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CONSTRUCTING SEXTIC SURFACES WITH A GIVEN NUMBER d OF NODES

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Introduction

From the theory of double covers of $\mathbb{P}^3$ ('double solids', see [4]), and from the problem of classification of vector bundles on $\mathbb{P}^3$, new interest has arisen about the problem of the existence and the construction of surfaces in $\mathbb{P}^3$ of a given degree n having a certain number d of nodes as the only singularities. Our main result is the construction, for each $1 \leq d \leq 64$, of sextic surfaces with exactly d nodes.

In this note we restrict ourselves to consider surfaces with a trihedral symmetry, i.e. surfaces G whose equation can be written, in a suitable coordinate system, as $F(x_0^2, x_1^2, x_2^2, x_3^2)$: it is a beautiful idea of B. Segre [8] to relate the singularities of G with those of F and with the analysis of the position of the coordinate tetrahedron with respect to F.

Segre chose for F the symmetric (four-nodal) cubic and a particular tetrahedron to construct a G with 63 nodes: here we use different specializations of the tetrahedron in order to get any d with $32 \leq d \leq 64$, and other nodal cubics for $1 \leq d \leq 31$.

Also, we show that 64 is the maximum allowed for trihedral sextics. In this respect, if one denotes by $\mu(n)$ the maximum value of d for which there exists a surface of degree n with d nodes, then $\mu(n)$ is known only for $n \leq 5$ ($\mu(5) = 31$ has been proven recently by Beauville [2]), and for larger values of n only some lower and upper bounds for $\mu(n)$ are known (see [10, 8, 2] for an history of the problem).

For sextics, Basset's bound [1] gives $\mu(6) \leq 66$, while Segre's bound [8] $\mu(6) \leq 63$ and Stagnaro's bound [9] $\mu(6) \leq 65$ are valid only under some generality assumptions; moreover, a sextic with 64 nodes has been constructed by Stagnaro by a different method (see [9], also for more references), no one is known for $d \geq 65$.

This note is divided into two parts: in Section 1 we study the geometry of the

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symmetric cubic surface D_3 which is related to its duality with the Steiner surface, and in particular we give a geometric construction for the inverse of the birational map $\psi : \mathbb{P}^2 \to D_3$ given by the web of cubic curves passing through the vertices of a complete quadrilateral; in Section 2 we apply these results considering the trihedral sextics.

1. The symmetric (four-nodal) cubic surface D

Let $(z) = (z_0, z_1, z_2, z_3)$ be homogeneous coordinates in $\mathbb{P}^3$, then D is given by the equation

$$\sigma_3(z) = \sum_{i=0}^{3} \frac{z_0 z_1 z_2 z_3}{z_i} = 0.$$

The four nodes of D are $e_0, e_1, e_2, e_3 = (1,0,0,0), \ldots, (0,0,0,1)$ and D contains exactly 9 lines: the 6 lines $l_{ij} = \langle e_i, e_j \rangle$ joining pairs of nodes, and the 3 lines in which the plane $\tilde{\pi}$ of equation $\sigma_1(z) = \sum_{i=0}^{3} z_i = 0$ intersects D, $l_2 = \{z_0 + z_1 = z_2 + z_3 = 0\}$, $l_1 =$

