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Babbage's Conjecture, Contact of Surfaces, Symmetric Determinantal Varieties and Applications

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Introduction

It is well known that if C is a smooth curve of genus $g \geqq 3$, and Φ_K is the canonical map of C, two possibilities occur: either

i) Φ_K is an embedding in $\mathbb{P}^{g-1}$ or

ii) Φ_K is of degree 2 and $\Phi_K(C)$ is a rational normal curve in $\mathbb{P}^{g-1}$.

Suppose now S is a surface with $p_g(S) \geqq 3$, and that Φ_K is a rational map of degree 2 onto a surface F: in [2] Babbage asserted, in analogy with ii), that under these assumptions $p_g(F)=0$.

It was later realized that his argument contained a gap, and that under these hypotheses one could predict one of the following two different behaviours (see [26] for a discussion of this):

a) $p_g(F)=0$

b) F is canonically embedded.

It was however dubious whether an example of case b) would actually occur.

The first motivation for this research was to find an example for case b), and D. Mumford suggested to look at quintic surfaces admitting a double cover ramified only on the singular points, hence everywhere tangent to some surface of even degree, and D. Gallarati pointed out his construction of quintic surfaces with 20 nodes and tangent to quartic surfaces with 10 nodes ([13]).

We then proved that these quintics were canonical images of some surfaces S with $K^2=10$, $q=0$, $p_g=4$, which formed a 20-dimensional subvariety in their local moduli space (of dimension 30).

Soon after this, a paper [34] by Van der Geer and Zagier appeared in which it was shown that the minimal model S of a Hilbert modular surface for the field $\mathbb{Q}(\sqrt{21})$ is such that $K_S^2=10$, $p_g(S)=4$, and $\Phi_K(S)=F$ is a quintic surface with 20

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nodes: later we heard from Mumford that A. Beauville had independently discovered the Gallarati quintics as the locus of singular quadrics in a generic web (a 3-dim. linear system) of quadrics in $\mathbb{P}^4$.[1]

In [13], Gallarati studies the situation of contact of surfaces in $\mathbb{P}^3$, and constructs his quintics by an inductive procedure: he starts from a quadric cone tangent to a cubic with 4 nodes, constructs a quartic with 10 nodes tangent to the cubic, and then a quintic tangent to the quartic: in this paper we show that contact is the geometrical counterpart of algebraic identities between products of determinants of minors of symmetric matrices, and that Gallarati's construction can be easily understood and generalized by taking determinants of symmetric matrices with homogeneous forms as entries (this also generalizes other known constructions, as in [31]).

As shown in [13], [14], contact between surfaces F, G, imposes some singularities of the two surfaces along the curve of contact, but a subset of these singular points is "even".

To explain what this means, we concentrate on the simplest case, when the singularities of F along the curve of contact are at most nodes (ordinary quadratic singularities): if $\tilde{F}$ is the blow-up of F at these nodes and N is the set of nodes where G has odd multiplicity, then the sum of the exceptional curves corresponding to the nodes in N gives a divisor which is either divisible by 2 in $\mathrm{Pic}(\tilde{F})$ or divisible by 2 after adding the pull back on $\tilde{F}$ of the hyperplane section of F.

In the former case we shall say that the set N is "strictly even", in the latter that it is "weakly even".

Conversely, if N is an even set of nodes on F, there exists a surface G which is everywhere tangent to F and has the property that N is precisely the set of nodes of F through which G passes with odd multiplicity. The order of N is defined to be the minimal degree of a surface G satisfying this property.

We define a surface F to be a "symmetric surface" if its equation can be written as the determinant of a symmetric matrix of homogeneous forms (we exclude of course the matrices of order one). A general symmetric surface has, as its only singularities, nodes whose equation is given by the vanishing of the determinants of the minors of order one less than the order of the matrix. Moreover, they form an even set and their number depends only on the degrees of the homogeneous forms which are the entries of the matrix. If these forms are linear, and F has degree n, we define F to be a linearly symmetric surface of degree n: these surfaces can be characterized as the surfaces of degree n having an even set N of $\binom{n+1}{3}$ nodes and with order $(n-1)$.

We then want to give, in general, a characterization of the even sets of nodes N which arise, as explained above, from symmetric matrices (for the sake of simplicity we shall call them "symmetric" sets of nodes). To do this, we remark that the existence of a strictly even set N of nodes allows one to construct a double cover S of F which is a finite cover ramified only at the nodes of N (when N is weakly even one must take the cover S to be ramified also at a plane section of F).

1 This and some other results similar to ours are to be found in [4]

Let R be the full ring of sections on S of multiples of the pull-back on S of the hyperplane bundle. Then R splits naturally into an invariant part (which is nothing else but the coordinate ring of F) plus an antiinvariant part R^-.

R^- is completely determined by N and if one views it as a module over the ring of homogeneous polynomials in 4 variables, gives the following information: N is symmetric if and only if R^- is a Cohen-Macaulay module. In down to earth language, the sought for matrix is the one given by a minimal free resolution of R^-: in this way one also has a complete description of the ring R. The condition that R^- be Cohen-Macaulay is equivalent to the vanishing of certain first cohomology groups. These conditions are of a topological nature for n smaller than 5 (they just mean that S has to be a regular surface) and in this way we can see that, up to quintics, all even sets of nodes are symmetric, whith the exception of the set of 16 nodes on a quartic Kummer surface. This however is the union of two symmetric sets (in several ways): in this way we rediscover Gallarati's classification of contact in degree smaller than 5 ([13]).

We then show how the case of surfaces differs substantially from the case of curves in the plane, where all the situations of contact are given by symmetric matrices of homogeneous forms: in particular we extend to all thetacharacteristics the connection with symmetric determinants, known classically for the case of ineffective thetacharacteristics ([10]).

An extension of these results to higher dimension and further applications to the case of curves will appear in future papers. A first application of our analysis concerns the reducibility of the family of surfaces of a fixed degree with a given number of nodes, in particular of quintics with 20 nodes.

Then we focus our attention on the linearly symmetric quintics and their double covers S: we prove the above quoted results regarding Babbage's conjecture, rephrase our preceding results in terms of generators and relations for the canonical ring of S, show that these surfaces are simply connected. Then, by taking the 4-dimensional subfamily of quintics admitting a free action of $\mathbb{Z}/5\mathbb{Z}$ we construct numerical Campedelli surfaces X (i.e. surfaces with $K^2=2$, $q=p_g=0$) which have $\mathbb{Z}/5\mathbb{Z}$ as fundamental group: they again form a proper subvariety in their local moduli space.

As mentioned at the beginning, one of our purposes was to show that, as a particular case of Beauville's and our equivalent constructions, one obtains the surface exhibited by Van der Geer and Zagier. This is done by showing that the 20 nodes of the canonical image of this surface (a quintic) form an even set of order 4.

Actually we can do more: it is well known that there is an action of the symmetric group $\mathfrak{S}_5$ on the Van der Geer-Zagier surface, so R^+ and R^-, and their graded pieces, are representations of $\mathfrak{S}_5$. By using the explicit description by Young diagrams of the irreducible representations of $\mathfrak{S}_5$, we determine which representations occur in the canonical ring, compute the numerical coefficients for the relations of R, and thus for the symmetric matrix of which $\Phi_K(S)$ is the determinant. This on one side gives new identities for the symmetric functions, on the other can be used to describe the ring of modular forms for the modular group of $\mathbb{Q}(\sqrt{21})$.

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Conventions

Throughout the paper we will be working over the ground field $\mathbb{C}$ of the complex numbers, though many of our results are valid over any algebraically closed field K.

For any scheme F, $\mathrm{Pic}(F)=H^1(F,\mathcal{O}_F^*)$ is the group of Cartier divisors/linear equivalence (see [25, 17], for details on this and the subsequent points). If F is a divisor in a smooth 3-fold, the dualizing sheaf ω_F is invertible: when consisidering it as the class of a Cartier divisor, we will use the symbol K_F, and call it the canonical divisor class.

Let s be a section of a line bundle L: we shall denote by $\mathrm{div}(s)$ the Weil divisor of the zeros of s.

The symbol $\equiv$ will denote linear equivalence of (Cartier) divisors and also, when used with numbers, congruence modulo some integer.

The abbreviation R.R. will stand for the Riemann-Roch theorem, expressing the Euler-Poincaré characteristic $\chi(X,L)$ of a coherent sheaf L on a smooth variety X ([17]).

§ 1. Identities for the Products of Minors of a Symmetric Matrix and Contact of Hypersurfaces

Let $A=(a_{ij})$ $(i,j=1,\dots,m+1)$ be a square $(m+1)$ matrix.

We denote by $A(i_k,\dots,i_1|j_1,\dots,j_k)$ $(1\leqq i_h\leqq m+1,\ 1\leqq j_l\leqq m+1)$ the determinant of the $(k\times k)$ matrix (b_{hl}) $(h,l=1,\dots,k)$ such that $b_{hl}=a_{i_h j_l}$. The $A(i_k,\dots,i_1|j_1,\dots,j_k)$'s can be viewed as the Plücker coordinates of the rectangular $(m+1)\times(2m+2)$ matrix $C=(AI_{m+1})$, where I_{m+1} is the identity $(m+1)\times(m+1)$ matrix, and the quadratic relations between these coordinates give some identities which can be obtained in a systematic way by the "straightening formula" of Doubillet-Rota-Stein ([12], cf. also [8] for the interpretation via Plücker coordinates).

In particular, for $n\leqq m$ one has the basic identity $(+_n)$:

$$(+_n)\quad \begin{aligned}&A(i_n,\dots,i_1|j_1,\dots,j_n)\cdot A(i_{n+1},i_{n-1},\dots,i_1|j_1,\dots,j_{n-1},j_{n+1})\\ &-A(i_n,\dots,i_1|j_1,\dots,j_{n-1},j_{n+1})\cdot A(i_{n+1},i_{n-1},\dots,i_1|j_1,\dots,j_n)\\ &+A(i_{n+1},\dots,i_1|j_1,\dots,j_{n+1})\cdot A(i_{n-1},\dots,i_1|j_1,\dots,j_{n-1})=0.\end{aligned}$$

Denote further by $B=(B_{ij})$ the adjoint matrix of $A=(a_{ij})$, by F the determinant of A: so

$$B_{ij}=(-1)^{i+j}A(m+1,\dots,\hat{j},\dots,1|1,\dots,\hat{i},\dots,m+1),$$
$$AB=BA=FI_{m+1},\qquad \det(B)=F^m.$$

Remark 1.1. From now on we will suppose that A is symmetric: then if $i_h = j_h$ in $(+_n)$ the second term becomes a square. Moreover, if we set

$$D_{ik}^{jl} = (-1)^{i+k+j+l} A(m+1, \ldots, \hat{k} \ldots \hat{i} \ldots 1 | 1 \ldots \hat{j} \ldots \hat{l}, m+1)(i<k, j<l)$$

we have the following identity

$$FD_{ik}^{jl} = B_{kj} B_{il} - B_{kl} B_{ij}. \tag{1.2}$$

and in particular

$$FD_{lk}^{kl} = B_{kk} B_{ll} - B_{kl}^2 \tag{1.3}$$

(where now D_{ik}^{il} is defined for all indexes to be alternating both in the lower and upper indexes).

Remark 1.4. Suppose that the a_{ij}'s are elements of a graded ring $R = \bigoplus_{m=0}^{\infty} R_m$ without zero divisors. Then all the $A(i_k, \ldots, i_1 | j_1, \ldots, j_k)$'s are homogeneous elements of R iff there are integers d_i such that $d_i \equiv d_j \pmod 2$ and $a_{ij} (= a_{ji})$ is a homogeneous element of degree $(d_i + d_j)/2$.

Proposition 1.5. *Let X be a smooth complete variety (or a compact complex manifold), let $\mathscr{L}$ be a line bundle on it. Let d_i $(i = 1, \ldots, m+1)$ be integers such that $d_i \equiv d_j \pmod 2$.*

Setting $R(\mathscr{L})_m = H^0(X, \mathscr{L}^{\otimes m})$, suppose moreover that we are given elements of $R(\mathscr{L}) = \bigoplus_{m=0}^{\infty} R(\mathscr{L})_m$

$$F \neq 0, \; B_{ij} = B_{ji}, \; D_{ik}^{jl} = D_{jl}^{ik} \quad (1 \leqq i < k \leqq m+1, \; 1 \leqq j < l \leqq m+1)$$

homogeneous of degrees respectively equal to $\sum_{h=1}^{m+1} d_h = d, d - (d_i + d_j)/2, d - (d_i + d_j + d_l + d_k)/2$ which satisfy the following assumptions

i) *identify* (1.2) *is satisfied;*

ii) *the matrix (with entries in the field of rational functions of X) $(B_{ij}/F)(i,j = 1, \ldots, m+1)$ has non zero determinant;*

iii) *$\{F = 0\}$ is a positive irreducible divisor.*

Then there exist elements $a_{ji} = a_{ij} \in R(\mathscr{L})$ $(i,j = 1, \ldots, m+1)$, homogeneous of degree $(d_i + d_j)/2$ such that the previously given F, B's, D's are the determinants of the minors of the matrix (a_{ij}) of order $(m+1), m, (m-1)$ respectively.

Proof. Consider the symplectic form $J = \begin{pmatrix} 0 & I_{m+1} \\ -I_{m+1} & 0 \end{pmatrix}$ and the Grassmannian $\hat{G}(m+1)$ of maximal isotropic subspaces for the symplectic form J. $\hat{G}(m+1)$ is smooth and the open set where a Plücker coordinate coordinate p differs from zero is isomorphic to the affine space of symmetric $(m+1) \times (m+1)$ matrices (see e.g. [7]). The determinant of the symmetric matrix one thus obtains can be called the "opposite" Plucker coordinate p' of $p (p'' = p)$.

Setting $p'=1$, the symmetric matrix $(a'_{ij})=(B_{ij}/F)$ $(i,j=1,\dots,m+1)$ defines a rational map of X into $\hat{G}(m+1)$.

