PLURICANONICAL - GORENSTEIN - CURVES

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§ 0. Introduction

Some of the most classical results in algebraic geometry deal with the pluricanonical mappings of a complete smooth curve.

Classically, if X is a smooth curve of genus $p \geq 2$ the sections of $H^o(X, (\Omega_X^1)^{\otimes n})$ give the n^{th} pluricanonical map Φ_n : $X \to \mathbb{P}(H^o(X, (\Omega_X^1)^{\otimes n})^v)$ of X into the projective space associated to the dual of $H^o(X, (\Omega_X^1)^{\otimes n})$.

It is well-known that this map is an embedding if

i) $n \geq 3$

ii) $n = 2$ and $p \geq 3$

iii) $n = 1$, $p \geq 3$ and X is not hyperelliptic.

Indeed, iii) characterizes hyperelliptic curves because if X is hyperelliptic then the canonical map Φ_1 yields a double cover of a rational normal curve. The need for extending these results to singular and reducible curves appears if one studies families of smooth curves and the possible degenerations of the generic fibre.

The first result in this direction, namely the extension of i) above for certain curves with nodes, was proved by Deligne and Mumford ([4]) in their work on the irreducibility of the moduli space of curves of genus p. F. Sakai ([10]) encountered similar problems in his study of open surfaces and his work shows the usefulness of having results of this kind for reduced curves lying on a smooth surface.

P. Le Barz et al. (eds.), *Enumerative Geometry and Classical Algebraic Geometry*

The object of this paper is to investigate in more detail this problem, with a greater generality which we hope suffices for most applications.

The correct analogue of pluricanonical mappings in the case of reducible curves with singularities is obtained replacing the sheaf Ω^1_X of 1-forms by the dualizing sheaf ω_X ; in order that sections of this sheaf, or tensor powers of it, define a map to some projective space, one has to assume that ω_X be an invertible sheaf (cf. [8], lecture 5). Notice that ω_X is invertible if and only if X is Gorenstein. We call the mapping associated to the linear system $|\omega_X{}^{\otimes n}|$ the n^{th} pluricanonical mapping of X.

In order for these mappings to be well defined on X one should not have components of X along which all sections of $\omega_X{}^{\otimes n}$ vanish. This leads naturally to the following definition.

<u>Definition 0.1</u>: X is said to be semi-canonically positive (S.C.P. for short) if and only if, for each component Y of X, the degree of $\omega_X|_Y = \omega_X \otimes \mathcal{O}_Y$ is non negative. If, for each Y, this degree is positive, X is said to be canonically positive (C.P.).
It is clear that if some pluricanonical mapping is an embedding, then X must be C.P.

Now we can state our simplest results.

<u>Theorem A</u>: If X is S.C.P $|\omega_X{}^{\otimes n}|$ is base point free for each $n \geq 2$.

<u>Theorem B</u>: If X is C.P. $|\omega_X{}^{\otimes n}|$ gives an embedding of X for each $n \geq 3$.

We shall also study in more detail the structure of the maps associated with ω_X and $\omega_X{}^{\otimes 2}$ and indeed the greater part of this paper will be devoted to prove the analogue of ii), and of iii) above under suitable conditions of connectedness. We shall also show that our conditions of connectedness are close to being necessary and sufficient for the validity of our statements and we shall produce several explicit examples.

This paper is organized as follows: In § 1 we recall known basic facts about Gorenstein curves, we show how to obtain a S.C.P. Gorenstein curve out of an arbitrary Gorenstein curve by destroying some components that we call negative tails, and we describe S.C.P. curves of genus one. In § 2 we discuss the behavior of $|\omega_X^{\otimes n}|$, for $n \geq 2$, using Riemann Roch duality and some explicit interpretations of first cohomology groups: we prove the above Theorems A and B, and also (Theorem C) describe when $|\omega_X^{\otimes 2}|$ does not give an embedding. § 3 is devoted to the study of the canonical map $|\omega_X|$ and, in particular, we describe explicitly the "hyperelliptic curves" (the ones for which the canonical map is not birational). Finally, in § 4 we show by means of an example that even the simplest Theorems A and B do not carry over to the non-reduced case without additional hypotheses.

Our notation is as follows:

k is an algebraically closed field over which all the varieties in question are defined.

If V is a k-vector space, $V^{\vee}$ is its dual.

If X is a projective scheme, with structure sheaf $\mathcal{O}_X$, ω_X is the dualizing sheaf of X (see [7] p. 242); moreover, if $\mathcal{F}$ is a coherent sheaf on X, we denote by $\mathcal{F}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, by $h^i(\mathcal{F})$ the dimension of $H^i(X, \mathcal{F})$ as a k-vector space, by

$$\chi(\mathcal{F}) = \Sigma(-1)^i h^i(\mathcal{F}).$$

If X is a reduced curve, $X = Y \cup Z$, with $\dim(Y \cap Z) = 0$, we will denote Z by X - Y. Also, $Y \cdot Z$ is defined to be equal to the length of $\mathcal{O}_{Y\cap Z}$, and if $x \in X$, $(Y\cdot Z)_x$ is, by definition, the length of $\mathcal{O}_{Y\cap Z,x}$.

If Y is a subscheme of X, $\mathcal{F}$ is coherent on X, $\mathcal{F}|_Y$ stands for $\mathcal{F} \otimes \mathcal{O}_Y$.

If s is a section of $\mathcal{F}$, $s \equiv 0$ means that the stalk of s is 0 at any point of X; $\equiv$ is also used to denote linear equivalence of divisors. Without explicit mention we shall assume all the schemes under consideration to be complete.

R. R. is an abbreviation for the Grothendieck-Serre-Riemann-Roch duality theorem (see [7], [11]) which, in the case of curves, reads out as follows:

$$\mathrm{Hom}\ (\mathcal{F}, \omega_X)^{v} \cong H^1\ (X, \mathcal{F}), \quad \mathrm{Ext}^1\ (\mathcal{F}, \omega_X)^{v} \cong H^0\ (X, \mathcal{F}).$$

The arithmetic genus p (X) of a curve X is, by definition, equal to $1 - \chi\ (\mathcal{O}_X)$, and then $\deg\ (\omega_X) = 2 - p\ (X)$. If X is Gorenstein $\Phi_n : X \to \mathbb{P}(H^0\ (X, \omega_X^{\otimes n})^{v})$ is the n^{th} - (pluri) canonical map of X.

§ 1. Gorenstein Curves

Lemma 1.1: Let W be a projective variety (possibly non connected), F an invertible sheaf on W, $\mathcal{G}$ a torsion free sheaf on W such that for each invertible sheaf L on W Hom (L, F) $\cong$ Hom $(L, \mathcal{G})$: then $F \cong \mathcal{G}$.

Proof: Assume W to be connected. Then Hom (F, F) = k hence there is a non trivial homomorphism $a : F \to \mathcal{G}$. a must be injective since $\mathcal{G}$ is torsion free, hence, if K = coker a, we have, upon tensoring with $\mathcal{O}_W$ (n), the following exact sequence

$$0 \to F\ (n) \to \mathcal{G}\ (n) \to K\ (n) \to 0$$

For n large enough $H^i\ (W, F\ (n)) = 0$ for $i \geq 1$, and

$H^o(\mathcal{G}(n)) \cong H^o(F(n))$ by our hypothesis, since e.g. $H^o(F(n)) = \text{Hom}(\mathcal{O}_W(-n), F)$. Then $H^o(K(n)) = 0$ for all n large enough, hence $K = 0$ and $F \cong \mathcal{G}$. If W is not connected, it suffices to show that if Y is a connected component of W, then our hypothesis holds true for Y, $F|_Y, \mathcal{G}|_Y$. But, for any invertible sheaf L' on Y, consider L on W to be equal to L' on Y, and to $\mathcal{O}_W(n)$ on $W-Y$: then for n large enough $\text{Hom}(L', F|_Y) \cong \text{Hom}(L,F)$, and the same is true for $\mathcal{G}$.

Q.E.D.

<u>Proposition 1.2</u> (<u>Noether's Formula</u>): Let $\pi : Y \to X$ be a finite birational morphism of Gorenstein curves. Then $\omega_Y \cong \pi^*(\omega_X) \otimes \tilde{C}$, where $\tilde{C}$ is the conductor of π viewed as an ideal sheaf on Y.

<u>Proof</u>: By the previous lemma, it suffices to prove that, for every invertible sheaf L on Y,

$$\text{Hom}(L, \omega_Y) \cong \text{Hom}(L, \pi^*(\omega_X) \otimes \tilde{C}).$$

Taking the dual vector spaces, the left hand side is $H^1(Y,L)$, while the right hand side is

$$H^o(Y, \mathcal{H}om_{\mathcal{O}_Y}(L, \pi^*(\omega_X) \otimes \tilde{C}))^{\vee}.$$

The map being finite, $H^1(Y,L) = H^1(X, \pi_* L) = \text{Hom}(\pi_* L, \omega_X)^{\vee}$, therefore it is enough to show that

$$\pi_*(\mathcal{H}om_{\mathcal{O}_Y}(L, \pi^*(\omega_X) \otimes \tilde{C})) \cong \mathcal{H}om_{\mathcal{O}_X}(\pi_* L, \omega_X).$$

This equality indeed is of local nature and follows from the fact that, at the finite set of points x where π is not an isomorphism, the conductor ideal $C = \pi_* \tilde{C}$ is equal to $\mathcal{H}om_{\mathcal{O}_X}(\pi_* \mathcal{O}_Y, \mathcal{O}_X)$, by its very definition.

Q.E.D.

Assume now that X is Gorenstein and reduced, and that $Y = \tilde{X}$ is the normalization of X.

We have then the standard exact sequence

$$(1.3) \qquad 0 \to \mathcal{O}_X \to \pi_* \mathcal{O}_{\tilde{X}} \to \Delta \to 0$$

where $\Delta = \bigoplus_{x \text{ singular}} \Delta_x$ and one usually denotes by δ_x the length of Δ_x, by δ the one of Δ $(\delta = \Sigma_x \delta_x)$.

Applying the functor $\mathcal{H}om_{\mathcal{O}_X}(\ ,\mathcal{O}_X)$ one obtains the dual exact sequence

$$(1.3') \qquad 0 \to C \to \mathcal{O}_X \to \mathcal{E}xt^1_{\mathcal{O}_X}(\Delta,\mathcal{O}_X) \to$$

$$\to \mathcal{E}xt^1_{\mathcal{O}_X}(\pi_*\mathcal{O}_{\tilde{X}},\mathcal{O}_X) \to 0$$

Let $M = \mathcal{O}_{X/C}$, $m_x = \text{length } (M_x)$.

<u>Lemma 1.4</u>: $C^* = \pi_* \mathcal{O}_{\tilde{X}}$, hence $\Delta \cong \mathcal{E}xt^1(M,\mathcal{O}_X)$.

<u>Proof</u>: For any ideal $\mathcal{J} \subset \mathcal{O}_x$ of finite colength, $\mathcal{J}$ contains a non zero divisor f. Then $\mathcal{J}^* \subset K_x$, where K_x is the full ring of fractions of $\mathcal{O}_x$: in fact if $\psi \in \mathcal{J}^*$, $h \in \mathcal{J}$, $\psi(h)\, f = \psi(hf) = h\, \psi(f)$, and thus $\psi(h) = h\ (\psi(f)/_f)$ and we can write $\psi = (\psi(f)/_f) \in K_x$.

Consider now $\psi \in C^*$: since C is an ideal in $\pi_*\mathcal{O}_{\tilde{X}}$, $\forall g \in \pi_*\mathcal{O}_{\tilde{X}}$, $\forall f \in C$, we have $\psi g f \in \mathcal{O}_X$, then $\psi f \in C$.

So $\psi C \subset C$, and ψ is regular on $\tilde{X}$. The last statement follows by taking the dual of the exact sequence

$$0 \to C \to \mathcal{O}_X \to M \to 0$$

Q.E.D.

<u>Theorem 1.5</u>: ω_X is invertible at x if and only if the following equivalent conditions are satisfied:

a) $\mathcal{E}xt^1(\pi_* \mathcal{O}_{\tilde{X}}, \mathcal{O}_X) = 0$

b) $\delta_x = m_x$

c) for each coherent sheaf F with supp (F) = x,

length $(\mathcal{E}xt^1(F, \mathcal{O}_X))$ = length (F).

Morever, in general, if x is singular $1 \leq m_x \leq \delta_x$.

Proof: We defer the reader to Serre's book ([11]pp. 76-80) for the proof of the more difficult parts, a, b $\Longrightarrow \omega_X$ invertible, $m_x \leq \delta_x$. We shall prove instead that ω_X invertible $\Longrightarrow$ c $\Longrightarrow$ a,b.

