ON THE RATIONALITY OF CERTAIN MODULI SPACES RELATED TO CURVES OF GENUS 4

F. Catanese* - Dip. di Matematica
Università di PISA, via Buonarroti 2

INTRODUCTION

Let M_g be the coarse moduli space for complete smooth curves of genus g, let R_g be the "Prym moduli space" of unramified (connected) double covers of curves of genus g; a general problem is: what can be said about the birational structure of M_g, R_g?

From the point of view of birational geometry we can also talk about $M_{g,n}$, the moduli space of curves of genus g together with an ordered n-tuple of points, though this moduli functor is not representable in general (cf. [10]).

Our main results are:

<u>Theorem A</u>: R_4 is a rational variety.

<u>Theorem B</u>: $M_{4,1}$ is a rational variety.

<u>Theorem C</u>: M_4 admits a covering of degree 24 by a rational variety.

To put these results into perspective, we notice that, while the rationality of M_1, R_1 is classical and well-known, the rationality of M_2 has been proved by Igusa (cf. [8], also [17]).

For higher values of the genus g, the situation is as follows:

i) M_g is known to be unirational for $g \leq 10$, ([16], [1]), $g = 12$ ([14]), uniruled for $g = 11$ ([9]), whereas, for g odd ≥ 25 M_g is variety of general type ([7]), and D. Mumford and J. Harris announced a similar result also for g even ≥ 40

ii) the unirationality of R_g for $g = 5,6$ has been proven only recently ([4], [6]).

If the base field is of characteristic $\neq 2$, R_g is a covering of M_g of degree $2^{2g} - 1$, so that theorems A and C produce two rational coverings of

---

*) Part of this research was done when the author was at the Institute for Advanced Study, partially supported by NSF grant MCS 81-033 65.
The author is a member of G.N.S.A.G.A. of C.N.R.

M_4 of degrees, respectively, 255 and 24.

I should finally remark that it is conjectured that M_3, M_4 are rational varieties, but, to my knowledge, this is still unsolved.

Our method of proof does not use classical invariant theory: our strategy consists in constructing, using the geometry of curves of genus 4, some rational Galois covers by some rational variety, and then computing explicitly the subfield of invariants.
These covers are constructed with elementary arguments in the case of theorems B),C) For theorem A) I use a classical result of Wirtinger ([18]), which can also be found in [5], and about which I was told by S. Recillas, (cf. [13]), to whom I am indebted for noticing a mistake in an earlier proof of Theorem 1.5.

Our notation is as follows:
k is an algebraically closed field of char. $\neq 2$
X is a complete smooth curve of genus g defined over k
Pic(X) is the group of divisors on X modulo linear equivalence, here denoted by $\equiv$.
η is a divisor in $\mathrm{Pic}_2(X) - \{0\}$, i.e. $2\eta \equiv 0$, $\eta \not\equiv 0$.
K is a canonical divisor on any Gorenstein variety Y, i.e. $\mathcal{O}_Y(K_Y) \cong \omega_Y$.
If D is a divisor, $|D|$ is the linear system of effective divisors $D' \equiv D$.
$\underline{S}_n$ is the symmetric group in n letters, and
V_n is the standard (permutation) representation on k^n.
Given any coherent sheaf $\mathcal{F}$ on a complete variety Y, we denote by $h^i(\mathcal{F})$ the dimension of $H^i(Y,\mathcal{F})$ as a k-vector space.
If U is a k-vector space, we denote by U^* its dual space.
R.R. is an abbreviation for the Riemann-Roch theorem

§1. GEOMETRY OF CURVES OF GENUS 4.

Let X be a non-hyperelliptic curve of genus 4. Then the linear system $|K_X|$ gives an embedding of X in $\mathbb{P}^3$ such that the image of X is the complete intersection of a quadric Q and of a cubic G.

The quadric Q is uniquely determined, and, since it is normal, as well as G, there are only two possibilities:

i) Q is smooth

ii) Q is a quadric cone.

Case ii) occurs if and only if there exists a half canonical divisor ν (i.e. $2\nu \equiv K_X$) such that $h^0(\mathcal{O}_X(\nu)) = 2$: we say then that X has a vanishing thetanull.

It is well-known that M_4 is an irreducible variety of dimension 9 and that curves with a vanishing thetanull form an 8-dimensional subvariety, hyperelliptic curves form a 7-dimensional subvariety.

<u>Definition 1.1</u>. Let $\eta \in \mathrm{Pic}_2(X) - \{0\}$. We shall say that the pair (X,η) is bi-elliptic if there exist an elliptic curve E, a double covering $f:X \to E$, and a divisor $\eta' \in \mathrm{Pic}_2(E) - \{0\}$ such that $\eta \equiv f^*(\eta')$.

<u>Definition 1.2</u>. A normal cubic surface G in $\mathbb{P}^3$ is said to be symmetric if its equation can be written as the determinant of a symmetric 3×3 matrix of linear forms (cf. [2]). A symmetrization of G is the datum of such a matrix $(a_{ij}(y)) = (a)$, where $y = (y_0,y_1,y_2,y_3)$ are coordinates in $\mathbb{P}^3$, up to the action of $PGL(3)$ (such that, for $g \in GL(3)$, $(a) \longmapsto {}^t g(a)g$).

How many symmetric cubics with a symmetrization are there in $\mathbb{P}^3$, up to the action of $PGL(4)$?

The answer is: as many as there are pencils of conics in $\mathbb{P}^2$, up to the action of $PGL(3)$.

In fact, let U be the space $\mathrm{Sym}^2(k^3)$ of symmetric 3×3 matrices; then $\mathbb{P}(U)$ is the space of conics in $\mathbb{P}^2$, and $\mathbb{P}(U)$ contains the cubic determinantal hypersurface $\Delta = \{\det(a_{ij}) = 0\}$: Δ is the dual variety of the Veronese surface W^* in $\mathbb{P}(U^*)$, and its singular locus is the Veronese surface W in $\mathbb{P}(U)$.

Now, the datum of a symmetrization amounts to giving a $\mathbb{P}^3 \subset \mathbb{P}(U)$ such that $\mathbb{P}^3 \cap \Delta$ is a normal cubic. But giving a $\mathbb{P}^3 \subset \mathbb{P}(U)$ is equivalent to giving a $\mathbb{P}^1$ in $\mathbb{P}(U^*)$, i.e. a pencil of conics.

Notice that the number of base points in the pencil of conics is the cardinality of $\mathbb{P}^3 \cap W$, the number of degenerate conics in the pencil is the cardinality of $\mathbb{P}^1 \cap \Delta^*$.

The following is the list of pencils of conics (up to projective equivalence):

i) pencils of reducible conics: $\lambda x_1^2 + \mu x_1 x_2 = 0$, or $\lambda x_1 x_2 + \mu x_2 x_3 = 0$

ii) pencil with 4 base points: $\lambda x_1 x_2 + \mu x_3(x_1+x_2+x_3) = 0$

iii) pencil with 3 base points: $\lambda x_1 x_2 + \mu x_3(x_1-x_2) = 0$

iv) pencil with 2 base points, 2 degenerate conics: $\lambda x_1 x_2 + \mu x_3^2 = 0$

v) pencil with 2 base points, one reducible conic: $\lambda x_1 x_2 + \mu(x_1 x_3 - x_2^2) = 0$

vi) pencil with 1 base point: $\lambda x_1^2 + \mu(x_1 x_3 - x_2^2) = 0$.