By (ii) p is not identically zero on X, and by the assumptions made on the degrees, $p\cdot F$ is a rational function on X. If we prove that $p\cdot F$ is regular, then it is a non zero constant, which we can assume to be 1. Hence we can take (a_{ij}) to be F times the adjoint matrix of $(B_{ij}/F)=(a'_{ij})$. We then thave $F=\det(a_{ij})$, and the B's, D's are the determinants of the minors of $A=(a_{ij})$ of order m, $(m-1)$. So, setting $A'=(a'_{ij})$ we are going to prove by induction on k that $F\cdot A'(i_k,\dots,i_1/j_1,\dots,j_k)$ is a regular section. For $k=1$ this is clear, since the B_{ij}'s are regular sections, and for $k=2$ this follows from assumption *i*).

Multiplying both sides of $(+_k)$ by F^2, we conclude that the divisor of zeros of $A'(i_{k-1},\dots,i_1/j_1,\dots,j_{k-1})\cdot F$ is equal to the divisor of poles of $F\cdot A'(i_{k+1},\dots,i_1/j_1,\dots,j_{k+1})$ plus a positive (or zero) divisor.

On the other hand $A'=(a'_{ij})$ has poles only at $\{F=0\}$, so the polar divisor of every minor's determinant is a multiple of $\{F=0\}$. But $A'(i_{k-1},\dots,i_1|j_1,\dots,j_{k-1})$ has negative degree, so it can't be a regular section, hence it has a pole of order exactly one at $F=0$. By the irreducibility of F, $\{F=0\}\cap\{A'(i_{k-1},\dots,i_1|j_1,\dots,j_{k-1})\cdot F=0\}$ has codimension 2 in X and, X being smooth, $F\cdot A'(i_{k+1},\dots,i_1|j_1,\dots,j_{k-1})$ is regular.

Remark 1.6. Actually in the proof one only needs the normality of X, and a similar result holds for non symmetric matrices.

Conditions i), ii) are clearly necessary, but also iii) is, as it is shown by the easy example of

$$(a_{ij})=\begin{pmatrix} x/y & & & 0 \\ & y & & \\ & & y & \\ 0 & & & y \end{pmatrix}$$

where x, y are coprime.

Definition 1.7. We will say that two subvarieties Y, Z of X are tangent or have contact of order $m\geqq 1$ if for every component W of $Y\cap Z$ the intersection multiplicity of Y, Z at W is $\geqq m+1$ and there is a component of $Y\cap Z$ along which this intersection multiplicity is exactly equal to $m+1$.

Proposition 1.8. *In the hypotheses of Prop.* 1.5 *assume further that* $\{F=0\}\cap\{B_{ii}=0\}\cap\{B_{kk}=0\}$ *has codimension* 2 *in* $\{F=0\}$ $(1\leqq i,\ k\leqq m+1)$. *Then* $\{F=0\}$ *and* $\{B_{ii}=0\}$ *have contact.*

Proof. For $1\leqq i,j\leqq m+1$ let $\operatorname{div}_F(B_{ij})$ be the Weil divisor on F associated to B_{ij}. By (1.3) $\operatorname{div}_F(B_{ii})+\operatorname{div}_F(B_{kk})=2\operatorname{div}_F(B_{ik})$, so there are divisors W_i, W_k such that $\operatorname{div}_F(B_{ii})=2W_i$, $\operatorname{div}_F(B_{ik})=W_i+W_k$, $\operatorname{div}_F(B_{kk})=2W_k$ and the assertion follows.

For use later on we notice that, outside of the locus

$$\{B_{ij}=0\,|\,i,j=1,\dots,m+1\}=\{F=B_{ii}=0\,|\,i=1,\dots,m+1\},$$

one can choose W_i as a Cartier divisor.

§ 2. Symmetric Surfaces

In this paragraph we will consider the following special case of the situation considered in § 1: $X = \mathbb{P}^3$, $\mathscr{L} = \mathcal{O}_{\mathbb{P}^3}(1)$, $R(\mathscr{L}) = \mathbb{C}[x_0, x_1, x_2, x_3] = \mathbb{C}[X]$. Assume then that we are given an infinite sequence (d) of positive integers d_i such that $d_i \equiv d_j \pmod 2$ and for every i, j, a homogeneous form $a_{ij} = a_{ji}$ of degree $(d_i + d_j)/2$. Set

$$\rho_h = \sum_{i=1}^{h} d_i, \qquad t_h = \sum_{i=2}^{h} (d_i \rho_i \rho_{i-1})/2.$$

Consider then the infinite symmetric matrix (a_{ij}) and set $F_h^{(d)} = A(h, \ldots, 1 \mid 1, \ldots, h)$: by abuse of notation we will use the same symbol for a surface in $\mathbb{P}^3$ and for the homogeneous form which defines it.

Definition 2.1. A node is an ordinary quadratic singularity ($x^2 + y^2 + z^2 = 0$ in local holomorphic coordinates).

Theorem 2.2. *For general choice of the forms a_{ij} $F_h^{(d)}$ is a surface of degree ρ_h having as singularities exactly t_h nodes, whose equation is given by the determinants B_{ij} $(i, j = 1, \ldots, h)$ of the $(h-1) \times (h-1)$ minors of the matrix $(a_{ij})_{i,j=1,\ldots,h}$.*

Moreover $F_h^{(d)}$ and $F_{h+1}^{(d)}$ are tangent along a smooth curve C_h (i.e. $2C_h = \operatorname{div}_{F_h^{(d)}}(F_{h+1}^{(d)})$) *passing through all the nodes of $F_h^{(d)}$ and $F_{h+1}^{(d)}$.*

Before giving the proof, we state some auxiliary results.

Lemma 2.3. *Let C be a reduced curve in $\mathbb{P}^3$, F, G surfaces of degrees, n, m, respectively, such that C is the curve of contact between them* ($\operatorname{div}_G(F) = \operatorname{div}_F(G) = 2C$).

Suppose moreover that the only singularities of F and G along C are nodes. Let $P_1, \ldots, P_t$ be the points of C which are singular for F and not for G, $P_{t+1}, \ldots, P_{t+s}$ those which are singular for both, $P_{t+r+1}, \ldots, P_{t+r+s}$ those which are singular for G and not for F.

Then

$$nm = 0 \pmod 2$$

$$nm(n-m) = 2(t-r)$$

$$mn(m+2n) \equiv 2t \pmod 8.$$

Finally, if C is smooth, its genus is

$$p = 1 - nm + (mn(m+2n) - 2t)/8.$$

Proof. Blow up $\mathbb{P}^3$ at the P_i's, denote by $\tilde{F}, \tilde{G}$ the proper transforms of the two surfaces, by $\tilde{C}$ the proper transform of C, by π the blowing-up map, set $A_i = \pi^{-1}(P_i) \cap \tilde{F}$, $E_j = \pi^{-1}(P_j) \cap \tilde{G}$ $(i = 1, \ldots, t+s, j = t+1, \ldots, t+s+r)$.

The A_i's, E_j's are smooth rational curves with self-intersection $= -2$. Let α_i be the multiplicity of G at P_i, β_i the multiplicity of F at P_i.

On $\tilde{F}$ (smooth along $\tilde{C}$)

$$K_{\tilde{F}} = \pi^*(\mathcal{O}_F(n-4)), \qquad \pi^*(\mathcal{O}_F(m)) = \mathcal{O}_{\tilde{F}}\left(2\tilde{C} + \sum_{i=1}^{t+s} \alpha_i A_i\right)$$

hence $\tilde{C}\cdot A_i=-\frac{1}{2}\alpha_i A_i^2=\alpha_i$; similarly on $\tilde{G}$

$$K_{\tilde{G}}=\pi^*(\mathcal{O}_G(m-4)),\qquad \pi^*(\mathcal{O}_G(n))=\mathcal{O}_{\tilde{G}}\left(2\tilde{C}+\sum_{j=t+1}^{t+r+s}\beta_j E_j\right),\qquad \tilde{C}\cdot E_j=\beta_j.$$

By the adjunction formula (used both on $\tilde{F},\tilde{G}$), if p is the arithmetic genus of $\tilde{C}$

$$\begin{aligned}8p-8&=2\tilde{C}\cdot(2\tilde{C}+2K_{\tilde{F}})\\&=\left(\pi^*(\mathcal{O}_F(m))\otimes\mathcal{O}_{\tilde{F}}\left(-\sum_{i=1}^{t+s}\alpha_i A_i\right)\right)\cdot\left(\pi^*(\mathcal{O}_F(m+2n-8))\right.\\&\left.\otimes\,\mathcal{O}_{\tilde{F}}\left(-\sum_{i=1}^{t+s}\alpha_i A_i\right)\right)=nm(m+2n-8)-2\sum_{i=1}^{t+s}\alpha_i^2,\end{aligned}$$

and by entirely similar computation $8p-8=mn(n+2m-8)-2\sum_{j=t+1}^{t+r+s}\beta_j^2$. Our assertions easily follow.

Remark 2.4. The result of the preceding lemma can be given in a more general form (see e.g. [14], pp. 252–253).

Proof of Theorem 2.2. We are going to prove the result by induction on h, so assume it is true for $h\leqq m$. Considering then the matrix (a_{ij}) $(i,j=1,\dots,m+1)$ and using the notations of §1 (expecially Proposition 1.5), we have $F_{m+1}^{(d)}=F$, $F_m^{(d)}=B_{m+1,m+1}$. Let i be an integer with $1\leqq i\leqq m$: by (1.3),

$$(\sharp)\qquad FD_{i,m+1}^{i,m+1}=-(B_{i,m+1})^2+B_{m+1,m+1}B_{i,i},$$

so, on $B_{m+1,m+1}$,

$$\operatorname{div}(F)+\operatorname{div}(D_{i,m+1}^{i,m+1})=2\operatorname{div}(B_{i,m+1}).$$

As the forms $a_{l,m+1}$ $(l=1,\dots,m)$ vary, the surface $B_{i,m+1}$ moves in a linear system whose fixed part is given by $D_{i,m+1}^{l,m+1}=0$ for $l=1,\dots,m$.

By the inductive assumption this fixed part gives on $B_{m+1,m+1}$ the smooth curve C_{m-1}^i of tangency between $B_{m+1,m+1}$ and $D_{i,m+1}^{i,m+1}$; hence, after subtracting the fixed part, the only fixed points are the nodes of $B_{m+1,m+1}$ (lying on C_{m-1}^i), where $B_{i,m+1}$ is not tangent to $B_{m+1,m+1}$.

By Bertini's theorem then C_m, being this residual intersection, is smooth and irreducible.

In particular F is smooth at the singular points of $B_{m+1,m+1}$, i.e. when $D_{i,m+1}^{i,m+1}=0$ $\forall\, i=1,\dots,m$; moreover F is smooth outside C_m, since, varying the form $a_{m+1,m+1}$, F moves in a linear system with base locus given by $C_m(B_{m+1,m+1}=F=0)$.

Finally, at points where $F=B_{m+1,m+1}=0$ but $D_{i,m+1}^{i,m+1}\neq 0$ for some $i(1\leqq i\leqq m)$, equality $(\sharp)$ implies that the singularity is a node if we can prove that $B_{i,i}$, $B_{m+1,m+1}$, $B_{i,m+1}$ have a transversal intersection where they vanish but $D_{i,m+1}^{i,m+1}$ does'nt.

To this purpose take homogeneous coordinates $x_0,\dots,x_3$ on $\mathbb{P}^3$, set $N_{i,j}+1=\binom{(d_i+d_j)/2+3}{3}$ (N_{ij} is the dimension of the projective space parametrizing the forms of degree $(d_i+d_j)/2$), and write

$$a_{m+1,m+1}=\sum_{|I|=d_{m+1}} a^I_{m+1,m+1}x^I, \qquad a_{i,i}=\sum_{|J|=d_i} a^J_{i,i}x^J$$

$$a_{i,m+1}=\sum_{|H|=\frac{d_i+d_{m+1}}{2}} a^H_{i,m+1}x^H,$$

where I,J,H are multiindexes, and if $I=(i_0,i_1,i_2,i_3)$, $|I|=i_0+i_1+i_2+i_3$.

In

$$\mathbb{P}^{N_{m+1,m+1}}\times\mathbb{P}^{N_{i,i}}\times\mathbb{P}^{N_{i,m+1}}\times\mathbb{P}^3$$

(with coordinates $(a^I_{m+1,m+1}, a^J_{i,i}, a^H_{i,m+1}, x_0,x_1,x_2,x_3)$) consider the subvariety given by

$$B_{ii}=B_{m+1,m+1}=B_{i,m+1}=0.$$

Its singular locus is contained, for each choice of multiindexes I,J,H, in the locus given by the vanishing of the determinant of the (3×3) matrix of partial derivatives of the 3 equations B_{ii}, $B_{m+1,m+1}$, $B_{i,m+1}$ with respect to the 3 coordinates $a^I_{m+1,m+1}$, $a^J_{i,i}$, $a^H_{i,m+1}$.

The matrix is

$$\begin{bmatrix} x^I\, D^{i,m+1}_{i,m+1} & 0 & 0 \\ 0 & x^J\, D^{i,m+1}_{i,m+1} & 0 \\ 0 & 0 & x^H\, D^{i,m+1}_{i,m+1} \end{bmatrix},$$

hence the singular set of Γ is contained in $D^{i,m+1}_{i,m+1}=0$, and for general a_{lk}'s the intersection is tranvsersal where $D^{i,m+1}_{i,m+1}\neq 0$.

Since we proved that C_m passes through the nodes of $F^{(d)}_m$ and $F^{(d)}_{m+1}$ by Lemma 2.3 it follows that $F=F^{(d)}_{m+1}$ has t_{m+1} nodes.

Definition 2.5. Let F be a surface, $N=\{P_1,\dots,P_t\}$ be a set of nodes on F, $\pi\colon \tilde F\to F$ the blow-up of F along N and $A_i=\pi^{-1}(P_i)$ $(1\leq i\leq t)$.

Let H be the pull-back on $\tilde F$ of a hyperplane section of F.