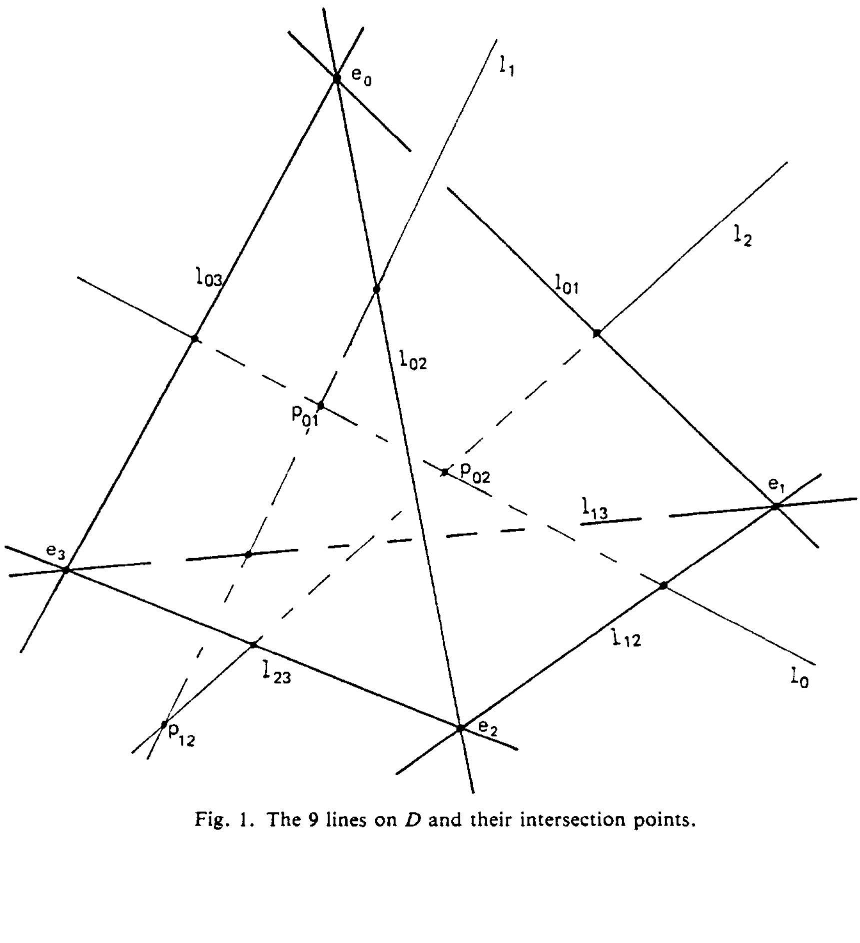

Fig. 1. The 9 lines on D and their intersection points.

$\{z_0+z_2=z_1+z_3=0\}$, $l_0=\{z_0+z_3=z_1+z_2=0\}$. Denote further by P_{ij} the point of intersection of l_i and l_j, i.e. $P_{12}=(1,-1,-1,1)$, $P_{02}=(1,-1,1,-1)$, $P_{01}=(1,1,-1,-1)$.

Let $(x)=(x_0,x_1,x_2)$ be homogeneous coordinates in $\mathbb{P}^2$, set for convenience $x_3=x_0+x_1+x_2$, and consider the quadrilateral whose sides are the lines L_i of equation $x_i=0$. The diagonals E_k $(k=0,1,2)$ of the quadrilateral have equation $x_k-x_3=0$, and, if $q_{ij}=L_i\cap L_j$, E_k joins q_{3k} with q_{ij} for (i,j,k) a permutation of $(0,1,2)$.

The map $\psi:\mathbb{P}^2\to\mathbb{P}^3$ such that $\psi(x)=(x_1x_2x_3, x_0x_2x_3, x_0x_1x_3, -x_0x_1x_2)$ gives the birational correspondence between $\mathbb{P}^2$ and D associated to the system of plane cubics passing through the q_{ij}'s. Clearly $\psi(E_k)=l_k$, ψ blows up q_{ij} to the line l_{ij}, and contracts L_i to the node e_i, $\psi^{-1}(\tilde{\pi})=E_0+E_1+E_2$.

Proposition 1. *Let π be a plane in $\mathbb{P}^3$ not passing through the nodes of D. Then π is simply tangent to D in $0\le n\le 3$ points, $n=3$ occurring iff $\pi=\tilde{\pi}$, and $n=2$ iff π contains one of the lines l_k.*

Proof. Remark first that the inverse image of a plane π containing l_k consists of E_k+C, where C is a conic passing through q_{3i}, q_{3j}, q_{ik}, q_{jk}: C is smooth unless it

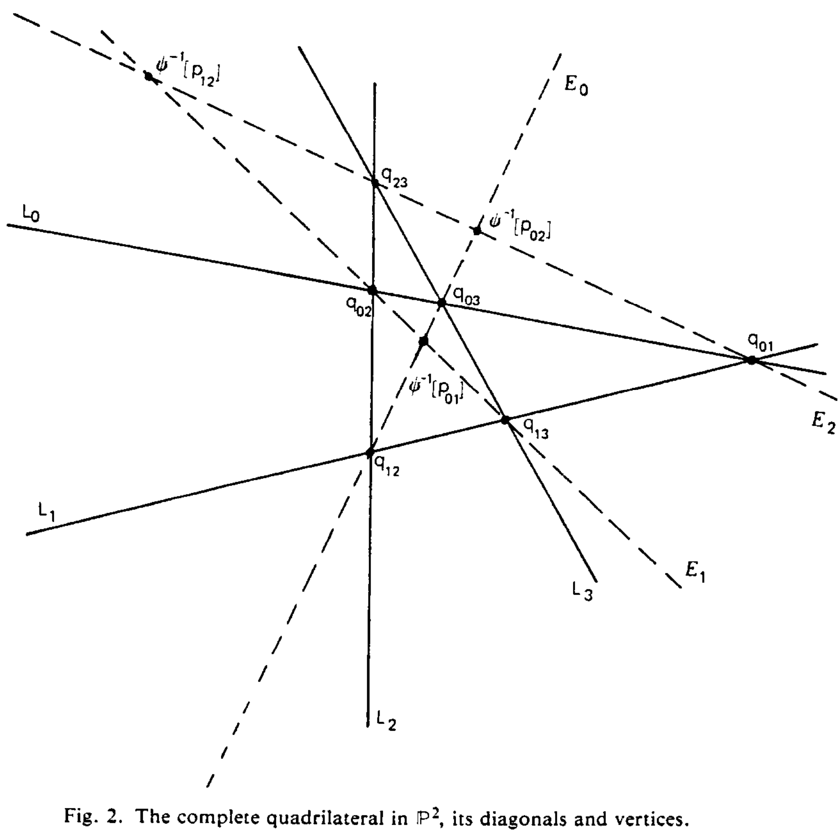

Fig. 2. The complete quadrilateral in $\mathbb{P}^2$, its diagonals and vertices.