Correspondingly we get the following symmetrizations:

i) $\begin{pmatrix} 0 & 0 & y_1 \\ 0 & y_0 & y_2 \\ y_1 & y_2 & y_3 \end{pmatrix}, \begin{pmatrix} y_0 & 0 & y_3 \\ 0 & y_1 & 0 \\ y_3 & 0 & y_2 \end{pmatrix}$ $\quad G \equiv \{y_0 y_1^2 = 0\}$, respectively

$G \equiv \{y_0 y_1 y_2 - y_1 y_3^2 = 0\}$, so G is reducible, and this case must be excluded,

ii) $\begin{pmatrix} y_0 & 0 & y_2 \\ 0 & y_1 & y_3 \\ y_2 & y_3 & (-y_2-y_3) \end{pmatrix}$ $\quad G \equiv \{y_0 y_1(y_2+y_3) + y_0 y_3^2 + y_1 y_2^2 = 0\}$

Here G has 4 singular points, and is also projectively equivalent to the 4-nodal cubic of Cayley of equation $\sigma_3(y) = \sum_{i=0}^{3} \frac{y_0 y_1 y_2 y_3}{y_i} = 0$.

iii) $\begin{pmatrix} y_0 & 0 & y_2 \\ 0 & y_1 & y_2 \\ y_2 & y_2 & y_3 \end{pmatrix}$ $\quad G \equiv \{y_0 y_1 y_3 - y_0 y_2^2 - y_1 y_2^2 = 0\}$.

G has three singular points, two nodes and a singularity of type A_3 at $\{y_0 = y_1 = y_2 = 0\}$.

iv) $\begin{pmatrix} y_0 & 0 & y_2 \\ 0 & y_1 & y_3 \\ y_2 & y_3 & 0 \end{pmatrix}$ $\quad G \equiv \{y_0 y_3^2 + y_1 y_2^2 = 0\}$.

The line $y_3 = y_2 = 0$ is singular, so this case must be excluded.

v) $\begin{pmatrix} y_0 & 0 & y_1 \\ 0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \end{pmatrix}$ $\qquad G \equiv \{y_0y_1y_3 - y_0y_2^2 - y_1^3 = 0\}.$

G has two singular points, one node at $\{y_1 = y_2 = y_3 = 0\}$, and a singular point of type A_5 at $\{y_0 = y_1 = y_2 = 0\}$.

vi) $\begin{pmatrix} 0 & y_0 & y_1 \\ y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \end{pmatrix}$ $\qquad G \equiv \{y_3y_0^2 + y_1^3 - 2y_0y_1y_2\}$

The line $y_0 = y_1 = 0$ is singular: moreover the plane $\{y_0 = 0\}$ is in the tangent cone at every point of the singular line so that this cubic is not projectively equivalent to the one in iv).

Theorem 1.3. Every symmetric cubic G has only one symmetrization. Moreover, G has exactly one finite irreducible double cover ramified exactly at the singular points of G.

Proof. The first statement follows from the above list. Now, let $Y \xrightarrow{f'} G$ be a double cover, and notice that G has only singularities of type A_n, $n = 1,3,5$. The fibre product $Z' = Y \times_G \tilde{G}$, where $\tilde{G}$ is a minimal desingularization of G, is an irreducible finite cover of $\tilde{G}$.
Therefore, if Z is the normalization of Z', there exists a reduced effective divisor E with support in the exceptional divisor of $p: \tilde{G} \to G$, and a divisor L on $\tilde{G}$ such that $2L \equiv E$, and $f: Z \to \tilde{G}$ is the double cover of $\tilde{G}$ in $\mathcal{O}_{\tilde{G}}(L)$ branched over E. Hence Z has only nodes as singularities, and $\omega_Z = f^*(-H) + f^*(L)$, where H is an hyperplane section of $\tilde{G}$. Z is then a rational variety, therefore $f'_*\mathcal{O}_Y$ is a Cohen-Macaulay sheaf on $\mathbb{P}^3$, with support on G (cf. [2], prop. 2.18). It follows also, by the Riemann-Roch theorem, that $h^0(\mathcal{O}_{\tilde{G}}(H-L)) = 3$. Applying theorem 2.19 of [2] we prove that a double cover as above gives a symmetrization.
Conversely, consider the sheaf F cokernel of $0 \to \mathcal{O}^3_{\mathbb{P}^3} \xrightarrow{(a_{ij}(y))} \mathcal{O}_{\mathbb{P}^3}(1)^3 \to F \to 0$, and define Y to be $\mathrm{Spec}(\mathcal{O}_G \oplus F)$, with algebra structure given as in [2], cor. 2.17.
Q.E.D.

We observe now that if we write $X = Q \cap G$, where G is symmetric, we are giving, as X is smooth, an unramified double cover $\tilde{X}$ of X, induced from the double cover $Y \to G$, or, equivalently a divisor $\eta \in \mathrm{Pic}_2(X)$. Now η is not trivial if $h^0(X,\mathcal{O}_X(K+\eta)) = 2$, and this follows from the exact sequence

$$(1.4) \qquad 0 \to H^0(\mathcal{O}_{\tilde{G}}(-H+L)) \to H^0(\mathcal{O}_{\tilde{G}}(H+L)) \to H^0(\mathcal{O}_X(K+\eta)) \to 0.$$

We are going now to prove a converse to this statement,

<u>Theorem 1.5</u>. (Wirtinger-Coble-Recillas).

Let X be a curve of genus 4, not hyperelliptic and with no vanishing theta null. Then giving a divisor $\eta \in \mathrm{Pic}_2(X) - \{0\}$, such that (X,η) is not bielliptic, is equivalent to writing X as the complete intersection of a smooth quadric Q and a symmetric cubic G.

<u>Proof</u>. We have already proven that giving G symmetric containing X determines an $\eta \in \mathrm{Pic}_2(X) - \{0\}$.
Conversely, consider the rational mapping $\Phi: X \to \mathbb{P}^2$ given by the linear system $|K_X + \eta|$. We break up the proof in several steps.

(1.6) $\quad |K + \eta|$ has no base points if X is not hyperelliptic.

<u>Proof</u>. Let $p \in X$: since $H^1(\mathcal{O}_X(K+\eta)) = 0$ p is not a base point if and only if $H^1(\mathcal{O}_X(K+\eta-p)) = 0$. By Roch's duality, this is equivalent to $|p - \eta| \neq \emptyset$. But if $q \in |p-\eta|$, then $2p \equiv 2q$, with $q \neq p$, and X is hyperelliptic.

<u>Q.E.D.</u>

Then Φ is a morphism. Denote by $C = \Phi(X)$, so that $\deg C \cdot \deg \Phi = 6$

(1.7) $\quad \deg C \geq 3$ if X is not hyperelliptic.

