MODULI OF SURFACES OF GENERAL TYPE

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Introduction

The present paper follows rather closely the text of the talk given at the Conference, and is therefore rather problem-oriented and of a mostly expository nature.

In the first part we give a very brief survey of the history of the problem of moduli for surfaces and at the very beginning we discuss with some detail a very elementary though important example, namely the deformations of rational ruled surfaces.

Later on we expose some recent results of ours (cfr. [5]) which shed some light on basic questions concerning moduli of surfaces of general type: these results are based on the theory of "bidouble" covers (i.e. Abelian covers with group $(\mathbb{Z}/(2))^2$) and their deformations, and on the application of M. Freedman's recent result on homeomorphisms of 4- manifolds ([7]). We give then a list of some problems, and in the formulation of one of them we are indebted to a private communication of A. Beauville ([1]).

While in [5] the examples we had considered were only bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$, we enlarge hare in the second part our consideration to bidouble covers of the rational ruled surfaces $\mathbb{F}_{2m}$: on the one hand we can thus explain better the meaning of a certain exact sequence (2.7., 2.18 of [5]), on the other we show how the deformations of the bidouble covers fit together smoothly when the base $\mathbb{P}^1 \times \mathbb{P}^1$ deforms to $\mathbb{F}_{2m}$.

Our notation is as follows:

For a complex space X, Ω^i_X is the sheaf of holomorphic i-forms, O_X is the sheaf of holomorphic functions.

If X is compact, and F is a coherent sheaf of O_X-modules we denote by $H^i(F)$ the finite dimensional $\mathbb{C}$-vector space $H^i(X,F)$, by $h^i(F)$ its dimension, by $\chi(F) = \sum_{i=0}^{\dim X}(-1)^i h^i(F)$.

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For Cartier divisors D,C on X, $O_X(D)$ is the invertible sheaf of sections of the associated line bundle; $\equiv$ will denote linear equivalence of divisors, $\sim$ algebraic equivalence, and $|D|$ will be the linear system of effective divisors linearly equivalent to D; $D\cdot C$ denotes the intersection product. If X is smooth T_X will denote the sheaf of holomorphic vector fields, and K_X, when it exists, will denote a canonical divisor, i.e. a divisor such that $O_X(K_X) \cong \Omega^n_X$, where $n = \dim_{\mathbb{C}} X$.

If X is an algebraic (compact, smooth) surface the geometric genus of X, p_g, is $h^\circ(\Omega^2_X)=h^2(O_X)$, the irregularity q is $h^\circ(\Omega^1_X)=h^1(O_X)$.

If M is a topological manifold of dimension 4, with a given orientation, τ is the signature of the quadratic form $q:H^2(M,\mathbb{Z})\to\mathbb{Z}$ given by Poincaré duality. As usual $b_i=\dim_{\mathbb{R}} H_i(M,\mathbb{R})$ is the i^{th} Betti number and $e = \sum_{i=0}^{\dim_{\mathbb{R}} M} (-1)^i b_i$ is the topological Euler-Poincaré characteristic of M.

§ 1. Moduli of surfaces: history and problems.

Let S be an algebraic compact smooth surface, which we assume to be minimal (i.e. S does not contain curves $E \cong \mathbb{P}^1$ such that $E^2 = -1$). Like the genus of a curve, the holomorphic invariants K^2_S, $\chi(O_S) = \chi(S)$ depend only on the topology and the orientation of S (this last being induced by the complex structure).

In fact

$$(1.0)\qquad \begin{cases} K^2 = 3\,\tau + 2\,e \\ 12\,\chi = 3\,\tau + 3\,e\,, \end{cases}$$ as a consequence of

the Hirzebruch - Riemann - Roch theorem (cf. [10]).

Assume that $\mathcal{S} \xrightarrow{p} B$ is a connected family of smooth surfaces, i.e.

a) B is a connected complex space

b) p is proper and $S_b=p^{-1}(\{b\})$ is smooth for each $b\in B$.

It is then a classical result that all the S_b's are diffeomorphic to each other.

According to Mumford ([17]) one has the following definition:

Definition 1.1. The complex space $\mathcal{M}$ is said to be a coarse moduli space for S if there exists a bijection g from the set of isomorphism classes $\{[S']\mid S'$ is homeomorphic to S by an orientation preserving homeomorphism$\}$

to M such that for each family $S \xrightarrow{p} B$ the induced mapping $f\colon B \longrightarrow M$ (such that $f(b) = g([S_b])$) is holomorphic.

A moduli space does not necessarily exist, as shows the following example, of rational ruled surfaces.

<u>Example 1.2.</u> Consider the rational ruled surfaces $\mathbb{F}_n = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(n))$, for $n \geq 2$.

Direct computations, which are well known, (cf. e.g. [12] pag. 42) give the result that for $n \geq 2$

$$(1.3) \qquad h^\circ(T_{\mathbb{F}_n}) = n + 5, \qquad h^1(T_{\mathbb{F}_n}) = n-1.$$

In particular, these surfaces are not biholomorphic to each other, but $\mathbb{F}_n$ and $\mathbb{F}_m$ are diffeomorphic iff $n \equiv m \pmod 2$. In fact, consider the rank 2 - vector bundles V on $\mathbb{P}^1$ which are extensions

$$(1.4) \qquad 0 \longrightarrow O_{\mathbb{P}^1} \longrightarrow V \longrightarrow O_{\mathbb{P}^1}(n) \longrightarrow 0$$

These are classified by the $(n-1)$ - dimensional vector space $B = H^1(O_{\mathbb{P}^1}(-n))$, and we get thus a family $F \xrightarrow{p} B$ of ruled surfaces where $p^{-1}(b) = \mathbb{P}(V_b)$, V_b being the vector bundle corresponding to $b \in B$. By (1.4) any homomorphism $\phi\colon O_{\mathbb{P}^1}(m) \longrightarrow V$ is trivial if $m>n$, and, if $m=n$, a non zero ϕ gives a splitting of the exact sequence (1.4).

For $b \in B$, let $m(b)$ the maximum m for which there exists a non trivial homomorphism $\phi : O_{\mathbb{P}^1}(m) \longrightarrow V_b$: by the maximality of $m(b)$ such ϕ determines a subline bundle of V_b, moreover $2\,m(b) \geq n$ by the Riemann-Roch theorem, hence $V_b \cong O_{\mathbb{P}^1}(m(b)) \oplus O_{\mathbb{P}^1}(n-m(b))$, and $\mathbb{P}(V_b) \cong \mathbb{F}_{2m(b)-n}$. Since $n \leq 2\,m(b) \leq 2n$, we get a decreasing filtration B_m of B, for $n/2 \leq m \leq n$ such that $B_m = \{b \mid$ exists a non trivial homomorphism $f\colon O_{\mathbb{P}^1}(m) \longrightarrow V_b\}$, Clearly $B_m - B_{m+1} = \{b \mid \mathbb{P}(V_b) \cong \mathbb{F}_{2m-n}\}$, and we have seen that B_n consists of the origin only.