equals E_i+E_j, L_3+L_k, L_i+L_j (the corresponding planes are then $\bar{\pi}$, the plane passing doubly through l_{3k}, the one passing doubly through l_{ij}). C moves in the pencil generated by L_3+L_k, L_i+L_j: since these two conics are tangent to E_k, if C is smooth it is not tangent to E_k.

To prove the last part of the statement we notice that any reducible cubic in the linear system contains either one side or one diagonal of the quadrilateral.

To prove the first part, we will show that the planes $\pi_{ij}=\langle l_{ij}, l_k\rangle$ (intersecting D in l_k+2l_{ij}) (where either (i,j,k) is a permutation of $(0,1,2)$ or $j=3$ and $i=k\neq 3$) are the only planes π which are tangent to D at a smooth point p where the plane section $\pi\cdot D$ has a non-ordinary singularity.

In fact, the parabolic curve of D (the locus of such points p) is the complete intersection of D with its Hessian surface H, of degree 4. So it suffices to prove that H is tangent to D along each l_{ij}; since π_{ij} is tangent to D along l_{ij}, this is equivalent to showing $H\cdot\pi_{ij}=2l_{ij}+\cdots$.

Now π_{ij} is defined by $z_k+z_h=0$ ((i,j,k,h) being a permutation of $(0,1,2,3)$), and l_{ij} in it is defined by $z_h=0$. Assume for simplicity $k=0$, $h=1$: then, $\bmod(z_0+z_1)$,

$$H=\det\begin{vmatrix} 0 & 0 & z_2-z_1 & z_1+z_2 \\ 0 & 0 & z_3-z_1 & z_3+z_1 \\ z_2-z_1 & z_3-z_1 & 0 & z_2+z_3 \\ z_1+z_2 & z_3+z_1 & z_2+z_3 & 0 \end{vmatrix}$$

which equals by adding and subtracting the last two rows and columns

$$4\det\begin{vmatrix} 0 & 0 & z_2 & z_1 \\ 0 & 0 & z_3 & z_1 \\ z_2 & z_3 & \frac{1}{2}(z_2+z_3) & 0 \\ z_1 & z_1 & 0 & -\frac{1}{2}(z_2+z_3) \end{vmatrix}$$

and is therefore divisible by $(z_1)^2$.

Let V be a surface in $\mathbb{P}^3$, V^* its dual variety (the closure in $\mathbb{P}^{3\cdot}$ of the locus of planes tangent to V at smooth points), $\tau: V\to V^*$ the (rational) Gauss map.

We recall here some basic facts about duality (cf. [7, p. 82 ff.] and [3, pp. 28–40]):

(i) $(V^*)^*=V$, so V^* is not a cone and V^* is a surface if V is not ruled,

(ii) if p is a smooth point of V, τ has maximal rank at p iff $\tau(p)$ is simply tangent to V at p,

(iii) if V is a surface of degree n with only d nodes as singularities, V^* has degree $n(n-1)^2-2d$.

Moreover, defining a line l to be transversal to V iff, at the points of $V\cap l$, V consists of smooth branches crossed transversally by l, also the following holds:

(iv) a line l is transversal to V iff l^* is transversal to V^*.

Proposition 2. *Let, on* $\mathbb{P}^{3\cdot}$, $(w)=(w_0,w_1,w_2,w_3)$ *be the coordinates dual to the* (z)'s, *and set* $\sigma=\sigma_1(w)=\sum_{i=0}^{3}w_i$, $\eta_1=w_0+w_1-w_2-w_3$, $\eta_2=w_0+w_2-w_1-w_3$, $\eta_3=w_0+w_3-w_1-w_2$.

Then the dual surface D^* *of* D *is the Steiner quartic surface of equation* $\eta_1\eta_2\eta_3\sigma-8(\eta_1^2\eta_2^2+\eta_1^2\eta_3^2+\eta_2^2\eta_3^2)$.

Proof. Since D is not ruled, by (i) D^* is a surface (and not a cone), by (iii) D^* has degree 4 Because of (ii) and Proposition 1, $(\bar{\pi})^*$ is a triple point of D^*, and, since the pencil of planes through l_k is a pencil of bitangent planes, $(l_k)^*$ is a double line of D^*. So D^* is the Steiner surface (cf. [5, pp. 81–82]).

To establish its equation we remark that this must also be a symmetric function of (w); furthermore, the triple point and the double lines are the vertex and the edges of the trihedron whose faces are the 3 planes of equation $\eta_i=0$.