The set N is said to be strictly even if the divisor $\sum_{i=1}^{t} A_i$ is divisible by 2 in $\operatorname{Pic}(\tilde F)$.

N is said to be weakly even if $H+\sum_{i=1}^{t} A_i$ is divisible by 2 in $\operatorname{Pic}(\tilde F)$.

In either one of the two preceding cases we shall say that N is an even set of nodes.

We remark that an even set of nodes cannot be both weakly an strictly even, at least if F has only rational double points as singularities (see the final remarks).

The next proposition explains how the notion of even sets of nodes is related to simple contact of surfaces.

Proposition 2.6. *Let F, G be surfaces in $\mathbb{P}^3$ such that $\operatorname{div}_F(G)=2C$, and assume that the singular points of F where C is not a Cartier divisor are just a set N of nodes: then N is an even set of nodes, strictly even iff G is a surface of even degree.*

Conversely, if N is an even set of nodes on F, there exists a surface G as above.

Proof. Using the notations of Definition 2.5., let m be the degree of G. Then, on $\tilde{F}$, mH is linearly equivalent to $2\tilde{C}+\sum_i \alpha_i A_i$, where $\tilde{C}$ is the proper transform of C on $\tilde{F}$ (and is a Cartier divisor), and α_i is the multiplicity of G at the node p_i.

Now, if α_i were even, C would be a Cartier divisor at p_i, hence

$$A=\sum_i A_i \equiv \delta H + 2([m/2]\,H - \tilde{C} - \sum_i [\alpha_i/2]\,A_i),$$

δ being 0 or 1 according to whether m is even or odd.

Conversely, let L be a divisor such that $2L \equiv A+\delta H$ ($\delta=0$ or 1): choose r such that $rH-L$ is linearly equivalent to an effective divisor $\tilde{C}$. Then $(2r-\delta)H \equiv 2\tilde{C}+A$, hence there exists a surface G of degree $(2r-\delta)$ with the required property.

Definition 2.7. Let N be an even set of nodes n F. Its order is defined to be the smallest degree of a surface G satisfying the requirements of the last proposition. A set of nodes N is said to be a "symmetric" set of nodes, and F a symmetric surface, if there exists a symmetric matrix (a_{ij}) of homogeneous forms such that $F=\det(a_{ij})$, and the determinants B_{ij} are homogeneous forms which define N as a reduced subscheme. (Note that, as a consequence of propositions 1.8 and 2.6, a symmetric set of nodes is an even set of nodes.)

Consider a symmetric set of nodes N on F, so that $F=\det(a_{ij})$ as in Definition 2.7. There exist integers d_i (see Remark 1.4) such that $\deg(a_{ij}) = \frac{d_i+d_j}{2}$: we want to see that some of the d_i's can be negative, but, for reasons that will become clear later, we impose the condition that a_{ij} be zero if $d_i+d_j \leqq 0$.

Assume that $d_1,\dots,d_h$ are in increasing order, and consider $F=\det(a_{ij})_{i,j=1,\dots,h}$.

Notice that, if there exist i,j such that $i+j>h$, $d_i+d_j \leqq 0$, then the first i rows of the matrix (a_{ij}) are linearly dependent and $F\equiv 0$.

If, moreover, there exist i,j such that $i+j=h$, $d_i+d_j \leqq 0$, then F is divisible by $(A(i,\dots,1\,|\,j+1,\dots,h))^2$.

Finally, if $d_1+d_{h-2} \leqq 0$, then F is singular in codimension 1 (where $a_{1,h-1}=a_{1,h}=0$).

If we want to generalize the result of Theorem 2.2 it is therefore a reasonable assumption on the d_i's that

(♮) the smallest k such that $d_{k+r}+d_{k+1-r}>0$ for every $r=1,\dots$ $\dots k$, does not exceed $(h/2-1)$.

Theorem 2.8. *If* (♮) *holds, then, for general choice of the* a_{ij}*'s (of degree* >0*),* F *has only isolated singularities, and the points where* $B_{ii}=0$ $\forall\, i$ (i.e. *where corank* $(a_{ij})\geqq 2$) *are a set* N *of* t_h *nodes of* F, *defined by the forms* B_{ij} *as a reduced subscheme.*

Proof. Obviously it is sufficient to prove the result when $h=2k+2$, since for $h>2k+2$ we can apply the same inductive approach used in the proof of Theorem 2.2 (and to use the same notations as there we set $h=m+1$).

Moreover we can prove the result under the stronger assumption that $a_{ij}=0$ for $i,j\leqq k$ or $j=k+r$, $i\leqq k-r$.

Then, by th. 2.2., the corank of $(a_{ij})_{i,j=k+1,\ldots,2k}$ is at most 2 at a finite number of points, where we can assume that the $a_{k+r,k+1-r}$'s $(r=1,\ldots,h)$ do not vanish $\left(\text{notice that } D^{m,m+1}_{m,m+1}=\prod_{r=1}^{k} a_{k+r,k+1-r}\right)$: therefore, since we also can assume $A(2k,\ldots,k+1\,|\,k,\ldots,2,1)$ not to vanish at the points where the corank of $(a_{ij})_{i,j=k+1,\ldots,2k}$ is 2, corank $(a_{ij})_{i,j=1,\ldots,2k}$ never exceeds 2, and equals 2 on a curve.

To finish the proof of theorem 2.8 we need to apply twice the following lemma.

Lemma 2.9. *Let $C=(c_{ij})$ be a symmetric $s\times s$ matrix such that the c_{ij}'s are general forms of positive degree, $(c_{ij})_{i,j=1,\ldots,(s-1)}$ has everywhere rank at least $s-3$, and rank equal to $s-1$ at some point.*

If Γ is a subvariety of $\mathbb{P}^3$ of dimension at most 1 (resp. equal to 2) then $\{x\in\Gamma\,|\,\text{corank } C\geqq 2\}$ is empty (resp. is at most a finite set of points).

Proof of the Lemma. Γ is covered by affine patchens where $\gamma=C((s-1), \hat{h},\hat{k},\ldots,1\,|\,1\ldots\hat{i},\hat{j},\ldots(s-1))$ does not vanish and, upon replacing Γ by a smaller subvariety, we can assume $C(s-1,\ldots,1\,|\,1,\ldots,s-1)=0$ on Γ.

Therefore

$$C(s,\ldots,\hat{h},\ldots,1/1,\ldots,\hat{i},\ldots,s)=c_{ks}c_{js}\cdot\gamma+\ldots$$
$$C(s,\ldots,\hat{k},\ldots,1/1,\ldots,\hat{j},\ldots,s)=c_{hs}c_{is}\cdot\gamma+\ldots$$

(where … stands for "up to terms not involving c_{ks}, c_{js}, c_{hs}, c_{is}"), and it follows immediately that the locus where these two terms both vanish has codimension 2 in Γ. This proves the lemma.

Returning to the proof of Theorem 2.8 we have:

1) the locus given by $B_{m+1,m+1}=D^{e,m+1}_{i,m+1}=0\ \forall\, i,e$ has dimension $\leqq 0$,

2) $N=\{B_{ii}=0,\ i=1,\ldots,m+1\}$ has dimension 0 and for each point $x\in N$ there exists $i\geqq k+1$ such that $D^{i,m+1}_{i,m+1}(x)\neq 0$.

In fact, in 2), since $B_{m+1,m+1}=0$ is irreducible, $\Gamma=\{B_{m+1,m+1}=D^{i,m+1}_{i,m+1}=0$ for $i\geqq k+1\}$ has dimension $\leqq 1$.

We can now finish our proof: varying $a_{m,m}, a_{m+1,m+1}$ we get (by Bertini's theorem) that $B_{m+1,m+1}$ is smooth when $D^{m,m+1}_{m,m+1}\neq 0$, F is smooth when $B_{m+1,m+1}\neq 0$.

By (‡), at points where $F=B_{m+1,m+1}=0$, F is smooth if $B_{m+1,m+1}$ is smooth there and $B_{ii}\neq 0$ for some $i\leqq m$.

By 1), 2), the locus $\{F=B_{m+1,m+1}=D^{m,m+1}_{m,m+1}=0\}$ is contained in

$$\{B_{m+1,m+1}=D^{m,m+1}_{m,m+1}=B_{i,m+1}=0\}$$

which has dimension $\leqq 0$, since, as the forms $a_{e,m+1}$ vary, the fixed locus of the linear system $|B_{i,m+1}|$ is given by $D^{e,m+1}_{i,m+1}$.

Hence F has only a finite number of singular points, and if x is a point of N where $D^{i,m+1}_{i,m+1}\neq 0$ (with $i\geqq k+1$) then x is a node and B_{ii}, $B_{m+1,m+1}$, $B_{i,m+1}$ have a transversal intersection at x (since a_{ii} is non zero, the same argument of theorem 2.2 applies).

By induction on h, B_{11} and F are tangent along a curve and have at most nodes as singularities on this curve, hence, by lemma 2.3, we deduce that N consists of t_h nodes.

Remark 2.10. The minimal degree of F for which some negative d_i can occur is 8, when one considers the integers $-1, 3, 3, 3$. N consists of 72 nodes, and F has one more node where $a_{12}=a_{13}=a_{14}=0$.

The sequence $(0, 2, 2, 2)$ gives a sextic surface with 33 nodes, 32 forming the strictly even set N.

Proposition 2.11. *Let F be a reduced surface of degree n in $\mathbb{P}^3$ with a "strictly even" set N of t nodes: then there exists a double cover $\Phi: S \to F$ where Φ is a finite map ramified only at the nodes $P_1, \dots, P_t$ of N and the dualizing sheaf ω_S is the pull-back $\Phi^*(\omega_F)$ of the dualizing sheaf of F.*

If F is smooth outside N, S is smooth, and minimal if $n \geqq 4$. One has $t \equiv 0 \pmod 4$, $\chi(S, \mathcal{O}_S) = 2\chi(F, \mathcal{O}_F) - t/4$. Moreover[2]*, if n is even, $t \equiv 0 \pmod 8$.*

Proof. Assume for simplicity F to be smooth outside N, and, with the notations of Definition 2.4 let L be a divisor such that $2L \equiv \sum_{i=1}^{t} A_i \overset{\text{def.}}{=} A$. In the line bundle $\mathcal{O}_{\tilde{F}}(L)$ let $\tilde{S} \xrightarrow{p} \tilde{F}$ be the double cover one obtains by taking the square root of the section of $\mathcal{O}_{\tilde{F}}(2L)$ corresponding to the divisor A. $\tilde{S}$ is smooth, and if $\tilde{A}_i = p^{-1}(A_i)$, $\tilde{A}_i$ is an exceptional curve of the first kind, so after contracting the $\tilde{A}_i$'s one obtains a smooth surface S, and a finite map $\Phi: S \to F$ ramified only at the P_i's. Denote by m the blowing up map $m: \tilde{S} \to S$. We have $\omega_{\tilde{F}} = \pi^* \omega_F$, $\omega_{\tilde{S}} = p^*(\omega_{\tilde{F}}(L)) = p^* \pi^* \omega_F \left(\sum_{i=1}^{t} \tilde{A}_i\right)$, $\omega_{\tilde{S}} = m^*(\omega_S) \left(\sum_{i=1}^{t} \tilde{A}_i\right)$, hence $\omega_S = \Phi^*(\omega_F) = \Phi^*(\mathcal{O}_F((n-4)H))$, H being the hyperplane section of F.

If $n \geqq 4$ then $\omega_S \cdot E \geqq 0$ for each curve and S is minimal.

One clarly has $\chi(S, \mathcal{O}_{\tilde{S}}) = \chi(\tilde{S}, \mathcal{O}_{\tilde{S}})$ but $p_*(\mathcal{O}_{\tilde{S}}) = \mathcal{O}_{\tilde{F}} \oplus \mathcal{O}_{\tilde{F}}(-L)$ so by the Leray spectral sequence for the map p plus R.R.

$$\begin{aligned}\chi(S, \mathcal{O}_S) &= 2\chi(\tilde{F}, \mathcal{O}_{\tilde{F}}) + \tfrac{1}{2}(-L)(\omega_{\tilde{F}}^{-1}(-L)) \\ &= 2\chi(F, \mathcal{O}_F) + A^2/8 = 2\chi(F, \mathcal{O}_F) - t/4.\end{aligned}$$

Notice that if $n=3$ there exist projective coordinates such that $F = x_0 x_1 x_2 + x_0 x_1 x_3 + x_0 x_2 x_3 + x_1 x_2 x_3$, and F contains 9 lines, 6 of which connect pairs of nodes. The remaining 3 lines form a triangle and on them the double cover splits: therefore S contains 6 exceptional curves of the first kind.

To prove the last assertion it suffices to show that $\chi(S, \mathcal{O}_S)$ is even when n is even.

Set $\Phi^* \mathcal{O}_F((n-4)/2 = G$: then $G \cdot G \equiv 0 (\mathrm{mod}\, 4)$, $G^2 \cong \omega_S$, therefore $\chi(S, G) \equiv 0 \pmod 2$ by virtue of Serre's duality theorem (in fact $h^0(S, G) = h^2(S, G)$ and $H^1(S, G)$ carries a non degenerate alternating bilinear form). By R.R. $\chi(S, G) = -1/2 G^2 + \chi(S, \mathcal{O}_S)$.

Remark 2.12. F is simply connected, hence $\tilde{F}$ is too; so $\mathcal{O}_F(L)$ is unique (up to isomorphism) as well as the double cover S.

2 This last remark is due to Miles Reid

Proposition 2.13. *Let F be a reduced surface of degree n with a weakly even set N of t nodes; then there exist double covers $\Phi\colon S \to F$, with Φ a finite map ramified only on N and a general hyperplane section H of F.*

Moreover

$$n \equiv 0 \pmod 2, \qquad \chi(S, \mathcal{O}_S) = 2\chi(F, \mathcal{O}_F) + (n(2n-7) - 2t)/8.$$

We omit the proof, since it is analogous to the one of Proposition 2.11.

From now on we shall freely use the notations introduced in 2.4, 2.6, 2.8 and 2.10.