In fact, if ω_X is invertible at x, then $\mathcal{E}xt^1(F, \mathcal{O}_X) \cong \mathcal{E}xt^1(F, \omega_X)$, hence its length is $h^0(\mathcal{E}xt^1(F, \omega_X)) = \dim(\mathrm{Ext}^1(F, \omega_X)) = h^0(F)$ by R.R.

If c) holds, by virtue of the exact sequence

$$0 \to M \to \mathcal{E}xt^1(\Delta, \mathcal{O}_X) \to \mathcal{E}xt^1(\pi_* \mathcal{O}_{\tilde{X}}, \mathcal{O}_X) \to 0$$

and of lemma 1.4., one obtains

$$\delta_x = m_x \quad \text{and} \quad \mathcal{E}xt^1(\pi_* \mathcal{O}_{\tilde{X}}, \mathcal{O}_X) = 0$$

Q.E.D.

Let $\tilde{X}_1, \ldots \tilde{X}_k$ be the connected (irreducible) components of $\tilde{X}$; then the long exact cohomology sequence associated to (1.3) gives

$$(1.6) \qquad (k-1) \leq \delta \quad , \quad p(X) = \Sigma_{h=1}^{k} p(\tilde{X}_h) + (\delta-k+1).$$

Actually, if X' is the disjoint union of the irreducible components X_i of X, one has an exact sequence analogous to (1.3)

$$(1.7) \qquad 0 \to \mathcal{O}_X \to \pi'_* \quad \mathcal{O}_{X'} \overset{\bigoplus_{i=1}^{k} \mathcal{O}_{X_i}}{\to} \Delta' \to 0$$

One associates to X a graph $|X|$ in the following way: take a segment $|X_i|$ for each component X_i, and mark a point $|y_i|$ in $|X_i|$ for each singular point y of X belonging to X_i; then, if

X_i and X_j meet at y, identify $|y_i|$ with $|y_j|$.

<u>Proposition 1.8</u>: Let X be a reduced connected curve; then $p(X) = 0$ if and only if

a) every component X_i is isomorphic to $\mathbb{P}^1$,

b) the singularities of X are given by r smooth branches with independent tangents,

c) the associated graph $|X|$ is contractible.

Moreover, if $p(X) = 0$, X is Gorenstein iff it has only nodes as singularities.

<u>Proof</u>: By (1.7) $p(X) = 0 \Longrightarrow p(X_i) = 0$. If X is irreducible, by (1.6), $p(X) = 0$ implies $p(\tilde{X}) = 0$, $\delta = 0$, hence X is smooth and $\cong \mathbb{P}^1$. One can assume clearly that X_1 is such that $Y = X - X_1$ is connected. Let Z be the disjoint union of Y and X_1, and consider the obvious morphism $p : Z \to X$. Again one has the exact sequence

$$0 \to \mathcal{O}_X \to p_* \mathcal{O}_Z \to \Delta'' \to 0,$$

therefore Δ'' has length 1, hence, first of all, Y and X_1 intersect in a single point y. Then $\mathcal{O}_{X,y}$ is a subring of $\mathcal{O}_{X_1,y} \oplus \mathcal{O}_{Y,y}$ contained in the subring $R = \{(f,g) \mid f(y) = g(y)\}$: since however Δ'' has length 1, $R = \mathcal{O}_{X,y}$, hence

$$\dim \mathfrak{m}_{y,Y}/\mathfrak{m}^2_{y,Y} + \dim \mathfrak{m}_{y,X_1}/\mathfrak{m}^2_{y,X_1} = \dim \mathfrak{m}_{y,X}/\mathfrak{m}^2_{y,X}$$

and b),c) are proven by induction on k.

The converse is also easy.

If X is Gorenstein at x, $\delta_x + 1$ components meet transversally at x, but here $C = \mathfrak{m}_x$, so $\delta_x = m_x = 1$, and x is thus a node.

Q.E.D.

<u>Definition 1.9</u>: X is said to be m - connected if, for each decomposition $X = Y \cup Z$, with $\dim Y \cap Z = 0$, one has $Y \cdot Z \geq m$. Y and Z are said to meet transversally at $x \in X$ if $(Y \cdot Z)_x = \dim \mathcal{O}_{Y \cap X, x} = 1$.

Recall now that we are assuming X to be connected, hence always 1-connected : if X is not 2-connected, then one can write $X = Y \cup Z$ with Y and Z intersecting transversally at a single point x.

<u>Proposition 1.10</u>: If $X = Y \cup Z$ and Y and Z meet transversally at x, then x is a node for X if ω_X is invertible at x.

<u>Proof</u>: The question is local, but, taking a normalization of X at the other points of intersection of Y and Z, and at the points where ω_X is not invertible, we can assume that X be Gorenstein and that $Y \cap Z = \{x\}$. Let $\pi : \tilde{X} \to X$ be the normalization of X at x, $\tilde{Z} = \pi^{-1}(Z)$, $\tilde{Y} = \pi^{-1}(Y)$.

We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_X & \to & \mathcal{O}_Z & \oplus & \mathcal{O}_Y & \to \mathcal{O}_{Z \cap Y} & \to 0 \\ & & \downarrow & & & \downarrow & & & \\ 0 & \to & \pi_* \mathcal{O}_{\tilde{X}} & \to & \pi_* \mathcal{O}_{\tilde{Z}} & \oplus & \pi_* \mathcal{O}_{\tilde{Y}} & \to 0 & \end{array}$$

Therefore, if $\Delta_X = \pi_* \mathcal{O}_{\tilde{X}}/\mathcal{O}_X$, $\delta_X = h^0(\Delta_X)$, and δ_Z, δ_Y are defined in the same way, $\delta_X = 1 + \delta_Z + \delta_Y$. Let C_X, C_Y, C_Z be the conductor ideals of the morphisms π, $\pi|_{\tilde{Y}}$, $\pi|_{\tilde{Z}}$, and let m_X, m_Y, m_Z be defined accordingly.

If Y, Z are smooth at x our claim is proven, otherwise, since $\mathfrak{m}_{x,X} = \mathfrak{m}_{x,Y} + \mathfrak{m}_{x,Z}$, and $C_X \subset \mathfrak{m}_{x,X}$, if they are both singular $C_X = C_Y \oplus C_Z$, if Y is smooth $C_X = C_Z \oplus \mathfrak{m}_{x,Y}$; in the first case $m_X = m_Y + m_Z - 1$, in the second $m_X = m_Z$.

In either case we have a contradiction, since $\delta_X = m_X, m_Z \leq \delta_Z$, $m_Y \leq \delta_Y$.

Q.E.D.

We are now going to show how a Gorenstein curve can fail to be S.C.P. (cf 0.1).

<u>Proposition 1.11</u>: Let $Y \subset X$ be a connected union of components X_i of X such that $\deg \omega_X|_{X_i} \leq 0$. Then $p(Y) = 0$ unless $Y = X$, $\deg \omega_X|_{X_i} = 0$ for each component X_i. In this last case $p(X) = 1$ and $\omega_X \cong \mathcal{O}_X$. Conversely, if X is S.C.P. and $p(X) = 1$, ω_X is trivial.

<u>Proof</u>: Y being connected, $p(Y) = h^1(Y, \mathcal{O}_Y) =$ (by R.R.) $=$ dim $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_Y, \omega_X) = h^0(X, \mathcal{J}_{X-Y}\,\omega_X)$ where $\mathcal{J}_{X-Y}$ is the ideal sheaf of $X-Y$: in fact, by the exact sequence

$$0 \to \mathcal{J}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) \to \mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}_Y, \mathcal{O}_X)$$

thus $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) = \{f \in \mathcal{O}_X \mid f\cdot g = 0 \ \forall g \in \mathcal{J}_Y\} = \mathcal{J}_{X-Y}$.

Assume now that $X-Y \neq \emptyset$: then every section s of $\mathcal{J}_{X-Y}\,\omega_X$ is identically zero on $X-Y$, so vanishes at some point of Y; but then, by the assumption made on $\deg \omega_X|_{X_i}$ for $X_i \subset Y$, s is identically zero. The same clearly holds if $\exists\, X_i \subset Y$ s.t. $\deg \omega_X|_{X_i} < 0$. Assume then $X = Y$; $p(X) = 0$ unless $\deg \omega_X|_{X_i} = 0$ for each i, and conversely if $p(X) = 0$, by prop. 1.8, X is not S.C.P. But then $p(X) = h^0(\omega_X) \geq 1$, so there exists a non zero section $s : \mathcal{O}_X \to \omega_X$. Since $\deg \omega_X = 2\,p(X) - 2$, if $\deg \omega_X|_{X_i} = 0$ for each i, s gives an isomorphism of ω_X with the trivial sheaf, and if $p(X) = 1$, X is S.C.P., then $\deg \omega_X|_{X_i} = 0$ for each i.

Q.E.D.

<u>Lemma 1.12</u>: Let $Z \subset X$ be both Gorenstein. Then the ideal $\mathcal{I}_{X-Z} \otimes \mathcal{O}_Z$ is invertible, $\omega_X|_Z = \omega_Z \otimes (\mathcal{I}_{X-Z} \otimes \mathcal{O}_Z)^{-1}$, and in particular $\deg \omega_{X|Z} = \deg \omega_Z + Z \cdot (X-Z)$

<u>Proof</u>: Let W be a normalization of $X-Z$ at the points where ω_{X-Z} is not invertible, let Y be the disjoint union of W and Z, $\pi : Y \to X$ the natural map. By proposition 1.2 we get that $\omega_Z = \pi^* \omega_X \otimes \tilde{C} = \omega_X \otimes C \otimes \mathcal{O}_Z$, and that $C \otimes \mathcal{O}_Z$ is invertible.

We have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \Delta_{X-Z} & & & & \\
 & & & & \uparrow & & & & \\
0 & \to & \mathcal{O}_X & \to & \mathcal{O}_Z \oplus \pi_* \mathcal{O}_W & \to & \Delta_X & \to & 0 \\
 & & \uparrow \wr & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{O}_X & \to & \mathcal{O}_Z \oplus \mathcal{O}_{X-Z} & \to & \mathcal{O}_{Z \cap (X-Z)} & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and, by dualizing (taking $\mathcal{H}om_{\mathcal{O}_X}(\ , \mathcal{O}_X)$) we obtain

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
0 & \to & C = \mathcal{I}_{X-Z} & \oplus & C_W & \to & \mathcal{O}_X \\
 & & & & \downarrow & & \downarrow \wr \\
0 & \to & \mathcal{I}_{X-Z} & \oplus & \mathcal{I}_Z & \to & \mathcal{O}_X
\end{array} .$$

Therefore $C \otimes \mathcal{O}_Z = \mathcal{I}_{(X-Z)} \otimes \mathcal{O}_Z$: finally we have an exact sequence

$$0 \to \mathcal{I}_{X-Z} \otimes \mathcal{O}_Z \to \mathcal{O}_Z \to \mathcal{O}_{X-Z} \otimes \mathcal{O}_Z \to 0$$

$$\mathcal{O}_{X-Z} \otimes \mathcal{O}_Z = \mathcal{O}_{(X-Z) \cap Z}$$

and the last assertion follows then.

Q.E.D.

<u>Proposition 1.13</u>: Let X be Gorenstein, $\pi : \tilde{X} \to X$ the normalization of a singular point x, Z an irreducible component of X containing x, $\tilde{Z} = \pi^{-1}(Z)$, $\delta_Z = h^0(\pi_* \mathcal{O}_{\tilde{Z}}/\mathcal{O}_Z)$. Then $\deg(\tilde{C}^{-1}|_{\tilde{Z}}) =$
$= 2\delta_Z + Z(X-Z)_x$.

<u>Proof</u>: We can clearly assume, as in 1.10, that $Z \cap (X-Z) = x$. We have then the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & 0 \to & \mathcal{O}_X & \to & \mathcal{O}_Z \oplus \mathcal{O}_{X-Z} & \to & \mathcal{O}_{Z\cap(X-Z)} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \pi_*\mathcal{O}_{\tilde{X}} & \cong & \pi_*\mathcal{O}_{\tilde{Z}} \oplus \pi_*\mathcal{O}_{(\tilde{X}-\tilde{Z})} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \to \mathcal{O}_{Z\cap(X-Z)} & \to & \Delta_X & \to & \Delta_Z \oplus \Delta_{X-Z} & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Taking the dual sequences we obtain

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & \pi_*(\tilde{C}|_{\tilde{Z}}) \oplus \pi_*(\tilde{C}|_{\tilde{X}-\tilde{Z}}) & \cong & C & & & \\
 & \downarrow & & \downarrow & & & \\
0 \to & \mathcal{I}_{X-Z} \oplus \mathcal{I}_Z & \to & \mathcal{O}_X & \to & \mathcal{E}xt^1(\mathcal{O}_{Z\cap(X-Z)}, \mathcal{O}_X) & \to 0 \\
 & \downarrow & & \downarrow & \nearrow & & \\
0 \to & \mathcal{E}xt^1(\Delta_Z, \mathcal{O}_X) \oplus \mathcal{E}xt^1(\Delta_{X-Z}, \mathcal{O}_X) & \to & \mathcal{E}xt^1(\Delta_X, \mathcal{O}_X) & & & \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array}$$

therefore length $(\mathcal{O}_{\tilde{Z}}/_{\tilde{C}|\tilde{Z}})$ = length $(\pi_* \mathcal{O}_{\tilde{Z}}/\mathcal{O}_Z)$ + length $(\mathcal{O}_Z/\mathcal{I}_{X-Z})$ +

$+ \text{length} (\mathcal{T}_{X-Z/ \pi_*(\tilde{C}|\tilde{Z})}) = \delta_Z + Z (X-Z)_x +$

$+ \text{length} (\mathcal{E}xt^1 (\Delta_Z, \mathcal{O}_X)) = 2 \delta_Z + Z (X-Z)_x$ by 1.5 c).