Then the equation of D^* must be of the form $\eta_1\eta_2\eta_3\sigma+f(\eta)$, where $f(\eta)$ is a form of degree 4 vanishing of order 2 at the three points $\eta_i=\eta_j=0$. Notice that $\eta_1\eta_2\eta_3$ is a symmetric function of (w), as is easy to check, and that every permutation of the η_i's can be obtained by a permutation of the w_j's: therefore f is a symmetric function of (η).

Thus there are constants λ, μ, such that $f(\eta)=\lambda(\eta_1^2\eta_2^2+\eta_1^2\eta_3^2+\eta_2^2\eta_3^2)+\mu(\eta_1+\eta_2+\eta_3)\eta_1\eta_2\eta_3$. Since the first term of the sum is a symmetric function of (w) but the other is not, we must have $\mu=0$, $\lambda\neq 0$. We omit the computation of $\lambda=-8$ since in the rest of the paper we will not need the precise value of λ.

Proposition 3. *For* $p\in D$, (i,j,k) *a permutation of* $(0,1,2)$, *let* $A_i(p)$ *be the third point of intersection of the line* $\langle p,p_{jk}\rangle$ *with* D. *Consider the map* $\varphi: D\to\mathbb{P}^{3\cdot}$ *such that* $\varphi(p)=\langle A_0(p),A_1(p),A_2(p)\rangle^*$. *Then* $\varphi(D)$ *is the plane given by* $\sigma_1(w)=0$, *and* φ *is the inverse of* ψ *in the sense that* $\varphi(\psi(y_0,y_1,y_2))=(y_0,y_1,y_2,-y_0-y_1-y_2)$.

Proof. Let $\psi(y)=p$, and $b_i(y)\in\mathbb{P}^2$ be the pre-image of $A_i(p)$. Since $\langle p,p_{jk}\rangle=\langle p,l_j\rangle\cap\langle p,l_k\rangle$, the $b_i(y)$'s are the points, other than (y) or the vertices of the quadrilateral, of pairwise intersection of the three conics $C_i(y)$ such that $\psi^{-1}(\langle p,l_i\rangle)=E_i+C_i(y)$. The equations of these conics are, respectively, in the (x) coordinates (and recalling that $x_3=\sigma_1(x)$, so $y_3=\sigma_1(y)$):

$$C_2=\{y_2y_3x_0x_1-y_0y_1x_2x_3=0\},\qquad C_1=\{y_1y_3x_0x_2-y_0y_2x_1x_3=0\},$$
$$C_0=\{y_0y_3x_1x_2-y_1y_2x_0x_3=0\}.$$

Note that the equations of C_1 and C_2 are permuted by exchanging the indexes 1 and 2, while the equation of C_0 remains invariant; therefore, knowing the coordinates of b_1 as rational functions of (y), we can get b_2 by permuting y_1 with y_2 in those functions and exchanging the coordinates x_1 and x_2.

In the same way the permutation of the indexes 0 and 1 allows to relate b_1 to b_0.

To compute $b_1(y)$, we rewrite the 2 equations C_2, C_0 as

$$x_1(y_2y_3x_0-y_0y_1x_2)=y_0y_1x_2(x_0+x_2),$$

$$x_1(y_0y_3x_2-y_1y_2x_0)=y_1y_2x_0(x_0+x_2).$$

Eliminating x_1 we get $(y_2y_3x_0-y_0y_1x_2)y_2x_0=(y_0y_3x_2-y_1y_2x_0)y_0x_2 \Leftrightarrow x_0=\varrho y_0$, $x_2=-\varrho y_2$, and finally, substituting in C_2 and setting $\varrho=y_3+y_1$, $x_1=y_1(y_2-y_0)$. Hence

$$b_2=(y_0(y_3+y_2),\, -y_1(y_3+y_2),\, y_2(y_1-y_0)),$$

$$b_1=(y_0(y_3+y_1),\, y_1(y_2-y_0),\, -y_2(y_3+y_1)),$$

$$b_0=(y_0(y_2-y_1),\, y_1(y_3+y_0),\, -y_2(y_3+y_0)).$$

The coordinate w_i of the plane spanned by $\psi(b_2)$, $\psi(b_1)$, $\psi(b_0)$, is given by $(-1)^i$ times the determinant of the 3×3 minor obtained by deleting the ith column of the matrix $(M_{ij})=(z_j(b_i))$.