<u>Proposition 1.5.</u> B_m is an algebraic cone of dimension equal to $\min(n-1, 2(n-m))$.

<u>Proof</u>. $\qquad B_m = \{b \mid H^\circ(V_b(-m)) \neq 0\}$

By virtue of the exact sequence

$$0 \longrightarrow H^\circ(V_b(-m)) \longrightarrow H^\circ(O_{\mathbb{P}^1}(n-m)) \xrightarrow{\beta(b)} H^1(O_{\mathbb{P}^1}(-m))$$

where $\beta(b)$ is given by cup product with $b\varepsilon B=H^1(O_{\mathbb{P}^1}(-n))$,

$$B_m = \{ b \mid \beta(b) \text{ is not injective} \} .$$

Fixing two points, 0 and ∞, in $\mathbb{P}^1$ and choosing an affine coordinate z on $\mathbb{P}^1 - \{\infty\}$ such that 0 corresponds to the origin, a basis for $H^1(O_{\mathbb{P}^1}(-n))$ is given by the Čech cocycles

$$z^{-1}, \ldots z^{-n+1} \varepsilon \ H^\circ(\mathbb{P}^1 - \{0\} - \{\infty\}, O_{\mathbb{P}^1}),$$

whereas a basis for $H^\circ(O_{\mathbb{P}^1}(n-m))$ is given by $1, z, \ldots z^{n-m}$.

Since $(z^{-i}) \vee z^j = \begin{cases} z^{j-i} & \text{or} \\ 0 & \text{if } j \geq i \text{ or } j-i \leq -m \end{cases}$

$$b = \sum_{i=1}^{n-1} b_i z^{-i} \quad \text{belongs to } B_m$$

i) if $2m \leq n+1$, or, for $2m \geq n+2$,

ii) if the following matrix $A_m(b)$ has rank strictly less than $(n-m+1)$, where

$$A_m(b) = \begin{vmatrix} b_1 & b_2 & \cdots & b_{n-m+1} \\ b_2 & b_3 & \cdots & b_{n-m+2} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-1} & b_m & \cdots & b_{n-1} \end{vmatrix}$$

It is immediate now that B_m is an algebraic cone, and we shall prove the assertion on its dimension by considering $V_i = \{b \mid$ the first i columns of $A_m(b)$ are linearly dependent$\}$ and proving, by increasing induction on i, that cod $V_i = m-i$, the case $i=1$ being immediate.

Now $V_i - V_{i-1}$ is covered by open sets where it is a complete intersection of $(m-i)$ hypersurfaces, hence cod $V_i \leq m-i$.

But if it were cod $V_i < m-i$, V_i would intersect the subspace

$P_i = \{b \mid b_j = 0$ for $j \leq i-1$ and $j \geq m\}$ in a locus of dimension at least 1, while it is easily seen that $V_i \cap P_i$ is just the origin.

<u>Q.E.D.</u>

The preceding example shows that the family $\mathcal{F} \xrightarrow{p} B$ contains as fibres all the $\mathbb{F}_k$'s with $n \geq k \geq 0$ and $k \equiv n \pmod 2$, and on a Zariski open set of B the fibre is $\cong \mathbb{F}_0$ ($\cong \mathbb{P}^1 \times \mathbb{P}^1$) for n even, and $\cong \mathbb{F}_1$ for n odd.
Now it is easy to see that a moduli space (according to definition 1.1.) cannot exist for these surfaces, because we would have a non costant holomorphic map $f: B \to M$ which should be constant on a dense open set, a contradiction since a complex space is a Haussdorff topological space.
The case when S is a surface of general type is a rather lucky one: in this case not only M exists but it is a quasi-projective variety by the theorem of Gieseker (1977, [8]).
The key feature here is given by the properties of the canonical divisor K_S: by the theorem of Bombieri ([2]), when $m \geq 5$ the complete linear system $|m K_S|$ yields a birational morphism $\Phi_m : S \longrightarrow \mathbb{P}^{P_m - 1}$, where

$$(1.6) \qquad P_m = \frac{m\,(m-1)}{2} K_S^2 + \chi(S), \quad \text{and such that}$$

the image X_m of Φ_m is a normal surface enjoying the following properties

a) $X_m \cong X_n$ for $n, m \geq 5$

b) the singularities of $X = X_m$ are R.D.P.'s (Rational Double Points) i.e. biholomorphic (locally) to the hypersurface singularities:

$$A_n = \{ z^2 + x^2 + y^{n+1} = 0\}$$

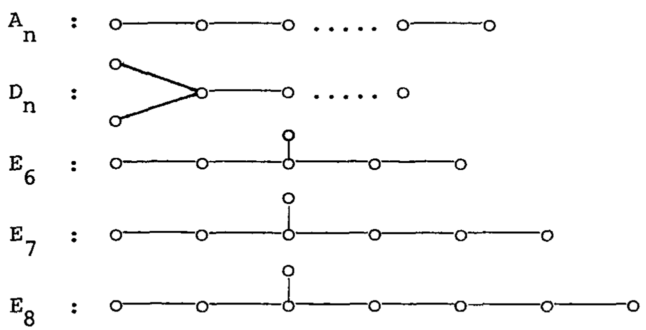
$$D_n = \{ z^2 + x(y^2 + x^{n-2}) = 0\} \qquad (n \geq 4)$$

$$E_6 = \{ z^2 + x^3 + y^4 = 0\}$$

$$E_7 = \{ z^2 + x\,(y^3 + x^2) = 0\}$$

$$E_8 = \{ z^2 + x^3 + y^5 = 0\}$$

c) $\Phi = \Phi_m$ is a minimal resolution of singularities of X, i.e. Φ is biholomorphic outside the singular points of X and the inverse image of a singular point is a (connected) union of curves $E \cong \mathbb{P}^1$ with $E^2 = -2$ intersecting transversally, whose structure is described by the Dynkin diagrams



whose vertices correspond to curves, and whose edges correspond to points of intersection of the two curves corresponding to the vertices of the edge. The index (e.g. n in A_n) denoter the number of vertices in the diagram.