Q.E.D.

Remark 1.14: Passing to the completion of the local rings in consideration, and considering branches of Z through x, one obtains an entirely analogous result (with the same proof) for the multiplicity of $\tilde{C}$ at a point q of $\tilde{X}$ s.t. $\pi(q) = x$ (see also [6] for a slightly different proof).

Definition 1.15: A negative tail Y contained in X is a maximal connected curve in X s.t. $\deg \omega_{X|Y} < 0$, and s.t., for each curve $X_i \subset Y$, $\deg \omega_{X|X_i} \leq 0$.

Proposition 1.16: A negative tail Y is Gorenstein with $p(Y) = 0$, and Y intersects $X-Y$ transversally at a single point.

Proof: $p(Y) = 0$ by 1.11, and if Y is Gorenstein, by 1.12 $\deg \omega_{X|Y} = -2 + \dim \mathcal{O}_{Y \cap (X-Y)} < 0$, hence $Y \cdot (X-Y) = 1$. Since a smooth $\mathbb{P}^1$ is Gorenstein, and $\deg \omega_{X|X_i} < 0$ for some component $X_i \subset Y$, there exists a maximal connected $Y' \subset Y$ such that Y' is Gorenstein, $\deg \omega_{X|Y'} < 0$: we claim that $Y' = Y$. In fact Y' intersects $X-Y'$ transversally in a point x which belongs to a component W of X : if $W \not\subset Y$, then Y' is a connected component of Y, hence $Y' = Y$, otherwise W is a smooth $\mathbb{P}^1 \subset Y$, and $Y' \cup W$ is Gorenstein $\subset Y$, a contradiction.

Q.E.D.

Remark 1.17: Let X be, as usual, reduced and Gorenstein: then X is S.C.P iff X contains no negative tails. By throwing away the negative tails, one can obtain from any X a new connected Gorenstein curve $X' \subset X$ which is S.C.P. It is clear that any section of ω_X^n, for $n \geq 1$, vanishes identically on the negative tails.

Let $x_1, \dots x_k$ be the nodes where X' intersects $X-X'$: then by 1.12

$$\omega_X^n|_{X'} = \omega_{X'}^n (\Sigma_{i=1}^k n\, x_i) \quad , \text{ and } \quad H^0 (X, \omega_X^n) \cong$$

$$\cong H^0 (X', \omega_{X'}^n ((n-1) \Sigma_{i-1}^k \; x_i)) \quad .$$

Then the rational maps $|\omega_X|$ and $|\omega_{X'}|$ coincide on X', while, for $n \geq 2$, $|\omega_{X'}^n|$ is obtained by $|\omega_X^n|$ followed by a projection.

To end this section, let me describe the S.C.P. Gorenstein curves X with $p(X) = 1$.

Proposition 1.18: A S.C.P. Gorenstein curve X with $p(X) = 1$ belongs to the following classes:

a1) - a5) (X lies on a smooth surface)

a1) X smooth, a2) X is rational with a node

a3) X is rational with an ordinary cusp a4) X consists of 2 $\mathbb{P}^1$'s tangent at a point (X tacnodal)

a5) X has only nodes and is a cycle of $\mathbb{P}^1$'s

b) X consists of k smooth $\mathbb{P}^1$'s meeting in a point x where the tangents to the branches are linearly dependent, but any $(k-1)$ of them are independent.

Proof: By (1.7) $\sum_{i=1}^{k} p(X_i) \leq 1$, if $X_1, \dots X_k$ are the irreducible components of X.

If $p(X_1) = 1$, since, by proposition 1.10, $\omega_X \cong \mathcal{O}_X$, if X_1 is Gorenstein, $\omega_{X_1} \cong \mathcal{O}_{X_1}$ and by 1.12 $X = X_1$. In fact X_1 is Gorenstein by

Lemma 1.19: An irreducible curve Y with $p(Y) = 1$ is Gorenstein and belongs to one of the classes a1), a2), a3).

Proof: Assume Y to be singular, and let $\pi : \tilde{Y} \to Y$ be the normalization. Then $\delta = 1$, hence Y has only one singular point x, and 1.5 implies that $1 \le m_x \le \delta_x = 1$, so Y is Gorenstein.
Then, since $\pi_* \mathcal{O}_{\tilde{Y}/C}$ has dimension 2, $\tilde{C}$ has degree -2; therefore either $\pi^{-1}(x) = p$, or $\pi^{-1}(x) = \{p_1, p_2\}$. In both cases $\mathcal{O}_{x,X} = k \oplus \mathfrak{m}_{x,X} = k \oplus C_{x,X}$ as a subring of $\pi_* \mathcal{O}_{\tilde{Y}}$, therefore in the first case x is an ordinary cusp, in the second x is a node.

Q.E.D.

End of proof of 1.18: Assume then that each X_i is a smooth $\mathbb{P}^1$. Then, by 1.12 $X_i(X-X_i) = 2$. Assume that X_1 intersects $X-X_1$ in 2 points (which are therefore nodes) : then it is easy to see that the same must hold for all X_i's (in fact there exists a maximal $Y \subset X$ such that X has only nodes along Y, and if $Y \neq X$, $\exists W \subset X$, $W \not\subset Y$, s.t. $W \cap Y \neq \emptyset$: then W intersects $X-Y-W$ at a point which is not a node, hence $W(X-W) \geq 3$, a contradiction). It is now obvious that the graph associated to X is a cycle, so that we are in case a5).

Otherwise we have that all the X_i's intersect in a single point x.

Then $\delta_x = m_x = k$: so $\mathfrak{m}_{x,X}$ has codimension 1 in $\mathfrak{m}' = \oplus_{i=1}^{k} \mathfrak{m}_{x,X_i}$ and, by Nakayama's lemma, $\mathfrak{m}^2_{x,X} = \oplus_{i=1}^{k} \mathfrak{m}^2_{x,X_i}$; in particular $C \supset \mathfrak{m}^2_{x,X}$, but then equality holds since $k = \dim \mathcal{O}_{x,X}/C = \dim \mathcal{O}_{x,X}/\mathfrak{m}^2_{x,X}$

Let t_i be a uniformizing parameter for X_i at x, e_i the function which is 1 on X_i and 0 on the other X_j's.
We know that $\mathfrak{m}_{x,X}/\mathfrak{m}^2_{x,X}$ is an hyperplane in $\oplus_{i=1}^{k} \mathfrak{m}_{x,X_i}/\mathfrak{m}^2_{x,X_i}$, so there exist $\alpha_i \in k$ s.t.

$$\mathfrak{m}_{x,X}/\mathfrak{m}^2_{x,X} = \left\{ \sum_{i=1}^{k} a_i t_i \;\middle|\; \sum a_i \alpha_i = 0 \right\} .$$

Clearly, also, $$C/\mathfrak{m}^2_{x,X} = \left\{ \sum_{i=1}^{k} b_i t_i \;\middle|\; b_j \alpha_j = 0 \;\; \forall j \right\}$$

since $f \in C$ iff $f \cdot e_j \in \mathcal{O}_{x,X}$ $\forall j = 1, \ldots k$.

The conclusion is that every α_j is $\neq 0$.

Q.E.D.

Remark 1.20: If X is S.C.P. and has only nodes as singularities, one has (cf. [4]) zero tails, i.e. chains of $\mathbf{P}^1$'s over which ω_X is trivial, and $|\omega_X^n|$ contracts these zero tails to points, so that the image of X is just the image of a C.P. X' obtained by taking off these tails and setting together the 2 "end points" of the tail to build a node.

If Y is a smooth $\mathbf{P}^1$ tangent to $X-Y$ at a smooth point x, one can throw away Y and obtain X' by putting a cusp in x (i.e. if t is a uniformizing parameter at x, one replaces $\mathcal{O}_{x,X-Y}$ by the subring generated by $1, t^2, t^3$). Analogously if Y crosses $X-Y$ in a node, one replaces the node by a tacnode to get X'.

Thus, there is also a natural way to obtain from a S.C.P X a C.P. X', such that $|\omega_X^n|$ has the same image of $|\omega_{X'}^n|$.

We won't however use this construction.

§ 2. The Pluricanonical Maps

To prove the first results (theorems A, B, C) we have to show the vanishing of some first cohomology groups: in turn, using R.R. duality, these are interpreted as certain homomorphisms, over which a rough hold is given by the following lemmas.

Lemma 2.1: Let x be a singular point of X, $\tilde{X}$ the normalization of X at x, $\hat{X}$ the blow-up of the maximal ideal $\mathfrak{m}_x$, $\pi: \tilde{X} \to X$, $p: \hat{X} \to X$ the natural maps. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathfrak{m}_x, \mathcal{O}_X)$ is naturally embedded in $\pi_* \mathcal{O}_{\tilde{X}}$, and actually in the subsheaf $p_* \mathcal{O}_{\hat{X}}$.

Proof: x being a singular point, $\mathfrak{m}_x \supset C$, hence there is a natural map $\mathcal{H}om_{\mathcal{O}_X}(\mathfrak{m}_x, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(C, \mathcal{O}_X) = \pi_* \mathcal{O}_{\tilde{X}}$.

The fact that this map is injective follows either from the arguments of lemma 1.4 or from the sharper statement that this sheaf embeds in $p_* \mathcal{O}_{\hat{X}}$. Let $f_1 \dots f_r$ be elements of $\mathcal{O}_{x,X}$ which induce a basis of $\mathfrak{m}_{x,X} / \mathfrak{m}^2_{x,X}$, and such that f_i is not a 0-divisor.
Let $\psi \in \mathcal{H}om_{\mathcal{O}_X}(\mathfrak{m}_x, \mathcal{O}_X)$. Assume that $\psi(f_i) = \varphi_i$: then, $\forall f \in \mathfrak{m}_{x,X}$ we have $\psi(f f_i) = f \psi(f_i) = f_i \psi(f)$, hence $\psi(f) = f \varphi_i / f_i$ (we are working in the full ring K_x of fractions of $\mathcal{O}_{x,X}$). The first thing to remark is that φ_i cannot be a unit, otherwise $f_j = \varphi_j f_i \varphi_i^{-1}$, contradicting the independence of the f_i's mod $\mathfrak{m}^2_{x,X}$.
But then ψ is given by multiplication by the rational function $\varphi_i / f_i = \varphi_j / f_j$ which is easily seen to be regular on $\hat{X}$.

Q.E.D.

Lemma 2.2: $\pi : \tilde{X} \to X$ being as in the previous lemma, let M be the (invertible) sheaf of ideals generated by $\pi^{-1}(\mathfrak{m}_x)$: then $\mathcal{H}om_{\mathcal{O}_X}(\mathfrak{m}^2_x, \mathcal{O}_X)$ embeds in $\pi_*(M^{-1})$.

Proof: If $\mathcal{J}$ is an ideal which contains a non 0-divisor h, we have seen (1.4) that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{O}_X)_x = \{ g \in K_x \mid g \cdot \mathcal{J} \subset \mathcal{O}_X \}$:

If $g \cdot \mathfrak{m}_x^2 \subset \mathcal{O}_X$, $\forall f' \in \mathfrak{m}_x$, $(g \cdot f')\ \mathfrak{m}_x \subset \mathcal{O}_X$, so, by 2.1., $g \cdot f' \subset \pi_* \mathcal{O}_{\tilde{X}}$, hence $g \in \pi_* (M^{-1})$.

Q.E.D.

<u>Theorem A</u>: Let X be a S.C.P. reduced Gorenstein curve, n an integer ≥ 2: then $|\omega_X{}^n|$ is free from base points.