To simplify the computations, we notice that

$$M_{ij}=\frac{\prod_{k=0}^{3} y_k}{y_j}\, S_i\cdot B_{ij},$$

where $S_2=(y_2+y_3)(y_1-y_0)$, $S_1=(y_1+y_3)(y_2-y_0)$, $S_0=(y_0+y_3)(y_2-y_1)$ and (B_{ij}) is the following matrix:

$$\begin{vmatrix} y_1-y_0 & y_0-y_1 & y_3+y_2 & y_3+y_2 \\ y_2-y_0 & y_3+y_1 & y_0-y_2 & y_3+y_1 \\ y_3+y_0 & y_2-y_1 & y_1-y_2 & y_3+y_0 \end{vmatrix}.$$

Let $\mu_j=(-1)^j\det Z_j$, where Z_j is the minor of (B_{ij}) obtained by deleting the jth column: by symmetry considerations, as before, we find that $\mu_0(y_0,y_1,y_2)=\mu_1(y_1,y_0,y_2)$, $\mu_1(y_0,y_1,y_2)=\mu_2(y_0,y_2,y_1)$, μ_3 is a symmetric function of (y) (every permutation of y_i with y_j has the effect of exchanging two rows and two columns of Z_3) and therefore, to prove that $(w)=(y_0,y_1,y_2,-y_3)$, we must show that $\mu_2=-\mu_3$. Now, subtracting the 3rd from the 1st and 2nd columns, then multiplying these last by (-1):

$$\det Z_2=\tfrac{1}{2}\det\begin{vmatrix} y_3+y_2+y_0-y_1 & y_3+y_2+y_1-y_0 & 2y_3+2y_2 \\ y_3+y_1+y_0-y_2 & 0 & 2y_3+2y_1 \\ 0 & y_3+y_1+y_0-y_2 & 2y_3+2y_0 \end{vmatrix}.$$

Subtract the first 2 columns from the 3rd:

$$=\tfrac{1}{2}\det\begin{vmatrix} (y_3+y_2+y_0-y_1) & (y_3+y_2+y_1-y_0) & 0 \\ (y_3+y_1+y_0-y_2) & 0 & (y_3+y_1+y_2-y_0) \\ 0 & (y_3+y_1+y_0-y_2) & (y_3+y_2+y_0-y_1) \end{vmatrix}.$$

Then, cyclically permuting the columns:

$$= \tfrac{1}{2}\det \begin{vmatrix} (y_3+y_2+y_1-y_0) & 0 & (y_3+y_2+y_0-y_1) \\ 0 & (y_3+y_2+y_1-y_0) & (y_3+y_1+y_0-y_2) \\ (y_3+y_0+y_1-y_2) & (y_3+y_0+y_2-y_1) & 0 \end{vmatrix},$$

and adding the 1st and subtracting the 2nd to the 3rd column:

$$= \det \begin{vmatrix} (y_3+y_2+y_1-y_0) & 0 & y_3+y_2 \\ 0 & (y_3+y_2+y_1-y_0) & y_0-y_2 \\ (y_3+y_1+y_0-y_2) & (y_3+y_2+y_0-y_1) & y_1-y_2 \end{vmatrix}.$$

After subtracting the 3rd column from the 1st, and then the 1st from the 2nd, we finally get $\det Z_3$.

Corollary 4. *If (y) is a general point of the irreducible quartic curve Γ of equation $(y_1+y_2)^2(y_0+y_2)^2+(y_0+y_1)^2(y_1+y_2)^2+(y_0+y_1)^2(y_0+y_2)^2=0$, then the plane $\varphi(\psi(y))$ is simply tangent to D at a single point.*

Proof. By Proposition 3, $\varphi(\psi(y))$ is such that $\sigma=0$, $\eta_1=2(y_0+y_1)$, $\eta_2=2(y_0+y_2)$, $\eta_3=-2(y_1+y_2)$, the result is thus gotten by substituting in the equation of D^*.

2. Sextic surfaces with nodes

Let u_i, $i=0,\dots,3$, be 4 general linear forms on $\mathbb{P}^3$, and denote by T_u the tetrahedron determined by the four planes $u_i=0$. Consider the map $\Omega:\mathbb{P}^3_v\to\mathbb{P}^3_u$ given by $u_i=v_i^2$: Ω is ramified simply on $T_v=\Omega^{-1}(T_u)$ and has degree 8. The degree of Ω reduces to 4 on the faces, to 2 on the edges, and to 1 on the vertices of T_u. By writing Ω in terms of local coordinates, one easily gets the following result (see [8]):

Proposition 5. *Let $F(u)$ be a surface of degree n, then $G(v)=F(\Omega(v))$ is a surface of degree $2n$ which has only nodes as singularities iff*

(a) *F has only nodes as singularities, and they lie outside T_u,*

(b) *if a face of T_u is tangent to F, it must be simply tangent, and the points of tangency must not lie on the edges,*

(c) *if an edge of T_u is tangent to F, it must be simply tangent in points which are not vertices.*

Moreover, if t is the number of nodes of F, r is the number of tangency points of the faces, s of the edges and m the number of vertices lying on F, then G has exactly $d=8t+4r+2s+m$ nodes.

Theorem 6. *A sextic surface of the form $F(\Omega(v))$ with only nodes as singularities can have at most 64 nodes.*

Proof. Using the notations of Proposition 5, we remark that:

(1) By (c), the 2 vertices of an edge of T_u tangent to F cannot both belong to F. Hence $2s+m$ is always ≤ 13, equality holding iff $s=6$, $m=1$.