<u>Proof</u>. If C is a smooth conic, then, let D be the inverse image of a general point in C: we have $h^0(\mathcal{O}_X(D)) \geq 2$, and D has degree 3. Let $D' = K - D$: by R.R. $h^0(\mathcal{O}_X(D')) = 2$, and $D' \not\equiv D$, since $2D \equiv K + \eta$, $\eta \not\equiv 0$. Since $|D|$ has no base points, by the "base point free pencil trick" (cf. [11]), $H^0(\mathcal{O}_X(K)) \cong$ $\cong H^0(\mathcal{O}_X(D')) \otimes H^0(\mathcal{O}_X(D))$. Since X is not hyperelliptic $H^0(\mathcal{O}_X(K))^{\otimes 2} \to H^0(\mathcal{O}_X(2K))$ is surjective. Since $2D' \equiv 2D$, it follows that then $H^0(\mathcal{O}_X(2D))^{\otimes 2} \to H^0(\mathcal{O}_X(2K))$ is surjective: this is anyhow absurd since $|2K|$ gives a birational morphism.

<u>Q.E.D.</u>

(1.8) $\quad$ If C is a singular cubic, then its normalization $\tilde{C}$ is $\mathbb{P}^1$, and, since Φ factors through $\tilde{C}$, X is hyperelliptic.

(1.9) C is a smooth cubic if and only if (X,η) is bielliptic.

Proof. If C is a smooth cubic, then $\Phi_*\mathcal{O}_X = \mathcal{O}_C \oplus \mathcal{O}_C(-E)$, $K_X = \Phi^*(E)$, where E is an effective divisor on C of degree 3. Let H be a hyperplane divisor on C. Then $\eta \equiv \Phi^*(H-E)$, but $\Phi^*:\mathrm{Pic}(C) \to \mathrm{Pic}(X)$ is injective ([12], pag. 332), hence (X,η) is bielliptic, with $\eta' \equiv H-E$.
Conversely, $K_X + \eta \equiv f^*(E+\eta')$, where $f_*\mathcal{O}_X \simeq \mathcal{O}_C \oplus \mathcal{O}_C(-E)$. But, by the Leray spectral sequence for the map f, $H^o(X,\mathcal{O}_X(K_X+\eta)) \cong f^*(H^o(C,\mathcal{O}_C(E+\eta'))$, therefore Φ factors through f and an embedding of C as a plane cubic.

Q.E.D.

Remark 1.10. An easy computation shows that bielliptic pairs form a six-dimensional subvariety of $\mathcal{R}_4$.

We are at the last step of the proof: $\Phi:X \to C$ is a birational morphism, therefore can be factored through a finite sequence of blow-ups.
We are therefore in the following situation: we are given a surface S obtained from $\mathbb{P}^2(\Phi:S\to\mathbb{P}^2)$ by a finite sequence of blow ups of the (possibly infinitely near) singular points of C, of multiplicities $r_1 \geq r_2 \geq \dots r_k$.
Let $E_1, \dots E_k$ be the total transforms of the exceptional curves of each blow up, H the total transform of a line in $\mathbb{P}^2$.
Then, on S, we have

(1.11) $H^2 = 1$, $H\cdot E_i = 0$, $E_i^2 = -1$, $E_i\cdot E_j = 0$ for $i \neq j$,

(1.12) $K_S \equiv -3H + \sum_{i=1}^{k} E_i$, $X \equiv 6H - \sum_{i=1}^{k} r_iE_i$.

(1.13) Let Δ be $\sum_{i=1}^{k}(r_i-1)E_i$; by the adjunction formula $\mathcal{O}_X(K_X) = \mathcal{O}_X(3H-\Delta)$, and, on X, $\Delta \equiv 3H-K_X \equiv 2H+\eta$.

Therefore i) $H^o(\mathcal{O}_X(2H-\Delta)) = 0$, hence $H^o(\mathcal{O}_S(2H-\Delta)) = 0$

ii) $\Delta\cdot K_X = 12 = \sum_{i=1}^{k} r_i(r_i-1)$.

Since $H^1(S,K_S) = H^1(S,\mathcal{O}_S) = 0$, we have an isomorphism of $H^o(\mathcal{O}_S(3H-\Delta)) \to H^o(\mathcal{O}_X(K_X))$ given by restriction.

Let H' be the inverse image of a line not passing through the singular points of C. Since $H'(3H-\Delta) = 3$, the exact sequence

$$(1.14) \qquad 0 = H^o(\mathcal{O}_S(2H-\Delta)) \to H^o(\mathcal{O}_S(3H-\Delta)) \to H^o(\mathcal{O}_{H'}(3H-\Delta))$$

says that the rational map $\psi : S \to \mathbb{P}^3$ given by the linear system $|3H - \Delta|$ embeds H' as a twisted cubic.

Therefore, if $G = \psi(\mathbb{P}^2)$, G contains a 2-parameter family of twisted cubics, two of which intersect in only one point, so that G is not a smooth quadric.

If X has no vanishing thetanull, G must be a cubic surface.

Let F be the fixed part of the linear system $|3H - \Delta|$, so that $|3H-\Delta| = F + |M|$, $M^2 = \deg(G)$, $M\cdot F \geq 0$. Since ψ embeds a general line in $\mathbb{P}^2$, F is a sum of exceptional curves, each of which can be written either in the form E_i, or in the form $E_i - E_j$, $i > j$.

Then $M^2 = (F+M)^2 - (F+M)\cdot F - FM \leq (3H-\Delta)^2 - (3H-\Delta)\cdot F$.

Now $(3H-\Delta)\cdot F = -\Delta F = -\sum_{i=1}^{k} (r_i-1)E_i\cdot F$, and for each component of F, $-\Delta\cdot F = 0$ or 1 $(-\Delta\cdot E_i = r_i - 1 \geq 1$, $-\Delta(E_i-E_j) = r_i - r_j \geq 0$ since $i > j)$.

$$\text{Hence} \quad M^2 \geq (3H-\Delta)^2 = 9 - \sum_{i=1}^{k} (r_i-1)^2 = 9 - \sum_{i=1}^{k} r_i(r_i-1) - \sum_{i=1}^{k} (r_i-1) = -3 + \sum_{i=1}^{k} (r_i-1).$$

If $\deg(G) = M^2 = 3$, then $r_1 = 2$; if $\deg(G) = 2$, then the only other possibility is $r_1 = 3$, $r_2 = 2$.

We can assume from now on $r_1 = 2$. By the exact sequence

$$(1.15) \qquad 0 \to \mathcal{O}_S(4H-2\Delta-X) \to \mathcal{O}_S(4H-2\Delta) \to \mathcal{O}_X \to 0$$

since $\quad H^1(\mathcal{O}_S(4H-2\Delta-X)) = H^1(\mathcal{O}_S(-2H)) = 0$, we conclude that

(1.16) $\quad |4H - 2\Delta|$ has dimension 0.

Let D be the unique divisor in $|4H-2\Delta|$: since $D \cap X = \emptyset$, D is mapped to the singular points of G, and, since $D \equiv 2(2H-\Delta)$, $|2H - \Delta| = \emptyset$, S admits a double covering ramified exactly on D; hence G admits a double covering ramified at most on the singular points, so that G is a symmetric cubic.

Since $\mathcal{O}_X(2H-\Delta) = \mathcal{O}_X(\eta)$, we have proven that G induces on X the double cover associated to η.