<u>Proof</u>: By 1.11, we can clearly assume $p(X) \geq 2$. Consider the standard exact sequence (k_x being the residue field at x)

(2.3) $$0 \to \mathfrak{m}_x \omega_X{}^n \to \omega_X{}^n \to k_x \to 0.$$

By R.R. $H^1(\omega_X{}^n) = H^0(\omega_X{}^{1-n}) = 0$ since X is S.C.P and $p(X) \geq 2$. Then x is not a base point iff $H^1(\mathfrak{m}_x\ \omega_X{}^n) = 0$. By R.R., again, we have to show that

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathfrak{m}_x, \omega_X{}^{1-n}) = 0$$

Assume now that x is a singular point of X. Let $\pi : \tilde{X} \to X$ be the normalization at x : by 2.1 it suffices to show that

$H^0(\tilde{X}, \pi^*(\omega_X{}^{1-n})) = 0.$

Let $\tilde{Y}$ be a connected component of $\tilde{X}$; on each irreducible component $\tilde{Y}_i$ of $\tilde{Y}$ $\pi^*(\omega_X) \otimes \mathcal{O}_{\tilde{Y}_i}$ has degree ≥ 0, so it is enough to prove that for some i this degree is > 0.

But if this were not to hold, $Y = \pi(\tilde{Y})$ would have $p(Y) = 0$ by 1.10 and $\omega_X|_Y$ would be trivial : hence Y would be contained in the base locus of $|\omega_X{}^n|$, in particular there would be smooth base points.

Let's then prove that a smooth point x cannot be a base point. Denote by Z the irreducible component to which x belongs, and set for commodity $F = \omega_X^{1-n} \otimes \mathcal{m}_x^{-1}$. Since $p(X) \geq 2$, $\deg F = 1 + 2(1-n)(p-1) < 0$, hence clearly $H^0(X,F) = 0$ if $\deg \omega_X|_Z \geq 1$. On the other hand $\deg \omega_X|_Z = 0 \Longrightarrow p(Z) = 0$ (1.11). By 1.12 $\dim \mathcal{O}_{Z \cap (X-Z)} = 2$, and, since $Z \cong \mathbb{P}^1$,

$$H^0(Z, F|_Z) = H^0(Z, \mathcal{O}_Z(x)) \to H^0(\mathcal{O}_{Z \cap X-Z} \otimes F)$$

is an isomorphism.

By the exact cohomology sequence associated to the sequence

$$0 \to F \to F|_Z \oplus F|_{X-Z} \to F \otimes \mathcal{O}_{Z \cap X-Z} \to 0$$

we obtain $H^0(X, F) = H^0(X-Z, F|_{X-Z}) = H^0(X-Z, \omega_X^{1-n}|_{X-Z})$.

By the previous argument his vector space is 0 if on every connected component Y of $X-Z$ $\deg \omega_X|_Y > 0$. But $Z(X-Z) = 2$ implies that there are at most 2 connected components.

If $X-Z$ is connected, clearly $\deg \omega_X|_{X-Z} > 0$.

If $X-Z$ has two connected components Y_1, Y_2 (thus meeting Z transversally at two distinct points Y_1, Y_2) and say, $\deg \omega_X|_{Y_1} = 0$, then $p(Y_1) = 0$, hence $\deg \omega_X|_{Y_1} = -1$, an evident contradiction.

Q.E.D.

Definition 2.4: An elliptic tail of a C.P. curve X is an irreducible component Y of X such that $p(Y) = 1$, $Y(X-Y) = 1$.

Theorem B: If X is C.P. $|\omega_X^n|$ gives an embedding of X for $n \geq 3$.

Theorem C: If $p(X) = 2$ or X has elliptic tails $|\omega_X^2|$ does not give a birational map Φ_2. If X is C.P., $p(X) \neq 2$, X has no elliptic tails, Φ_2 is an embedding unless (possibly, cf. 3.23) at a point x if

a) $X \supset W$ with $W \cap (X-W) = \{x\}$, $W \cdot (X-W) = 2$, $p(W) = 1$, W is either rational irreducible with a cusp at x or a cycle of $2\mathbf{P}^1$'s meeting at x, and morever $C \not\subset \mathfrak{m}^2_{x,X}$

b) $X \supset W \cong \mathbf{P}^1$, $W \cap (X-W) = \{x\}$, $W \cdot (X-W) = 3$.

Proof of Theorems B,C: Let x,y be 2 points of X and consider the exact sequence

$$(2.5) \qquad 0 \to \mathfrak{m}_x \mathfrak{m}_y \omega_X^n \to \omega_X^n \to \omega_X^n/\mathfrak{m}_x \mathfrak{m}_y \omega_X^n \to 0$$

where, if $x = y$, $\mathfrak{m}_x \mathfrak{m}_y$ has to be understood as $\mathfrak{m}_x^2$. Since, for $n \geq 2$, $H^1(X, \omega_X^n) = 0, |\omega_X^n|$ gives an embedding if and only if

$$(2.6) \qquad \mathrm{Hom}_{\mathcal{O}_X}(\mathfrak{m}_x \mathfrak{m}_y, \omega_X^{1-n}) = 0 \quad .$$

We have to consider separately the following cases:

i) x, y smooth

ii) x singular, y smooth

iii) $x \neq y$, x, y both singular

iv) $x = y$ singular

i: Let F be the invertible sheaf $\omega_X^{1-n} \otimes (\mathfrak{m}_x \mathfrak{m}_y)^{-1}$: we have to prove that $H^0(X,F) = 0$. For every component X_i of X, $\deg F|_{X_i} =$ $= (1-n) \deg \omega_X|_{X_i} + \rho$, where $\rho = 2$ if $x, y \in X_i$, $\rho = 0$ if $x, y \notin X_i$, $\rho = 1$ in the remaining case.

If $n \geq 3$, X being C.P., this degree is ≤ 0, and < 0 on at least one component of X : in fact $\deg \omega_X \geq 2$.

Let n be equal to 2, and let x, y belong to 2 different components:

then deg $F|_{X_i} \leq 0$, and deg $F = 4 - 2p(X)$; therefore 2.6 fails if and only if $p(X) = 2$ and $\mathcal{O}_X(x+y) = \omega_X$; but then X consists of 2 components with deg $\omega_{X|X_i} = 1$.

It is then easy to see that either X consists of 2 elliptic tails, or X consists of two $\mathbf{P}^1$'s X_1, X_2 with $X_1 \cdot X_2 = 3$.

The former case though gives a contradiction, since then x should be a singular point of X ($\omega^{-1}_{X|X_1} = \mathcal{I}_{X_2} \otimes \mathcal{O}_{X_1}$), in the latter case we get a curve of genus 2.

If x, y belong to the same component Z, either $Z = X$ and $p(X) = 2$, or deg $(\omega_{X|Z}) = 1$; in this case every section of $F|_{X-Z}$ is identically zero, so we can apply Proposition A of [2], namely the following result

(2.7) Let L be an invertible sheaf on a curve X, s a non zero section of $H^0(X,L)$ such that s is identically 0 ($s \equiv 0$) on $Y \subset X$, $s \not\equiv 0$ on any component of $Z = X-Y$: then $Y \cdot Z \leq \deg L|_Z$

to obtain that Z is an elliptic tail.

<u>ii:</u> Let $\pi : \tilde{X} \to X$ be the normalization at x. Then $\mathrm{Hom}_{\mathcal{O}_X}(\mathfrak{m}_x \mathfrak{m}_y, \omega_X^{1-n})$ embeds in $H^0(\tilde{X}, L)$, where L is the invertible sheaf $\pi^* \omega_X^{1-n} \otimes \mathfrak{m}_y^{-1}$.

Clearly deg $L|_{\tilde{X}_i} < 0$ for every component $\tilde{X}_i$ of $\tilde{X}$, provided $n \geq 3$; if $n = 2$ this degree is ≤ 0 and $H^0(\tilde{X},L)$ can be non zero only if the irreducible component $\tilde{Y}$ of $\tilde{X}$ containing $\pi^{-1}(y)$ is a connected component of $\tilde{X}$, and deg $\pi^* \omega_{X|\tilde{Y}} = 1$. But then $H^0(\tilde{Y}, L|_{\tilde{Y}}) \neq 0$ implies that $L|_{\tilde{Y}}$ is trivial, hence $\tilde{C}\,\omega_{\tilde{Y}}(-y)$ is trivial: by a degree argument $p(\tilde{Y}) \leq 1$, and actually $p(\tilde{Y}) = 0$ since $y \neq x$. Then deg $\tilde{C} = -3$ and $\pi: \tilde{Y} \to Y$ must be an isomorphism. Then $Y \cap (X-Y) = \{x\}$, $Y \cdot (X-Y) = 3$, therefore, from the fact that $|\omega_X^2|$ has no base points, either $H^0(X, \omega_X^2)$ maps onto $H^0(Y, \omega_X^2|_Y)$, so that x and y are separated by the bicanonical map, or this map restricted to Y is a double cover of $\mathbb{P}^1$, hence we go back to case i).

iii: Let $\pi: \tilde{X} \to X$ be the normalization at both x,y. Then $\mathrm{Hom}_{\mathcal{O}_X}(\mathfrak{m}_x \mathfrak{m}_y, \omega_X^{1-m})$ embeds in $H^0(\tilde{X}, \pi^* \omega_X^{1-n})$ which is 0 for $n \geq 2$ since X is C.P.

iv: Let $\pi: \tilde{X} \to X$ be the normalization at x. By lemma 2.2 $\mathrm{Hom}(\mathfrak{m}^2_x, \omega_X^{1-m})$ is a subspace of $H^0(\tilde{X}, \mathcal{L})$, where $\mathcal{L}$ is the invertible sheaf $\pi^* \omega_X^{1-m} \otimes M^{-1}$.

Let $\tilde{Y}$ be a connected component of $\tilde{X}$, $\tilde{W}$ an irreducible component of $\tilde{Y}$. If $W = \pi(\tilde{W}) \not\ni x$, then $\deg \mathcal{L}|_{\tilde{W}} < 0$; if $W \ni x$, and x is smooth for W, $\deg \mathcal{L}|_{\tilde{W}} = 1 + (1-m) \deg \omega_{X}|_W$: this degree is then < 0 for $m \geq 3$, and, for $m = 2$, it is ≤ 0. If equality holds, $\deg \omega_X|_W = 1$, hence either $p(W) = 0$ or W is an elliptic tail (apply 1.12 to $\tilde{W} \subset \tilde{Y}$).

If x is a singular point of W, let C' be the conductor of $\mathcal{O}_{\tilde{W}}$ in $\mathcal{O}_W$, $\tilde{C}' = \pi^{-1}(C')$. We can write $\mathcal{L}$ as $\omega_{\tilde{Y}}^{-1} \otimes \pi^*(\omega_X^{2-m}) \otimes (\tilde{C}^{-1} \otimes M)^{-1}$, and let $d = \deg \omega_{\tilde{Y}}|_{\tilde{W}}$, $t = \deg (\tilde{C}^{-1} \otimes M)|_{\tilde{W}}$.

Since $C' \subset \mathfrak{m}_{x,W}$, $\tilde{C}' \subset M|_{\tilde{W}}$, $\dim \mathcal{O}_{\tilde{W}}/_{\tilde{C}} \leq 2\,\delta_W$, by 1.13 we conclude that $t \geq W(X-W)_x$. Hence either $(X-W) \not\ni x$, or $t \geq 2$ (in fact $W(X-W)_x = 1$ $\Rightarrow x$ is a node for X, but then, W being singular at x, $(X-W) \not\ni x$). In any case $\deg \mathcal{L}|_{\tilde{W}} \leq -t-d$, and if $t = 0$, then $C = M$, hence x is either a node or a cusp for W (and for X).

Consider the case when $d \geq 0$: then $\deg \mathcal{L}|_{\tilde{W}} \leq 0$, equality holding iff $d = t = 0$, $m = 2$ $(X-W \not\ni x)$.

Then, by 1.11, either $\tilde{W} = \tilde{Y}$ is elliptic, or $p(\tilde{W}) = 0$. In the former case $X = W$ is of genus 2, in the latter there exists a component

$\tilde{Z}$ of $\tilde{Y}$ on which $\deg \mathcal{L}|_{\tilde{Z}} < 0$, so that $H^0(\tilde{Y}, \mathcal{L}) = 0$.

If $d = -1$, $p(\tilde{W}) = 0$, $\tilde{W} \neq \tilde{Y}$, hence either $\deg \mathcal{L}|_{\tilde{W}} < 0$, or there exists $\tilde{Z} \subset \tilde{Y}$ with $\deg \mathcal{L}|_{\tilde{Z}} < 0$: if $\deg \mathcal{L}|_{\tilde{W}} = 0$, $H^0(\tilde{Y}, \mathcal{L}) = 0$, if $\deg \mathcal{L}|_{\tilde{W}} > 0$, then $t = 0$ and therefore W is an elliptic tail.

Finally, in the case when $d = -2$, $\tilde{W} = \tilde{Y} = \mathbb{P}^1$.