(2) By (b), an edge tangent to F cannot lie in a tritangent face, so, if there are k tritangent faces, then $s\leq 3$ for $k=1$, $s\leq 1$ for $k=2$, $s=0$ for $k\geq 3$.

We refer to ([6, pp. 640–646]) for the plane representation of the cubic surfaces with nodal singularities, from which it follows that:

(3) If $t=3$ (as for $t=4$) there are only 3 lines on F not passing through the nodes and they lie on a plane (thus if a face of T_u is tritangent to F, the remaining ones are not bitangent).

(4) If $t=2$ any 2 tritangent planes (not passing through the nodes) meet in a line of F.

Moreover, we have, when $t=4$:

(5) If a face is bitangent to $F=D$, another one is tangent, then their common edge l cannot be tangent to D. For otherwise l^* would join two points of D^*, one of them lying on a double line, so l^*, being tangent to D^*, would have to be tangent at one of these points. This, again by duality, would contradict (b).

(6) If an edge l is tangent to D and belongs to 2 faces π_1, π_2, also simply tangent to D, then l cannot have a vertex v on $\tilde{\pi}\cap D$. In fact, dually, v^* would be a plane containing a double line, so $v^*\cdot D^*=2$ (double line) + conic, $l^*\cdot D^*=(\pi_1)^*+(\pi_2)^*+$ point of tangency: hence l^* would be tangent at a point of the double line. By duality, l would be tangent to D in a point whose tangent plane contains a side of $\tilde{\pi}\cap D$, hence in a point of $\tilde{\pi}\cap D$, i.e. v, contradicting (c).

Now, if $t=0,1$, it follows easily by (2) that d is ≤ 60. If $t=2$, then (4) implies $r\leq 9$ and if $r=9$, $s\leq 3$ by (2), so $d\leq 61$.

When $t=4$, (2) and (5) imply $s\leq 4$ if $r=5$, $s\leq 3$ if $r=6$, $s=0$ if $r=7$. Hence, for $r\leq 5$ d is at most 62, for $r=7$ d is at most 64 (our example), while, for $r=6$, if a face is tritangent we get, by (6), $2s+m\leq 7$ (Segre's example), in the other case we get, using (1), $2s+m\leq 8$.

Finally, if $t=3$, we use (3) to get $r\leq 7$ and (1) to conclude that d is less than or equal to 64 unless $m=1$, $s=6$, $r=7$.

Let's show that this last case cannot occur. In fact, by (2), there should be 3 bitangent faces and a simply tangent one; so, if P is the vertex of the tetrahedron lying on F, there must be at least two bitangent faces, π_1 and π_2, through P. Let P' be the point of tangency of the line $l=\pi_1\cap\pi_2$ with F; since $\pi_i\cap F$ consists of a conic C_i plus a line l_i, l is tangent to C_i at P' and l_i passes through P. From this it follows that P must be a vertex of the unique tritangent plane, since, by (a), the lines l_i cannot pass through any node. Let Φ be the projection of F with centre P onto the plane π spanned by the three nodes of F, and let A, B be the images of the lines l_1, l_2. It is easy to see that, by blowing up P and then contracting the proper transforms of l_1, l_2, we get a double cover F' of π branched on a quartic curve Γ with 5 double points: A, B and the three nodes of F.

Thus Γ consists of two lines and a conic Q and we can choose coordinates (x, y, z) in π such that

(i) $A=(1,0,0)$, $B=(0,1,0)$,

(ii) the equation of Γ is $xy(xy+tz(x+y+z))=0$, where t is a nonzero constant.

The equation of F' in $O_{\mathbb{P}^2}(2)$ is given by:

$$w^2-xy(xy+tz(x+y+z))=0.$$

Therefore, if (x, y, z, u) are coordinates in $\mathbb{P}^3$ such that $P=(0,0,0,1)$, $\pi=\{u=0\}$, x, y, z are the above mentioned coordinates on π, the birational map between F' and F is given by

$$(x, y, z, w) \mapsto (x, y, z, (w-xy)/z),$$

and the equation of F is therefore

$$u(uz+2xy)-txy(x+y+z)=0.$$

Let E, C, D be, respectively, the projections on π of l and of other two edges of the tetrahedron passing through the vertex P. By our assumptions, we have that E, D, C belong to Γ, the line $\overline{EC}$ passes through B, $\overline{ED}$ passes through A, and $\overline{DC}$ is tangent to Γ. Therefore E belongs to Q, C is the intersection of $\overline{BE}$ with $\{y=0\}$, D is the intersection of $\overline{AE}$ with $\{x=0\}$.

Let $E=(a,b,c)$: since E is different from A and B, we can assume $c=1$ and we have $ab+t(a+b+1)=0$. Then $C=(1,0,-a)$, $D=(0,1,-b)$.

