ; then $\Omega^1_S \otimes \mathcal{O}_C \cong \mathcal{O}_C(-C) \oplus \omega_C$, and C being of genus 2 $h^{\circ}(\Omega^1_S \otimes \mathcal{O}_C)=h^{\circ}(\omega_C)=2$.

THEOREM 4. Suppose S is a smooth w.c.i. of type (6,6) in $\mathbb{P}$, with canonical equations (where $Y_o=X_o^2$):

$$\begin{cases} f=Z_3^2+X_oZ_4\left(\sum_{i=0}^{2} a_iY_i\right) + \sum_{0\leq i\leq j\leq k} f_{ijk}Y_iY_jY_k=0 \\ g=Z_4^2+X_oZ_3\left(\sum_{i=0}^{2} b_iY_i\right)+ \sum_{0\leq i\leq j\leq k} g_{ijk}Y_iY_jY_k=0 \ . \end{cases}$$

Then the Kuranishi family of S is smooth of dimension 18. Moreover there exists a non zero polynomial $\Delta(a_1,a_2,b_1,b_2,f_{111},f_{112},f_{122},f_{222}, g_{111},g_{112},g_{122},g_{222})$ such that μ is injective iff $\Delta\neq 0$ for S.

<u>Proof</u>. One has the following exact sequences, where $T_{\mathbb{P}}$ is the tangent sheaf to $\mathbb{P}$, $e_o=1, e_1=e_2=2, e_3=e_4=3$ are the weights of $\mathbb{P}$:

$$\text{i)}\quad 0\rightarrow \mathcal{O}_S \xrightarrow{\alpha} \bigoplus_{i=0}^{4} \mathcal{O}_S(e_i) \xrightarrow{\beta} T_{\mathbb{P}}\otimes\mathcal{O}_S \longrightarrow 0$$

$$\text{ii)}\quad 0\rightarrow T_S \xrightarrow{\gamma} T_{\mathbb{P}}\otimes \ _S \xrightarrow{\delta} \mathcal{O}_S(6)\oplus\mathcal{O}_S(6)\longrightarrow 0$$

Here $\mathcal{O}_S(6)\oplus\mathcal{O}_S(6)$ is the normal bundle to S in $\mathbb{P}$, α is given by the transpose of $(X_o,2Y_1,2Y_2,3Z_3, 3Z_4)$, $\beta({}^t(f_o,\dots f_4))=$

$$= f_o\frac{\partial}{\partial X_o} +\dots f_4\frac{\partial}{\partial Z_4} \ .$$

We know from [15] prop. 3.3 that $h^1(S,\mathcal{O}_S(n))=0\ \forall n$, and $H^o(S,\mathcal{O}_S(n))=R'_n$, where $R'=R/_I$ (I the ideal of S).

Therefore from the exact sequence of cohomology of i) we infer that $H^1(S,T_{\mathbb{P}}\otimes\mathcal{O}_S)=0$, provided $H^2(\mathcal{O}_S)\longrightarrow\bigoplus_{i=0}^{4}H^2(\mathcal{O}_S(e_i))$ is injective: this is equivalent to the surjectivity of the dual mapping $\bigoplus_{i=0}^{4}H^{\circ}(\mathcal{O}_S(1-e_i))\xrightarrow{t_\alpha}H^{\circ}(\mathcal{O}_S(1))$. This last can be written as $R'_{\circ}\xrightarrow{\text{mult by } x_{\circ}}R'_1$, hence α is an isomorphism ($R'_{\circ}=\mathbb{C}$, $R'_1=\mathbb{C}X_{\circ}$): thus we also obtain that $H^2(S,T_{\mathbb{P}}\otimes\mathcal{O}_S)=0$, which, together with the exact cohomology sequence of ii), and the above cited vanishing of $H^1(S,\mathcal{O}_S(6))$, gives $H^2(S,T_S)=0$.