Definition 2.14. Let N be an even set of nodes of F, and let R be the graded ring $\bigoplus_{m=0}^{\infty} H^0(S, \Phi^* \mathcal{O}_F(m))$. R splits into the $(+1)$ and (-1) eigenspaces for the involution determined by Φ. So $R = R^+ \oplus R^-$, where R^+ is nothing else but the coordinate ring of F, while R^- (by the Leray spectral sequences) is isomorphic to

$$\bigoplus_{m=0}^{\infty} H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(mH - L)) = \bigoplus_{m=0}^{\infty} H^0(F, (\pi_* \mathcal{O}_{\tilde{F}}(-L))(m)).$$

R^-, viewed as a graded module over the graded ring

$$\mathbb{C}[x_0, x_1, x_2, x_3] \overset{\text{def.}}{=} \mathscr{A},$$

is called the associated module of the even set of nodes N.

Remark 2.15. If, as in 2.6, δ is such that $2L \equiv A + \delta H$, and m is the smallest integer for which R_m^- is not zero then (prop. 2.6.) N has order $2m - \delta$.

Also, for further use, we notice that

$$H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(mH - L)) \cong H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((m-\delta) H + L)).$$

This isomorphism comes out of the long cohomology sequence of the exact sequence

$$0 \to \mathcal{O}_{\tilde{F}}(mH - L) \to \mathcal{O}_{\tilde{F}}((m-\delta) H + L) \to \bigoplus_{i=1}^{t} \mathcal{O}_{A_i}(L) \to 0$$

since $L \cdot A_i = -1$.

Given a graded module $M = \bigoplus_{m \in \mathbb{Z}} M_m$, we will use the standard notation $M[r]$ for the module obtained from M by shifting degrees according to the following rule: $M[r]_m = M_{r+m}$ (cf. [29]).

Theorem 2.16. *If N is a symmetric set of nodes on the reduced surface F, then the associated module is a Cohen-Macaulay $\mathscr{A}$-module. More precisely, using the notations of Th. 2.2 and assuming that the integers d_i $(i = 1, \ldots, h)$ are taken in increasing order, set $r_i = 1/2(n + \delta - d_i)$, $l_j = 1/2(n + \delta + d_j)$ where $n = \rho_h =$ degree of F, $\delta \equiv n - d_i \pmod 2$.*

Then there exists a minimal set of homogeneous generators $w_1, \ldots, w_h$ of R^- of degrees $r_1, \ldots, r_h$ respectively such that

$$w_i w_j = B_{ij} \in R^+.$$

Moreover R admits a minimal free resolution

$$0 \to \bigoplus_{j=1}^{h} \mathscr{A}[-l_j] \xrightarrow{(a_{ij})} \bigoplus_{i=1}^{h} \mathscr{A}[-r_i] \xrightarrow{\gamma} R^- \to 0.$$

where the map γ corresponds to the choice of generators $w_1, \ldots, w_h$.

The order of N is $n-d_h$.

Proof. Equations 1.2. and 1.3. imply immediately that each B_{ii} vanishes of odd order at the nodes of N, so that on $\tilde{F}$ we have $\operatorname{div}(B_{ii})=A+2C_i$, and we can therefore choose w_i in $R^-_{r_i}$ such that $C_i=\operatorname{div}(w_i)$, and $w_i w_j = B_{ij}$.

First of all, we want to prove that $\sum_{j=1}^{h} a_{ij} w_j = 0$ in R^-.

Since we are assuming that the ideal generated by the B_{ij}'s defines N scheme-theoretically, it follows that the curves C_k have an empty intersection on $\tilde{F}$. Therefore an element of R is zero iff, for each $k=1,\ldots,h$, we get zero after multiplying it by w_k. We have $\sum_{j=1}^{h} a_{ij} w_j w_k = \sum_{j=1}^{h} a_{ij} B_{jk} = F\,\delta_{ik} = 0$ in R (here δ_{ik} is the Kronecker symbol).

Let's now prove that these h relations generate over $\mathscr{A}$ all the relations among the w_i's.

Suppose in fact that $f_1, \ldots, f_h$ are elements of $\mathscr{A}$ such that $\sum_{j=1}^{h} f_j w_j = 0$ in R^-: after multiplying by w_i, we get $\sum_{j=1}^{h} f_j w_j w_i = 0$ in R^+, hence there exist elements g_i in $\mathscr{A}$ such that $g_i F = \sum_{j=1}^{h} f_j B_{ji}$, or, in matrix notation, $F(g_i) = (B_{ij})(f_j)$. We multiply both sides of the equality by the matrix (a_{ki}) on the left to obtain $F(a_{ki})(g_i) = F(f_k)$: this is an equality in $\mathscr{A}^h$, so we can cancel F and our assertion is proven.

Next, we are going to see that R^- is generated by the w_i's. In fact, if w is an element of R^-, $ww_i \in R^+ = \mathscr{A}/(F)$, therefore we can choose, for each i, $b_i \in \mathscr{A}$ representing the class of $ww_i \pmod F$.

We use again the same trick: multiply the relations $\sum_{j=1}^{h} a_{ij} w_j$ by w to infer the existence of elements f_i in $\mathscr{A}$, such that, in $\mathscr{A}^h$, $(a_{ij})(b_j) = F(f_j)$. Then multiply on the left by the matrix (B_{ki}) and cancel F, in order to get $(b_k) = (B_{ki})(f_i)$. This means that $(w_k w) = (w_k w_i)(f_i)$, and again this implies that $w = \sum_{i=1}^{h} f_i w_i$.

We have therefore proven that the above sequence is exact, and thus R^- is a Cohen-Macaulay $\mathscr{A}$-module, having a free resolution of length $1 = \dim \mathscr{A} - \dim R^-$ (see [30]).

Corollary 2.17. *If N is a symmetric set of nodes, the ring R is generated by $x_0, \ldots, x_3, w_1, \ldots, w_h$ subject to the only relations:*

$$\sum_{j=1}^{h} a_{ij}(x) w_j = 0 \qquad (i=1,\ldots,h)$$

$$w_i w_j = B_{ij}(x) \qquad (i,j=1,\ldots,h)$$

$\left(\textit{note that } F(x) = \sum_{j=1}^{h} a_{ij} B_{ji} = \left(\sum_{j=1}^{h} a_{ij} w_j\right) w_i = 0\right).$

Proposition 2.18. *Suppose F is reduced. If we do not assume that N is a symmetric set of nodes, the module R^- has projective dimension at most two.*

R^- is a Cohen-Macaulay $\mathscr{A}$-module if and only if

$$H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(mH+L))=0 \quad \text{for } 0 \leqq m \leqq \frac{n-4}{2}.$$

Proof. In terms of the Ext functor, the condition for R^- to be Cohen-Macaulay is given by $\operatorname{Ext}^i(R^-, \mathscr{A}[-4])=0$ for $i \geqq 2$ ($i \geqq 3$ for R^- to have projective dimension $\leqq 2$).

Since R^- is the full module of sections of the associated sheaf on $\mathbb{P}^3$, twisted by $\mathcal{O}_{\mathbb{P}^3}(m)$, these groups automatically vanish when $i \geqq 3$, by virtue of th. 1 of [29], pag. 265: this theorem also gives the isomorphism

$$\operatorname{Ext}^2(R^-, \mathscr{A}[-4]) = H^1(R^-).$$

Therefore one must have $H^1(\tilde{F}, mH-L)=0$ $\forall m \in \mathbb{Z}$, or, equivalently by Serre duality on $\tilde{F}$, $H^1(\tilde{F}, mH+L)=0$ $\forall m \in \mathbb{Z}$.

Let H be a general hyperplane section of $\tilde{F}$, notice that L is not trivial when restricted to H and consider the exact cohomology sequence of

$$(\sharp_m) \qquad 0 \to \mathcal{O}_{\tilde{F}}((m-1)H+L) \to \mathcal{O}_{\tilde{F}}(mH+L) \to \mathcal{O}_H(mH+L) \to 0.$$

We have $H^1(\mathcal{O}_H(mH+L))=0$ when the degree is at least $2p(H)-2$, i.e. when $m \geqq n-3$, and $H^0(\mathcal{O}_H(mH+L))=0$ when $\delta+m \leqq 0$.

Since $h^0(\mathcal{O}_A(mH+L))=h^1(\mathcal{O}_A(mH+L))=0$, then, using Serre duality on $\tilde{F}$, one obtains

$$\begin{aligned} H^1(\mathcal{O}_{\tilde{F}}(mH+L)) &= H^1(\mathcal{O}_{\tilde{F}}(mH+L-A)) = H^1(\mathcal{O}_{\tilde{F}}((m+\delta)H-L)) \\ &= H^1(\mathcal{O}_{\tilde{F}}((n-4-m-\delta)H+L)) \end{aligned}$$

and the statement of the proposition immediately follows.

We are now in a position to give several equivalent characterizations of symmetric sets of nodes

Theorem 2.19. *Suppose F is reduced and irreducible and that N is an even set of nodes of F. N is symmetric iff one of the following three equivalent conditions hold:*

I) *let $w_1, \ldots, w_h$ be a minimal set of homogeneous generators for the $\mathscr{A}$-module R^-, $B_{ij}(x)=w_i w_j$. Then $\det(B_{ij})$ is a (non zero) polynomial of degree $n(h-1)$*

II) *R^- is a Cohen-Macaulay $\mathscr{A}$-module*

III) $H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(mH+L))=0$ *for* $0 \leqq m \leqq \frac{n-4}{2}$.

Proof. If I holds we set $\delta=0,1$ according to whether N is strictly or weakly even; we also set $d_i=n-2r_i+\delta$, where $r_i=\deg w_i$.

Since $B_{kj}B_{il}-B_{kl}B_{ij}$ vanishes identically on F, there exist homogeneous forms D_{ik}^{jl} such that (1.2) holds: therefore we can apply prop. 1.5 to infer that, setting $(a_{ij})=F \cdot (B_{ij})^{-1}$, (a_{ij}) is a polynomial matrix with determinant equal to F.

Since, for $m \gg 0$, $mH-L$ is very ample on $\tilde{F}$, the equations $B_{ij}(x)=0$ define the nodes of the set N, and N is symmetric.

In general, by Hilbert's syzigies theorem (see e.g. [35] p. 240, or [1] pp. 575–588) R^- has a minimal free resolution of length equal to the projective dimension of R^- (which is either 1 or 2, by Proposition 2.15).

So II) $\Leftrightarrow$ III) $\Leftrightarrow$ R^- has a minimal free resolution of length one.

Assuming this, since R^- has dimension 3, the two free modules appearing in the resolution must have equal rank, say h. Let

$$0 \to \bigoplus_{j=1}^{h} \mathscr{A}[-l_j] \xrightarrow{(a_{ij})} \bigoplus_{i=1}^{h} \mathscr{A}[-r_i] \to R^- \to 0$$

be such a minimal resolution, and consider the long cohomology sequence of modules ([29] p. 254)

$$(\natural) \qquad 0 \to H^2(R^-) \xrightarrow{d} \bigoplus_{j=1}^{h} H^3(\mathscr{A}[-l_j]) \xrightarrow{(a_{ij})} \bigoplus_{i=1}^{h} H^3(\mathscr{A}[-r_i]) \to 0.$$

We apply to this latter the exact contravariant functor $*$, such that $(M^*)_n$ $=$dual vector space of M_{-n} (ibidem p. 263), to get

$$0 \to \bigoplus_{i=1}^{h} H^3(\mathscr{A}[-r_i])^* \xrightarrow{{}^t(a_{ij})} \bigoplus_{j=1}^{h} H^3(\mathscr{A}[-l_j])^* \xrightarrow{d^*} H^2(R^-)^* \to 0.$$

We notice that Serre duality gives again an isomorphism of $H^2(R^-)^*$ with $R^-[n-4]^*$, so that we can tensor by $\mathscr{A}[-n+4-\delta]$ to obtain

$$0 \longrightarrow \bigoplus_{i=1}^{h} (H^3(\mathscr{A})^*)[r_i-n+4-\delta] \xrightarrow{{}^t(a_{ij})} \bigoplus_{j=1}^{h} (H^3(\mathscr{A})^*)[l_j-n+4-\delta]$$
$$\xrightarrow{d^*} H^2(R^-)^*[-n+4-\delta] \to 0.$$

Now the last module is isomorphic to R^- under an isomorphism τ: $R^- \to H^2(R^-)^*[-n+4-\delta]$, and since $H^3(\mathscr{A})^* \cong \mathscr{A}[-4]$ we have obtained another minimal free resolution of R^-.

Now, two minimal free resolutions are isomorphic ([35], p. 238 or by the arguments of [24] lemma 8 p. 136) so, if we assume that we have ordered the r's and l's in such a way that $r_1 \leqq r_2 \leqq \dots r_h$, $l_1 \geqq l_2 \dots \geqq l_h$, first of all we must have $l_i - n - \delta = -r_i$.

Moreover, if we set $d_{ij} = \deg(a_{ij})$, we have

$$d_{ij} = l_i - r_j = n + \delta - r_i - r_j.$$

Therefore, if $d_i = \deg(a_{ii})$, we have

$$d_{ij} = \frac{d_i + d_j}{2}$$

$$r_i = \tfrac{1}{2}(n+\delta-d_i), \qquad l_j = \tfrac{1}{2}(n+\delta+d_j).$$

Let's now look at the leading term of the Hilbert polynomial of R^-: for $m \gg 0$

$$\dim R_m^- = \chi(\tilde{F}, \mathcal{O}_{\tilde{F}}(mH-L)) = \binom{n}{2} m^2 + \dots$$

by R.R., while the above resolution gives

$$\dim R_m^- = \sum_{i=1}^{h} \dim \mathscr{A}_{m-r_i} - \dim \mathscr{A}_{m-l_i} = \sum_{i=1}^{h} \binom{m-r_i+3}{3} - \binom{m-l_i+3}{3}$$
$$= \frac{m^2}{2}\left(\sum_{i=1}^{h} l_i - r_i\right) + \ldots = \frac{m^2}{2}\left(\sum_{i=1}^{h} d_i\right) + \ldots$$

Since R^- has support on $F=0$, $\det(a_{ij})$ is divisible by F, but its degree is exactly n, hence $F=\det(a_{ij})$ and we only need to show that we can assume that (a_{ij}) is a symmetric matrix.