Assume that $(X-W) \not\ni x$, i.e. $W = X$; then, since $p(X) \geq 3$, $\delta_x \geq 3$. But then $C \subset \mathcal{M} \subset M \subset \pi_* \mathcal{O}_{\tilde{W}}$ and all the inclusions are strict ($\mathcal{M}$ is not an ideal in $\pi_* \mathcal{O}_{\tilde{W}}$), and since $\dim \mathcal{M}/C = \delta_x - 1$, $t = \dim M/C \geq 3$, and we are done.

If, on the other hand, $(X-W) \ni x$ and $t = 2$, by our previous argument $W (X-W)_x = 2$, $M = \tilde{C}'$, hence x is either a node or a cusp for W, which is thus Gorenstein with $\omega_W \cong \mathcal{O}_W$.

In this case, though $\deg \mathcal{L}|_{\tilde{W}} = 0$ for $m = 2$, we prove that, unless $C \not\subset \mathcal{M}^2_x$, any section of $\mathcal{H}om(\mathcal{M}^2_{x,X}, \omega_X^{-1})$ is $\equiv 0$ on W. Let $Z = X - W$, and consider the standard exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_W \oplus \mathcal{O}_Z \to \mathcal{O}_{Z \cap W} \to 0$$

Since $\mathcal{O}_{W \cap Z}$ has length 2, by Nakayama's lemma $\mathcal{M}^2_x \otimes \mathcal{O}_{W \cap Z} = 0$, and, if we set $\mathcal{M} = \mathcal{M}_x$, we have an isomorphism $\mathcal{M}^2 \mathcal{O}_X \cong \mathcal{M}^2 \mathcal{O}_W \oplus \mathcal{M}^2 \mathcal{O}_Z$: in fact $\mathcal{M}^2 \mathcal{O}_X$ injects into $\mathcal{M}^2 \mathcal{O}_W \oplus \mathcal{M}^2 \mathcal{O}_Z$, and clearly the projection on each factor is surjective; however $\mathcal{M}^2 \mathcal{O}_W = \pi_* (\tilde{C}|_{\tilde{W}})$ is contained in $C = \pi_* (\tilde{C})$, hence $\mathcal{M}^2 \mathcal{O}_W \subset \mathcal{M}^2 \mathcal{O}_X$ (and our assertion is thus proven) unless exists $f \in C - \mathcal{M}^2$ (this cannot hold if conjecture 3.23 is true).

Tensoring by ω_X^2, we get

$$H^1(X, \mathcal{M}^2 \omega_X^2) \cong H^1(W, \mathcal{M}^2 \mathcal{O}_W \otimes \omega_X^2) \oplus H^1(Z, \mathcal{M}^2 \mathcal{O}_Z \otimes \omega_X^2).$$

Use now R.R. duality on W, Z, respectively: this vector space is dual to $\mathrm{Hom}(\mathcal{M}^2 \mathcal{O}_W, \omega_X^{-2} \otimes \omega_W) \oplus \mathrm{Hom}(\mathcal{M}^2 \mathcal{O}_Z, \omega_X^{-2} \otimes \omega_Z)$.

We want to prove that the first summand is 0 : but here ω_W is trivial, hence this vector space embeds into $H^o(\tilde{W}, M^{-1}\pi^*\omega_X^{-2}|_{\tilde{W}}) = 0$ (these are the sections of a line bundle of degree -2).

Thus, given a section of $Hom(\mathfrak{m}^2_x, \omega_X^{-1})$, we have proven that it is 0 on any connected component $\tilde{Y}$ of $\tilde{X}$, except possibly if all irreducible components $\tilde{W}_i$ of $\tilde{Y}$ satisfy the following conditions (i=1,...r):

a) $x \in W_i$ and x is smooth for W_i

b) $p(W_i) = 0$, $W_i(X-W_i) = 3$.

It is easy to see that $r \leq 2$, so that either $W \cap (X-W) = \{x\}$, or W_1 intersects W_2 transversally at a node y of X.

In the latter case, if $X = Y$, then $p(X) = 2$, otherwise $2 = W_1(X-W_1)_x =$ = (by [6]) = $(W_1 \cdot W_2)_x + W_1(X-Y)_x$, hence $p(Y) = 1$, $Y(X-Y) = 2$ and we can repeat the argument given above.

We are left out with the case $W = \mathbb{P}^1$, $W \cap (X-W) = \{x\}$, $W \cdot (X-W) = 3$, i.e. b).

v: To end the proof of theorem C, let's prove the first statement . Namely, let W be an elliptic tail, and let x be the node of X such that $W \cap (X-W) = \{x\}$.

Then $\omega_{X|W} = \mathcal{O}_W(x)$, and, since x is not a base point for the bicanonical map of X, $H^o(X, \omega_X)$ restricts onto $H^o(W, \mathcal{O}_W(2x))$, which has dimension 2, by R.R.

Therefore, under the bicanonical map of X, W is a double cover of $\mathbb{P}^1$.

If, finally, $p(X) = 2$, let s_o, s_1 be a basis of $H^o(X, \omega_X)$. Assume that $s_o \cdot s_1 \equiv 0$: then, if X_i is the largest curve $\subset X$ s.t s_i does not vanish identically on any component of X_i, dim $X_o \cap X_1 = 0$, and, by 2.7, since X is connected, $1 \leq X_i \cdot (X-X_i) \leq \deg \omega_{X|X_i}$.

But $p(X) = 2 \Longrightarrow \deg \omega_X = 2$, hence $\deg \omega_{X|X_i} = 1$, so X_o, X_1 are elliptic tails; since moreover X is C.P. $X = X_o \cup X_1$.

If, on the other hand, $\forall s, \sigma \in H^o(X, \omega_X)$, $s \cdot \sigma \not\equiv 0$, $s_o^2, s_o s_1, s_1^2$ constitute a basis for $H^o(X, \omega_X^2)$, and the bicanonical map of X is a double cover of a smooth conic in $\mathbb{P}^2$.

Q.E.D.

§ 3. The Canonical Map

Throughoutthis section we will continue to assume that X is a complete reduced C.P. Gorenstein curve.

In order to discuss the behavior of the canonical map of X, we need some definitions.

Remark 3.1: If X is not 2 connected according to def. 1.3, there exists, by 1.10, a Z such that $Z \cap (X-Z) = x$, and x is a node for X: such an x is called a disconnecting node.

Definition 3.2: An irreducible component Y of X with $p(Y) = 0$ is said to be a loosely connected rational tail (L.C.R.T.) if $Y (X-Y)$ equals the number of connected components of $X-Y$.

Remark 3.3: If Y is a L.C.R.T., Y intersects $(X-Y)$ in disconnecting nodes. The next result gives necessary and sufficient conditions in order that the canonical map be a morphism.

Theorem D: If X is C.P. the base locus of $|\omega_X|$ consists exactly of the L.C.R.T.'s and of the disconnecting nodes. So $|\omega_X|$ is free from base points if and only if X is 2-connected.

Proof: Consider the exact sequence

$$0 \to \mathcal{M}_x\, \omega_X \to \omega_X \to k_x \to 0 .$$

Then x is a base point if $H^1(\mathcal{M}_x\, \omega_X) \to H^1(\omega_X) \to 0$ is not an isomorphism, i.e. if and only if $h^1(\mathcal{M}_x\, \omega_X) = 2$, i.e. $\dim \mathrm{Hom}(\mathcal{M}_x, \mathcal{O}_X) = 2$.

Step I: Assume now x to be a singular point of X. By 2.1, if $\hat{X}$ is the blow-up of X at x, $\mathrm{Hom}(\mathcal{M}_x, \mathcal{O}_X) \to H^0(\hat{X}, \mathcal{O}_{\hat{X}})$, hence if x is a base point $\hat{X}$ is not connected, and a fortiori $\tilde{X}$ is not connected (π: $\tilde{X} \to X$ being the normalization at x).

Let $\pi^{-1}(x) = \{p_1, \dots p_k\}$, and let t_i be a uniformizing parameter at p_i, m_i the multiplicity of $\tilde{C}$ at p_i, D the divisor $\Sigma_{i=1}^{k} m_i p_i$ on $\tilde{X}$.

Let W be the vector space $\omega_{\tilde{X}}(D)/\omega_{\tilde{X}}$: every element η of W can be written in an unique way as $\Sigma_{i=1}^{k} \Sigma_{j_i=1}^{m_i} a_{i,j_i} \left(d\, t_i \big/ t_i^{j_i} \right)$, and the dimension of W is $2\,\delta_x$.

W contains the vector subspace $V = \{\eta \mid \forall f \in \mathcal{O}_x \mid \Sigma_{i=1}^{k} \mathrm{Res}_{p_i}(f \cdot \eta) = 0\}$, of dimension δ_x, and a local section of ω_X around x is a local section of $\omega_{\tilde{X}}(D)$ around the p_i's such that its image in W belongs to V.

Moreover a local generator ω_X lifts to a differential with pole of order exactly m_i at each p_i, (so that $a_{i,m_i} \neq 0$), and via this choice, one can identify $\mathcal{O}_{X/C}$ with V.

Let us denote by U the image of $H^0(\omega_X)$ in $\mathcal{O}_{X/C}$: by what we just said, we can view U as a subspace of V.

Consider now the exact sequence

$$(3.4) \qquad 0 \to H^0(\omega_{\tilde{X}}) \to H^0(\omega_{\tilde{X}}(D)) \to \underset{\displaystyle W}{\overset{}{\omega_{\tilde{X}}(D)/\omega_{\tilde{X}}}} \xrightarrow{d} H^1(\omega_{\tilde{X}}) \to 0$$

Let K be the kernel of d. It is clear then that $U = K \cap V$, and proving that x is not a base point amounts to proving that there exists a vector in U with some $a_{i,m_i} \neq 0$. We have described the linear equations which define V: we claim now that the elements of K are those who satisfy the following equations

(3.5) $\sum_{p_i \in \tilde{Y}} a_{i,1} = 0$, for each connected component $\tilde{Y}$ of $\tilde{X}$.

In fact (3.4) is the direct sum of the exact sequences on each $\tilde{Y}$, and then there is a canonical isomorphism of $H^1(\omega_{\tilde{Y}})$ with k, given by the trace map.

Take $w = \sum_{i=1}^{k} w_i$, where $w_i = \sum_{1 \le j_i \le m_i} a_{i,j_i} (dt_i/t_i^{j_i})$. Then $w|_{\tilde{Y}} = \sum_{p_i \in \tilde{Y}} w_i$, and if you take A_i an open set in $\tilde{Y}$ where the above expression for w_i gives a section of $\omega_{\tilde{Y}}(D)$ (assume $A_i \not\ni p_j$ for $i \neq j$), and set $A_o = \tilde{Y} - \pi^{-1}(x)$, $d|_{\tilde{Y}}(w)$ is given by the cocycle $(w_i - w_j)$ on $A_i \cap A_j$.

Following the same argument given in [7], page 248, we see that we get the zero element in $H^1(\tilde{Y}, \omega_{\tilde{Y}})$ if $\sum \text{Res } w_i = 0$, i.e. if (3.5) holds.

By 1.14 $m_i \geq 2$ unless x is a node; assume then that x is not a node.

We can decompose $W = W' \oplus W''$, where W' is the span of the (dt_i/t_i)'s, W'' is the span of the (dt_i/t_i^j), for $j \geq 2$.

Consider the equations defining V: if $f = 1$ we get the equation $\sum_{i=1}^{k} a_{i,1} = 0$, if $f \in \mathfrak{m}_x$ we get an equation involving only the $(a_{i,j})$'s with $j \geq 2$.

We can therefore conclude that $K = K' \oplus W''$, $V = V' \oplus V''$, $V' \supset K'$, hence $U = K \cap V = K' \oplus V''$.

Since there is a vector in V'' with $a_{i,m_i} \neq 0$, we infer that x is not a base point for $|\omega_X|$. If, instead, x is a node which disconnects, we have $\pi^{-1}(x) = \{p_1, p_2\}$, $m_i = 1$ and must be, for vectors in K, $a_{1,1} = a_{2,1} = 0$, so x is a base point of $|\omega_X|$.

Step II: Let x be a smooth point of X, Z the irreducible component of X to which x belongs. Let L be the line bundle $\mathcal{O}_X(x)$: if x is a base point $h^0(X,L) = 2$, in particular $h^0(Z, \mathcal{O}_Z(x)) = 2$, hence $p(Z) = 0$. Let y be a point of $Z \cap (X-Z)$, and s a non zero section of L vanishing at y: then s vanishes identically exactly on a curve W which is a union of connected components of $(X-Z)$. By (2.7) $Z \cdot W = 1$, so y is a disconnecting node and W is connected. Therefore Z is a L.C.R.T.