So the Kuranishi family of S is smooth of $\dim=h^1(S,T_S)=$ $=-\chi(S,T_S)$ (cf. [12], [13]) $=-\frac{7}{6}K^2+\frac{5}{6}C_2=18$ (this formula, where $C_2=$ the Euler number of S, is obtained applying Hirzebruch's Riemann-Roch theorem to T_S).

As $H^1(\mathcal{O}_S)=0$ $\bigoplus_{i=0}^{4}R'_{e_i}\xrightarrow{\beta}H^{\circ}(T_{\mathbb{P}}\otimes\mathcal{O}_S)\longrightarrow 0$ is exact, as well as $H^{\circ}(T_{\mathbb{P}}\otimes\mathcal{O}_S)\xrightarrow{\delta}R'_6\oplus R'_6\qquad\xrightarrow{\partial}H^1(T_S)\longrightarrow 0$: it is important to notice that $\delta\circ\beta$ is given by the matrix

$$\begin{pmatrix}\partial_{\circ}f & \dots\dots\dots\dots & \partial_4 f\\ \partial_{\circ}g & \dots\dots\dots\dots & \partial_4 g\end{pmatrix}\qquad\text{where }\ \partial_{\circ}f=\frac{\partial f}{\partial X_{\circ}},\ \dots\dots .$$

We can tensor the two above sequences by $\mathcal{O}_S(1)$, and multiplication by $X_{\circ}$ gives a morphism of the former to the latter: again $\bigoplus_{i=0}^{4}R'_{e_i+1}\xrightarrow{\beta}H^{\circ}(T_{\mathbb{P}}\otimes\mathcal{O}_S(1))\longrightarrow 0$ is exact (because $H^1(\mathcal{O}_S(1))=0$) while $H^2(\mathcal{O}_S(1))$ is 1-dimensional,

$H^2(\mathcal{O}_S(e_i+1))=0$, being dual to $H^0(\mathcal{O}_S(-e_i))$, hence $h^1(T_{\mathbb{P}} \times \mathcal{O}_S(1))=1$, and one has the following commutative diagram, which is exact in the rows, and where the vertical arrows are given by multiplication by X_o

$$\begin{array}{ccccccccc}
\bigoplus_{i=0}^{4} R'_{e_i} & \xrightarrow{\delta\circ\beta} & R'_6 \oplus R'_6 & \xrightarrow{\partial} & H^1(T_S) & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow \mu & & & \searrow & \\
\bigoplus_{i=0}^{4} R'_{e_i+1} & \xrightarrow{\delta\circ\beta} & R'_7 \oplus R'_7 & \xrightarrow{\partial'} & H^1(T_S\otimes\mathcal{O}(1)) & \longrightarrow & H^1(T_{\mathbb{P}}\otimes\mathcal{O}_S(1)) & \longrightarrow & 0
\end{array}$$

We remark that $h^1(\Omega^1_S)=19$, so $\mathrm{Im}(\partial')$ has dimension 18: as $\mu(H^1(T_S))\subset\mathrm{Im}(\partial')$, μ is injective iff it is surjective on $\mathrm{Im}(\partial')$.

This last condition means that $R'_7 \oplus R'_7=(X_oR'_6 \oplus X_oR'_6)+ + \delta\circ\beta\,(\bigoplus_{i=0}^{4} R'_{e_i+1})$. Now $R'_m = R_m/I_m$, and $I_m=0$ for $m \leq 5$, $I_7=\mathbb{C}X_of+\mathbb{C}X_og$, and we observe that we have the splitting $R_m=W_m+X_oR_{m-1}$.

Given an element $\alpha \in R$, one can use the splitting $R=W \oplus X_oR$ to write $\alpha = \alpha_W+X_o\,\alpha_R$ in an unique way, and the same can be done for the matrix of $\delta\circ\beta : \bigoplus_{i=0}^{4} R_{e_i+1} \longrightarrow R_7 \oplus R_7$.