This follows by the functoriality of the above isomorphism with respect to Serre duality.

To be precise, we are going to show that, under the canonical isomorphism of $H^3(\mathscr{A}^*)$ [4] with $\mathscr{A}$ ([29], p. 264) a lifting $\tilde{\tau}\colon \bigoplus_{i=1}^{h} \mathscr{A}[-r_i] \to \bigoplus_{i=1}^{h} \mathscr{A}[-r_i]$ is given by the matrix of polynomials $\tilde{\tau} = \frac{1}{F}({}^t a \cdot B)$, where, for brevity, we have set $a = (a_{ij})$.

One this is proven, either we apply part I of the theorem, or simply change basic in the first free module so that instead of (a_{ij}) we take a ${}^t\tilde{\tau}^{-1} = F \cdot B^{-1}$, which is a symmetric matrix.

Here we adopt the notations of [29] pp. 253 and foll., denote by V the vector space with free basis $dx_0, \ldots, dx_3$ and choose $k \gg 0$ such that the exact cohomology sequence (♮) is induced by the exact sequence of complexes $C_k(\ldots)$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda^2 V \otimes \mathscr{A}^h & \xrightarrow{(a)} & \Lambda^2 V \otimes \mathscr{A}^h & \longrightarrow & \Lambda^2 V \otimes R^- & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda^3 V \otimes \mathscr{A}^h & \xrightarrow{(a)} & \Lambda^3 V \otimes \mathscr{A}^h & \longrightarrow & \Lambda^3 V \otimes R^- & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda^4 V \otimes \mathscr{A}^h & \xrightarrow{(a)} & \Lambda^4 V \otimes \mathscr{A}^h & \longrightarrow & \Lambda^4 V \otimes R^- & \longrightarrow & 0
\end{array}$$

where the vertical maps are given by exterior product on the left with $\alpha_k = \sum_{j=0}^{3} x_j^k dx_j$, and we forget about the gradings to simplify the notations.

If ω is a cocycle in $\Lambda^3 V \otimes R^-$, write $\omega = \sum_{i=1}^{h} \eta_i \otimes w_i$. Then $d\omega$ is given by $(a)^{-1}(\alpha_k \wedge \eta)$ where, if $e_1, \ldots, e_h$ is standard basis of $\mathscr{A}^h (e_s \to w_s)$, $\eta = \sum_{i=1}^{h} \eta_i \otimes e_i$. We must verify that, with the above definition of $\tilde{\tau}$, we have $\langle \tilde{\tau}(e_s), d\omega \rangle = \langle \tau(w_s), \omega \rangle$.

To do this, notice that τ is given by Serre duality on $\tilde{F}$, and instead of taking values of $\langle\,,\rangle$ in $\mathbb{C}$ it is more convenient to take values, under fixed canonical isomorphisms, in the one-dimensional vector space $H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4))$ (recall that $H^2(F, \mathcal{O}_F((n-4)H))$ and $H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4))$ are canonically isomorphic via the coboundary operator in the cohomology sequence associated to

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^3}(n-4) \to \mathcal{O}_F(n-4) \to 0.$$

Then

$$\langle \tau(w_s), \omega\rangle = \left\langle \tau(w_s), \sum_{i=1}^{h} \eta_i \otimes w_i \right\rangle$$
$$= \partial \left(\sum_{i=1}^{h} \eta_i \otimes (w_s w_i)\right) = \frac{1}{F} \sum_{i=1}^{h} B_{is} \alpha_k \wedge \eta_i,$$

while

$$\langle \tilde{\tau}(e_s), d\omega\rangle = \left\langle \frac{1}{F}\, {}^t a \left(\sum_{i=1}^{h} B_{is} e_i\right), a^{-1} \left(\alpha_k \wedge \sum_{i=1}^{h} \eta_i \otimes e_i\right) \right\rangle$$
$$= \left\langle \frac{1}{F} \sum_{i=1}^{h} B_{is} e_i, \sum_{i=1}^{h} \alpha_k \wedge \eta_i \otimes e_i \right\rangle$$
$$= \frac{1}{F} \sum_{i=1}^{h} B_{is} (\alpha_k \wedge \eta_i).$$

Remark 2.20. i) The number t of nodes of N can also be computed by equating the constant terms in the two expressions for the Hilbert polynomial of R^-. In particular if N is symmetric $t \leqq \binom{n+1}{3}$, equality holding iff $d_i = 1$ for each i.

ii) No one of the a_{ij}'s can be a non zero constant, since $w_1, \ldots, w_h$ have been chosen to be a minimal set of generators.

iii) In Theorem 2.19 to show that II, III are equivalent to N being symmetric it is actually not necessary to assume that F is irreducible.

Proposition 2.21. *If F is a surface of degree n with an "even" set N of t nodes, and $t \leqq \binom{n+1}{3}$, then N has order $\leqq n-1$.*

Proof. It suffices to show that $h^0(\mathcal{O}_{\tilde{F}}(rH - L)) \neq 0$ for $2r \geqq n-1$.

By Serre duality $h^2(\mathcal{O}_{\tilde{F}}(rH - L)) = h^0(\mathcal{O}_{\tilde{F}}((n-4-r)H + L))$ but $rH - L - (n-4-r)H - L \geqq (2r - n + 4)H - H - A \geqq 2H - A$: now if $D \in |mH + L|$, $D \cdot A_i = -1 \Rightarrow D \geqq A_i$, so $h^0(\mathcal{O}_{\tilde{F}}(rH - L)) > h^2(\mathcal{O}_{\tilde{F}}(rH - L)$ (for $2r \geqq n-1$!) and it is enough to prove that $\chi(\mathcal{O}_{\tilde{F}}(rH - L)) > 0$.

This inequality follows from R.R. and in particular we have that $\chi(\mathcal{O}_{\tilde{F}}(rH - L)) = n$ if $t = \binom{n+1}{3}$ and $2r = n-1$ for n odd, or $2r = n$ for n even and the set N "weakly even".

Definition 2.22. Let N be a symmetric set of nodes: we will call N a linearly symmetric set of nodes (and hence F a linearly symmetric surface) iff the integers d_i are all equal to 1.

Thus a linearly symmetric set of nodes N has $\binom{n+1}{3}$ elements and order $n-1$ (the maximum allowed, in view of Proposition 2.21, and Theorem 2.16).

Theorem 2.23. *Let N be an even set of nodes on a reduced surface F of degree n: then N is linearly symmetric if and only if N has $\binom{n+1}{3}$ nodes and order $(n-1)$.*

Proof. We clearly need to prove only the "if" part of the statement.

Set $r=\left[\frac{n}{2}\right]$: then, by Proposition 2.21, $\chi(\mathcal{O}_{\tilde{F}}(rH-L))=n$ (because, if n is even, N is "weakly even", its order being $(n-1)$), and $\chi(\mathcal{O}_{\tilde{F}}((r-1)H-L)=0$.

Since $h^2(\mathcal{O}_{\tilde{F}}((r-1)H-L))=\dim R^-_{r-2}$, and $R^-_m=0$ for $m<r$,

$$h^i(\mathcal{O}_{\tilde{F}}((r-1)H-L)=0 \qquad \text{for } i=0,1,2.$$

By looking at the exact cohomology sequence of

$$0\to\mathcal{O}_{\tilde{F}}((m-1)H-L)\to\mathcal{O}_{\tilde{F}}(mH+L)\to\mathcal{O}_H(mH-L)\to 0$$

for $m=r$, $r-1$ we deduce that $H^1(\mathcal{O}_H((r-1)H-L)=H^2(\mathcal{O}_{\tilde{F}}((r-2)H-L)=0$, hence $R^-_r\cong H^0(\mathcal{O}_H(rH-L))$, where both spaces of sections are dimensional.

One can now either easily show that condition III of Theorem 2.19 is verified, or, in case F is reduced and irreducible, use the following "elementary" argument: let H' be another hyperplane section, cutting H transversally in n points $y_1,\dots,y_n$, and consider the restriction mapping $R^-_r\to\bigoplus_{i=1}^{n}\mathbb{C}_{y_i}$.

Since $H^0(\mathcal{O}_H((r-1)H-L))=0$ this map is an isomorphism: hence one can choose a basis $w_1,\dots,w_n$ of R^-_r such that $w_i(y_j)=\delta_{ij}$. If $B_{ij}(x)=w_iw_j$, then $B_{ii}(y_j)=\delta_{ij}$, $B_{ij}(y_s)=0$ for $i\neq j$; therefore, for $i\neq j$, $B_{ij}(x)$ is identically zero on the line $H\cap H'$, since it has degree $(n-1)$ and n distinct zeros there. Hence $\det(B_{ij})$ is not identically zero on the line, and we can apply the arguments of I) of Theorem 2.19.

The next proposition shows that the hypothesis on the order of N cannot be completely dropped. In fact, taking $h=6$, we get a surface of degree $n=16$ with a strictly even set of $\binom{17}{3}$ nodes of order at most 14.

Proposition 2.24. *There exist surfaces F_h of degree $4+2h$ with a strictly even set of $t=8\left(2+h(h+2)+\binom{h+1}{3}\right)$ nodes, of order less than or equal to $2+2h$.*

Proof. For $h=0$ take a Kummer surface, a quartic with a strictly even set of 16 nodes (see [13] p. 49). We are going to apply Gallarati's inductive procedure in this case to construct, given F_h, a surface F_{h+1}, tangent to F_h along a smooth curve passing through the nodes, and with only nodes as singularities.

Step 1. On $\tilde{F}_h$ let L be a divisor such that $2L\equiv\sum_{i=1}^{t}A_i=A$. We claim that $|(h+3)\cdot H-L|$ contains a smooth irreducible curve C.

For $h=0$ see [13]: for $h\geqq 1$ by the inductive hypothesis there exists a smooth irreducible curve C' in $|(h+1)H-L|$, and since $C'+|2H|\subset|(h+3)H-L|$, this last linear system can have at most C' as fixed part, and if C' is not a fixed part, then there are no base points which are singular for the system, and the assertion follows by Bertini's theorem.

However if $|(h-3)H-L|\neq\emptyset$ then C' is not a fixed part. If $|(h-3)H-L|=\emptyset$, by Serre duality $H^2(\mathcal{O}_{\tilde{F}_h}((h+3)H-L))=0$; in this case one thus has using R.R.

$$\dim |(h+3)H-L| \geqq \tfrac{1}{2}((h+3)H-L)((3-h)H-L)+\chi(\mathcal{O}_{F_h})-1$$
$$=(h+2)(9-h^2)-\frac{t}{4}+\binom{2h+3}{3}=9h+15,$$

therefore

$$|(h+3)H-L| \neq C'+|2H|.$$

Step II. Let G be a surface of degree $2h+6$ such that $\operatorname{div}_{F_h}(G)=2C$. We only need to prove that if Q is a general quadratic form in the x's, $F_{h+1}=G+QF_h$ has as singularities only nodes, lying on C. By Bertini's theorem it follows that F_{h+1}, which moves in a linear system with C as base locus, is smooth outside C.

Moreover, C being smooth, F_{h+1} has at most double points as singularities and is smooth where F_h is singular. Let $P \in C$ be a smooth point of F_h: then there exist holomorphic coordinates (x, y, z) around P such that

$$F_h=\{z=0\}, \qquad C=\{z=y=0\}, \qquad G=y^2+zg(x,y,z)=0, \qquad Q=q(x,y,z).$$

In these local coordinates $F_{h+1}=y^2+z(g+q)$.

We notice that a point p' of C is singular for F_{h+1} if $(g+q)$ vanishes at P' and is a node if the zero is simple: for general q it is easily seen that all the zeros are simple.

For linearly symmetric surfaces the matrix a_{ij} has a nice geometrical interpretation.

Proposition 2.25. *If F is a general linearly symmetric surface the $\binom{n+1}{3}$ nodes (forming an even set) are independent, i.e. the B_{ij}'s span the subspace of surfaces of degree $(n-1)$ passing through them. This subspace is thus isomorphic to* $\operatorname{Sym}^2(H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(rH-L))$ *(notations as in Theorem 2.23) and via this isomorphism the map $x \to a_{ij}(x)$ is the first polarity map.*

Proof. Let $a_{ij}=a_{ji}$ $(i,j=1,\dots,n)$ be homogenous coordinates in the projective space $\mathbb{P}^N$, where $N+1=n(n+1)/2$.

Consider the determinantal hypersurface $\det(a_{ij})=0$: a general linearly symmetric surface is the section of this hypersurface with a general 3-dimensional linear subspace of $\mathbb{P}^N$.

By ([9] Theorem 5.7, p. 349) the determinants of $(n-1)\times(n-1)$ minors generate a prime homogeneous ideal, so the restriction to a general 3-dimensional linear subspace gives a radical ideal.

For the second statement observe that if $F=\det\left(\sum_{k=0}^{3} a_{ij}^k x_k\right)$, (x_0, x_1, x_2, x_3) being homogeneous coordinates in $\mathbb{P}^3$, if $y=(y_0,\dots,y_3)$ is a point in $\mathbb{P}^3$, the first polar of F with respect to y, F_y, is given by

$$F_y=\sum_{k=0}^{3} y_k \frac{\partial F}{\partial x_k}=\sum_{k=0}^{3} y_k \sum_{i,j=1}^{n} a_{ij}^k B_{ij}(x)=\sum_{i,j=1}^{n} a_{ij}(y)\, w_i w_j.$$

Proposition 2.26. *The symmetric surfaces of degree n depend on $n^2+2n-15$ moduli for $n \geqq 5$, or $n=3$.*

Proof. If $n=3$ actually all the cubics with 4 nodes are projectively equivalent. If $n \geqq 5$ and $F=\det(a_{ij}(x))$, $F'=\det(a'_{ij}(x))$ $(i,j=1,\ldots,n)$, F, F' have only rational double points as singularities and there is an isomorphism $g: F \to F'$, g is induced by a projective automorphism of $\mathbb{P}^3$: in fact $g^*(K_{F'})=K_F$ and the surfaces $\tilde{F}$, $\tilde{F}'$ are simply connected, hence $g^*(H_{F'})=H_F$ (H being the hyperplane bundle).