The importance of the pluri-canonical model X_m lies in the fact that any morphism $f: S \longrightarrow S'$ induces a projectivity between X_m and X'_m, therefore $\mathcal{M}$ appears as a quotient of a locally closed subscheme H of the Hilbert scheme parametrizing surfaces of degree $m^2K_S^2$ in $\mathbb{P}^{P_m-1}$ by the projective group PGL (P_m). In particular one has:

(1.7) Surfaces of general type with fixed K^2,χ belong to a finite number of families (cf. [3], p. 395).

For later use we show another application of the previous result.

<u>Theorem 1.8</u>. Let G be a finite group and let $\mathcal{M}^G$ be the subset corresponding to the isomorphism classes of surfaces S for which G acts faithfully as a group of automorphisms of S. Then $\mathcal{M}^G$ is a closed subvariety of $\mathcal{M}$.

<u>Proof.</u> Let $m\geq 5$, and set $N=P_m(S)-1$. If G acts faithfully on S, G acts linearly on the vector space $H^\circ(S, \mathcal{O}_S(m\,K_S))$, hence on its dual space, and we get an $(N+1)$-dimensional representation ρ of G inducing a faithful projective representation of G on $\mathbb{P}^N$ leaving X_m invariant. Conversely any faithful representation ρ of G on $\mathbb{P}^n$ leaving X_m invariant, since any automorphism of X_m lifts to an automorphism of S (S being a minimal desingularization), gives an injective homomorphism of G into Aut(S), hence in particular lifts to a linear $(N+1)$- dimensional representation of G. Now there is only a finite number of isomorphism classes of such projective representations ρ, and correspondingly $\mathcal{M}^G$ can be expressed as a finite

union of subsets M^{ρ}. Fix therefore such a (linear) representation ρ inducing a faithful action on $\mathbb{P}^N$, and let H^{ρ} be the locally closed sub scheme of the Hilbert scheme H parametrizing m-canonical images X_m of surfaces S, such that H^{ρ} is the locus of fixed points for the action of G on H. Clearly M^{ρ} is the projection to the quotient of the image of $H^{\rho} \times PGL(N+1)$ in H.

To prove that such image is closed we use the valuative criterion of properness (cf. e.g. [9], theorem 4.7.): assume that we have a 1-parameter family $S \xrightarrow{p} B$, and that, setting $B^* = B - \{b_o\}$, $S^* = p^{-1}(B^*)$, we have a faithful action of G an S^*, such that, for $g \in G$, $p \circ g = p$.

By the m^{-th} canonical mapping we get a family $X \xrightarrow{f} B$, with $X \longrightarrow B \times \mathbb{P}^N$, and it suffices to prove that for every g in G its action extends from X^* to X and in such a way that g does not act as the identity on $X_o = f^{-1}(b_o)$: in fact, X^* being dense, such extension is then unique, hence we get a homomorphism of G into $\mathrm{Aut}(X_o)$ which is injective by the second property.

Now, for $g \in G$, we get an invertible matrix $a(t)$, for $t \in B^*$, which we can assume to be given by a regular function on B with $a(b_o) \neq 0$: clearly it suffices to prove that $a(b_o)$ is invertible and is not a multiple of the identity. By continuity, the eigenvalues of $a(b_o)$ are limit of the eigenvalues of $a(t)$, whose ratios are certain fixed roots of unity, therefore, not all the eigenvalues of $a(b_o)$ are equal and if 0 were an eigenvalue of $a(b_o)$, then $a(b_o)$ would be zero, a contradiction.

Q.E.D.

As we heard from D. Mumford's lecture, much is known about the moduli spaces M_g of curves of genus g, the basic fact being that M_g is quasi-projective normal irreducible variety of dimension 3g-3 ($g \geq 2$); on the other side, not many general results are known about the moduli spaces of surfaces of general type, and we shall show here that too optimistic expectations have a negative answer: e.g. these moduli spaces are "in general" highly reducible, with a lot of components of different dimension. But, in order to explain all this more precisely, let's introduce some notation and let's make same historical remark.

Definition 1.9. Let S be a surface of general type: then the number of

moduli of S, denoted by M(S), is the dimension of the moduli space $\mathcal{M}$ at the point [S] corresponding to S.
M. Noether ([18]) in 1888, under very special hypotheses, postulated for M a formula which in our terminology reads out as $M = 10\chi - 2K^2$.
This formula is verified quite seldom (especially since M is a positive integer, whereas the right side can be very negative, even for complete intersections), but it is the merit of F. Enriques to understand that $10\chi - 2K^2$ should give a lower bound for M in the case of non ruled surfaces.
In fact Enriques gave two proofs (see e.g. his book [6], p.204-215, especially the historical note on page 213) which were both incomplete, and in fact relying on some assumptions which did not hold true. In the first proof Enriques assumed to have a surface $F \subset \mathbb{P}^3$ with ordinary singularities, of degree n, and with double curve C: he assumed that the characteristic system (cut on the normalization of F by adjoint surfaces of degree n) should be complete, and this is not true in general as was shown by Kodaira in 1965 ([11]); similarly in the second proof it was assumed that the characteristic system of plane curves with cusps and nodes should be complete, an assertion which was disproven by Wahl in 1974 ([22]), relying on the examples of Kodaira (we defer the reader, for a more thorough discussion, to the appendix to Chapter V of Zariski's book [25], written by D. Mumford).
A proof finally came in 1963, through the theorem of Kuranishi ([13]) culminating the theory of deformations of complex structures due to Kodaira and Spencer.
Let $\mathcal{X} \xrightarrow{p} B$ be a connected family of smooth manifolds and $b_0 \in B$: then the fibres $X_b = p^{-1}(\{b\})$ are said to be deformations of $X_0 = X_{b_0}$.
Any holomorphic map f of a complex space T into B, with $f(t_0) = b_0$, induces another family of deformations of X_0, namely the fibre product $T \times_B \mathcal{X}$.
A family of deformations $(\mathcal{X}, X_0) \xrightarrow{p} (B, b_0)$ is said to be semi-universal if, for every other deformation $(Y, X_0) \xrightarrow{g} (T, t_0)$ the restriction to a sufficiently small neighbourhood of t_0 is induced by a holomorphic map $f : T \to B$ whose differential at t_0 is uniquely determined; it is said to be universal if moreover such a f is always unique.
The theorem of Kuranishi asserts that a semiuniversal deformation exists