Conversely, given X as $Q \cap G$, let η be the induced divisor: then Φ is induced by the linear system $|H - L|$ on $\tilde{G}$. If H'' is a general hyperplane section of G, one has a restriction isomorphism of $H^o(\mathcal{O}_{\tilde{G}}(H-L)) \to H^o(\mathcal{O}_{H''}(H-L))$, therefore, if we denote still by Φ the rational map $\Phi:\tilde{G} \to \mathbb{P}^2$ given by $|H - L|$, Φ embeds H'' as a smooth plane cubic. Since through any two general points x,y of G there passes a plane section H'' as above, Φ is birational, and the inverse map $\psi:\mathbb{P}^2 \to G$ is given by a system of plane cubics. Since $\Phi_{|X}$ is a morphism, clearly $|K + \eta|$ gives a birational morphism and (X,η), by (1.9), is not bi-elliptic.

Q.E.D.

Just for completeness, we indicate, for the three types of symmetric cubics, which are the systems of plane cubics giving the rational map ψ.

In case ii) we consider the six points of intersection of four independent lines in $\mathbb{P}^2$, and we blow then up to get $S \cong \tilde{G}$, with $\Delta = \sum_{i=1}^{6} E_i$, and $D \in |4H - 2\Delta|$ given by the of the proper transforms of the four lines (cf. e.g. [3]).

In case iii): take three lines L_1, L_2, L_3 in general position in $\mathbb{P}^2$ and blow up $\mathbb{P}^2$ at the three points $L_i \cap L_j$, at a fourth point $P_4 \in L_3$, and then at the 2 infinitely near points P_{4+i} lying over $L_i \cap L_3 = P_i$ $(i=1,2)$ in the direction of L_i. Let $P_3 = L_1 \cap L_2$.

Here you obtain S where $D \in |4H - 2\Delta|$ is given by the proper transform of $2L_3 + L_1 + L_2$ together with $E_1 - E_5$, $E_2 - E_6$, and $S \cong \tilde{G}$.

The double cover Z of S is smooth, being branched on the proper transforms of L_1, L_2, and $(E_1 - E_5)$ $(E_2 - E_6)$, i.e. on a smooth divisor consisting of four (-2) rational curves, while the finite cover Y has just a node as singularity, lying over the A_3 singular point of G.

Since we believe that case v) is the least known, we explain how to obtain the mapping ψ.

Choose w_0, w_1, w_2 a basis of $H^0(\mathcal{O}_{\tilde{G}}(H-L))$ such that (cf. [2], cor. 2.17) the following relations hold:

$$(1.17)\qquad \begin{cases} y_0 w_0 + y_1 w_2 = 0 \\ y_1 w_1 + y_2 w_2 = 0 \\ y_1 w_0 + y_2 w_1 + y_3 w_2 = 0 \end{cases}$$

We can solve these as linear equations in $y_0, \dots y_3$ and express then as homogeneous polynomials in (w_0, w_1, w_2).

We get $y_0 = w_2^3$, $y_1 = -w_0 w_2^2$, $y_2 = w_0 w_1 w_2$, $y_3 = w_0(w_0 w_2 - w_1^2)$, and this is an expression of ψ in appropriate coordinates on $\mathbb{P}^2$ and $\mathbb{P}^3$.

The system of cubics has 2 base points, namely $\{w_2 = w_0 = 0\} = P$, and $\{w_2 = w_1 = 0\} = P'$, and a general cubic of the system is smooth at P, P':

but to obtain a system free of base points one has to blow up three times over P at the points where the line $\{w_0 = 0\}$ (whose proper transform will be denoted by L_0) passes and three times over P' at the points where the conic $\{w_0 w_2 - w_1^2 = 0\}$ passes through.
Denote by L_2 the proper transform of the line $\{w_2 = 0\}$.
We get thus E_1, E_2, E_3, E_1', E_2', E_3' on S, and we notice that $L_2^2 = L_0^2 = -2$, L_2 intersects transversally in exactly one point $(E_1 - E_2)$, $(E_2' - E_3')$; L_0 intersects E_3 transversally in exactly one point.
The total transform of the quartic $\{w_0 w_2^3 = 0\}$ is thus
$L_0 + (E_1 + E_2 + E_3) + 3(E_1' + E_2' + E_1) + 3L_2$ i.e. $3L_2 + L_0 + 2\Delta + (E_1' - E_2') + 2(E_2' - E_3') + 2(E_1 - E_2) + (E_2 - E_3)$.
The normal double cover Z of $S = \tilde{G}$ is thus ramified on $L_2 + L_0 + (E_2 - E_3) + (E_1' - E_2')$, hence Z is smooth, and the finite cover Y of G has just a singular point of type A_2 lying over the singular point of G of type A_5.

The meaning of theorem 1.5 in terms of $\mathcal{R}_4$ is the following

Theorem 1.18. $\mathcal{R}_4$ is an irreducible variety, birational to the quotient $\mathbb{P}(\mathrm{Sym}^2(V_4))/\underline{S}_4$, where V_4 is the standard representation of

Proof. Since $\mathcal{R}_4$ is a finite cover of $\mathcal{M}_4$, it is pure dimensional.

Let A be the open set of $\mathcal{R}_4$ corresponding to pairs (X,η) such that:

i) X is not hyperelliptic

ii) X has no vanishing thetanull

iii) (X,η) is not bielliptic.

By remark 1.10 and the considerations made at the beginning of the paragraph A is dense.

Let Q be a fixed smooth quadric in $\mathbb{P}^3$, and let B be the open set in the space of symmetric 3×3 matrices of linear forms such that, if $(a_{ij}(y)) \in B$, $G = \det(a_{ij}(y))$ is a normal cubic and $X = G \cap Q$ is a smooth curve of degree 6.

In view of theorem 1.5, there is a morphism of B onto A which is a quotient by the previously described action of $GL(3)$ on B. Hence $\mathcal{R}_4$ is irreducible (actually this was known already).

Moreover, let B' be the open subset of B such that G is a 4-nodal cubic (case ii)), and A' its image in $\mathcal{R}_4$: A' is again dense, being non-empty.

Assume that (X,η) corresponds to giving generators Q, G of the ideal of X in $\mathbb{P}^3$ such that G is a symmetric cubic, and analogously (X',η') corresponds to (Q',G'); if $f: X \to X'$ is an isomorphism such that $f^*(\eta') = \eta$, then

f is induced by a projectivity $g:\mathbb{P}^3 \to \mathbb{P}^3$ such that $Q = g^*(Q')$, $G = g^*(G')$, by theorem 1.5, and, conversely, such a projectivity induces an isomorphism of the pair (X,η) with the pair (X',η').

Since all 4-nodal cubics are projectively equivalent, we can fix the 4-nodal cubic to be G_0, the cubic of equation $\sigma_3(y) = \sum_{i=1}^{4} \frac{y_1y_2y_3y_4}{y_i} = 0$.

Consider now the open set Λ in $\mathbb{P}(\mathrm{Sym}^2(V_4))$ corresponding to the quadrics Q in $\mathbb{P}(V_4)) = \mathbb{P}^3$ such that $Q \cap G_0$ is a smooth sextic curve X. We get thus a morphism f of Λ into R_4, with $f(\Lambda) \supset A'$, such that Q, Q' map to the same pair (X,η) if and only if there exists $g \in PGL(4)$ such that $g(G_0) = G_0$, $g(Q) = Q'$. We conclude the proof since it is well-known that $\underline{S}_4$ is the group of projective automorphisms of G_0.