$R'_7 \oplus R'_7 = (X_oR'_6 \oplus X_oR'_6)+\delta\circ\beta\,(\bigoplus_{i=0}^{4} R'_{e_i+1})$ is equivalent to

$R_7 \oplus R_7 = (X_oR_6 + X_oR_6) + (\delta\circ\beta)_W(\bigoplus_{i=0}^{4} \cdot W_{e_i+1}) \Longleftrightarrow (\delta\circ\beta)_W :$

$: \bigoplus_{i=0}^{4} W_{e_i+1} \longrightarrow W_7 \oplus W_7$ is an isomorphism.

Now we can write the matrix associated to $(\delta \circ \beta)_W$: it is

$$\begin{pmatrix} Z_4(a_1Y_1+a_2Y_2), & 3f_{111}Y_1^2+2f_{112}Y_1Y_2+f_{122}Y_2^2, & f_{112}Y_1^2+2f_{122}Y_1Y_2+3f_{222}Y_2^2, & 2Z_3, & 0 \\ Z_3(b_1Y_1+b_2Y_2), & 3g_{111}Y_1^2+2g_{112}Y_1Y_2+g_{122}Y_2^2, & g_{112}Y_1^2+2g_{122}Y_1Y_2+3g_{222}Y_2^2, & 0, & 2Z_4 \end{pmatrix}.$$

To simplify the computations, observe that $W_7=Z_3W_4 \oplus Z_4W_4$, and one can write $(W_2 \oplus W_3 \oplus W_3) \oplus (W_4 \oplus W_4) = \bigoplus_{i=0}^{4} W_{e_i+1}$,

$W_7 \oplus W_7 = (Z_4W_4 \oplus Z_3W_4) \oplus (Z_3W_4 \oplus Z_4W_4)$: $(\delta \circ \beta)_W$ is also a direct sum map, and its second summand, being given by

$$(W_4 \oplus W_4) \xrightarrow{\begin{pmatrix} 2Z_3 & 0 \\ 0 & 2Z_4 \end{pmatrix}} Z_3W_4 \oplus Z_4W_4,$$

is an obvious isomorphism.

So we conclude that $(\delta \circ \beta)_W$ is an isomorphism iff the first summand is such: to see this last map we pick bases of these 2 6-dimensional vector spaces over $\mathbb{C}$, and write the associa ted matrix A.

We choose for $(W_2 \oplus W_3 \oplus W_3)$ the ordered basis

$$\begin{pmatrix} Y_1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} Y_2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ Z_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ Z_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ Z_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ Z_4 \end{pmatrix},$$

and for $(Z_4W_4 \oplus Z_3W_4)$ the ordered basis

$$\begin{pmatrix} Y_1^2 Z_4 \\ 0 \end{pmatrix} \begin{pmatrix} Y_2^2 Z_4 \\ 0 \end{pmatrix} \begin{pmatrix} Y_1 Y_2 Z_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ Y_1^2 Z_3 \end{pmatrix} \begin{pmatrix} 0 \\ Y_2^2 Z_3 \end{pmatrix} \begin{pmatrix} 0 \\ Y_1 Y_2 Z_3 \end{pmatrix} :$$

then A is the matrix (where an empty array means that the correspon_ ding entry is zero)

a_1			$3f_{111}$		f_{112}
	a_2		f_{122}		$3f_{222}$
a_2	a_1		$2f_{112}$		$2f_{122}$
b_1		$3g_{111}$		g_{112}	
	b_2	g_{122}		$3g_{222}$	
b_2	b_1	$2g_{112}$		$2g_{122}$	

Call Δ the determinant of A: then μ is injective iff $\Delta(a_1,\ldots g_{222})\neq 0$ for S, and it remains only to show that Δ is not identically zero. But the monomial $a_2 b_1 f_{111} f_{222} g_{111} g_{222}$ appears in Δ with coefficient -36.