(it is then unique by its defining property), and moreover that its base B is a germ of analytic subset of $(H^1(X_o, T_{X_o}), 0)$ defined by $h^2(X_o, T_{X_o})$ equations vanishing of order at least two at the origin.
Later Wavrik ([23]) proved that, if $H^o(T_{X_o})=0$, then the deformation is universal, what implies that if a moduli space M exists for X_o, then the germ of M at $[X_o]$ is biholomorphic to the quotient $B/\mathrm{Aut}(X_o)$ (though, e.g. in the case of Galois covers whose deformations are all Galois covers, the action of $\mathrm{Aut}(X_o)$ on $H^1(T_{X_o})$ need not be effective).
Now, when S is a surface of general type, Aut(S) is a finite group (this is another application of pluricanonical embeddings), hence $H^o(T_S)=0$, being the Lie algebra of a finite group. Deformation theory gives a solution to Enriques' inequality via the Hirzebruch - Riemann - Roch theorem: if a surface X is not ruled, then $H^o(T_X)=0$, and M = dim B, if M exists.
Clearly one has, by the previous remarks on B,

$$(1.10) \qquad h^1(T_X) - h^2(T_X) \leq \dim B = M \leq h^1(T_X)$$

but, since $h^o(T_X)=0$, the left hand side is $-\chi(T_X)$, i.e. $10\chi - 2K^2$ by the Hirzebruch R.-R. theorem, hence (1.10) is exactly Enriques' inequality.
One drawback of (1.10) is that the upper bound for M does not depend only on topological invariants: hovever, since by Serre duality

$$h^2(T_X) = h^o(\Omega^1_X \otimes \Omega^2_X),$$

the right hand side is $10\chi - 2K^2 + h^o(\Omega^1_X \otimes \Omega^2_X)$, so it is enough to give an upper bound on $h^o(\Omega^1_X \otimes \Omega^2_X)$, and in the case e.g. of surfaces of general type this can be done via exact sequences, restricting the sheaf $\Omega^1_S \otimes \Omega^2_S$ to a smooth curve in $|K|$ or $|mK|$.
One gets (theorems B and C of [5]) the upper bounds

$$(1.11) \qquad M \leq 10\chi + 3K^2 + 108 \qquad \text{(in general)}$$

$$(1.12) \qquad M \leq 10\chi + q + 1 \qquad \text{if S contains a smooth canonical curve C.}$$

These extimates appear to be too crude and an interesting question is, roughly speaking:

(1.13) what is asymptotically the best upper bound for M?

I will return later to better bounds for irregular surfaces, for the moment let me remark that, for a surface of general type S, by Castelnuovo's theorem, Noether's inequality, and the inequality of Bogomolov - Miyaoka - Yau, the topological invariants K^2, χ are subject to the following inequalities:

$$(1.14) \qquad \begin{cases} K^2 \geq 1, \chi \geq 1 \\ K^2 \geq 2\chi - 6 \\ K^2 \leq 9\chi. \end{cases}$$

It is possible therefore that, as $K^2, \chi \longrightarrow +\infty$ one may have different best upper bounds according to the limiting value of the ratio K^2/χ between 2 and 9.

One may ask however whether the moduli space is pure-dimensional: we proved recently that this is not true, and that M can attain arbitrarily many different values for orientedly homeomorphic surfaces.

More precisely, we proved ([5] theorem A)

(1.15) for each positive integer n there exist integers $0<M_1<M_2< \dots M_n$ and homeomorphic simply-connected surfaces of general type $S_1, \dots S_n$, such that $M(S_i)=M_i$.

An important remark is that the surfaces we consider are such that the canonical map is a biregular embedding, and their invariants K^2, χ are quite "spread" in the region defined by (1.14), so that these examples should be considered the rule rather than the exception.

Let me sketch briefly the idea of the proof, which consists of 3 basic ingredients.

<u>Step I:</u> If S_1, S_2 are simplyconnected, have equal K^2, χ, and $K^2 \neq 9$, they are orientedly homeomorphic if and only if either

a) $K_{S_i} \in 2\, P_{ic}(S_i)$ $\qquad$ (i=1,2)

b) $K_{S_i} \notin 2\, P_{ic}(S_i)$ $\qquad$ (i=1,2)

<u>Step II:</u> Find families of surfaces, with the properties stated in step I, depending on many integral parameters, and compute in terms of those K^2, χ, M.

Step III: Show that one can fix K^2,χ and obtain different values $M_1,\dots M_n$ for M: this is a number theoretic problem, solved by E. Bombieri (cf. the appendix to [5]), so that I will not talk about this in a Conference on Algebraic Geometry.

Step I was suggested by B. Moishezon and depends almost entirely on the recent deep theorem of M. Freedman (cf. [7]).

(1.16) If S_1, S_2 are simply-connected compact oriented differentiable 4-manifolds with the same intersection form on $H^2(S_i,\mathbb{Z})$, then they are (orientedly) homeomorphic.

and on the theorem of Yau ([24])

(1.17) if $K^2 = 9\chi$ and K is ample, then the universal cover of S is the unit ball in $\mathbb{C}^2$.

In fact, by (1.0), K^2,χ determine the rank and the signature of the unimodular quadratic form $q: H^2(S,\mathbb{Z}) \longrightarrow \mathbb{Z}$, and it is known that indefinite unimodular integral quadratic forms are classified only by the rank, signature and parity (q being even iff for each x, $q(x) \equiv 0 \pmod 2$, being odd otherwise).

Therefore Freedman's result applies provided that q is not negative or positive definite.

But q is negative definite only for surfaces of class VII which have $b_1=1$, while it is positive definite if and only if $S=\mathbb{P}^2$, as an easy corollary of Yau's theorem (1.17) (cf. [14], [21]).

Step II consists in considering bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$ and studying their small deformations: we will return to this point in the second paragraph, where we shall consider, using the results of [5], the more general case of bidouble covers of $\mathbb{F}_{2n}$.

Anyhow 1.15 shows in particular that, fixing the homeomorphism type of S, and varying the complex structure (which necessarily gives a surface of general type if $K^2 \geq 10$), the number of moduli M varies in an interval whose size grows to infinity with K^2,χ.

(1.18) How many irreducible components does $M_{K^2,\chi}$, (the union of the moduli spaces of surfaces with K^2,χ fixed), have at most?

(1.19) Is it true also that the number of connected components of M is unbounded, as $K^2, \chi \longrightarrow \infty$?