Q.E.D.

We want to find out now a dominant rational map of $\mathbb{P}^{10}$ to $M_{4,1}$.
To do this, recall that a curve of genus 4 X which is not hyperelliptic and has no vanishing thetanull is a smooth divisor of bidegree $(3,3)$ on $Q = \mathbb{P}^1 \times \mathbb{P}^1$.

Fix three points $\infty, 0, 1$ in $\mathbb{P}^1$ and let $p \in Q$ be the point (∞,∞), M be the (unordered) set of five points $\{(\infty,\infty), (\infty,0), (\infty,1), (0,\infty)\ (1,\infty)\}$.

Given a general $[C',p'] \in M_{4,1}$ we can assume to have chosen coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $p = (\infty,\infty)$, and that the two lines $\{\infty\} \times \mathbb{P}^1$, $\mathbb{P}^1 \times \{\infty\}$ intersect C' in three distinct points. Let I_M be the ideal sheaf of M on Q. Therefore if we take the linear system $|G| = |I_M(3,3)|$ we obtain a rational dominant map of $|G|$ onto $M_{4,1}$ just by sending $C \in |G|$ to the pair $[C,(\infty,\infty)]$.

Assume now that two pairs $C, C' \in |G|$ are isomorphic: then there exists an automorphism g of $\mathbb{P}^1 \times \mathbb{P}^1$ which leaves (∞,∞) fixed and such that $g(C) = C'$, since all the automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ are induced, via the Segre embedding, by automorphisms of $\mathbb{P}^3$.
But now g leaves the set $A = (\mathbb{P}^1 \times \{\infty\}) \cup (\{\infty\} \times \mathbb{P}^1)$ invariant, and, since $M = A \cap C = A \cap C'$, $g(M) = M$.

Let us choose affine coordinates (x,y) on $\mathbb{P}^1 \times \mathbb{P}^1 - A$: then g belongs to the group generated by the involution g_3 such that $g_3(x,y) = (y,x)$, and by the two involutions g_1, g_2 such that $g_1(x,y) = (1-x,y)$, $g_2(x,y) = (x,1-y)$.

Let $r = g_3g_1$: then r has period 4; if we set $s = g_3$, then $s^2 = 1$, $r^4 = 1$, $sr^3 = rs = g_2$, $r^2 = g_1g_2$, and our group is the dihedral group D_4.
We can thus reformulate our discussion with the following

<u>Theorem 1.19</u>. $M_{4,1}$ is the quotient of $\mathbb{P}^{10}$ by a suitable action of the dihedral group D_4.

For the geometrical construction underlying theorem C, consider again a non-hyperelliptic curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1 = Q$. In this picture we have in Q a family of lines of the form $\mathbb{P}^1 \times \{a\}$, $a \in \mathbb{P}^1$, and another of the form $\{b\} \times \mathbb{P}^1$, $b \in \mathbb{P}^1$, which we visualize as being orthogonal to the first.

Definition 1.20. A rectangle R in Q is the union of four distinct lines in Q, of the form $R = (\mathbb{P}^1 \times \{b\}) \cup (\mathbb{P}^1 \times \{b'\}) \cup (\{a\} \times \mathbb{P}^1) \cup (\{a'\} \times \mathbb{P}^1)$. Its vertices are the four points (a,b), (a,b'), (a',b), $(a'b')$ and if they all belong to C we shall say that R is inscribed into C.

Theorem 1.21. A general curve C of genus 4 admits 6 inscribed rectangles (lying in the unique quadric Q containing the canonical image of C).

Proof. Consider C^4 and the four projections $f_i : C^4 \to C$ $(i=1,\dots 4)$ on the four factors of the product. Let moreover $p,p' : Q \to \mathbb{P}^1$ be the two natural projections: they define two divisors of degree 3 on C, which we denote, respectively, by D and D'.

Let D_i be the divisor on C^4 such that $D_i = f_i^*(D)$ (resp. $D_i' = f_i^*(D')$); let moreover $\Delta_{ij} \subset C^4$ be $\{(y_1,y_2,y_3,y_4) \ \ y_i = y_j\}$

Consider in C^4 the subvariety $W = \{(y_1,y_2,y_3,y_4) \mid p(y_1) = p(y_2)$, $p(y_3) = p(y_4)$, $p'(y_1) = p'(y_4)$, $p'(y_2) = p'(y_3)\}$.

Given an inscribed rectangle R and a vertex x of R one determines a unique point $y = (y_1,y_2,y_3,y_4)$ in W with $y_1 = x$, and such that $y \in W - \bigcup_{i \le j} \Delta_{ij}$.

Conversely, if $y \in W - \Delta_{12} - \Delta_{34} - \Delta_{14} - \Delta_{23}$, then also $y_1 \neq y_3$ since otherwise $p'(y_1) = p'(y_3) = p'(y_2)$, and, since $p(y_1) = p(y_2)$, one would have $y_1 = y_2$; analogously one has $y_2 \neq y_4$.
Therefore the points of $W - \Delta_{12} - \Delta_{34} - \Delta_{14} - \Delta_{23}$ are in a bijection with the pairs (R,x) where R is a rectangle inscribed into C, x is a vertex of R.
Now the above mentioned set is the complete intersection of four divisors.
In fact, consider in C^2 the divisor $B = \{(y_1,y_2) \mid p(y_1) = p(y_2)\}$. $B = \Delta + \Gamma$, where Δ is the diagonal of $C \times C$, and Γ is smooth away from Δ since p is a covering of degree equal to three.
B is the pull back of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ under the morphism $p \times p : C^2 \to (\mathbb{P}^1)^2$, therefore its class as a divisor on C^2, using our previous notations $(f_i : C^2 \to C$, $i=1,2$, being the two projections), is just $D_1 + D_2$. Since C has genus four $\Delta^2 = -6$, moreover $B \cdot \Delta = 6$, so that $\Gamma \cdot \Delta = 12$.

Consider the monodromy of $p: C \to \mathbb{P}^1$: if C is general, then p has only ordinary ramification, i.e.

a) Γ and Δ intersect transversally

b) the monodromy of p is generated by transpositions

Since C is connected b) implies that the monodromy is the full symmetric group, hence, in general, Γ is smooth, irreducible, transversal to Δ in the points corresponding to the ramification points of p.

Considering the projection p' instead of p, we define analogously $\Gamma' \subset C^2$.

Let $\Gamma_{ij} = (f_i \times f_j)^*(\Gamma)$, $\Gamma'_{hk} = (f_h \times f_k)^*(\Gamma')$.

Then we claim that $W - \Delta_{12} - \Delta_{34} - \Delta_{14} - \Delta_{23} = \Gamma_{12} \cap \Gamma_{34} \cap \Gamma'_{14} \cap \Gamma'_{23}$, and the intersection is transversal, for C general.

In fact, if $(y_1,y_2) \in \Delta_{12} \cap \Gamma_{12}$, then $y_1 = y_2$ and y_1 is a ramification point of p: since $y = (y_1,\dots y_4) \in W$ it follows that $y_3 = y_4$, hence y_3 is a second ramification point of p, and $p'(y_1) = p'(y_3)$.