Moreover the matrix a_{ij} is determined by F up to a change of basis in the vector space $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(rH-L))$. By ([23], Corollary 1) any such F has at most a finite set of projective automorphisms for $n \geqq 4$, so if one takes the quotient of $\mathbb{P}(\mathrm{Hom}(\mathbb{C}^4, \mathrm{Sym}^2(\mathbb{C}^n)))$ by $\mathbb{P}GL(4) \times \mathbb{P}GL(n)$ with the obvious action one gets a projective variety an open set of which represents the coarse moduli space for symmetric surfaces with only nodes. The dimension of this moduli space is then

$$\left(\frac{n(n+1)}{2}\cdot 4-1\right)-15-(n^2-1)=n^2+2n-15$$

(in fact no projective transformation in $\mathbb{P}GL(n)$ stabilizes four general quadratic forms).

Remark 2.27. For $n=4$ the above formula gives the number of projective moduli.

We end this section by showing how, in the case of plane curves, contact is always given by symmetric determinants. Suppose C is a reduced curve of degree n in $\mathbb{P}^2$, and that $\mathcal{L}$ is a non trivial line bundle on C such that $\mathcal{L}^2 \equiv \mathcal{O}_C(\delta)$, $\delta=0$ or 1.

As in the surface case we define the associated module R^- of L as the $\mathcal{A}$ $=\mathbb{C}[x_0, x_1, x_2]$-module $\bigoplus_{m=0}^{\infty} H^0(C, \mathcal{L} \otimes \mathcal{O}_C(m))$.

Proposition 2.28. *R^- is a Cohen-Macaulay $\mathcal{A}$-module. Moreover, for each choice $w_1, \ldots, w_h$ of a minimal set of homogeneous generators for R^-, there exists a symmetric $(h \times h)$ matrix $a_{ij}(x)$, which has rank at least $(h-1)$ everywhere on $\mathbb{P}^2$, with the following properties:*

1) $\det(a_{ij}(x))$ *is the equation F of C*

2) *the rows of (a_{ij}) give relations $\sum_{j=1}^{h} a_{ij}(x) w_j$ which generate the relations among the w_i's*

3) $w_i w_j = B_{ij}(x)$ *((B_{ij}) being, as usual, the adjoint matrix of (a_{ij})). Conversely, any matrix (a_{ij}) with the above properties determines a line bundle $\mathcal{L}$ in such a way that there exists a minimal set of generators $w_1, \ldots, w_h$ of R^- with $(w_i w_j)=B_{ij}(x)$, $(a_{ij})(B_{ij})=F \cdot$ Identity.*

Proof. By Theorem 1 of [29] p. 265, $\mathrm{Ext}^i(R^-, \mathcal{A})=0$ for $i \geqq 2$, hence R^- has projective Dimension 1.

The rest of the proof is entirely analogous to the one of Theorem 2.19: the fact that the B_{ij}'s have no common zero on $\mathbb{P}^2$ is a consequence of the fact that R^-_m has no base points for $m \gg 0$.

Remark 2.29. In particular, when $\delta \equiv n-1 \pmod 2$, i.e. the d_i's are odd, $\mathcal{L} \otimes \mathcal{O}_C\left(\left[\frac{n-3}{2}\right]\right)=\vartheta$ is a thetacharacteristic on the curve C (this means that

$\vartheta^{\otimes 2} \cong \mathcal{O}_C(n-3) = \omega_C$), and any such ϑ, different from $\mathcal{O}_C\left(\frac{n-3}{2}\right)$ when n is odd, occurs in this way. Also, $h^0(\vartheta)=0$ iff all the d_i's are equal to 1; so ineffective thetacharacteristics correspond to a way of writing C as the determinant of a symmetric matrix of linear forms: K. Hulek pointed out to us that this fact had been proven a long time ago by A. Dixon ([10]).

In this connection the linearly symmetric sets of nodes on a surface in $\mathbb{P}^3$ are those for which the restriction of $\mathcal{L}$ to an hyperplane section C not passing through the nodes gives rise to an ineffective thetacharacteristic on C.

Finally we want to mendion that if in Proposition 2.28 one allows the B_{ij}'s to have some common zero (hence at singular points of C), then one has a correspondence with thetacharacteristics or with 2-torsion bundles on a partial normalization of C: we are going to discuss this in a future paper, giving applications to the theory of curves.

§ 3. Quartic and Quintic Surfaces and the Counterexample to Babbage's Conjecture

Let N be an "even" set of nodes on a surface F of degree $n \leqq 5$: then, by Theorem 2.16, N is symmetric iff $H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(L))=0$, i.e. the double cover S of Propositions 2.11, 2.13 has $q=0$.

The case of $n=2, 3$ being trivial, we want to discuss even sets of nodes when $n=4, 5$, rediscovering Gallarati's classification ([13]).

Theorem 3.1. *If F is a quartic surface with an even set N of t nodes and only rational double points as singularities, F is symmetric, and N is symmetric except for the case when $t=16$: if this occurs F is a Kummer surface.*

Proof. Let $\Phi: S \to F$ be the associated double cover. If N is weakly even S is a minimal model of a surface of general type with $K_S^2=2$. Hence $H^1(S, 2K_S)=0$ ([6] p. 185) and therefore, since

$$h^1(\tilde{S}, 2K_{\tilde{S}}) = t = h^1(\tilde{F}, H+A) + h^1(\tilde{F}, L) = t + h^1(\tilde{F}, L),$$

$h^1(\tilde{F}, L)=0$ and Theorem 2.19 applies.

If N is strictly even, the possible values for t are only 8 or 16 by Proposition 2.11 since $t \leqq 16$ (see [33]).

When $t=8$, then $\chi(\tilde{F}, H-L)=2$, $\chi(\tilde{F}, L)=0$ (they are, respectively, equal to 0, -2 when $t=16$).

Since $h^0(\mathcal{O}_H(H-L))=2$, $h^1(\mathcal{O}_H(H-L))=0$, from the exact sequence

$$\begin{aligned} 0 \to R_1^- \to H^0(\mathcal{O}_H(H-L)) \to H^1(\mathcal{O}_{\tilde{F}}(-L)) \\ \to H^1(\mathcal{O}_{\tilde{F}}(H-L)) \to H^1(\mathcal{O}_H(H-L)) \to 0 \end{aligned}$$

we deduce that for $t=8$, $\dim R_1^- = 2$, $h^1(\tilde{F}, \mathcal{O}(L)) = h^1(\tilde{F}, \mathcal{O}(-L))=0$.

When $t=16$ $\dim R_1^- = 0$, $h^1(\tilde{F}, L)=2$ so that the double cover S is an abelian surface and N is not symmetric.

In this last case, however, F possesses 16 weakly even sets of 6 nodes, the 16 complementary ones (with 10 nodes), and 30 strictly even sets of 8 nodes (cf. e.g. [16], pp. 762 and foll., exp. p. 787).

Lemma 3.2. *If a quintic surface F has an even set N of t nodes, with $t \geqq 20$, N has order $\geqq 4$.*

Proof. If $H^0(\mathcal{O}_{\tilde{F}}(H-L)) \neq 0$ there exists a quadric Q tangent to F along a curve C, and having odd multiplicity at the nodes $P_1, \ldots, P_t$ of N: hence Q is smooth at them.

Also, $\operatorname{div}_Q(F) = 2C$ implies that Q has to be singular since otherwise $\operatorname{div}_Q(F)$ would not be divisible by 2 in $\operatorname{Pic}(Q)$ (its bidegree being $(5,5)$). Q cannot be a plane π counted twice, because Q is smooth at the P_i's, and if $Q = \pi_1 \cup \pi_2$, $l = \pi_1 \cap \pi_2$, l does not pass through any node, so $F \cdot \pi_i = k_i l + 2C_i$ $\left(C = C_1 + C_2 + \frac{k_1+k_2}{2} l\right)$.

One can assume that C_1 passes through more than 10 nodes of N, $P_1, \ldots, P_s$: but

$$\begin{aligned} 2p(C_1) - 2 &= -2 = (C_1 \cdot (C_1 + H))_{\tilde{F}} \\ &= \tfrac{1}{2} C_1 \cdot \left(3H - k_1 l - \sum_{i=1}^{s} A_i\right) = \frac{(3-k)\deg C_1 - s}{2}. \end{aligned}$$

As $k_1 =$ either 1 or 3, $s = 8$ or 4, absurd. Finally if Q were a cone with vertex P, since $P \notin N$, by Lemma 2.3 we would get $t = 16$.

Theorem 3.3. *If N is an even set of t nodes on a quintic surface F, N is symmetric when $t \leqq 20$ ($t = 16$ or 20).*

Proof. Consider the exact cohomology sequence of

$$0 \to \mathcal{O}_{\tilde{F}}(L) \to \mathcal{O}_{\tilde{F}}(H+L) \to \mathcal{O}_H(H+L) \to 0.$$

By Proposition 2.28 $h^0(\mathcal{O}_H(H+L)) = h^1(\mathcal{O}_H(H+L))$ can only be 0 or 1: by R.R. $\chi(\mathcal{O}_{\tilde{F}}(H+L)) = 5 - \frac{t}{4}$, and since $h^2(\mathcal{O}_{\tilde{F}}(H+L)) = 0$, it follows that $t \geqq 16$.

If $t = 16$ $h^0(\mathcal{O}_{\tilde{F}}(H+L)) = 1$ and $h^1(\mathcal{O}_{\tilde{F}}(L)) = 0$, if $t = 20$, $h^0(\mathcal{O}_{\tilde{F}}(H+L)) = 0$ by Lemma 3.2 and again $h^1(\mathcal{O}_{\tilde{F}}(L)) = 0$, so that Theorem 2.19 applies in both cases.

Remark 3.4. Quintics with an even set of 16 nodes are thus the zero set of the determinant of a symmetric matrix corresponding to the sequence of integers $(1,1,3)$. They are tangent to a quadric cone along a smooth curve of genus 2.

These surfaces F have been considered by Togliatti as the branch locus of the projection of a cubic hypersurface Γ in $\mathbb{P}^5$, with centre a general line $l \subset \Gamma$, to a skew $\mathbb{P}^3$. Togliatti proves that if Γ has r nodes (where $r \leqq 15$ by [32]) and the line l is general, then the quintic F has exactly $16 + r$ nodes: for $r = 15$ one obtains a quintic surface with 31 nodes. A Beauville ([5]) has proved rigorously that 31 is the maximum number of nodes that a quintic surface can have, and in the course of proving this he shows also that there do not exist on a quintic even sets with more than 20 nodes (thus confirming a conjecture of Gallarati in [13]) and that a quintic F with 31 nodes possesses 31 even sets with 16 nodes: it follows then that Theorem 3.3 can be stated without the restriction $t \leqq 20$, and as a consequence one gets that a quintic with 31 nodes is a Togliatti quintic.

We now come to the counterexample to the conjecture of Babbage, by simply restating Corollary 2.17 in a particular case:

Theorem 3.5. *If F is a quintic surface whose singular set consists of a linearly symmetric set N of nodes, $F = \det(a_{ij}(x))$, and $\Phi: S \to F$ is the double cover ramified on N, then:*

i) *Φ is the canonical map of S.*

ii) *the canonical ring of S is generated by linearly independent elements $x_0, x_1, x_2, x_3 \in R_1^+$, $w_1, \ldots, w_5 \in R_2^-$ subject to the following relations:*

a) $\sum_{k=0}^{3} \sum_{j=1}^{5} a_{ij}^k w_j x_k = 0$ *for* $i = 1, \ldots, 5$

b) $w_i w_j - B_{ij}(x) = 0$ *for* $i, j = 1, \ldots, 5$. *((B_{ij}) being the adjoint of (a_{ij}) as usual).*

Theorem 3.6. *If S is as in Theorem 3.5, its local moduli space has dimension bigger than* 30, *and for a general deformation the canonical map is a birational morphism on a surface of degree* 10.

Proof. By R.R., if T_S is the tangent bundle to S, $\chi(T_S) = \frac{7}{6} K_S^2 - \frac{5}{6} c_2(S)$, and here $c_2(S) = 50$, hence $\chi(T_S) = -30$, and $h^1(T_S) = 30 + h^2(T_S)$ $(h^0(T_S) = 0$, S being of general type). By Kuranishi's theorem ([22]) (cf. also [21] p. 165 and foll.) there exists an analytic subspace X of a neighbourhood of 0 in $H^1(S, T_S)$, defined by $h^2(T_S)$ equations, hence of dimension $\geqq 30$, and a family $f: \mathscr{S} \to X$ which is the univeral deformation of S (we set $S_x = f^{-1}(x)$, so $S \approx S_0$). Let $\omega_{\mathscr{S}|X}$ be the relative dualizing sheaf ($\omega_{\mathscr{S}|X}$ restricted to S_x is K_{S_x}): $h^1(S, K_S) = 0$, and by base change $f_*(\omega_{\mathscr{S}|X})$ is a trivial bundle of rank 4 (see e.g. [17] Theorem 12, 11, p. 290).

Fix 4 independent sections of $f_*(\omega_{\mathscr{S}|X})$: they lift to 4 sections of $\omega_{\mathscr{S}|X}$ which span $H^0(S_x, K_{S_x})\,\forall x$, and do not have common zeros on $S = S_0$. Hence we can assume, by shrinking X, that they do not have common zeros on $\mathscr{S}$; so there is defined a morphism $\Phi_X: \mathscr{S} \to \mathbb{P}^3 \times X$, and $\Phi_X|_{S_x \times \{x\}}$ is the canonical map of S_x.