We remark here that connected components of M correspond to connected components of the subscheme H of the Hilbert scheme mentioned before (1.7), hence two surfaces S,S' are such that their classes [S] ,[S'] belong to the same connected component of M if and only if they are deformation of each other: in particular they must be diffeomorphic.
One could have therefore made a different choice for the "moduli space", considering only M_{diff}, i.e. the union of the connected components of M corresponding to diffeomorphic surfaces, and ask, regarding M_{diff}, similar questions to those posed for M, i.e. pure-dimensionality, etc..
As a matter of fact, though bidouble covers can be deformed until the branch locus is a union of lines (one gets then surfaces with only A_1-singularities), even in this last case it is not easy to tell directly whether homeomorphic surfaces are diffeomorphic: we have not pursued this, also there is some hope that Freedman's result can be made stronger as to imply that the two given 4-manifolds should be diffeomorphic.
Let's go back now to the last piece of history: in 1949 G. Castelnuovo ([4]) claimed that

(1.20) For an irregular surface S without irrational pencils, the number M of moduli is $\leq p_g + 2q$.

To explain what this means, we recall the classical theorem of Castelnuovo - De Franchis

(1.21) Assume that η_1, η_2 are independent sections of $H^\circ(\Omega^1_S)$ such that $\eta_1 \wedge \eta_2 \equiv 0$: then there exists a morphism $f: S \to B$, where B is a smooth curve possessing two 1-forms $\xi_1, \xi_2 \in H^\circ(\Omega^1_B)$ such that $\eta_i = f^*(\xi_i)$.

Now, such a map f: S → B is called a pencil, whose genus is, by definition, the genus of B, and an irrational pencil is just a pencil of genus at least one.
So, if an irregular surface does not have irrational pencils, first of all its image under the Albanese map $\alpha: S \to A = H^\circ(\Omega^1_S)^\vee / H_1(S, \mathbb{Z})$ is a surface, (hence $q \geq 2$!).

Conversely, it is easy to see that if the Albanese variety A is simple and is not the Jacobian of a curve then S does not have irrational pencils. Unfortunately Castelnuovo's claim is false, as we showed in [5], exhibiting counterexamples where M grows, keeping q fixed, like 4 p_g; anyhow we want to show here (cf. [5], th. D) that it is possible to rescue in some sense his assertion, in view of the Castelnuovo - De Franchis theorem.

<u>Theorem 1.22.</u> If $q \geq 3$ and there exist $\eta_1, \eta_2 \in H^\circ(\Omega^1_S)$ such that C = div $(\eta_1 \wedge \eta_2)$ is a reduced irreducible curve, then $M \leq p_g + 3q - 3$ and $K^2 \geq 6\chi$, this last equality holding if and only if the Albanese map is unramified into an Abelian 3-fold.

<u>Proof.</u> η_1, η_2 define an exact sequence

$$(1.23) \qquad 0 \longrightarrow O^2_S \longrightarrow \Omega^1_S \longrightarrow F \longrightarrow 0$$

with supp (F) = C, and, since $h^\circ(\Omega^1_S) \geq 3$, we have a non zero section of F, hence a sequence

$$(1.24) \qquad 0 \longrightarrow O_C \longrightarrow F \longrightarrow \Delta \longrightarrow 0$$

where the support of Δ has dimension zero.

By the multiplicativity of global Chern classes with respect to exact sequences, we obtain

$$c_2(\Delta) = - \text{length}(\Delta) = - (c_1^2 - c_2).$$

Hence $c_1^2 \geq c_2$, i.e. $K^2 \geq 6\chi$, and if equality holds $\Delta=0$, $F \cong O_C \Longrightarrow q=3$ and Ω^1_S is generated by global sections. The assertion about M follows by tensoring (1.23) and (1.24) with Ω^2_S, bounding h° of the middle term with the sum of the h°'s of the two other terms, and $h^\circ(O_C(K))$ with p_g+q-1.

<u>Q.E.D.</u>

<u>Remark 1.25.</u> The hypotheses of 1.22 are verified e.g. if Ω^1_S is generated by global sections outside a finite set of points.

Castelnuovo's error in fact was based on some wrong results of Severi ([19]): e.g. Severi claimed that for a surface S without irrational pencils of genus q the sections of $H^\circ(\Omega^1_S)$ would have no common zeros,

what is not true (see [5] for a discussion and counterexamples).
In the same paper Severi deduced from these incorrect assertions the following statement, whose validity we have not checked and we pose then as a problem

(1.26) Is it true that for an irregular (minimal) surface without irrational pencils $K^2 \geq 4\chi$?

(1.27) Also, it is an interesting question for us whether, under the hypotheses of 1.22, Castelnuovo's inequality $M \leq p_g+2q$ holds: looking at the proof we see that it would be indeed the case if $h^\circ(O_C(K))$ could be bounded by p_g+2. This is not true if S has irrational pencils, and this inequality is related to a question posed by Enriques ([6] page 354):

(1.28) when is the dimension of the paracanonical system $\{K\}$ less than or equal to p_g?

We recall that the paracanonical system can be defined as follows: consider the subscheme $[K]$ of the Hilbert scheme consisting of curves in S algebraically equivalent to a canonical divisor K, and consider the irreducible component $\{K\}$ of $[K]$ which contains the complete linear system $|K|$.
At the conference we posed the problem whether the hypothesis "S without irrational pencils" would imply $\dim\{K\} \leq p_g$, and ideed we asked also more, i.e. whether, under those assumptions, for $\eta \in \mathrm{Pic}^\circ(S)-\{0\}$ it should be $H^1(S,\eta)=0$, a fact which implies $\dim[K] \leq p_g$.
This latter has been answered negatively by A. Beauville ([1]) who gave an example where $[K]$ has dimension bigger than p_g. His example is as follows:

(1.29) Let B, A, be Abelian varieties of respective dimensions g and q, ω an element of $A-\{0\}$ with $2\omega=0$, and let i be the fixed point free involution on B x A such that $i(b,a) = (-b, a+\omega)$.

Let X be the quotient manifold B x A/i: the direct image of $\mathcal{O}_{B\times A}$ splits as $\mathcal{O}_X \oplus \mathcal{O}_X(\eta)$, where $2\eta\equiv 0$ but η is not a trivial divisor.
It is easily seen that $h^1(\mathcal{O}_X) = q$, $h^1(\mathcal{O}_X(\eta))=g$, and that A/ω is the

Albanese variety of X.
Taking an embedding of X by a sufficiently very ample linear system, and intersecting X with a general linear subspace of codimension $(g+q-2)$, one gets a surface S whose Albanese variety is just A/ω, and with $h^1(\mathcal{O}_S(\eta))=g$.
But then, if $g>q$, the dimension of the linear system $|K_S+\eta|$ is $p_g+(g-q)$, $> p_g$.
Clearly, as we remarked before, if A is not isogenous to a Jacobian and it is simple, S has no irrational pencils.
In this example, the system $[K]$ consists of $|K+\eta|$ and $\{K\}$, which has dimension p_g, in fact $H^1(\mathcal{O}_S(\varepsilon))=0$ for $\varepsilon \in \mathrm{Pic}^{\tau}(S)$, $\varepsilon \neq 0,\eta$, since on an Abelian variety Y the only divisor δ in $\mathrm{Pic}^{\circ}(Y)$ with $H^1(\mathcal{O}_Y(\delta)) \neq 0$ is $\equiv 0$ (cf. [16]).
To end with this first part, let me mention two more problems whose solution I'd like to see.