Let π_3 be the face of the tetrahedron opposite to P: π_3 is bitangent to F, so it contains the line of $F\{z=0, 2u-t(x+y)=0\}$ mapped by Φ onto $\overline{AB}$. The equation of π_3 is then of the form

$$vz+(2u-tx-ty)=0$$

where v is a constant.

Since the three edges of the tetrahedron lying on π_3 are tangent to F, the projection on π of $\pi_3 \cap F$ must be tangent to the triangle DEC. This projection is easily seen to be the curve Δ of equation

$$z((tx+ty-vz)^2-4(v+t)xy)=0.$$

The parametric equations of the three lines $\overline{AD}=\overline{EA}$, $\overline{CB}=\overline{CE}$, $\overline{CD}$ are given by, respectively,

$$(\sigma, \tau, -\tau b), \qquad (\tau, \sigma, -\tau a), \qquad (\sigma, \tau, -\sigma a-\tau b),$$

where (σ, τ) are homogeneous coordinates in $\mathbb{P}^1$.

Imposing the condition of tangency of the first two lines with Δ we get the equations

$$(v+t)(t^2+vtb-v-t)=0, \qquad (v+t)(t^2+vta-v-t)=0.$$

Since π_3 is not tritangent, Δ consists of the line $\{z=0\}$ plus a nonsingular conic, therefore $(v+t)$ and v are different from zero. It follows that $a=b$, and the condition that $\overline{CD}$ be tangent to Δ boils down to

$$t^2+v^2a^2+2tva-v-t=0.$$

Using this and the previous equation

$$t^2+vta-v-t=0, \tag{$*$}$$

we get

$$va(t+va)=0.$$

Since $a=0$ implies $C=A$, $D=B$, which is absurd, we must have $t+va=0$, and $(*)$ gives again $v+t=0$, which is a contradiction. This ends the proof.

Theorem 7. *For every d with $1 \leq d \leq 64$ there exist sextic surfaces of the form $F(\Omega(v))$ with exactly d nodes as singularities.*

Proof. We will consider the smooth manifold M of the (u)'s such that the faces of T_u do not pass through the nodes of F and the edges are not tangent to F. We remark that if $F=D$, by Proposition 1, for (u) in M T_u satisfies the conditions of Proposition 5; otherwise we must impose the conditions that the faces are simply tangent.

Let us consider first $d \geq 32$ and $F=D$. Our strategy will consist in constructing a lattice of subvarieties of M ordered by proper inclusion, and with a minimum element N: these subvarieties are then not empty if we prove that N is not empty. To start with, we define N as the set of tetrahedra in M with vertices p, $A_0(p)$, $A_1(p)$, $A_2(p)$, where p is a point of $\psi(\Gamma)$, and show that N is not empty. We need the following lemma:

Lemma 8. *For a general $p \in \psi(\Gamma)$ no one of the $A_i(p)$'s is the point p' of tangency of the plane $\phi(p)$ with D.*

Proof. Supposing the contrary, the lines $\langle p,p' \rangle$ would intersect the plane $\tilde{\pi}$ in a vertex of the triangle $D \cap \tilde{\pi}$, this vertex being fixed since Γ is irreducible: let it be p_{ij}. The natural action of $\mathscr{S}_3$ on $\mathbb{P}^2$ induces a linear action on $\mathbb{P}^3$ (permuting the coordinates z_0, z_1, z_2) and Γ is invariant by this action.

Denote by (y') $\psi^{-1}(p')$: (y') is the singular point of the cubic curve of equation $x_3(y_0x_1x_2+x_0y_1x_2+x_0x_1y_2)-x_0x_1x_2y_3$, which is a covariant of (x) and (y). Therefore if $\sigma \in \mathscr{S}_3$, $(\sigma y)'=\sigma(y')$, and the line $\langle \sigma p, (\sigma p)' \rangle$ passes through $\sigma(p_{ij})$. This is a contradiction because $\mathscr{S}_3$ acts transitively on the (p_{jk})'s.

Now, with this choice of the vertices, T_u has 1 tangent and 3 bitangent faces, and the vertices on D. If p is a general point of $\psi(\Gamma)$, it is clear that the faces do not pass through the nodes of D and, since the $\langle p, p_{jk} \rangle$'s have 3 points of intersection with D, they cannot be tangent to D. In the proof of Theorem 6 we have already seen that, if