We want to prove that Φ_X is generally injective. If the contrary holds, we can assume it to be a finite map of degree 2, and let σ be the involution determined by Φ_X on $\mathscr{S}$: since σ has only isolated fixed points on S_0, σ has no divisor of fixed points on $\mathscr{S}$, and on $S_x \sigma$ has only isolated fixed points, at worse. Then $\forall x$ F_x is a quintic with an even set of nodes; since $H^0(\tilde{F}_x, \mathcal{O}_{\tilde{F}_x}(H-L)) = 0$, by R.R. $t \geqq 20$ so $t = 20\,\forall x$ by shrinking X again.

The result then follows by Proposition 2.26.

Proposition 3.7. *All the double covers S of linearly symmetric quintics are diffeomorphic.*

Proof. Let V be the open set of $\operatorname{Hom}(\mathbb{C}^4, \operatorname{Sym}^2(\mathbb{C}^5))$ of linear maps $(x_0, \ldots, x_3) \to (a_{ij}(x))$ such that $\det(a_{ij}(x))$ is a symmetric quintic, and let $Y \hookrightarrow V \times \mathbb{P}^3$ be the family of quintics parametrized by V, $q: \mathscr{S} \to V$ the family of double covers. Then q is a differentiable fibre bundle with connected base.

An interesting application of the notion of even sets of nodes concerns the family of surfaces in $\mathbb{P}^3$ of degree n with exactly d nodes as singularities.

This family is clearly irreducible for $n \leqq 3$ (for $n=3$ one exploits the plane representation of the cubic surfaces) but fails to be so for quartic and quintic surfaces for certain values of d.

The reason is the following: first of all, if N, N' are two even sets of nodes on F, their symmetric difference is also an even set. Moreover, if $\mathscr{F} \xrightarrow{f} M$ is a family of surfaces, with d nodes (i.e. f is a topological fibre bundle), where M is smooth and connected, if one particular surface $F_m = f^{-1}(m)$ has an even set of t nodes, all the other do. In fact all the $\tilde{F}_m$ are then homeomorphic, and, by Lefschetz's (1, 1) theorem, the condition for a set of nodes to be even is merely topological.

We also observe that if the d nodes of F impose independent conditions on the surface of degree n, then F is a smooth point of the variety of surfaces of degree n with d nodes: in view of Theorem 3.3, Remark 3.4, Proposition 2.25 we have for instance (besides other, classically known, cases [18])

Proposition 3.8. *The variety V of quintic surfaces in $\mathbb{P}^3$ with 20 nodes is reducible.*

Proof. There exist quintics F with 20 independent nodes and an even set of 16 nodes by Remark 3.4, and these cannot be linearly symmetric by Theorem 3.3.

§ 4. Some Numerical Campedelli Surfaces with $\pi_1 = \mathbb{Z}/5\mathbb{Z}$.

In this paragraph, as in the following, we consider the standard representation of the symmetric group $\mathfrak{S}_5$ on $\mathbb{C}^5$, by permutation of the indexes $\{0, \ldots, 4\}$: i.e. if $\sigma \in \mathfrak{S}_5$, $y=(y_i)$ $\sigma(y)_j = y_{\sigma^{-1}(j)}$, or in terms of the canonical basis $e_0, \ldots, e_4$, $\sigma(e_i) = e_{\sigma(i)}$. The hyperplane $\sum_{i=0}^{4} y_i = 0$ is an irreducible representation of $\mathfrak{S}_5$, we call it V. Identify $\mathbb{Z}/5\mathbb{Z}$ with the cyclic subgroup of $\mathfrak{S}_5$ generated by the cycle $\tau = (0, 1, 2, 3, 4)$. Let W be the regular representation of $\mathbb{Z}/5\mathbb{Z}$: W has a basis $w_0, \ldots, w_4$ where $\tau(w_i) = w_{i+1}$ (i, $i+1$ are to be understood as elements of $\mathbb{Z}/5\mathbb{Z}$). We want to determine the $\mathbb{Z}/5\mathbb{Z}$ equivariant linear maps $A\colon V \to \mathrm{Sym}^2(W)$.

$$A\begin{pmatrix} y_0 \\ \vdots \\ y_4 \end{pmatrix} = \left(\sum_{k=0}^{4} a_{ij}^k y_k\right)_{i,j=0,\ldots,4}$$

and

$$A(\tau(y)) = \tau(A(y)) \Leftrightarrow \sum_{k=0}^{4} a_{ij}^k y_k = \sum_{k=0}^{4} a_{i+1,j+1}^k y_{k-1},$$

i.e. $a_{ij}^k = a_{i+1,j+1}^{k+1}$ (remember that also one has $a_{ij}^k = a_{ji}^k$).

Therefore $\mathrm{Hom}_{\mathbb{Z}/5\mathbb{Z}}(V \oplus \mathbb{C}, \mathrm{Sym}^2 W)$ has Dimension 15.

$$\mathrm{Hom}_{\mathbb{Z}/5\mathbb{Z}}(\mathbb{C}, \mathrm{Sym}^2 W) = \{(a_{ij}) \mid a_{ij} = a_{i+1,j+1}\}$$

has dimension 3, so, if $U' = \mathrm{Hom}_{\mathbb{Z}/5\mathbb{Z}}(V, \mathrm{Sym}^2(W))$, $\dim U' = 12$. For $a \in U'$, $a = (a_{ij}^k)$, let F_a be $\det\left(\sum_{k=0}^{4} a_{ij}^k y_k\right) = 0$: this gives a symmetric quintic on the hyperplane $\sum_{i=0}^{4} y_i=0$, provided that the equation defines a surface with only nodes as singularities.

This requirement, plus the one that F_a does not contain the 4 fixed points for the action of $\mathbb{Z}/5\mathbb{Z}$ on $\mathbb{P}^3=\mathbb{P}(V)$, determine an open set $\hat{U}$ in U' (that $\hat{U}$ is not empty is shown e.g. by the example in the next §).

Proposition 4.1. *For $a\in\hat{U}$ the action of $\mathbb{Z}/5\mathbb{Z}$ lifts to the double cover S_a, and the quotient $S_{a/\mathbb{Z}/5\mathbb{Z}}$ is a numerical Campedelli surface X,* (i.e. $K_X^2=2$, $q(X)=p_g(X)=0$), *with fundamental group $\mathbb{Z}/5\mathbb{Z}$. The surfaces X thus obtained form a proper subvariety in their local moduli space.*

Proof. $\tilde{S}_a$ is given in L by the square roots of a section $s\in H^0(\mathcal{O}_{\tilde{F}}(2L))$ such that $\operatorname{div}(s)=A$. Since $\pi_1(\tilde{F}_a)=0$, from $\tau^*(2L)\equiv 2L$, it follows that $\tau^*(L)\cong L$; multiplying by a constant if necessary we can arrange things so that this isomorphism φ: $L\to\tau^*(L)$ gives a lifting to L of the action of $\mathbb{Z}/5\mathbb{Z}$ on $\tilde{F}_a$: finally, since $H^0(\mathcal{O}_{\tilde{F}_a}(2L))$ is 1-dimensional, we can again multiply φ by a 5th root of unity so that $\varphi(s)=s$.

Notice that in this way the action of $\mathbb{Z}/5\mathbb{Z}$ on S_a is without fixed points, thus the quotient X is smooth, and has $K^2=2$, $\chi(\mathcal{O}_X)=1$; the assertion on the fundamental group comes from the fact that S_a is simply connected (Proposition 3.5 and Proposition 5.1).

The local moduli space of X has dimension $\geq -\chi(T_X)=-(\frac{7}{6}\cdot 2-\frac{5}{6}\cdot 10)=6$.

On the other hand if X, X' are obtained from F_a, $F_{a'}$ and are isomorphic, there is given an isomorphism ψ of S_a, $S_{a'}$ compatible with the $\mathbb{Z}/5\mathbb{Z}$ action, inducing an isomorphism ψ^*: $R(S_{a'})\to R(S_a)$.

In particular

$$\psi^*: R_1^+(S_{a'})\xrightarrow{\approx} R_1^+(S_a)\cong V,\qquad \psi^*: R_2^-(S_{a'})\xrightarrow{\approx} R_2^-(S_a)\cong W;$$

a, a' being given by the first polarity map, there are automorphisms (of $\mathbb{Z}/5\mathbb{Z}$ representations) g: $V\to V$, $h=W\to W$ such that $(a')=\operatorname{sym}^2(h)(a)g$.

So, if $G=\operatorname{Aut}_{\mathbb{Z}/5\mathbb{Z}}(V)\times\operatorname{Aut}_{\mathbb{Z}/5\mathbb{Z}}(W)\cong(\mathbb{C}^*)^4\times(\mathbb{C}^*)^5$, G contains the subgroup $H=\{(g,h)|g=\lambda I_4,\ h=\mu I_5$ with $\lambda,\mu\in\mathbb{C}^*$, and $\lambda\mu^2=1\}$ which acts trivially on $\hat{U}$, and if we set $\hat{G}=G/H\cong(\mathbb{C}^*)^8$, every point in $\hat{U}$ has a finite stabilizer in $\hat{G}$ (by Theorem 3.3, e.g.). Therefore the quotient $\hat{U}/\hat{G}$ exists and has dimension 4.

§ 5. A Hilbert Modular Surface for the Field $\mathbb{Q}(\sqrt{21})$ and its Canonical Ring

We first summarize some results and constructions contained in ([34]).

Let K be the field $\mathbb{Q}(\sqrt{21})$, $\mathcal{O}_K$ the ring of algebraic integers of K, $\mathcal{O}_K=\left\{\frac{m+n\sqrt{21}}{2}\,|\,m,n\in\mathbb{Z}\ m\equiv n\ (\mathrm{mod}\ 2)\right\}$, G the Hilbert modular group $\mathbb{P}SL_2(\mathcal{O}_K)$, Γ the 2-congruence subgroup of G,

$$\Gamma=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathbb{P}SL_2(\mathcal{O}_K)\,\middle|\,\begin{pmatrix}a&b\\c&d\end{pmatrix}\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}(\mathrm{mod}\ 2)\right\}.$$

$\mathcal{O}_{K/2\mathcal{O}_K} \cong \mathbb{F}_4$, (the field with 4 elements), so that $G/\Gamma = SL_2(\mathbb{F}_4) = \mathfrak{A}_5$ (the alternating group in five letters), where the last isomorphism is obtained by the representation $SL_2(\mathbb{F}_4) \to \operatorname{Aut}(\mathbb{P}^1(\mathbb{F}_4))$ (see e.g. [11] p. 309).

The extended modular group $\hat{G} \to \mathbb{P}GL_2(\mathcal{O}_K)$ is the subgroup of matrices with determinant a totally positive unit, and if $U \subset \mathcal{O}_K$ is the group of units, U^+ the subgroup of totally positive units ($U^+/U^2 \cong \mathbb{Z}/2\mathbb{Z}$), we have the following exact sequence $1 \to G \to \hat{G} \to U^+/U^2 \to 1$.

Analogously to Γ, $\hat{\Gamma}$ is the 2-congruence subgroup of $\hat{G}$

$$\hat{\Gamma} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \hat{G} \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\operatorname{mod} 2) \right\}.$$

Denote by $a \to a'$ conjugation in $\mathcal{O}_K$, by H the Siegel upper half plane, $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$: Γ acts freely on $H \times H$ via the action

$$(z_1, z_2) \to \left(\frac{a z_1 + b}{c z_1 + d}, \frac{a' z_2 + b'}{c' z_2 + d'} \right).$$

Compactifying $H \times H/\Gamma$ and resolving its singularities one gets a non minimal smooth surface Y; after blowing down 20 exceptional curves of the first kind via a map $f\colon Y \to S$, one obtains a minimal surface S with $\chi(\mathcal{O}_S) = 5$, $K_S^2 = 10$.

From a theorem of Schvartsman ([27] p. 188–189) one derives the following

Proposition 5.1. *S is simply connected.*

The canonical map $\Phi\colon S \to \mathbb{P}^3$ is a finite map of degree 2 and $F = \Phi(S)$ is a quintic surface with 20 nodes, coming from the 20 isolated fixed points for the involution determined by Φ and induced by the action of $\hat{\Gamma}/\Gamma$ on Y.

If $\hat{Y}$ is obtained by first compactifying and then resolving $H \times H/\hat{\Gamma}$, $\hat{Y}$ has 10 exceptional curves of the first kind and by a blow down map $\hat{f}\colon \hat{Y} \to \tilde{F}$ one obtains the desingularization of F. The situation is perhaps best illustrated by the following diagram:

$$\begin{tikzcd}
& \tilde{Y} \arrow[dl, "\hat{m}"'] \arrow[dr, "\hat{p}"] \arrow[dd, "g"] & \\
Y \arrow[dd, "f"'] & & \hat{Y} \arrow[dd, "\hat{f}"] \\
& \tilde{S} \arrow[dl, "m"'] \arrow[dr, "p"] & \\
S \arrow[dr, "\Phi"'] & & \tilde{F} \arrow[dl, "\pi"] \\
& F &
\end{tikzcd}$$

where $p, \hat{p}$ are double covers ramified on 20 smooth rational curves with self intersection -2, $m, \hat{m}$ are blowing ups of the 20 isolated fixed points of the involution. S is acted on by $\hat{G}/\Gamma = \mathfrak{S}_5$, and $V = H^0(\mathcal{O}_S(K_S))$ has 5 special sections, corresponding to the cycles coming from the resolution of the 5 cusp singularities; denote them by $x_1, \ldots, x_5$: one has $\sum_{i=1}^{5} x_i = 0$ and $\mathfrak{S}_5$ acts on V by $\sigma(x_i) = x_{\sigma(i)}$ (V is thus the irreducible self dual representation of $\mathfrak{S}_5$ considered in §4). In [34] it is also shown that the equation of F in the dual basis is

$$\sum_{i=1}^{5} y_i^5 - 5/4 \left(\sum_{j=1}^{5} y_j^2 \right) \left(\sum_{i=1}^{5} y_i^3 \right),$$

and the 20 nodes of F are the $\mathfrak{S}_5$-orbit of the point

$$P_0 = (-2, -2, -2, 3+\sqrt{-7}, 3-\sqrt{7}).$$

We are going to apply the results of § 3 to this particular surface S, using the remark that, R being the canonical ring of S, R_m^+, R_m^- are representations of $\mathfrak{S}_5$.