(1.30) It is known (cf. e.g. [20], page 402 and foll.) that, given any finite group G, one can find, for each $n \geq 2$, a variety X of dimension n with $\pi_1(X)=G$. In the case where G is abelian I have proved ([5], Cor. 1.9) the stronger statement that for any simply-connected variety Y of dimension $n \geq 2$, there exists an abelian cover of Y with group G^n such that $\pi_1(X)=G$. I guess that something similar could be done for any finite group G, so that, in particular, "every finite group is the fundamental group of infinitely many surfaces".

This last question is a recurrent one when one wants to describe explicitly some particular classes of surfaces.
We recall that the pluricanonical model X of a surface S of general type is isomorphic to S if and only if the canonical bundle of X is ample, i.e. if and only if there are no curves $E \cong \mathbb{P}^1$ with $K \cdot E=0$ ($\Leftrightarrow E^2=-2$) (these are the curves coming from the resolution of R.D.P.'s).
It is not clear to me whether these curves can be stable by deformation, i.e..

(1.31) Do there exist irreducible components Z of some moduli space of surfaces of general type such that for each $[S] \in Z$ the canonical bundle K_S is not ample?

R. Klotz has announced the result that $K^2 < 9\chi$ if K is not ample: this result in particular says that there are no discrete cocompact subgroups Γ of automorphisms of the unit ball D in $\mathbb{C}^2$ with D/Γ not smooth and with only R.D.P. as singularities (these subgroups Γ are rigid, by the theorem of Mostow [15]).

§ 2. Bidouble covers of rational ruled surfaces.

Def. 2.1. A bidouble cover $\pi: S \longrightarrow X$ is a Galois finite cover with group $G=(\mathbb{Z}/2)^2$. A bidouble cover is said to be smooth if, moreover, S, X, are smooth varieties.

Let $\pi: S \longrightarrow X$ be a smooth bidouble cover where S, X, are surfaces, and let $\sigma_1, \sigma_2, \sigma_3$ be the 3 non trivial involutions in the group $(\mathbb{Z}/2)^2$. Let $X_i = S/\sigma_i$, and let $\pi_i: X_i \longrightarrow X$ be the induced double cover. The locus $\mathrm{Fix}(\sigma_i)$ of fixed points for σ_i consists of a smooth divisor R_i, and a finite set N_i': it is clear that $R = R_1 + R_2 + R_3$ is the ramification divisor of π, that $\pi(R_i) = D_i$ is a smooth divisor, and $D=D_1+D_2+D_3$ is the branch locus of π.
$X_i \xrightarrow{\pi_i} X$ is branched on $D_j + D_k$ ($\{i,j,k\} = \{1,2,3\}$, here and in the following); therefore since the only singularities of X_i are A_1-points (nodes), corresponding to the points in N_i', we have:

(2.2) the divisor D has normal crossings, $N_i' = \pi^{-1}(D_j \cap D_k)$, and there exist divisors L_i on X s.t. $2L_i \equiv D_j+D_k$, so that X_i is the double cover of X in $\mathcal{O}_X(L_i)$ branched on D_j+D_k.

In [5] it is proven then that $\pi_* \mathcal{O}_S \cong \mathcal{O}_X \oplus (\bigoplus_{i=1}^{3} \mathcal{O}_X(-L_i))$, and that on X

(2.3) $$D_k + L_k \equiv L_i + L_j.$$

To describe more explicitly the algebra structure of $\pi_* \mathcal{O}_S$ we use the following notation: x_i is a section of $\mathcal{O}_X(D_i)$ such that $\mathrm{div}(x_i)=D_i$, z_i

is a section of $\mathcal{O}_S(R_i)$ with div $(z_i)=R_i$.
Then $(z_j\ z_k)^2 = x_j\ x_k$, and, setting $w_i = z_j\ z_k$, w_i is a section of $\mathcal{O}_S(\pi^*\ L_i)$ with

$$w_i^2 = x_j\ x_k$$

and w_i is precisely the square root extracted through the cover π_i. Conversely, in the rank-3 bundle $V = \bigoplus_{i=1}^{3} \mathcal{O}_X(L_i)$ one can consider the bidouble cover described by the equations

$$(2.4)\qquad \begin{cases} w_i^2 = x_j\ x_k \\ x_k\ w_k = w_i\ w_j \end{cases}$$

and ([5], prop. 2.3) all smooth bidouble covers arise in this way from divisors D_i, L_i, satisfying (2.2), (2.3).

<u>Def. 2.5.</u> A surface S' is called a natural deformation of S if there exist sections γ_i of $\mathcal{O}_X(D_i-L_i)$, x'_j of $\mathcal{O}_X(D_j)$ (i,j=1,2,3) such that S' is defined in V by the following equations

$$(2.6)\qquad \begin{cases} w_i^2 = (\gamma_j\ w_j + x'_j)(\gamma_k\ w_k + x'_k) \\ w_j\ w_k = x'_i\ w_i + \gamma_i\ w_i^2. \end{cases}$$

Since natural deformations are parametrized by a smooth variety, it is important to know to which subspace of $H^1(T_S)$ they give rise: the answer is given by the following result (thm. 2.19 of [5]).

<u>Theorem 2.7.</u> There exists an exact sequence

$$0 \longrightarrow H^\circ(T_S) \longrightarrow H^\circ(\pi^* T_X) \longrightarrow \bigoplus_{i=1}^{3} H^\circ(\mathcal{O}_{D_i}(D_i) \oplus \mathcal{O}_{D_i}(D_i - L_i)) \xrightarrow{\partial}$$

$$\xrightarrow{\partial} H^1(T_S) \longrightarrow H^1(\pi^*\ T_X)$$

and Im ∂ = Kodaira Spencer image of the natural deformations.

<u>Remark 2.6.</u> In [5] it is also proved that S is simply connected if X is such and the D_i's move in a pencil with transversal base points.