It is easy to see that curves C of type $(3,3)$ in Q such that the above situation can hold form a proper subvariety in the linear system $|\mathcal{O}_Q(3,3)|$

To show that $\Gamma_{12} \cap \Gamma_{34} \cap \Gamma'_{14} \cap \Gamma'_{23}$ gives a transversal intersection for C general, we consider the variety $\Lambda \subset |\mathcal{O}_Q(3,3)| \times Q^4$ defined by

$$\Lambda = \{(C, y_1,y_2,y_3,y_4) \mid y_i \in C,\ i=1,\dots 4,\ p(y_1) = p(y_2),\ p(y_3) = p(y_4),$$
$$p'(y_2) = p'(y_3),\ p'(y_1) = p'(y_4)\}$$

Λ is of dimension 15 and smooth at the general point, hence our assertion is proven if the projection of Λ on $|\mathcal{O}_Q(3,3)|$ is surjective: but if this were not the case, for C general, $\Gamma_{12} \cap \Gamma_{34} \cap \Gamma'_{14} \cap \Gamma'_{23}$ would be empty.

Finally we compute: $\Gamma_{12} \cdot \Gamma_{34} \cdot \Gamma'_{14} \cdot \Gamma'_{23} = (D_1+D_2-\Delta_{12}) \cdot (D_3+D_4-\Delta_{34}) \cdot (D'_1+D'_4-\Delta_{14}) \cdot$ $\cdot\ (D'_2+D'_3-\Delta_{23}) = 2 \cdot 3^4 - 2 \cdot 3^3 \cdot 4 + 2 \cdot 3^2 \cdot 6 - 2 \cdot 3 \cdot 4 - 6 = 3^3 \cdot 2 - 30 = 24$; in fact $\Delta_{12} \cdot \Delta_{34} \cdot \Delta_{14} \cdot \Delta_{23}$ equals the self-intersection of Δ in $C \times C$.

Q.E.D.

Theorem C is now a straightforward consequence of theorem 1.21.

Namely, consider in $Q = \mathbb{P}^1 \times \mathbb{P}^1$ the following set of six points:

$M' = \{(\infty,\infty), (0,0), (0,\infty), (\infty,0), (1,\infty), (\infty,1)\}$.

Let $|G'|$ be the linear system $|\mathcal{I}_{M'}(3,3)|$: we can choose affine coordinates (x,y) on $\mathbb{P}^1 \times \mathbb{P}^1 - \{\infty\} \times \mathbb{P}^1 - \mathbb{P}^1 \times \{\infty\}$. Then $|G'|$ is the projective space associated with the vector space U spanned by the monomials

$$\begin{cases} x,\ x^2,\ x^2y,\ x^3y(1-y) \\ y,\ y^2,\ y^2x,\ y^3x(1-x) \\ xy,\ x^2y^2 \end{cases}$$

These monomials are permuted by the action on $|G'|$ induced by the automorphism $s:Q \to Q$ such that $s(x,y) = (y,x)$. It is then obvious that $|G'|_{/s} = \mathbb{P}(U)/_s$ is birational to $\mathbb{P}^9$. We conclude then this paragraph with

<u>Theorem C.</u> M_4 has covering of degree 24 by a rational variety. More precisely, the rational map of $|G'| \to M_4$ is a covering of degree 48 which factors through the action of s on $|G'|$, and a general point of $|G'|_{/s}$ corresponds to the datum of a triple (C,R,p) where C is a curve of genus 4, R is a rectangle inscribed into C, p is a vertex of R.

<u>Proof.</u> To a curve $C \in |G'|$ we associate the triple $(C, (\mathbb{P}^1 \times \{0,\infty\} \cup (\{0,\infty\} \times \mathbb{P}^1), (\infty,\infty))$.

Assume now that C, C' give isomorphic triples: then there exists $g \in \mathrm{Aut}(Q)$ such that $g(\infty,\infty) = (\infty,\infty)$, $g(C) = C'$, and for the rectangle $R = (\mathbb{P}^1 \times \{0,\infty\}) \cup (\{0,\infty\} \times \mathbb{P}^1)$ one has $g(R) = R$. In particular, $g(M') = M'$, so that necessarily g is either the identity or the involution s. The fact the degree of the rational map of $|G'|$ onto M_4 is 48 follows immediately from theorem 1.21.

Q.E.D.

§2. RATIONALITY OF THE INVARIANT SUBFIELDS.

Before turning to prove the rationality of R_4, we first state a more general auxiliary result.

Let V_n be the standard permutation representation of the symmetric group $\underline{S}_n$, V_n^m the direct sum of m copies of V_n. Then the field of rational functions on V_n^m, $k(V_n^m)$, can be written as $k(x_{11},\dots x_{1n}, x_{21},\dots x_{2n},\dots,x_{m1},\dots x_{mn})$ and a permutation τ acts on x_{ij} by sending it to $x_{i\tau(j)}$. Consider the following invariant rational functions, where σ_i denotes the i-th elementary symmetric function, and a variable with a cap has to be omitted:

$$(2.1)\quad \begin{cases} \sigma'_i = \sigma_i(x_{11},\dots x_{1n}) & i = 1,\dots n \\ \sigma_1^{(h)} = \sum_{j=1}^{n} x_{hj}x_{1j} & h = 2,\dots m \\ \sigma_i^{(h)} = \sum_{j=1}^{n} x_{hj}\ \sigma_{i-1}(x_{11},\dots,\hat{x}_{1j},\dots x_{1n}) & h = 2,\dots m \\ & i = 2,\ \dots m. \end{cases}$$

<u>Lemma 2.2</u>. The invariant subfield $k(V_n^m)^{\underline{S}_n}$ is a rational field: more precisely the nm functions given by 2.1 form a basis of the purely transcendental extension over k.

<u>Proof</u>. σ', $\sigma^{(2)}$, ..., $\sigma^{(m)}$ determine a morphism $\psi : V_n^m \to (\mathbb{A}^n)^m$ and to prove that ψ induces a birational map of $V_n^m/\underline{S}_n$ onto the affine space $(\mathbb{A}^n)^m$ it is enough to prove that on a Zariski open set of V_n^m $\psi(x) = \psi(y)$ if and only if there exists $\tau \in \underline{S}_n$ such that $\tau(x) = y$.
The "if" part being obvious, let's assume that $\psi(x) = \psi(y)$: then, in particular, $\sigma'(x) = \sigma'(y)$.

By virtue of the fundamental theorem on symmetric functions, we can assume, acting on y by a suitable $\tau \in \underline{S}_n$, that $x_{1j} = y_{1j}$ for $j = 1,\dots n$.

Let us set for convenience $z_j = x_{1j}$ $(j=1,..,n)$. Then the variables x_{hj}, y_{hj} $(h=2,\dots m,\ j=1,\dots n)$ are solutions, by 2.1, of the same system of $n(m-1)$ linear non-homogeneous equations, hence they are equal if the determinant of the system is non-zero.