Conversely, if Z is a L.C.R.T., $y_i, \ldots y_r$ are the disconnecting nodes belonging to Z, $\omega_X|_Z = \omega_Z(y_1 + \ldots + y_r)$, but since every section of ω_X vanishes at the y_i's by Step I, every section of ω_X vanishes identically on Z.

Q.E.D.

Remark 3.6: If X is C.P. and connected, but not 2-connected, one can take the normalization of X at the disconnecting nodes, to obtain $\pi : Y \to X$, where $Y = \bigcup_{i=1}^{m} Y_i$ consists of (say) m connected components. It is straight forward to verify that $H^0(X, \omega_X) \cong H^0(Y, \omega_Y) = \bigoplus_{i=1}^{m} H^0(Y_i, \omega_{Y_i})$, and that the Y_i's are 2-connected curves; in other words the rational canonical map $|\omega_X|$ consists of π^{-1} followed by the canonical morphisms of the Y_i, whose images span projective subspaces in a skew position. Therefore it is not restrictive to consider only the canonical map of a 2-connected curve.

In the rest of the paragraph we are going to examine necessary and sufficient conditions in order that the canonical map be an embedding, and we shall often start with some example just to explain some definitions and results. The first question is whether $|\omega_X|$ is injective, and we have the following

Definition 3.7: X is strongly connected if there do not exist two nodes x, y of X such that $X - \{x\} - \{y\}$ is disconnected. In particular, if X is 3-connected, then X is strongly connected.

Theorem E: If X is 2-connected, C.P., and the canonical map is injective, then X is strongly connected. More precisely, if x, y are two singular points of X, they have the same image under $|\omega_X|$ if and only if x, y are nodes and $X - \{x\} - \{y\}$ is disconnected.

Proof: If x, y are not nodes, we can repeat the argument given in Step I of Theorem D. Namely, let $\tilde{X}$ be the normalization of X at x, y, $\pi: \tilde{X} \to X$, $\tilde{C} = \mathcal{O}_{\tilde{X}}(-D_1 - D_2)$, where D_1, D_2 are effective divisors with supp $(D_1) = \pi^{-1}(x)$, supp $(D_2) = \pi^{-1}(y)$.

Let $W_1 = \omega_{\tilde{X}}(D_1 + D_2) / \omega_{\tilde{X}}(D_2) \supset \mathcal{O}_{X,X}/C = V_1$, and let W_2, V_2 be defined in an analogous way.

Again we can decompose V_i as $V_i' \oplus V_i''$, and if $V = V_1 \oplus V_2$, U is the image of $H^0(\omega_X)$ in $W = \omega_{\tilde{X}}(D_1 + D_2) / \omega_{\tilde{X}}$, $U = K \cap V = K' \oplus V''$, where $K' \subset V'$ and $U \supset V'' = V_1'' \oplus V_2''$, so that there exists sections of ω_X vanishing at x but not at y, and conversely.

Assume instead that x is a node, and let X' be the normalization of X at x: then $C = \mathfrak{m}_{x,X}$, therefore $H^0(X, \omega_X \mathfrak{m}_{x,X}) = H^0(\omega_{X'})$, and x, y have the same image under $|\omega_X|$ if and only if y is a base point for $|\omega_{X'}|$. The result follows then immediately from Theorem D.

Q.E.D.

<u>Remark 3.8</u>: Let X be 2-connected, C.P., but not strongly connected, x_1 a node of X such that the normalization X' of X at x is not 1-connected, but has $(r-1)$ disconnecting nodes $x_2, \ldots x_r$. Then, if Φ is the canonical map, $\Phi(x_i)$, for $i=1,..r$, is a fixed point p of $\Phi(X) = C$.

Let $\tilde{X}$ be the normalization of X at the x_i's: then the effect of projecting C from p is the same than to consider the canonical map of $\tilde{X}$. Therefore we obtain easily in this way examples where the canonical map is not injective, though being birational.

We are now going to discuss hyperelliptic curves, i.e. those for which $|\omega_X|$ is not birational.

<u>Definition 3.9</u>: X is hyperelliptic if there exist 2 smooth points x, y (possibly $x = y$) such that $H^0(\mathcal{O}_X(x+y)) = 2$.

<u>Proposition 3.10</u>: Let X be 2-connected. X is hyperelliptic if and only if $|\omega_X|$ is not birational, and also if and only if two smooth points have the same image, or $|\omega_X|$ is not an embedding at a smooth point.

<u>Proof</u>: The second part follows immediately by the exact sequence

$$0 \to H^0(\omega_X(-x-y)) \to H^0(\omega_X) \to \omega_X / \mathfrak{m}_x \mathfrak{m}_y \omega_X \to$$

$$\to H^1(\omega_X(-x-y)) \to H^1(\omega_X) \to 0,$$ since the dual space

of $H^1(\omega_X(-x-y))$ is $H^0(\mathcal{O}_X(x+y))$.

For the first part, notice that $H^0(\mathcal{O}_X(x+y))$ defines a morphism $f : X \to \mathbb{P}^1$, so that, for a general $p \in \mathbb{P}^1$, $f^{-1}(p)$ consists of two smooth points x', y', which have the same image under $|\omega_X|$.

Q.E.D.

<u>Example 3.11</u>: Let G be a cubic surface in $\mathbb{P}^3$ with an ordinary quadratic singularity at P, and containing exactly 6 lines through P. Let π_1, π_2 be two planes tangent to G at P, and such that $\pi_i \cdot G = Y_i$ is an irreducible cubic curve. Let Q be the point where Y_1, Y_2 intersect transversally ($\pi_1 \cdot \pi_2 \cdot G = 2P + Q$), and blow up $\mathbb{P}^3$ at Q.

The strict transform X of $Y = Y_1 \cup Y_2$ is a genus 3 curve, and it is easy to see that the canonical map of X is given by projection with center Q, hence the canonical map has as its image two lines in $\mathbb{P}^2$, and has degree 2 on each component.

<u>Example 3.12</u>: Notice first that the union of 2 conics in $\mathbb{P}^2$ is canonically embedded. Here the cross ratio of the 4 points in a conic through them determines uniquely the conic in the pencil determined by the 4 base points. Consider now, on $\mathbb{P}^1 \times \mathbb{P}^1$, two irreducible curves of type $(1,n)$, $(1,m)$ respectively: they have $p = 0$, and intersect in $(n+m)$ points (possibly infinitely near). It is easy to see that the canonical map is induced by the complete linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, n+m-2)|$, hence it is given by the projection on the second factor of $\mathbb{P}^1 \times \mathbb{P}^1$, followed by the embedding of $\mathbb{P}^1$ as a rational normal curve of degree $(n+m-2)$. Here the cross ratio of any 4- of the $(n+m)$ points is the same on both curves.

<u>Remark 3.13</u>: Let Y be an irreducible hyperelliptic curve: thus there exists a morphism $f : Y \to \mathbb{P}^1$ of degree 2. Then f is finite, and exists n such that $f_*\mathcal{O}_Y = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. In particular Y is a

divisor in a smooth surface (a line bundle over $\mathbb{P}^1$), hence Y is Gorenstein, and has at most double point as singularities.

Proposition 3.14: Let X be 2-connected, and let x, y be smooth points of X such that $h^0(\mathcal{O}_X(x+y)) = 2$.

Then either

a) x, y belong to 2 different components Y_1, Y_2 with $p(Y_i) = 0$, and such that for every connected component Z_j of $X-Y_1-Y_2$ $Z_j \cdot Y_i = 1$,

or

b) x, y belong to an irreducible hyperelliptic curve Y such that, for each connected component Z of $X-Y$ the invertible sheaf $(\mathcal{J}_Z \otimes \mathcal{O}_Y)^{-1}$ is isomorphic to the hyperelliptic sheaf $\mathcal{O}_Y(x+y)$.

Proof: Let L be the invertible sheaf $\mathcal{O}_X(x+y)$. By assumption $h^0(L) = 2$ hence, for every $z \neq x, y$, $h^0(\mathfrak{m}_{z,X}L) = 1$.

In case a), pick up z either on $Y_1 \cap Y_2$ or, if $Y_1 \cap Y_2 = \phi$, in a connected component Z of $X-Y_1-Y_2$ such that $Z \cap Y_i \neq \phi$.

Let s be a non zero section of $H^0(X, L)$ vanishing at z: since $L|_{X-Y_1-Y_2}$ is trivial, s vanishes at some point of Y_i else than x, or y.

The section s cannot vanish identically on any of the Y_i's : in fact it cannot vanish on both Y_1 and Y_2, so assume $s|_{Y_1} \equiv 0$, $s|_{Y_2} \not\equiv 0$.

Let W be the union of connected components of $X - Y_2$ where $s \equiv 0$: by 2.7 $Y_2 \cdot W \leq 1$, hence X would not be 2-connected $(W \cdot (X-W) \leq 1)$, a contradiction.

Therefore the restriction map $H^0(X, L) \to H^0(Y_i, L|_{Y_i})$ is an isomorphism and $p(Y_i) = 0$.

By the same argument, for each connected component Z of $X - Y_1 - Y_2$, $Z \cdot Y_i \leq 1$, and since X is 2-connected $Z\,(Y_1 \cup Y_2) \geq 2$, hence $Z \cdot Y_i = 1$. In case b), if Z is a connected component of X - Y, $Z \cdot Y \leq 2$ by 2.7, so equality holds by 2-connectedness. Moreover $h^0(Y, L|_Y) = 2$, so Y is hyperelliptic; hence Y is Gorenstein and by 1.12 $\mathcal{I}_Z \otimes \mathcal{O}_Y$ is invertible, of degree -2.

Since there exists a non zero section s of $H^0(X, L)$ such that $s \in H^0(\mathcal{I}_Z L)$, we get an inclusion $0 \to \mathcal{O}_Y \to \mathcal{I}_Z L \otimes \mathcal{O}_Y$, and the cokernel of the map is a skyscraper sheaf of length 0, therefore

$$\mathcal{I}_Z \otimes \mathcal{O}_Y \cong L^{-1}|_Y .$$

Q.E.D.

To sharpen the result of the last proposition, and also prove a converse statement characterizing hyperelliptic reducible curves, it is convenient to have a digression on cross-ratios and rational normal curves (cf. 3.12).

Definition 3.15: An n-tuple of points on a smooth curve X consists of the following data: and ideal sheaf $\mathcal{I}$ of $\mathcal{O}_X$ such that length $\mathcal{O}_{X/\mathcal{I}} = n$, together with isomorphisms α_i for each $p_i \in \operatorname{supp}(\mathcal{O}_{X/\mathcal{I}})$, of $(\mathcal{O}_{X/\mathcal{I}}) \to k[t]/_{(t^{m_i})}$, (where $m_i =$ = length $(\mathcal{O}_{X,p_i/\mathcal{I}})$).

Definition 3.16: Two n-tuples of points on $\mathbb{P}^1$, $(\mathcal{I}, \alpha_i)$, $(\mathcal{I}', \alpha'_i)$ are said to have the same cross ratios if there exists an automorphism of $\mathbb{P}^1$ such that $g^*(\mathcal{I}) = \mathcal{I}'$, and $g^* : \mathcal{O}_{\mathbb{P}^1/\mathcal{I}} \to \mathcal{O}_{\mathbb{P}^1/\mathcal{I}'}$ is such that $\alpha_i' \circ g^* = \alpha_i$.

Let now Y_1, Y_2 be two smooth rational curves of the same degree d in

$\mathbb{P}^N$, and $X = Y_1 \cup Y_2$. Then they have an n-tuple of points in common if length $\mathcal{O}_{Y_1 \cap Y_2} = n$, because, if $p_i \in Y_1 \cap Y_2$, any isomorphism α_i of $\mathcal{O}_{Y_1 \cap Y_2, p_i}$ to $k[t]/(t^{m_i})$ induces an n-tuple of points on Y_1 and Y_2. It makes therefore sense to say that Y_1 and Y_2 have n points in common with the same cross ratios.

Lemma 3.17: Let Y_1, Y_2 be two rational normal curves of degree d in $\mathbb{P}^N$ with n points in common. If $n \geq d + 3$, or $n = d + 2$ and the 2 n-tuples have the same cross-ratios, then $Y_1 = Y_2$.

Proof: Let's prove the result by induction on d.

For $d = 2$ the result is elementary and well known (one has only to remark that the hypothesis implies that the 2 conics lie in the same plane). So assume the theorem to be true for d-1.