COROLLARY 17. The subfamily of special surfaces is contained in a 14 dimensional family where $\operatorname{Ker} \mu$ is 2 dimensional.

Proof. It is evident that the rank of μ drops by 2 iff the rank of A drops by 2: but 2 columns of A vanish in the 3 following cases:

$$\begin{cases} a_1 = a_2 = b_1 = b_2 = 0 \\ g_{111} = g_{112} = g_{122} = g_{222} = 0 \\ f_{111} = f_{112} = f_{122} = f_{222} = 0 \end{cases}$$

We have only to recall to our mind that S is special iff $a_o=a_1=a_2=b_o=b_1=b_2=0$.

REMARK 18. When the w.c.i. Y has rational double points, one considers as usual $T_Y = \underline{\mathrm{Hom}}(\Omega^1_Y, \mathcal{O}_Y)$, and by [3] prop. 1.2. one has, if $S \xrightarrow{\pi} Y$ is the desingularization, $\pi_*(T_S)=T_Y$, $h^2(T_S)=h^2(T_Y)$.
The analogous of the exact sequence ii) is now

$$0 \longrightarrow T_Y \longrightarrow T_{\mathbb{P}} \otimes \mathcal{O}_Y \rightarrow \underbrace{\mathcal{O}_Y(6) \oplus \mathcal{O}_Y(6)}_{= N_Y} \xrightarrow{\gamma} \underbrace{\underline{\mathrm{Ext}}^1(\Omega^1_Y, \mathcal{O}_Y)}_{= M} \longrightarrow 0$$

where M is supported only at the singular points.
If $N'_Y=\ker \gamma$, we have 2 short exact sequences

$$0 \longrightarrow T_Y \longrightarrow T_{\mathbb{P}} \otimes \mathcal{O}_Y \longrightarrow N'_Y \longrightarrow 0$$

$$0 \longrightarrow N'_Y \longrightarrow N_Y \longrightarrow M \longrightarrow 0$$

As in the proof of theorem 4 $h^i(T_{\mathbb{P}} \otimes \mathcal{O}_Y)=0$ for $i=1,2$,

hence $H^2(T_Y)\cong H^1(N'_Y)$, and (as $H^1(N_Y)=0$), one has

$$H^o(N_Y) \xrightarrow{r} H^o(M) \longrightarrow H^1(N'_Y) \longrightarrow 0 .$$

So the Kuranishi family of S is obstructed iff r is not surjective: we have however not yet found Y's for which r is not onto.

§ 4. THE RESTRICTION OF THE LOCAL PERIOD MAPPING TO SPECIAL SURFACES.

We begin by fixing a rather general situation: X a smooth compact manifold of dimension k, L a line bundle on X $p:L \to X$ the projection map, s,σ two independent sections in $H^{o}(X,L^{\otimes n})$, S_t the n^{th} cyclic covering of X in L branched over $D_t = \mathrm{div}(s+t\sigma)$, which we assume smooth for $|t|<\varepsilon$. Let moreover M be another line bundle on X, m a non zero section of $H^{o}(S,p^{*}M)$, (where $S=S_{o}$), $\left(\frac{\partial S_t}{\partial t}\right)_{t=0} = \theta \in H^{1}(S,T_S)$ the infinitesimal deformation of S (see e.g. [12] pag. 36 and following).

<u>THEOREM 5</u>. $\theta \cdot m \in H^{1}(T_S \otimes M)$ is non zero if T_X does not split on $D=D_o$ as $T_D \oplus N_D$.