The system being given by the matrix

$$(2.3)_n\quad \begin{pmatrix} z_1, & \dots ,z_n \\ \sigma_1(z_2,\dots z_n) & \dots ,\sigma_1(z_1,\dots,z_{n-1}) \\ \vdots & \vdots \\ \sigma_{n-1}(z_2,\dots z_n) & \dots ,\sigma_{n-1}(z_1,\dots,z_{n-1}) \end{pmatrix}$$

it suffices to verify that the determinant of the matrix $(2.3)_n$ is not identically zero. We prove this by induction on n, since for $n = 2$ we get $\det\begin{pmatrix} z_1 & z_2 \\ z_2 & z_1 \end{pmatrix} = z_1^2 - z_2^2$. For bigger n, the determinant of $(2.3)_n$ modulo z_n, is given, up to sign, by the product of $z_1 \dots z_{n-1} = \sigma_{n-1}(z_1, \dots z_{n-1})$ times the determinant of $(2.3)_{n-1}$.

Q.E.D.

Theorem A. R_4 is a rational variety.

Proof. In view of theorem 1.18 we have to show the rationality of $\mathbb{P}(\mathrm{Sym}^2(V_4))/\underline{S}_4$. We use here the fact that $\underline{S}_4$ has a normal subgroup $G \cong (\mathbb{Z}/2)^2$ given by the double cycles in $\underline{S}_4$; the quotient $\underline{S}_4/G$ is isomorphic to $\underline{S}_3$ and in this way any representation of $\underline{S}_3$ induces canonically a representation of $\underline{S}_4$ that we shall denote by the same symbol.
Since the action of $\underline{S}_4$ on $\mathrm{Sym}^2(V_4)$ is linear, it is clearly sufficient to prove the rationality of the quotient $\mathrm{Sym}^2(V_4)/\underline{S}_4$.
We subdivide the proof in four steps, noticing that we have the following chain of inclusions

$$(2.4) \qquad k(\mathrm{Sym}^2(V_4)) \supset k(\mathrm{Sym}^2(V_4))^G \supset k(\mathrm{Sym}^2(V_4))^{\underline{S}_4} = \left(k(\mathrm{Sym}^2(V_4))^G\right)^{\underline{S}_3} .$$

Let W_4 be the irreducible $\underline{S}_4$-submodule of V_4 generated by $x_1 - x_2$, $x_2 - x_3$, $x_3 - x_4$: $V_4 = \mathbb{1} \oplus W_4$, $\mathbb{1}$ being the trivial one dimensional representation spanned by $\sigma_1(x_1, \dots x_4)$.

Step I. $\mathrm{Sym}^2(V_4) \cong \mathbb{1} \oplus W_4^2 \oplus V_3$

Proof. $\mathrm{Sym}^2(V_4) \cong V_4' \oplus V_3' \oplus W_4'$ where V_4' is spanned by $x_1^2, x_2^2, x_3^2, x_4^2$, V_3' is spanned by $y_1 = x_1x_2 + x_3x_4$, $y_2 = x_1x_3 + x_2x_4$, $y_3 = x_1x_4 + x_2x_3$, W_4' is spanned by $w_1 = x_1x_2 - x_3x_4$, $w_2 = x_1x_3 - x_2x_4$, $w_3 = x_1x_4 - x_2x_3$.
V_4' is clearly isomorphic to V_4; also, since G acts trivially on V_3', V_3' is induced by a representation of $\underline{S}_3$.
V_3' has as basis three vectors corresponding to the three non-trivial double cycles of $\underline{S}_4$, and the action of $\underline{S}_4$ on the basis is given by conjugation in $\underline{S}_4$ (G acts trivially being abelian).
Observing that the transposition $(1,4)$ permutes y_1 with y_2 and leaves y_3 fixed, $(1,2)$ leaves y_1 fixed and permutes y_2 with y_3, we conclude that V_3' is isomorphic to V_3.

On W_4' we have the following actions:

$$(12)(34) \text{ acts by } \begin{cases} w_1 \mapsto w_1 \\ w_2 \mapsto -w_2 \\ w_3 \mapsto -w_3 \end{cases}, \quad (12) \text{ by } \begin{cases} w_1 \mapsto w_1 \\ w_2 \mapsto -w_3 \\ w_3 \mapsto -w_2 \end{cases}$$

$$(123) \text{ by } \begin{cases} w_1 \mapsto w_3 \\ w_2 \mapsto w_1 \\ w_3 \mapsto -w_2 \end{cases}, \quad (1234) \text{ by } \begin{cases} w_1 \mapsto -w_3 \\ w_2 \mapsto -w_2 \\ w_3 \mapsto w_1 \end{cases}$$

Let χ' be the character of W_4': the character of W_4 equals the character χ of V_4 minus 1, hence we conclude that $\chi - 1 = \chi'$ by computing explicitly the table of characters

Conjugacy classes	id	(12)(34)	(12)	(123)	(1234)
χ'	3	-1	1	0	-1
χ	4	0	2	1	0

If the characteristic of k is different from 2,3, this implies that $W_4' \cong W_4$; in characteristic 3, this is also true, because both representations are irreducible: in fact (cf. [15], pag. 155) their modular characters are indecomposable. □

To compute $k(\mathrm{Sym}^2(V_4))^G$, in view of step I, suffices

Step II. $k(W_4^2)^G = L$, if $w_1, w_2, w_3, w_1', w_2', w_3'$ are coordinates on $W_4 \oplus W_4$, is generated by

(2.5) $\quad w_1^2, w_2^2, w_1w_2w_3, w_iw_i' \quad (i=1,2,3)$

Proof. The six given functions are G-invariant, and $k(W_4^2) = L(w_1,w_2)$, so we have an extension of degree 4 and L is the whole subfield of G-invariants. □

Step III. Let F be the subfield of L generated by $w_1^2, w_2^2, w_3^2, w_iw_i'$. Then L is a quadratic extension of F given by $F(t)$, where $t = w_1w_2w_3$. $F = k(V_3^2)$ as a representation of $\underline{S}_3 = \underline{S}_4/G$.

Proof. Clearly $t \notin F$, $t^2 = (w_1^2)(w_2^2)(w_3^2)$. Also the action of $\underline{S}_4$ on V_3 differs from the one of $\underline{S}_4$ on W_3 only up to sign, i.e., as it is easy to verify, $\underline{S}_4$ acts by permuting the basis given by y_1, y_2, y_3, and if $\tau(y_i) = y_j$, then $\tau(w_i) = \pm w_j$, hence $\tau(w_i^2) = w_j^2$, $\tau(w_iw_i') = w_jw_j'$.
□

Step IV. Let M be the field generated by w_i^2, $w_i'w_i$, y_i $(i=1,2,3)$. M is a purely transcendental extension of k, $M \cong k(V_3^3)$, $k(\mathrm{Sym}^2V_4)^{\underline{S}_4} = M^{\underline{S}_3}(t,\sigma)$, where $\sigma = \sigma_1(x_1,\dots x_4)$, $t = w_1w_2w_3$.

Proof. $k(\mathrm{Sym}^2V_4)^{\underline{S}_4} = (M(t,\sigma))^{\underline{S}_3}$, but σ is an invariant for $\underline{S}_4$ from the very beginning, while t is an $\underline{S}_3$ invariant by the formulas written in step I. That M is isomorphic to $k(V_3^3)$ follows by step III.
□

End of the proof. $k(\mathbb{P}(\mathrm{Sym}^2V_4))^{\underline{S}_4} = M^{\underline{S}_3}(t)$. But, by lemma 2.1, $M^{\underline{S}_3}$ is a rational field with basis of transcendency $\sigma_1', \sigma_2', \sigma_3', \sigma_1^{(2)}, \sigma_2^{(2)}, \sigma_3^{(2)}, \sigma_1^{(3)}, \sigma_2^{(3)}, \sigma_3^{(3)}$. We conclude observing that $\sigma_i' = \sigma_i(w_1^2,w_2^2,w_3^2)$, hence $t^2 = \sigma_3'$.
Q.E.D.