We are going to apply (2.5), in the case when $X = \mathbb{F}_{2m}$. We consider then

the family $F \xrightarrow{p} B$ obtained from (1.4) for n=2m. In the exact sequence (1.4) the trivial line subbundle of V_b determines a relative Cartier divisor $S \subset F$ which is a section of the projection of F onto $B \times \mathbb{P}^1$. For each b in B, $S|\mathbb{F}_b = S_b$ is a section of $\mathbb{F}_b$ with normal bundle $\cong \mathcal{O}_{\mathbb{P}^1}(2m)$, hence $S_b^2 = 2m$.
The projection g of F onto $\mathbb{P}^1$ also induces a relative (w.r.t. to p)Cartier divisor $Y = g^*(\mathcal{O}_{\mathbb{P}^1}(1))$.Clearly $Y_b^2 = 0$, and Pic(F) is a free abelian group with basis given by Y, S: moreover $Y_b \cdot S_b = 1$.

Consider now a divisor $\mathcal{D} \equiv a_1 S + a_2 Y$, with $a_1, a_2 \in \mathbb{Z}$: when restricting it to $\mathbb{F}_b \cong \mathbb{F}_{2k}$, $k \leq m$, since $S_b - (m+k)Y_b$ is an effective smooth section of $\mathbb{F}_b$, if the divisor D_b is effective, then either $D_b \cdot Y_b =$ $= a_1 > 0$, or $a_1 = 0$, $a_2 > 0$ and D is a union of fibres of the ruling of $\mathbb{F}_b$ onto $\mathbb{P}^1$. If $a_1 > 0$, also, since $D_b(S_b - (m+k)Y_b) = a_1((2m) - (m+k)) + a_2 =$ $= a_1(m-k) + a_2$, if $(S_b - (m+k)Y_b)$is not a fixed part of $|D_b|$ then $D_b \equiv a_1 (S_b - (m-k)Y_b) + a_3 Y_b$, with $a_3 \geq 0$. We set for convenience $S_b' = S_b - (m-k)Y_b$: this is a smooth section with $(S_b')^2 = 2k$.

<u>Lemma 2.7.</u> If D_b is an effective divisor on $\mathbb{F}_b \cong \mathbb{F}_{2k}$ $(0 \leq k \leq m)$, and $D_b \equiv a_1 S_b' + a_3 Y_b$, then the linear system $|D_b|$ has no base points if and only if $a_1, a_3 \geq 0$.

<u>Proof.</u> $|S_b'|$ has no base points by the following exact sequence (notice that $S_b' \cong \mathbb{P}^1$)

$$(2.8) \qquad 0 \longrightarrow H^\circ(O_{\mathbb{F}_b}) \longrightarrow H^\circ(O_{\mathbb{F}_b}(S_b')) \longrightarrow H^\circ(O_{\mathbb{P}^1}(2k)) \longrightarrow 0.$$

Moreover, clearly $|Y_b|$ has no base points.

<u>Q.E.D.</u>

In the previous discussion we have also seen that, given $\mathcal{D} \equiv a_1 S + a_2 Y$, $H^\circ(\mathbb{F}_b, O_{\mathbb{F}_b}(D_b)) \geq 1$ if and only if $a_1 \geq 0$, $a_2 \geq -a_1(m+k(b))$, where $k(b) = m(b)-m$ (cf. 1.4. and foll.), $0 \leq k(b) \leq m$ (and k(b)=m only for b=0).
Let K be the relative canonical divisor of $p: F \longrightarrow B$: since $K_b \cdot Y_b =$ $= (K_b + S_b) S_b = -2$,

$$(2.9) \qquad K \equiv -2S + (2m-2)Y.$$

By Serre duality then $H^2(O_{\mathbb{F}_b}(D_b))=0$ if D_b is an effective divisor. Moreover, if D_b is effective, by the exact sequence

$$0 \longrightarrow O_{\mathbb{F}_b}(-D_b) \longrightarrow O_{\mathbb{F}_b} \longrightarrow O_{D_b} \longrightarrow 0,$$

it follows that $H^1(O_{\mathbb{F}_b}(-D_b))=0$ if $|D_b|$ has no base points or it has a reduced and connected general member (i.e., in view of 2.7, $a_1=0$, $a_2=1$ or $a_1 > 0$, $a_2 \geq -a_1(m - k(b)) - 2\,k(b)$.

Again by Serre duality $H^1(O_{\mathbb{F}_b}(D_b)) = H^1(O_{\mathbb{F}_b}(-D_b - K_b)))=$

$= H^1(O_{\mathbb{F}_b}(-((a_1+2)S_b + (a_2+2-2m)Y_b)$ and is therefore $= 0$ if $a_1 \geq 0$, $a_2 + 2 + a_1(m-k(b)) + \geq 2(m-k(b))$.

<u>Corollary 2.10.</u> Let $\mathcal{D}$ be the divisor $a_1\mathcal{S} + a_2\mathcal{Y}$, and assume $a_1 \geq 0$, $a_2 \geq -2$. Then $R^i p_*(O_{\mathcal{F}}(\mathcal{D}))=0$ for $i=1,2$, $p_*(O_{\mathcal{F}}(\mathcal{D}))$ is locally free of rank equal to $\quad ma_1(a_1 + 1) + (a_1 + 1)(a_2 + 1)$.

<u>Proof:</u> By the Riemann - Roch theorem and the previous considerations, $h^i(O_{\mathbb{F}_b}(D_b))=0$ for $i=1,2$, hence for $i=0$ one obtains $h^\circ =$

$= \chi(O_{\mathbb{F}_b}(D_b)) = 1 + \frac{1}{2}(D_b\cdot(D_b-K_b)) = 1+\frac{1}{2}(a_1\mathcal{S} + a_2\mathcal{Y})\,((a_1+2)\mathcal{S} +$

$+ (a_2 +2-2m)\mathcal{Y}) = 1+\frac{1}{2}[a_1(a_1+2)\cdot 2m + a_2(a_1+2) + a_1(a_2+2-2m)] =$

$= m\,a_1(a_1+1) + (a_1+1)(a_2+1)$.

The result follows then from the Base change theorems (cf. e.g. [9], chap. III, 12.11, page 290).

<u>Q.E.D.</u>

Let $g\colon X=\mathbb{F}_n \longrightarrow \mathbb{P}^1$ be the canonical projection: then the tangent bundle T_X can be written as an extension of two line bundles, where T_v is the subbundle of vectors tangent to the fibres of g

$$(2.11) \qquad 0 \longrightarrow T_v \longrightarrow T_X \longrightarrow g^*(T_{\mathbb{P}^1}) \longrightarrow 0.$$

In the case of $X = \mathbb{F}_b$, an easy computation gives

$T_v \cong O_{\mathbb{F}_b}(2\,S_b - 2\,m\,Y_b)$, hence, if $L \equiv d_1\,\mathcal{S} + d_2\,\mathcal{Y}$, then

$$h^i(T_V(-L_b)) = h^i(\mathcal{O}_{\mathbb{F}_b}((2-d_1)S_b - (d_2 + 2m) Y_b))$$

is 0, for $i=0,1$, as soon as $d_1 \geq 3$, $d_2 \geq -2m$.