<u>Proof</u>. Take an acyclic covering $\{V_{o\alpha}, V_{1\alpha}, V_{2\alpha}\}_{\alpha\in A}$ of X satisfying the following properties (here we consider the Haussdorff topology of X)

i) $V_{2,\gamma} \cap V_{o,\alpha} = \emptyset \quad \forall \alpha,\gamma$

ii) $X - \bigcup_{\alpha\in A}\{V_{1\alpha}, V_{2\alpha}\} = F$ is a closed neighbourhood of D

iii) $D \cap V_{o,\alpha} \neq \emptyset$, $V_{o\alpha} \cap V_{1\alpha} \neq \emptyset$

iv) $V_{o,\alpha} \cap V_{o\beta} \cap D \neq \emptyset \iff V_{1\alpha} \cap V_{1\beta} \neq \emptyset$

v) $\{V_{o\alpha} \cup V_{1\alpha}, V_{2\alpha}\}_{\alpha\in A}$ is a trivializing cover for L,M,T_X.

This can be achieved by taking a Stein covering (V'_α) of D, trivializing L, local coordinates $(z^1_\alpha, \ldots z^{k-1}_\alpha, s_\alpha)$ near the points of V'_α, and choosing conveniently $\delta_{1\alpha} < \eta_\alpha < \delta_{2\alpha}$ so that $V_{0\alpha} = \{(z_\alpha, s_\alpha) z_\alpha \in V'_\alpha, \ |s_\alpha| < \eta_\alpha\}$, $V_{1\alpha} = \{(z_\alpha, s_\alpha) z_\alpha \in V'_\alpha \ \delta_{1\alpha} < |s_\alpha| < \delta_{2\alpha}\}$ satisfy iii), iv).

On $L|_{V_{0\alpha} \cup V_{1\alpha}}$ S_t is defined by $y_\alpha(t)^n = s_\alpha + t\sigma_\alpha$.

If $U_{i\alpha} = p^{-1}(V_{i\alpha})$, on $U_{0\alpha}$ take as local coordinates $(z^1_\alpha, \ldots, z^{k-1}_\alpha, y_\alpha(t))$, on $U_{1\alpha}$ $(z^1_\alpha, \ldots, y_\alpha(0))$ (because we assume ε so small that for $|t| < \varepsilon$ D_t is contained in the interior of F), on $U_{2\alpha}$ the lifting of any coordinates on X.

Observe that the coordinate changes which depend on t are consequences of the coordinate change from $U_{0\alpha}$ to $U_{1\alpha}$: we get that $\theta_{1\alpha, 0\alpha} = -\frac{1}{n} \frac{\sigma_\alpha}{y_\alpha^{n-1}} \frac{\partial}{\partial y_\alpha}$.

Now $(U_{i\alpha})$ is an acyclic covering, so if $\theta \cdot m$ were a coboundary, there would exist vector fields $\eta_{i\alpha}$ such that

$$\eta_{0\alpha} - \eta_{1\alpha} = m_\alpha \frac{\sigma_\alpha}{y_\alpha^{n-1}} \frac{\partial}{\partial y_\alpha}, \quad \eta_{j\alpha} = \frac{m_\alpha}{m_\beta} \eta_{j\beta} \ (j=0,1).$$

These equations, plus condition iv) imply that

$$\frac{\sigma_\alpha}{y_\alpha^{n-1}} \frac{\partial}{\partial y_\alpha} = \frac{\sigma_\beta}{y_\beta^{n-1}} \frac{\partial}{\partial y_\beta}$$: being $(y_\alpha)^n = s_\alpha$, $\frac{\partial}{\partial y_\alpha} = n y_\alpha^{n-1} \frac{\partial}{\partial s_\alpha}$, then

$$\frac{\sigma_\alpha}{\sigma_\beta} \cdot \frac{\partial}{\partial s_\alpha} = \frac{\partial}{\partial s_\beta}.$$

This equality says that T_X in a neighbourhood of D has a 1-dimensional subbundle, isomorphic to $\mathcal{O}(D)$, giving on D a direct summand for T_D.

We can apply this criterion to obtain the following :

THEOREM 6. If S is a general "special" surface, in the Kuranishi family of S the special surfaces form a submanifold P and the restriction of the local period mapping to P is an embedding .