Theorem B. $M_{4,1}$ is a rational variety.

Proof. By theorem 1.19 and the arguments preceding it we have a linear representation $\rho: D_4 \to \mathrm{Aut}(U)$ where U is 11-dimensional, and we know that $M_{4,1}$ is birational to $\mathbb{P}(U)/D_4$.
U has a basis given by the polynomials $1, x, x^2, y, y^2, xy, x^2y, xy^2, x^2y^2, xy^3(1-x), x^3y(1-y)$, and, if r, s are the generators of D_4, such that $s^2 = r^4 = 1$, $sr^3 = rs$,

$$s(x,y) = (y,x), \qquad r(x,y) = (y,1-x). \tag{2.6}$$

To decompose U as a direct sum of irreducibles, since D_4 has order 8 and we assume $\mathrm{char}(k) \neq 2$, we compute the character χ of ρ.
For s we observe that $\rho(s)$ permutes the elements of the basis, leaving $1, xy, x^2y^2$ fixed: hence $\chi(s) = 3$.
For sr, $sr(x,y) = (1-x,y)$ and choosing for U the new basis $x, (1-x), x(1-x), yx, y(1-x), yx(1-x), y^2x, y^2(1-x), y^2x(1-x), y^3x(1-x), x^3y(1-y)$, we see that the

trace of $\rho(sr)$ is 3 since sr permutes the first 10 elements of the basis, leaving 4 of them fixed, while $sr(x^3y(1-y)) = (1-x)^3(1-y)y = -x^3y(1-y)$ + terms of lower degree in x.
For $\chi(r)$, x, x^2, y, y^2, x^2y, xy^2, $xy^3(1-x)$, $x^3y(1-y)$ are easily seen to give a zero contribution (by degree considerations), while 1 is invariant, $xy \longmapsto y - xy$, $x^2y^2 \longmapsto y^2 - 2xy^2 + x^2y^2$, therefore $\chi(r) = 1$.
Since $r^2(x,y) = (1-x, 1-y)$, the trace of $\rho(r^2)$ is easily seen (in the first given basis) to be equal to $1 - 1 + 1 - 1 + 1 + 1 - 1 - 1 + 1 - 1 - 1 = -1$.
We now put together the character χ of ρ and the characters of the irreducible representations of D_4.

(2.7)

Conjugacy classes		1	$\{r, r^3\}$	$\{s, sr^2\}$	sr, sr^3	r^2
characters	ψ_1	1	1	1	1	1
	ψ_2	1	1	-1	-1	1
	ψ_3	1	-1	1	-1	1
	ψ_4	1	-1	-1	1	1
	χ'	2	0	0	0	-2
	χ	11	1	3	3	-1

Since χ', and the ψ_i's are an orthogonal basis for the space of class functions, by computing scalar products we obtain that $\chi = 3\psi_1 + \psi_3 + \psi_4 + 3\chi'$. Now ψ_1 is the trivial representation, hence we conclude:

Step I. $k(\mathbb{P}(U)/D_4)$ is a purely transcendental extension of degree 2 of the invariant subfield $k(V)^{D_4}$, where V is the representation with character $3\chi' + \psi_3 + \psi_4$.

Step II. The cyclic subgroup generated by r is normal, hence $k(V)^{D_4} = (k(V)^r)^{\mathbb{Z}/2}$.

Now, if i is a square root of -1, then the representation λ corresponding to χ' is given by

$$(2.8) \qquad \lambda(r) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \quad \lambda(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Therefore we can choose coordinates x_j, y_j $(j=1,2,3)$, z_1, z_2 on V such that r acts by $x_j \mapsto ix_j$, $y_j \mapsto iy_j$, $z_h \mapsto -z_h$ while s acts by permuting x_j with y_j, while $s(z_1) = z_1$, $s(z_2) = -z_2$.

Step III. $k(V)^r$ is a purely transcendental extension of k, K, generated by x_1^4, x_1/x_2, x_1/x_3, x_1y_1, y_1/y_2, y_1/y_3, $z_1(x_1^2+y_1^2)$, $z_2(x_1^2+y_1^2)$.

Proof. $K \subset k(V)^r$, and clearly $k(V) = K(x_1)$, but $x_1^4 \in K$, so we have equality. Unfortunately in this way the action of s is not linear any more: to avoid this we replace first in the basis x_1^4 by $u = x_1^2/y_1^2 = x_1^4/x_1^2y_1^2$.

Then $s(u) = 1/u$: finally we replace u by $(u-1)/(u+1) = w$ so that $s(w) = -w$.

End of the proof. In this way we have a linear action of $\mathbb{Z}/2$ on an 8-dimensional vector space, and with 4 eigenvalues equal to $(+1)$, 4 equal to (-1). The quotient is obviously rational.

Q.E.D.

REFERENCES

[1] Arbarello, E. - Sernesi, E.: The equation of a plane curve, Duke Math.J. 46, 2(1979), 469-485.

[2] Catanese, F.: Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications, Inv.Math. 63(1981), 433-465.

[3] Catanese, F. - Ceresa, G.: Constructing sextic surfaces with a given number d of nodes, J.Pure & Appl.Alg. 23(1982), 1-12.

[4] Clemens, H.: Double solids, Advances in Mathematics, to appear.

[5] Coble, A.B.: Algebraic geometry and theta functions, Coll.Publ. of the A.M.S. vol.10 (1929) (reprint 1961).

[6] Donagi, R.: The unirationality of A_5, to appear.

[7] Harris, J. - Mumford, D.: On the Kodaira dimension of the moduli space of curves, Inv.Math.67, 1(1982) 23-86.

[8] Igusa, J.I.: Arithmetic variety of moduli for genus two, Annals of Math. 72(1960), 612-649.

[9] Mori, S.: The uniruledness of M_{11}, to appear.

[10] Mumford, D.: Geometric invariant theory, Springer Verlag (1965).

[11] Mumford, D.: Varieties defined by quadratic equations, in "Questions on algebraic varieties", C.I.M.E. Varenna, 1969, Cremonese.

[12] Mumford, D.: Prym varieties I, in "Contributions to Analysis", Academic Press (1974), 325-350.

[13] Recillas, S.: La variedad de los modulos de curvas de genero 4 es unirracional, Ann.Soc.Mat. Mexicana (1971).

[14] Sernesi, E.: L'unirazionalitá della varietá dei moduli delle curve di genere dodici, Ann. Scuola Norm.Sup. Pisa, 8(1981), 405-439.

[15] Serre, J.P.: Linear representations of finite groups, G.T.M. 42, Springer Verlag (1977).

[16] Severi, F.: Vorlesungen über Algebraïschen Geometrie, Teubner, Leipzig (1921).

[17] Van der Geer, G.: On the geometry of a Siegel modular threefold, Math.Ann. 260, (1982) 317-350.

[18] Wirtinger, W.: Untersuchungen über Thetafunktionen, Teubner, Leipzig (1895)