Take a point $p \in Y_1 \cap Y_2$ and consider the projection $g: \mathbb{P}^N \to \mathbb{P}^{N-1}$ with centre p. Then $g(Y_1)$, $g(Y_2)$ satisfy the hypotheses of the lemma, hence $g(Y_1) = g(Y_2)$, so Y_1, Y_2 are contained in the cone Γ_{d-1} over the rational normal curve of degree (d-1) (in particular $\Gamma_{d-1} \subset \mathbb{P}^d \subset \mathbb{P}^N$). Consider $F = F_{d-1}$ the rational ruled surface obtained by blowing up p in $\Gamma = \Gamma_{d-1}$, $\pi: F \to \Gamma$ being the resolution of singularities. In Pic (F), let f be the class of a fibre, e the class of the exceptional divisor E; $\pi^*(\mathcal{O}_{\mathbb{P}^d}(1)) \equiv (d-1) f + e$, and we have $e^2 = -d + 1$, $e\, f = 1$, $f^2 = 0$. Let's denote by Y'_i the proper transform of Y_i. Since $Y'_i \cdot ((d-1) f + e) = d$, $Y'_i \cdot e = 1$, $Y'_i \equiv d f + e$, hence $Y'_1 \cdot Y'_2 = d + 1$. However $Y'_1 \cdot Y'_2 \geq n - 1$, hence it is a contradiction to assume $Y_1 \neq Y_2$ if $n \geq d + 3$. If, on the other hand, $n = d + 2$ and the cross-ratios of the n-tuple of points is the same, then the equations of Y'_1, Y'_2 induce the same element of $H^0(E, \mathcal{O}_F(Y'_i)/\mathcal{O}_F(Y_i - mE))$, where m is the length of $\mathcal{O}_{Y_1 \cap Y_2}$ at p: hence again $Y'_1 \cdot Y'_2 \geq n$ and we have a contradiction.

Q.E.D.

<u>Definition 3.18</u>: An honestly hyperelliptic curve is a 2-connected Gorenstein curve Y with a finite morphism $f: Y \to \mathbb{P}^1$ of degree 2.

<u>Theorem F</u>: A C.P., 2-connected Gorenstein curve X is hyperelliptic if and only if X contains an honestly hyperelliptic curve Y with an hyperelliptic invertible sheaf L on Y such that, for each connected component Z of $X-Y$, $(\mathcal{J}_Z \otimes \mathcal{O}_Y)^{-1}$ is isomorphic to L. Moreover, if $f : Y \to \mathbb{P}^1$ is the morphism associated to $H^o(Y, L)$, then the canonical map Φ_1 maps Y to a rational normal curve and factors through f. If Y is not irreducible, the above condition is equivalent to : $Y = Y_1 \cup Y_2$ with $p(Y_i) = 0$, and s.t. for every connected component Z_j of $X - Y$, $Z_j \cdot Y_i = 1$, and moreover, if we set $P_{ij} = Y_i \cap Z_j$, the n-tuples $(Y_1 \cap Y_2, P_{1j})$, $(Y_2 \cap Y_1, P_{2j})$ have the same cross-ratios.

<u>Proof</u>: Assume X to be hyperelliptic, and let x, y be to smooth points which have the same image under Φ_1. Following the arguments of 3.14., the invertible sheaf $L' = \mathcal{O}_X(x+y)$ defines a morphism $f': X \to \mathbb{P}^1$, which is non constant on a curve Y ($= Y_1 \cup Y_2$ in case a)). $f = f'|_Y$ makes Y a honestly hyperelliptic curve, and it is easy to see that $\Phi_{1|Y}$ factors through f.

Since $L' = f'^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and a connected component Z of $X - Y$ is f'^{-1} (point), the argument of 3.14 gives $L'|_Y = L \cong (\mathcal{J}_Z \otimes \mathcal{O}_Y)^{-1}$.

Conversely, we claim that we can extend L to an invertible sheaf L' on X such that $L'|_Y = L$, $L'|_{X-Y} \cong \mathcal{O}_{X-Y}$: in fact we have the exact sequence

$$0 \to \mathcal{O}_{Y \cup Z} \to \mathcal{O}_Z \oplus \mathcal{O}_Y \to \mathcal{O}_{Y \cap Z} \to 0,$$

so choose a section s_Z of L not vanishing at $Y \cap Z$ and identify it with $1 \in H^0(\mathcal{O}_Z)$; in this way we have defined an $\mathcal{O}_{Y \cup Z}$ invertible sheaf, so, repeating the operation for each Z, we obtain L with the desired property.

Clearly $H^0(X, L) = H^0(Y, L')$, therefore $\Phi_{1|Y}$ factors through f and X is hyperellptic.

Let $Y \cdot (X - Y) = 2k$, $p = p(Y)$: then $\omega_{X|Y} = f^*(\mathcal{O}_{\mathbb{P}^1}(k + p - 1))$. It remains to prove that, via f^*, $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k + p - 1)) =$
$= H^0(X, \omega_X)|_Y$.

Observe that $X - Y = Z_1 \cup \dots \cup Z_k$, and that by R.R. $H^0(Z_j, \omega_{X|Z_j}) =$
$= p_j + 1$, where $p_j = p(Z_j)$.

In other words, $H^0(Z_j, \omega_{X|Z_j}) \to \omega_X \otimes \mathcal{O}_{Y \cap Z_j}$ has a 1-dimensional image giving a local generator of ω_X at the points of $Y \cap Z_j$.

From the exact sequence

$$0 \to H^0(X, \omega_X) \to H^0(Y, \omega_X|_Y) \oplus H^0(X-Y, \omega_{X|X-Y}) \to$$

$$\to H^0(Y \cap (X-Y), \omega_X \otimes \mathcal{O}_{Y \cap (X-Y)}) \to H^1(X, \omega_X) \to 0$$

if follows easily that $H^0(\omega_X)|_Y$ has dimension $k + p$.

The last assertion follows by definition 3.16: in fact there exist isomorphisms $g_i : Y_i \to \mathbb{P}^1$ and ideals $\mathcal{J}, \mathcal{J}_i$ on $\mathbb{P}^1$ such that

a) $g_1^*(\mathcal{J}) = \mathcal{O}_{Y_1} \otimes \mathcal{J}_{Y_2}$ (resp. $g_2^*(\mathcal{J}) = \dots$), $g_i^*(\mathcal{J}_j) = \mathcal{O}_{Y_i} \otimes \mathcal{J}_{Z_j}$

b) $(g_1^*)^{-1}(g_2^*)$ induces the identity on $\mathcal{O}_{\mathbb{P}^1}/\mathcal{J}$, and on $\mathcal{O}_{\mathbb{P}^1}/\mathcal{J}_j$

when $P_{1j} = P_{2j}$ hence g_1, g_2 glue to give a finite morphism $f : Y \to \mathbb{P}^1$ of degree 2 such that $f^*(\mathcal{J}_j) = \mathcal{J}_{Z_j} \otimes \mathcal{O}_Y$.

Q.E.D.

Let us assume, for the rest of the paragraph, that X is not hyperelliptic and that X is strongly connected: then the canonical map Φ_1 is a birational morphism, is an embedding at smooth points, and separates pairs of smooth points as well as pairs of singular points. The next proposition ensures that Φ_1 is an injective morphism.

Proposition 3.19: Let X be C.P. and 2-connected. Then if x is a singular point, and y is a smooth point, $\Phi_1(x) \neq \Phi_1(y)$.

Proof: Let $\tilde{X}$ be the normalization at x. Since $H^0(\omega_{\tilde{X}}) = H^0(C\omega_X)$, if $H^0(\mathfrak{m}_x \mathfrak{m}_y \omega_X) = H^0(\mathfrak{m}_x \omega_X)$, y would be a base point for $|\omega_{\tilde{X}}|$. If Γ is the component of X containing y, it follows that $\tilde{\Gamma}$ is a L.C.R.T. (or contained in a negative tail).

Take now the normalization of $\tilde{X}$ at the points of $\tilde{\Gamma} \cap (\tilde{X} - \tilde{\Gamma})$, to get $\pi\colon \bar{X} \to X, \bar{\Gamma} \to \Gamma$, and let $\pi^*(\omega_X) = \omega_{\bar{X}}(\bar{D})$.
Choose t an affine coordinate on $\bar{\Gamma} \simeq \mathbf{P}^1$ such that $p_1,\dots p_k$ are the coordinates of the points in $\bar{\Gamma} \cap \pi^{-1}(x)$, $q_1,\dots q_s$ the ones of the points lying over $\tilde{\Gamma} \cap (\tilde{X} - \tilde{\Gamma})$ (they do not lie in $\pi^{-1}(x)$!).

Let also $z_1,\dots z_r$ be the points in $(\bar{X} - \bar{\Gamma}) \cap \pi^{-1}(x)$, t_i be a local coordinate at z_i, let $u_1,\dots u_s$ be the points of $\bar{X} - \bar{\Gamma}$ lying over $\tilde{\Gamma} \cap (\tilde{X} - \tilde{\Gamma})$, and let τ_h be a local coordinate at u_h.

The multiplicity of $\bar{D}$ at u_h, q_h, is one, and let m_i be the multiplicity of $\bar{D}$ at p_i, n_j the multiplicity of $\bar{D}$ at z_j.

Consider the usual exact sequence:

$$0 \to H^0(\omega_{\bar{X}}) \to H^0(\omega_{\bar{X}}(\bar{D})) \to W = \omega_{\bar{X}}(\bar{D})/\omega_{\bar{X}} \xrightarrow{\partial}$$

$$\xrightarrow{\partial} H^1(\omega_{\bar{X}}) \to 0$$

An element η in W can be written in the form

$$\sum_{\substack{i=1 \\ 1\le j\le m_i}}^{k} a_{ij}\, dt\, (t-p_i)^{-j} + \sum_{h=1}^{s} a_h\, dt\, (t-q_h)^{-1} +$$

$$+ \sum_{h=1}^{s} b_h\, d\tau_h\, (\tau_h)^{-1} + \sum_{\substack{e=1 \\ 1\le n\le n_e}}^{r} c_{e,n}\, dt_e\, (t_e)^{-n}$$

We can clearly assume that $m_i \geq 2$ for each i (otherwise x would be a node and X would not be either C.P. or 2-connected). Remark also that $H^o\,(\omega_{\overline{X}})\,|\,_{\overline{\Gamma}} = 0$. An element $\eta \in W$ is in the image U of $H^o\,(\omega_X)$ if and only if it belongs to the intersection of two subspaces, K' and V.

K' is defined by the equations of $K = \operatorname{Ker}\partial$ plus the local equations given by the nodes in $(\tilde{X} - \tilde{\Gamma}) \cap \tilde{\Gamma}$: since $\overline{X}$ has at least $s+1$ connected components, there is given, for each $h = 1,\ldots s$, a subset I_h of $\{1,\ldots r\}$, and also are given subsets $J_{h'}$, $h'=1,\ldots p$, such that the I_h's and $J_{h'}$'s give a partition of $\{1,\ldots r\}$ and K' is defined by the following equations

$$\sum_{i=1}^{k} a_{i1} + \sum_{h=1}^{s} a_h = 0 \quad ,$$

$$a_h + b_h = 0 \quad (h=1,..s), \quad b_h + \sum_{e\in I_h} c_{e,1} = 0 \,(h=1,..s)$$

$$\sum_{e\in J_{h'}} c_{e,1} = 0 \quad (h'=1,..p).$$

The subspace V is defined by the equations $\Sigma \operatorname{Res}(f\eta) = 0$ for $f \in \mathfrak{m}_x$, and the variables $a_{i1}, a_h, b_h, c_{e,1}$ do not appear in these equations.

The conditions that η vanishes at x is given by any of the equations $a_{i\,m_i} = 0$ (these are all equivalent to each other modulo the equations defining V) : again the above mentioned variables do not appear.

An easy computation around $y = \infty$ gives that η vanishes at y if and only if

$$(\#) \qquad \sum_{i=1}^{k} a_{i1}\, p_i + \sum_{h=1}^{s} a_h\, q_h + \sum_{i=1}^{k} a_{i2} = 0.$$

If then $H^{o}(\mathfrak{m}_x \mathfrak{m}_y \omega_X) = H^{o}(\mathfrak{m}_x \omega_X)$, the equation (#) should be a linear combination of the equations defining K', V, and of the equations $a_{im_i} = 0$.

By looking at the coefficient of the a_{i1}'s, we get that all the p_i's should be equal. This however is possible only if $k = 1$, and then we can assume $p_1 = 0$.

Look now at the coefficient of a_{12} : a_{12} appears in the equations defining V if and only if Γ is smooth at x.

If Γ is singular at x, a_{12} appears in (#) with coefficient 1, and it has non zero coefficient in the other equations only if $m_1 = 2$. But then, by 1.13., $X - \Gamma \not\ni x$ and x is an ordinary cusp. In this case we have a contradiction again since either $X = \Gamma$ has genus 1, or X is not 2-connected. Assume finally Γ to be smooth at x.