As a consequence we obtain:

(2.12) $\quad H^i(T_X(-L_b))=0$ if $d_1 \geq 3$, $d_2 \geq 0$, $i=0,1$.

Proposition 2.13. The family $F \xrightarrow{p} B$ induces the germ of the semi-universal deformation of $\mathbb{F}_n$.

Proof. It suffices to show that the Kodaira - Spencer map $\rho\colon T_{B,0} \longrightarrow H^1(\mathbb{F}_\circ, T_{\mathbb{F}_\circ})$ is an isomorphism. Let V be the vector bundle on $B \times \mathbb{P}^1$ such that $F = \mathbb{P}(V)$ (cf. (1.4)). The relative tangent bundle of p, $T_{F|B}$ fits into an exact sequence

$$0 \longrightarrow T_V \longrightarrow T_{F|B} \longrightarrow g^*(T_{\mathbb{P}^1}) \longrightarrow 0$$

g being the projection on $\mathbb{P}^1$.

Now, in concrete terms, choose an affine coordinate z on $\mathbb{P}^1 - \{\infty\}$, and then on $F|_{\mathbb{P}^1-\{\infty\}}$ we have coordinates

$(y_0, y_1, b_1, \ldots, b_{n-1}, z)$, whereas on $F|_{\mathbb{P}^1-\{0\}}$ we have coordinates $(y'_\circ, y'_1, b_1, \ldots b_{n-1}, z')$, with

$$\begin{cases} z' = 1/z \\ y'_\circ = y_\circ + y_1 \sum_{i=1}^{n-1} b_i z^{-i} \\ y'_1 = y_1 \cdot z^{-n} \end{cases}$$

hence $\quad y_\circ = y'_\circ + y'_1 \cdot \sum_{i=1}^{n-1} b_i z^{n-i}$

Since $\rho(\frac{\partial}{\partial b_i})$ is the difference of the two liftings of $\frac{\partial}{\partial b_i}$ according to the two given coordinate patches, we obtain

(2.14) $$\rho\left(\frac{\partial}{\partial b_i}\right) = (y'_1 \cdot z^{n-i}) \frac{\partial}{\partial y_\circ} = (y_1 z^{-i}) \frac{\partial}{\partial y_\circ}.$$

These are $(n-1)$ elements in $H^1(\mathcal{U}, T_V)$, $\mathcal{U}$ being the cover given by the two open sets above.

An easy computation shows that, for $b=0$, these elements are a basis of $H^1(\mathbb{F}_\circ, T_{\mathbb{F}_\circ})$.

Q.E.D.

Let now X be a smooth bidouble cover of $\mathbb{F} = \mathbb{F}_{2m}$ corresponding to the divisors L_1, L_2, L_3 and branched on the divisors D_1, D_2, D_3.
If L_i is $\equiv a_i S + b_i Y$, we shall say that X is of type (a_1,b_1) (a_2,b_2), (a_3,b_3).
Now, if $a_i \geq 3$, $b_i \geq 0$ for each i, by (2.12) $H^1(\pi^* T_{\mathbb{F}}) \cong H^1(\pi_*\pi^* T_{\mathbb{F}})$ is $\cong H^1(T_{\mathbb{F}})$.
Moreover, consider then the family $\mathcal{F} \xrightarrow{p} B$ of deformations of $\mathbb{F}$, and, on it, the divisor $\mathcal{L}_i \equiv a_i \mathcal{S} + b_i \mathcal{Y}$, $\mathcal{D}_i \equiv \frac{1}{2}((a_j+a_k)\mathcal{S} + (b_j+b_k)\mathcal{Y})$: then the direct images of the associated invertible sheaves in $\mathcal{F}$ are locally free (and $R^i p_* = 0$ for $i \geq 1,2$).
On the other hand $p_* O_{\mathcal{F}}(\mathcal{D}_i - \mathcal{L}_i)$ is locally free if

$$(2.15.i) \quad \begin{cases} a_j+a_k - 2a_i \geq 0 \\ b_j+b_k - 2\, b_i \geq 2, \end{cases}$$

but also (it is then equal to zero) if

$$(2.15.ii) \quad a_j + a_k - 2a_i < 0.$$

If, for each i=1,2,3, either (2.15.i) or (2.15.ii) holds, then one can choose a trivialization of $p_* O_{\mathcal{F}}(\mathcal{D}_i)$, $p_* O_{\mathcal{F}}(\mathcal{D}_i - \mathcal{L}_i)$ on B.
Then one has a vector space U and, for each $b \in B$, $u \in U$, sections γ_i of $O_{\mathcal{F}}(\mathcal{D}_i - \mathcal{L}_i)$, x_j' of $O_{\mathcal{F}}(\mathcal{D}_j)$: according to (2.6) one defines a family of deformations $\mathcal{X} \xrightarrow{f} B \times U$ which, restricted to $\{0\} \times U$, gives the natural deformations of X. In view of theorem 2.7 and of proposition 2.13, the associated Kodaira - Spencer map is surjective.
Thus we get the following.

<u>Theorem (2.16)</u> Let X be a smooth bidouble cover of $\mathbb{F} = \mathbb{F}_{2m}$ of type (a_i,b_i) (i=1,2,3) with (2.15) i) or ii) holding for each i. Then the moduli space of X contains only one (unirational) irreducible component passing through X, and its dimension equals $\sum_i h^\circ(O_{\mathbb{F}}(D_i)) + h^\circ(O_{\mathbb{F}}(D_i - L_i)) - 6$.

<u>Proof.</u> In view of the preceding discussion the family $\mathcal{X} \xrightarrow{f} B \times U$, which is induced by a morphism h of $B \times U \longrightarrow H^1(T_X)$ from the semi-universal deformation,is such that h is of maximal rank at the origin of the vector space $B \times U$. Therefore the semi-universal deformation has as basis an open neighbourhood of the origin in $H^1(T_X)$, moreover then

$B \times U$ dominates an affine neighbourhood of $[X]$ in its moduli space M. The assertion regarding the dimension follows from theorem 2.7, since, by (2.12), $h^1(\pi^* T_{\mathbb{F}}) - h^\circ(\pi^* T_{\mathbb{F}}) = h^1(T_{\mathbb{F}}) - h^\circ(T_{\mathbb{F}}) = -6$.

Q.E.D.

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