Theorem 5.2. *R_2^- is an irreducible representation of $\mathfrak{S}_5$.*

Lemma 5.3. *In $\mathbb{P}^3 = \mathbb{P}(V)$ there exists precisely an $\mathfrak{S}_5$-invariant quartic B passing through the point $P_0 = (-2, -2, -2, 3+\sqrt{-7}, 3-\sqrt{-7})$, and B is smooth.*

Proof. As in [34], set $S_h = \sum_{i=1}^{5} y_i^h$, and denote by σ_h the h-th elementary symmetric function. Since $S_1 = \sigma_1 = 0$ on $\mathbb{P}^3$, any $\mathfrak{S}_5$-invariant quartic B belongs to the pencil $\lambda S_4 + \mu S_2^2$. If we impose the condition $P_0 \in B$, we obtain $\lambda = 4$, $\mu = 7$. A point $(y_i) \in \mathbb{P}^3$ is singular for B iff there is a constant c such that $\partial B / \partial x_i (y) = c$ for $i = 1, \ldots, 5$, or, more explicitly,

$$(+_i) \qquad 4y_i^3 + 7S_2(y)\, y_i = c/4 \qquad \text{for } i = 1, \ldots, 5.$$

Add the 5 equations $(+_i)$, noting that $\sum_{i=1}^{5} y_i = 0$: then $4S_3(y) = 5/4c$, and $(+_i)$ is equivalent to

$$(\#_i) \qquad y_i^3 + \tfrac{7}{4} S_2(y)\, y_i - \tfrac{1}{5} S_3(y) = 0.$$

As in [34] we note that the 5 coordinates $y_1, \ldots, y_5$ of (y) satisfy the same equation of third degree, therefore they can take at most 3 distinct values (and at least 2).

Let α, β, γ be the 3 roots of equation $\#$: then for the 5 coordinates of (y), up to permutation, one has the following possibilities

1) $(\alpha, \alpha, \beta, \beta, \gamma)$
2) $(\alpha, \alpha, \alpha, \beta, \gamma)$
3) $(\alpha, \alpha, \alpha, \beta, \beta)$
4) $(\alpha, \alpha, \alpha, \alpha, \beta)$

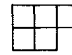
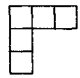
They however can all be excluded as absurd because α, β, γ are the roots of $(\#)$ so we must have

i) $\alpha + \beta + \gamma = 0$
ii) $\alpha\beta + \alpha\gamma + \beta\gamma = 7/4 S_2(y_1, \ldots, y_5)$
iii) $\alpha\beta\gamma = 1/5 S_3(y)$ and moreover we must have $\sum_{i=1}^{5} y_i = 0$.

For example let us show how possibility 1) is excluded:

$$\sum_{i=1}^{5} y_i = 0 \Leftrightarrow 2\alpha + 2\beta + \gamma = 0,$$

and this, with i), gives $\beta = -\alpha$, and then $\gamma = 0$. ii) then cannot hold, because the left hand side is $-\alpha^2$, the right hand one is $7\alpha^2$, and the solution $\alpha = 0$ must be excluded.

Proof of Theorem 5.2. The symmetric group $\mathfrak{S}_5$ has two 1-dimensional representations which we label here by 1 and χ, and which correspond to the trivial and the signature character. As for the other irreducible representations, they are given by the linear spaces freely generated by the standard Young tableaux of a given shape (see [19], [20] for the definitions and the proofs of the statement). V corresponds to the shape , to the shape  corresponds a representation we call W (as it will turn out to be isomorphic to R_2^-), to $U=\Lambda^2 V$ corresponds the shape : in this correspondence tensoring a representation with χ has the effect of exchanging rows with columns in the shape.

W is 5-dimensional, U is 6-dimensional. If R_2^- were reducible, it would contain a 1-dimensional subrepresentation: so there would exist a section $w \in R_2^-$ such that $\sigma(w)$ = either w or $\chi(\sigma)w$ (for $\sigma \in \mathfrak{S}_5$), and in any case w^2 would be an invariant section in R_4^+, corresponding to an invariant quartic passing through the 20 nodes and tangent to F. This however is impossible by Lemmas 5.3, 2.3.

Remark 5.4. Since $\chi^2=1$, $\mathrm{Sym}^2(W) \cong \mathrm{Sym}^2(W \otimes \chi)$.

Given representations W, W' of a finite group it is well known that for the corresponding characters the following rules hold:

$$\chi_{W\otimes W'} = \chi_W \cdot \chi_{W'}, \qquad \chi_{W\otimes W'} = \chi_W + \chi_{W'},$$

and the characters of irreducible representations are an orthonormal basis for the central functions on the group.

Using these rules and the character table for the representations of $\mathfrak{S}_5$, that we reproduce here for the reader's convenience:

Representations	Conjugacy classes						
	1	(12)(34)	(123)	(12345)	(1234)	(12)	(12)(345)
χ	1	1	1	1	-1	-1	-1
V	4	0	1	-1	0	2	-1
W	5	1	-1	0	1	-1	-1
U	6	-2	0	1	0	0	0

one can then compute (as $\Lambda^2 V \cong U$) and find that

$$\mathrm{Sym}^4(V) = 1 \oplus 1 \oplus (V)^3 \oplus W \oplus (W \otimes \chi)^2 \oplus U.$$

By Remark 5.4 and Theorem 5.2, $\mathrm{Sym}^2(R_2^-) \cong \mathrm{Sym}^2(W)$, moreover $\mathrm{Sym}^4(R_1^+) \cong \mathrm{Sym}^4(V)$, and by Corollary 2.25 $\mathrm{Sym}^2(W) \hookrightarrow \mathrm{Sym}^4(V)$.

By computing the character again, we get

$$W \otimes W \cong 1 \oplus W \oplus V \oplus U \oplus (V \otimes \chi) \oplus (W \otimes \chi).$$

Since $V\otimes\chi$ is not contained in $\mathrm{Sym}^4(V)$ we have $\Lambda^2 W\cong(V\otimes\chi)\oplus U$ (for reasons of dimension and because $1\subset\mathrm{Sym}^2 W$), $\mathrm{Sym}^2(W)=1\oplus V\oplus W\oplus(W\otimes\chi)$, and this last 15-dimensional subrepresentation of $\mathrm{Sym}^4(V)$ corresponds to the space of quartic surfaces passing through the nodes.

1 is given by the line generated by the quartic $4S_4+7S_2^2$ (the unique invariant quartic passing through the nodes), and V is the image of the first polarity map:

$$(y)\to\sum_{i=1}^{5} y_i\frac{\partial F}{\partial x_i}=\sum_{i=1}^{5} y_i\big(5x_i^4-\tfrac{15}{4}S_2(x)x_i^2 -\tfrac{5}{2}S_3(x)x_i-S_4(x)+\tfrac{3}{4}S_2^2(x)\big).$$

Theorem 5.5. *The canonical image F of the Hilbert modular surface S is given by the vanishing of the determinant of the following symmetric matrix* $\Big($*where* $x_1,\dots,x_5$, *with* $\sum_{i=1}^{5} x_i=0$, *are coordinates in* $\mathbb{P}(V)$ *such that* $\mathfrak{S}_5$ *acts by* $(\sigma(x))_i=x_{\sigma^{-1}(i)}\Big)$.

$$\begin{matrix}
(-2x_1+2x_2+2x_3+2x_4+2x_5) & (x_1+x_2-x_3-3x_4-x_5) & (-x_1-x_2+x_3+x_4+3x_5) & (-x_1+x_2-x_3+3x_4+x_5) & (x_1-x_2+x_3-x_4-3x_5)\\
(x_1+x_2-x_3-3x_4-x_5) & (4x_4) & (2x_2-2x_4-2x_5) & (-2x_4) & (x_1-x_2-x_3+x_4+x_5)\\
(-x_1-x_2+x_3+x_4+3x_5) & (2x_2-2x_4-2x_5) & (4x_5) & (x_1-x_2-x_3+x_4+x_5) & (-2x_5)\\
(-x_1+x_2-x_3+3x_4+x_5) & (-2x_4) & (x_1-x_2-x_3+x_4+x_5) & (4x_4) & (2x_3-2x_4-2x_5)\\
(x_1-x_2+x_3-x_4-3x_5) & (x_1-x_2-x_3+x_4+x_5) & (-2x_5) & (2x_3-2x_4-2x_5) & (4x_5)
\end{matrix}$$

Remark 5.6. Theorems 3.3 and 5.5 combine to give a complete description of the canonical ring of S.

Proof of 5.5. W corresponds to the representation associated to the Young diagram $\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; & \; \\ \cline{1-2}\end{array}$. So, in concrete terms, a basis of W is given by:

$$w_1=\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5\\ \cline{1-2}\end{array},\quad w_2=\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5\\ \cline{1-2}\end{array},\quad w_3=\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2}\end{array},$$

$$w_4=\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2}\end{array},\quad w_5=\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4\\ \cline{1-2}\end{array}.$$

Pick as generators of $\mathfrak{S}_5$ the following transpositions: $\sigma_1=(12)$, $\sigma_2=(2\,3)$, $\sigma_3=(34)$, $\sigma_4=(4\,5)$. Using the straightening formulas of Doubillet-Rota-Stein [12] we can compute the action of the σ_j's on W.

(4 5) acts by sending $w_1\to-w_1$, by permuting w_2 with $w_3\,(w_2\leftrightarrow w_3)$, w_4 with $w_5\,(w_4\leftrightarrow w_5)$.

(3 4) acts by: $w_3\to-w_3$, $w_4\to-w_4$, $w_1\leftrightarrow w_2$, $w_5\to w_5-w_4-w_1-w_3+w_2$

(2 3) acts by: $w_1\to-w_1$, $w_2\leftrightarrow w_4$, $w_3\leftrightarrow w_5$.

(1 2) acts by: $w_1\to-w_1$, $w_2\to-w_2$, $w_3\to-w_3$, $w_4\to w_4-w_2+w_1$, $w_5\to w_5-w_3-w_1$.

$\mathrm{Sym}^2(W)$ has the monomials $w_1 w_j$ $(i,j=1,\ldots,5\ i\leqq j)$ as a basis, and we know that there exists one and only one $\mathfrak{S}_5$-equivariant map $a\colon V\oplus 1\to\mathrm{Sym}^2(W)$, and we must then only find its coefficients in the given bases.

Suppose then that

$$(x_1,\ldots,x_5)\to\sum_{\substack{i\leqq j\\ k=1}}^{5} a_{ij}^k w_i w_j x_k$$

$$a(\sigma(x))=\sigma(a(x)),\qquad \forall\,\sigma\in\mathfrak{S}_5.$$

Then, $e_1,\ldots,e_5$ being the basis of the coordinates $(x_1,\ldots,x_5)$ we have:

$$a(e_k)=\sum_{i\leqq j} a_{ij}^k w_i w_j,\qquad \sigma(e_i)=e_{\sigma(i)}.$$

so the condition of equivariance is that

(*) $$\sum_{i\leqq j} a_{ij}^k\,\sigma(w_i w_j)=\sum_{i\leqq j} a_{ij}^{\sigma(k)}\,w_i w_j,\qquad (k=1,\ldots,5),\ \forall\,\sigma\in\mathfrak{S}_5.$$

Of course it is enough to verify (*) when σ is one of the four above mentioned generators of $\mathfrak{S}_5$.

Taking $(e_1,\ldots,e_5)$ as a basis for $V\oplus 1$, $(w_1^2,\ w_1w_2,\ w_1w_3,\ w_1w_4,\ w_1w_5,\ w_2^2,\ w_2w_3,\ w_2w_4,\ w_2w_5,\ w_3^2,\ w_3w_4,\ w_3w_5,\ w_4^2,\ w_4w_5,\ w_5^2)$ as an (ordered!) basis for $\mathrm{Sym}^2(W)$, the only matrix (up to a constant multiple) for which the equations (*) are satisfied is found, after a tedious computation, to be equal to the transpose of

$$\begin{pmatrix}
-1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & -1 & 1 & -1 & 0 & 2 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0\\
1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 2 & 0\\
1 & -3 & 1 & 3 & -1 & 2 & -2 & -2 & 1 & 0 & 1 & 0 & 2 & -2 & 0\\
1 & -1 & 3 & 1 & -3 & 0 & -2 & 0 & 1 & 2 & 1 & -2 & 0 & -2 & 2
\end{pmatrix}$$

Finally we observe that the monomial $w_i w_j$ corresponds to $\dfrac{w_i\otimes w_j+w_j\otimes w_i}{2}$ and we multiply the matrix by 2 not to lose integral coefficients: the result then follows.

Further Remarks

Our construction has been generalized by W. Barth ([3]) by considering discriminant surfaces F of a quadratic form (a) defined on a vector bundle V and with values in a line bundle L (the case of symmetric surfaces being the one where V is a direct sum of line bundles). Again, the locus where corank (a) = 2 is an even set of nodes, if (a) is "general" in the appropriate sense, and it is an interesting question whether all even sets of nodes can be obtained in this way.

We also wonder if Theorem 2.23 holds under the weaker assumption that N be strictly even for n odd, weakly for n even.

Moreover, some bounds for the number of nodes of an even set on a surface F of degree n might be useful to extend the result of [5] in degree $\geqq 6$.

Referring to 2.5, an even set N can be both strictly and weakly even iff the hyperplane class H of F is divisible by 2 as a Cartier divisor.

The question, posed by the referee, is whether this can happen: we believe that the answer is no, for any surface F, and we give now a short proof for the case when F has only rational double points as singularities.

If n is the degree of F, there exists a smooth surface G of degree n in $\mathbb{P}^3$ and containing a line: therefore the hyperplane class of G is not divisible by 2 in cohomology. By G.N. Tjurina's result (Funk. Anal. i Pril. 4, n. 1 77–83) the minimal resolution F' of F is a deformation of G, so F' is diffeomorphic to G under a diffeomorphism which takes the canonical class into the canonical class, and thus the hyperplane class of G into the pull-back to F' of the hyperplane class of F; hence this last is not divisible by 2 in Pic(F).

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