Restrict the linear form (#) to the subspace where

$$a_{1j} = 0 \text{ for } j \geq 2, \quad c_{e,j} = 0 \text{ for } j \geq 2.$$

Then the linear form $\sum_{h=1}^{s} a_h q_h$ should be a linear combination of the linear forms

$$a_{11} + \sum_{h=1} a_h \,, \quad a_h + b_h \quad (h=1,..s), \quad b_h + \sum_{e \in I_h} c_{e,1}$$

$$(h=1,...s), \quad \sum_{e \in J_{h'}} c_{e,1} \quad (h' = 1,..p).$$

This is however easily seen to be impossible.

Q.E.D.

<u>Example 3.20</u>: Let X_1, X_2 be two smooth non hyperelliptic curves meeting in a point x such that $(X_1 \cdot X_2)_x = 2$ (a tacnode), set $X = X_1 \cup X_2$. Then, if s is a section of ω_X vanishing at x, $s|_{X_i} \in H^{o}(X_i, \omega_{X_i}(2x))$ hence it vanishes to second order on X_i at x.

This shows that X is non hyperelliptic, and the canonical map is not an embedding at x.

This motivates the following

Definition 3.21: A (C.P.) Gorenstein curve X is said to be very strongly connected if

a) X is strongly connected

b) there does not exist a decomposition $X = X_1 \cup X_2$ where $X_1 \cap X_2$ is a single point.

Proposition 3.22: Assume that X is very strongly connected and that x is a double point (i.e. formally isomorphic to the plane singularity $y^2 - x^k = 0$). Then the canonical map is not an embedding at x if and only if X is hyperelliptic with $f: X \to \mathbf{P}^1$ not constant on the components of X passing through x.

Proof: Let's prove first the "if" part of the statement.

Let Γ be the union of the components passing through x (they are at most 2).

Then by Theorem F the restriction to Γ of the canonical map of X factors through f, hence is not an embedding at x.

Conversely, let $\tilde{X}$ be the normalization of X at x: then by our hypothesis $\tilde{X}$ is connected.

Moreover, dim Hom $(\mathcal{m}^2_x, \mathcal{O}_X) = 2$, and, by lemma 2.2, $\tilde{X}$ is such that $h^o(\tilde{X}, M^{-1}) \geq 2$, hence $\tilde{X}$ is hyperelliptic, with $L = M^{-1}$ as hyperelliptic bundle.

We have in fact that if $h^o(\tilde{X}, L) = 3$, then $\tilde{X}_1, \tilde{X}_2$ must be negative tails (if they were L.C.R.T. X would not be 2-connected), and then X

is clearly hyperelliptic with the desired properties (according to Theorem F). So we can assume $h^0(\tilde{X}, L) = 2$, and that $p(\tilde{X}) \geq 1$. Observe that there exists an integer r s.t. $\pi^*(\omega_X) = \omega_{\tilde{X}} \otimes L^r$ (in fact $r = [k/2]$).

Let σ be a section of L vanishing on $\pi^{-1}(x)$, τ a section of L vanishing at two smooth points p,q of $\tilde{\Gamma}$ but not on $\pi^{-1}(x)$, η a section of $H^0(\omega_{\tilde{X}})$ such that $\eta|_{\tilde{\Gamma}} =$ a power of τ .

By the exact sequence

$$0 \to H^0(\omega_{\tilde{X}}) \to H^0(\omega_X) \to \omega_{X/C\omega_X} \to 0$$

we see that $\eta\tau^h\sigma^{r-h}$ $(h = 1,\ldots r)$ are sections of $\pi^*(\omega_X)$ which, in $\omega_{\tilde{X}}/_{M^{-r}\omega_{\tilde{X}}}$, give a basis of $\omega_{X/C\omega_X}$.

Therefore the pull-back of sections of ω_X, when restricted to $\tilde{\Gamma}$, are linear combinations of $\tau^{h'}\sigma^{h''}$, hence p and q have the same image under Φ_1 and X is hyperelliptic.

Q.E.D.

<u>Theorem G</u>: Let X be very strongly connected, not hyperelliptic, and such that for each singular point x of X where $C \not\subset \mathfrak{m}_x^2$, x is a double point. Then the canonical map Φ_1 is an embedding.

<u>Remark 3.23</u>: The hypotheses in Theorem G would only be that X be very strongly connected and not hyperelliptic if the following conjecture were true: any Gorenstein singular point x where $C \not\subset \mathfrak{m}_x^2$ is a double point. This is obvious if $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 = 2$ and we shall later give a proof of this fact when the singularity is unibranch, i.e. formally irreducible. Also case a) of Theorem C would be vacuous if the conjecture were true.

Proof of Theorem G: In view of 3.19, 3.22, we are only left to prove that Φ_1 is an embedding at a singular point x where $\mathfrak{m}_x^2 \supset C$.

Consider the exact sequence

$$0 \to H^0(C\omega_X) \to H^0(\omega_X) \to \omega_X/_{C\,\omega_X} \to$$

$$\to H^1(C\omega_X) \to H^1(\omega_X) \to 0 : \text{ since } H^1(C\omega_X) =$$

$$= H^1(\pi_* \omega_{\tilde{X}}) = H^1(\omega_{\tilde{X}}), \text{ and } \tilde{X} \text{ is connected, the}$$

restriction homomorphism $H^0(\omega_X) \to \omega_{X/C\,\omega_X}$ is surjective. Also $\mathcal{O}_{X/C}$ surjects onto $\mathcal{O}_{X/\mathfrak{m}_x^2}$, and we are done.

Q.E.D.

Proposition 3.26: Let (X,x) be a reduced Gorenstein unibranch singularity (i.e. if $\pi: \tilde{X} \to X$ is the normalization at x, $\pi^{-1}(x)$ is a single point p). If $C \not\subset \mathfrak{m}_x^2$, then x is a double point.

Proof: Let t be a uniformizing parameter in $\tilde{\mathcal{O}} = \mathcal{O}_{\tilde{X},p}$, and let M be the semigroup $M = \{\text{ord}_t f \mid f \in \mathfrak{m}_x\}$. Notice that $M \not\ni 1$, and we can assume that $M \not\ni 2$, otherwise then x is a double point. Take a function $g \in C - \mathfrak{m}_x^2$ such that $m = \text{ord}_t g$ is maximum (observe that $\mathfrak{m}_x^2 \supset C^2 = (t^{4\delta})$, so $\text{ord}_t g \leq 4\delta$).

Then we claim that $m \notin M + M$. Otherwise if $m = m_1 + m_2$, $m_i = \text{ord}_t f_i$, $f_i \in \mathfrak{m}_x$, there would exist a constant λ such that $\text{ord}_t (g - \lambda f_1 f_2) > m$, but, since $(g - \lambda f_1 f_2) \in C - \mathfrak{m}^2$ (in fact $\text{ord}_t f \geq 2\delta \Leftrightarrow f \in C$), this contradicts the maximality of m.

If $1 \leq n_1, \dots n_r \notin M$, then the r-dimensional subspace $\sum_{i=1}^{r} \lambda_i t^{n_i}$ intersects $\mathcal{O} \subset \tilde{\mathcal{O}}$ only in 0, and since $\dim \tilde{\mathcal{O}}/\mathcal{O} = \delta$, it follows easily that $\delta \geq \text{card}(\mathbb{N} - M)$ (actually one has equality). Moreover, by definition of C, $(2\delta - 1) \notin M$. Consider now the $[m/2]$ pairs $\{1, m-1\}$, $\{2, m-2\}, \dots$: since $m \notin M + M$ at least one element for each pair does not belong to M, therefore $\delta \geq [m/2]$, hence $1 + 2\delta \geq m$. Since $m \geq 2\delta$, either $m = 2\delta$, but then we have noticed that $1, 2\delta-1 \notin M$, or $m = 2\delta + 1$ but then, $2, 2\delta - 1 \notin M$.

Q.E.D

§ 4. Some Remarks on the Non-Reduced Case

Let C be a smooth curve of genus g, L, N line bundles on it, and consider C as the zero section of $V = L \oplus N$ (a smooth non complete threefold with a projection $p: V \to C$).

The sheaves of sections of L, N, pull back, via p, to invertible sheaves $\mathcal{L}, \mathcal{N}$ on V. The normal sheaf to C in V is clearly $(\mathcal{L} \oplus \mathcal{N}) \otimes \mathcal{O}_C$, hence $\omega_{V|C} = \omega_C \otimes \mathcal{L}^{-1} \otimes \mathcal{N}^{-1}$. Let $X \to V$ be the curve (locally complete intersection) defined by the ideal I_X spanned by $\mathcal{L}^{-2} + \mathcal{N}^{-2}$ (here $\mathcal{L}^{-1}, \mathcal{N}^{-1}$ are viewed as given by linear forms on the fibres of $p: V \to C$). Therefore $X_{red} = C$, and the conormal sheaf to X in V is given by $(\mathcal{L}^{-2} \oplus \mathcal{N}^{-2}) \otimes \mathcal{O}_X$, hence $\omega_X = (\omega_V \otimes \mathcal{L}^2 \otimes \mathcal{N}^2)|_X =$
$= (\omega_C \otimes \mathcal{L} \otimes \mathcal{N})|_X$ (again here ω_C stands for the pull back via p).
Let d be the degree of $(\omega_C \otimes \mathcal{L} \otimes \mathcal{N})_C$, then it follows that $\deg \omega_X = 4d$, for instance since we have the exact sequence

$$0 \to (\mathcal{L}^{-1} \oplus \mathcal{N}^{-1} \oplus \mathcal{L}^{-1}\mathcal{N}^{-1})|_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

and we can tensor it by ω_X^n to compute $\chi(\omega_X^n)$. Since C is a subscheme of X, if $|\omega_X^n|$ is free from base points or embeds, the analogous statement must hold true a fortiori for $|\omega_X^n|_C|$.
But, if $\mathcal{L}, \mathcal{N}$ are chosen to be general in Pic (C), one needs $n d \geq 2g$ (respectively $n d \geq 2g + 1$). This is a lower bound on n which however depends on $\deg(\omega_{X|C})$, compared to $\deg(\omega_C) = 2g - 2$, i.e. on the negativity of the normal bundle to C.

But consider now the following (non closed) double point of X :

a tangent vector sticking out of a point $x \in C$ in the direction of N, together with x.

In other words, we consider the subscheme of X defined by the ideal $J = p^* (\mathcal{M}_{x,C}) + \mathcal{L}^{-1}$.

This double point is not embedded if $H^0 (J \omega_X^n)$ has codimension ≤ 1 in $H^0 (\omega_X^n)$.

This clearly happens if x is a base point of $|\omega_X^n|_C|$, and, in the other case, one can consider the following diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & \omega_X^n \otimes & (\mathcal{L}^{-1} \oplus \mathcal{L}^{-1}\mathcal{N}^{-1} \oplus \mathcal{M}_x \mathcal{N}^{-1})|_C & \to & J \omega_X^n & \to & \mathcal{M}_x (\omega_X^n|_C) & \to & 0 \\ & & \downarrow \gamma & & \downarrow \alpha & & \downarrow \beta & & \\ 0 \to & \omega_X^n \otimes & (\mathcal{L}^{-1} \oplus \mathcal{L}^{-1}\mathcal{N}^{-1} \oplus \mathcal{N}^{-1})|_C & \to & \omega_X^n & \to & (\omega_{X|C}^n) & \to & 0 \end{array}$$

Assume that γ induces an isomorphism of H^0's: then

$\mathrm{cod}\ \mathrm{Im}\ H^0 (\alpha) \leq \mathrm{cod}\ \mathrm{Im}\ H^0 (\beta) \leq 1$.

$H^0 (\gamma)$ is clearly an isomorphism iff

$$0 \to H^0 (\omega_X^n \mathcal{N}^{-1} \otimes \mathcal{M}_{x,C}) \to H^0 (\omega_X^n \mathcal{N}^{-1} \otimes \mathcal{O}_C)$$

is an isomorphism.

A sufficient condition for this to hold is that

$$H^0 (C, \omega_X^n \mathcal{N}^{-1} \otimes \mathcal{O}_C) = 0, \text{ e.g. if}$$

$$\deg \mathcal{N}|_C > nd = n (2g - 2 + \deg \mathcal{L}|_C + \deg \mathcal{N}|_C).$$

This condition means that if $\deg \mathcal{N}|_C = m$, $\deg \mathcal{L}|_C = e$, m must be very positive, e very negative, and yet the degree of the normal bundle to C, $\delta = m + e$, can be positive. In fact the above inequality is then

$$m > n (2g - 2) + n \delta .$$

The conclusion is that the hypothesis of the normal bundle to C being positive still does not give any lower bound for n in order that $|\omega_X{}^n|$ be an embedding.

If X is a curve lying on a smooth surface, then one can define, according to Franchetta and Ramanujam (see [9], [5], [1]) a notion of numerical m-connectedness for X : it would be interesting to extend this notion for a Gorenstein curve, and to see whether some conditions of this kind can give some results of the type of Theorems A, B.

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