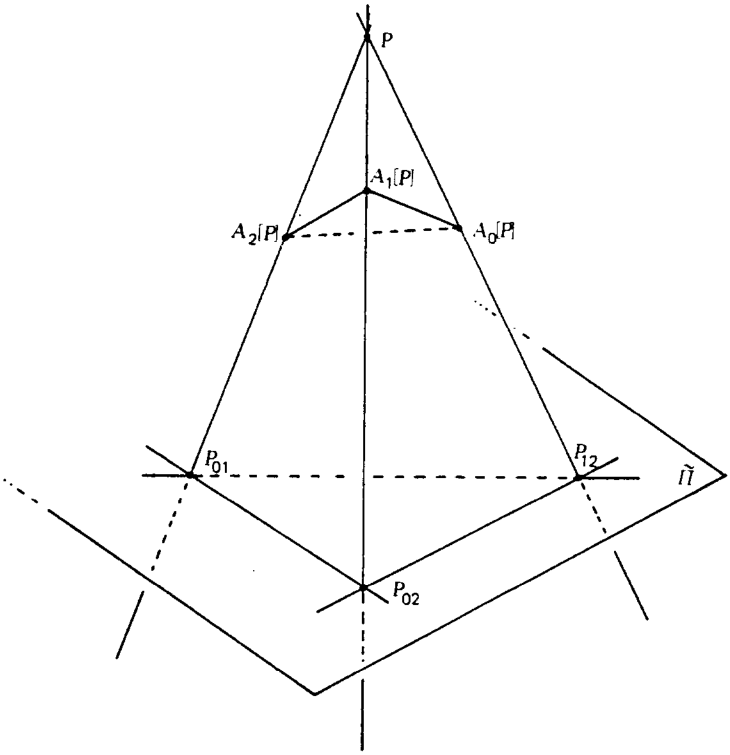

Fig. 3. The u-coordinates tetrahedron and the plane $\tilde{\pi}$.

$\langle A_i, A_j \rangle$ were tangent to D, it would be in the tangency point of the face $\varphi(p)$: since $\langle A_i, A_j \rangle$ has 2 intersections with D at A_i, A_j, one of these would be the above mentioned point, contradicting Lemma 8.

The other subvarieties of the lattice are defined by the following conditions:

$d = 63$: take as vertices $p \in D$, $A_1(p)$, $A_2(p)$, $q = \langle p, p_{12} \rangle \cap \pi'$, where π' is a tangent plane to D through $\langle A_1, A_2 \rangle$.

$d = 62$: $p \in D$, $A_1(p)$, q as before, with $\pi' \in D^* \cap A_1^*$, and $q' = \langle p, p_{10} \rangle \cap \pi'$.

$d = 61$: $p \in D$, q, q', q'' are the 3 intersections of the $\langle p, p_{ij} \rangle$'s with a $\pi' \in D^*$.

$d = 60$: $p \in D$, A_1, A_2, A_0. Proceed as before to get $d = 59, 58, 57$.

$d = 56$: $p \in \mathbb{P}^3$, q, q', q'' are points on the lines $\langle p, p_{ij} \rangle$.

$32 \leq d \leq 55$: write d as $32 + 8h + 4k + m$, where $m \leq 3$, $k \leq 1$; take h bitangent and k tangent faces, and m vertices on D.

To get sextic surfaces with d nodes, $d \leq 31$, write d as $8t + 4k + m$ (k and m as before), and take F with t nodes, T_u with k tangent faces and m vertices on F.

Remark. Of course, there are several ways of applying this construction. For instance, $d = 48$ can also be obtained from D by taking a T_u with 4 general tangent planes as faces, or 2 general bitangent planes, or the tritangent plane and a tangent plane, ... ; or from a smooth cubic and a T_u with 4 tritangent faces.

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References

[1] A.B. Basset, The maximum number of double points on a surface, Nature 73 (1906) 246.

[2] A. Beauville, Sur le nombre maximum de points doubles d'une surface dans P^3 ($\mu(5)=31$), in: Journées de Géométrie Algébrique d'Angers 1979 (Sijthoff & Noordhoff, Leyden, 1980) 207–215.

[3] M. Beltrametti and F. Catanese, Coomologia dei fasci coerenti e teoremi di Lefschetz sulle varietà proiettive, Centro Analisi Globale (Firenze, 1976).

[4] H. Clemens, Double solids, in: Journées de Géométrie Algebrique d'Angers 1979 (Sijthoff & Noordhoff, Leyden, 1980).

[5] F. Conforto, Le Superficie Razionali (Zanichelli, Bologna, 1939).

[6] P.H. Griffiths and J. Harris, Principles of Algebraic Geometry (Wiley Interscience, New York, 1978).

[7] B. Moishezon, Complex Surfaces and Connected Sums of Complex Projective Planes, Lecture Notes in Math., No. 603 (Springer, Berlin, 1977).

[8] B. Segre, Sul massimo numero di nodi delle superficie algebriche, Atti Acc. Ligure, 10 (1) (1952) 15–22.

[9] E. Stagnaro, Sul massimo numero di punti doppi isolati di una superficie algebrica, Rend. Sem. Mat. Univ. di Padova 59 (1978) 179–198.

[10] E.G. Togliatti, Sulle superficie algebriche col massimo numero di punti doppi, Rend. Sem. Mat. Torino 9 (1950) 47–59.