Proof. Here we will adopt the notations of § 2, expecially prop. 10 . $\tilde{S}$ is a double cover of the smooth K3 surface $\tilde{X}$, branched on the smooth curve $L + \sum E_j$.
As the holomorphic 2-form on $\tilde{S}$ is the pull back of the 2-form on $\tilde{X}$, and the 2-dimensional homology of $\tilde{S}$ is the direct sum of the (+1) and (-1) eigenspaces for the involution, by the local Torelli theorem for K3 surfaces ([19] pag. 202 and foll.) we can only limit ourselves to consider infinitesimal deformations arising from moving L in $|p^*(\mathcal{O}_{\mathbb{P}^2}(1))|$, and apply theorem 5 with $m = x_0 \in H^0(\mathcal{O}_S(K))$: we have then only to prove that on L the exact sequence

$$0 \longrightarrow \mathcal{O}_L(-L) \longrightarrow \Omega^1_X \otimes \mathcal{O}_L \longrightarrow \mathcal{O}_L(L) \longrightarrow 0$$

does not split, or that the extension class in $H^1(\mathcal{O}_L(-2L))$ is not zero .
To this purpose we first choose the following covering of L :
$V_1 = \{Y_1 FG \neq 0\}$, $V_2 = \{Y_2 FG \neq 0\}$, V_3 a neighbourhood of the six points of L where $FG = 0$.
Denote by $x = Y_1 / Y_2$, $x' = Y_2 / Y_1$, and write $F =$
$= F'(Y_1,Y_2) + Y_0 F''$, $G = G'(Y_1,Y_2) + Y_0 G''$; F , G being general one may take as a basis of Ω^1_X on V_1 $d(Y_0/Y_1)$, dx' , on V_2 $d(Y_0/Y_2)$, dx , on V_3 $d(Y_0/Y_1)$, $d\zeta_1$.

One easily computes the extension class : $\tau_{12} = 0$, $\tau_{13} = $
$= \frac{F'G'' + F''G'}{2\zeta_1 Y_1^5}$ (τ_{23} is deduced by the cocycle condition) .

By Serre duality we can interpret (τ_{ij}) as an element in the dual of $H^0(\omega_L(2L))$: to prove that it is non zero we check its value on the form $\frac{dx}{\zeta_1} \in H^0(\omega_L(2L))$.

We associate a repartition to (τ_{ij}) by setting $\tau_1 = \tau_{13}$, $\tau_3 = 0$, $\tau_2 = \tau_{13}$, and taking in V_i the repartition corresponding to τ_i .

Then $\left\langle (\tau_{ij}) , \frac{dx}{\zeta_1} \right\rangle = \sum_{P \in V_1 \cup V_2} \mathrm{Res}_P \left(\frac{dx}{\zeta_1} \frac{F'G'' + F''G'}{2\zeta_1 Y_1^5} \right)$.

Now on V_2 dx is holomorphic , and $\frac{F'G'' + F''G'}{\zeta_1^2 Y_1^5} =$
$= \frac{G''Y_1}{G'} + \frac{F''Y_1}{F'}$ is holomorphic too.

On V_1 $dx \left(\frac{G''Y_1}{G'} + \frac{F''Y_1}{F'} \right) = -\frac{dx'}{x'^2} \left(\frac{\frac{G''F' + F''G'}{Y_1^5}}{\frac{F'G'}{Y_1^6}} \right)$

so that the sum of the residues is non zero iff
$\frac{G''F' + F''G'}{Y_1^5}$ has a non zero term of order 1 in x' , and this is clearly achieved for F,G general .

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Added: Prop. 4 has a shorter proof.

In fact in prop. 5 is proven that x_o, y_1, y_2, z_3, z_4 generate $H^o(5K)$, so a base point for $|3K|$ would be a base point for $|5K|$, against Th. 2 of [1] .

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