COMMUTATIVE ALGEBRA METHODS AND EQUATIONS OF REGULAR SURFACES

(Fabrizio CATANESE - Università di PISA)

0. INTRODUCTION

The aim of this paper is to describe a rather general method to write down explicit equations for regular surfaces of general type; the method applies also to the (much more restricted) class of weak Del Pezzo surfaces.

Though at first sight this method does not seem to be classical, on the other hand its geometrical counterpart is deeply related to the treatment of surfaces via generic projections to $\mathbb{P}^3$ using the theory of adjunction, a point of view appearing e.g. in the book of Enriques ([E]).

The basis of our analysis rests on two wider notions, the notion of a good (weighted) canonical projection, and of a quasi-generic canonical projection.

Given a surface S, a good canonical projection is a morphism $\phi: S \to \mathbb{P}$, where $\mathbb{P}$ is a weighted projective space of dimension 3, $\mathbb{P} = \mathbb{P}(e_0, e_1, e_2, e_3)$ (cf. [Do]), and ϕ is given by 4 sections y_0, y_1, y_2, y_3, with $y_i \in H^\circ(S, \mathcal{O}_S(e_i K_S))$; ϕ is said to be quasi-generic if moreover either

i) ϕ is birational onto $\Sigma = \phi(S)$ or

ii) ϕ is of degree 2 and $\Sigma = \phi(S)$ is a normal surface.

To illustrate our results, let's assume S to be a (minimal) regular surface of general type.

To S we attach its canonical ring $R = \bigoplus_{m=0}^{\infty} H^\circ(S, \mathcal{O}_S(mK_S))$ and its canonical model $X = \mathrm{Proj}\,(R)$.

Given a good canonical projection ϕ, R can be naturally viewed as a graded module over the polynomial ring $A = \mathbb{C}[Y_0, Y_1, Y_2, Y_3]$, graded in such a way that Y_i has degree equal to e_i (hence $\mathbb{P} = \mathrm{Proj}(A)$).

The assumption that S be regular implies that the A-module R has a

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minimal free resolution of length 1, given by a square matrix α of homogenous polynomials.
An application of Serre's duality theorem in the more general form given by Delorme (cf. [Se],[De]) allows us to show that the matrix α can be chosen to be symmetric when ϕ is quasi-generic. From these first results we deduce immediately some information about degrees of generators and relations for R (we refer the reader to [Ci 3] for more complete results in this direction). Later on we assume the projection ϕ to be quasi-generic. We show then that the matrix α is subject to the following (closed) rank condition:

(R.C.) *Let* h *be the size of* α, *and let* α' *be the matrix obtained by erasing the first row of* α: *then the entries of* $\Lambda^{h-1}(\alpha)$ *belong to the ideal* I *generated by the entries of* $\Lambda^{h-1}(\alpha')$.

It turns out that condition (R.C.) is necessary and sufficient in order to give a ring structure to the A-module R determined by the matrix α.
Our main result is that, given α satisfying (R.C.), and the (open) condition that X = Proj(R) have as singularities only Rational Double Points (R.D.P.'s for short), then X = Proj(R) is the canonical model of a surface of general type. This result on the one hand gives a way to show the existence of surfaces with given numerical invariants, on the other hand can be used to study the moduli spaces of these surfaces, since they can be parametrized by locally closed sets of matrices α.
As an easy application of this result, we show that the surfaces with $K^2 = 6,7$, $p_g=4$, $q=0$, such that $|K|$ is free from base points (plus a further condition in case $K^2=6$), have a unirational irreducible moduli space.
We should remark that the method presented here owes much to the ideas and work of several authors.
The use of projective resolutions to study equations of curves appears in the work of Arbarello and Sernesi ([A-S]) and was later clarified by Sernesi ([Sn]), whereas the idea that the resolution could be taken to be symmetric (since R is a commutative ring) appears in our paper [Ca 1], where it is mainly treated (though not in the greatest generality) the case when ϕ is a quasi-generic projection of degree 2.
Ciliberto ([Ci 1]) extended the methods of [Sn] and [Ca 1] to describe

some surfaces birationally mapped to $\mathbb{P}^3$ by the complete canonical system $|K|$, but under the assumption that the image Σ should have only ordinary singularities (i.e. the singularities of a generic projection).
The present method, which is a combination of the ideas appearing in [Ca 1] and [Ci 1], has been used in our joint paper with Debarre ([C-D]) to describe regular surfaces with $K^2=2$, $p_g=1$: we refer the reader to [C-D] also for a treatment of the non quasi-generic case.
We want finally to point out two directions in which the present investigations can be extended.
The first centers around the following remark: let α'' be the minor of α obtained by deleting the first row and the first column. $F=\det(\alpha'')$ is called the adjoint surface to Σ and, if $\mathbb{P}$ is smooth, for a generic choice of (α''), the only singularities of F are nodes, exactly at the points where rank $(\alpha'')=h-3$; in these points must be rank $(\alpha)=h-3$ and, once this condition is satisfied, then α_{11} is determined (modulo I) by $\alpha_{12},\ldots,\alpha_{1h}$.
When for generic α'', and generic $\alpha_{12},\ldots,\alpha_{1h}$ such that rank $(\alpha')=h-3$ when rank $(\alpha'')=h-3$, α_{11} exists, one sees that there is defined a natural irreducible unirational component of the moduli space, and it would be interesting to see whether there are other irreducible components.
Another interesting direction is towards the investigation of the relation between the determinantal equation of Σ and the deformation theory of S, thus generalizing Kodaira's work ([Ko]). We hope to return on these problems in the future.
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Notations

We shall work over the field $\mathbb{C}$ of complex numbers.
Given a projective variety X, $\mathcal{O}_X$ is the sheaf of regular functions, and ω_X the Grothendieck dualizing sheaf.
When ω_X is invertible, i.e. when X is a Gorenstein variety, we denote by K_X (or simply K when no confusion arises), an associated Cartier di-

visor, and call K a canonical divisor.

For a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we denote by $H^i(\mathcal{F})$ the cohomology group $H^i(X,\mathcal{F})$ and by $h^i(\mathcal{F})$ its dimension as a complex vector space; if D is a Cartier divisor on X, by abuse of notation we denote $H^i(X,\mathcal{O}_X(D))$ simply by $H^i(D)$. Let D,D' be Cartier divisors on X: $D'\equiv D$ means that D' is linearly equivalent to D, $D'\sim D$ means that D' is numerically equivalent to D, and $|D|$ denotes the linear system of all the effective divisors linearly equivalent to D.

Pic(X) is the group of Cartier divisors modulo linear equivalence, and if s is a section of an invertible sheaf, div(s) denotes the (Weil) divisor of zeros of s.

If C is an effective divisor on a smooth surface S, the arithmetic genus p(C) is by definition $p(C)=1-\chi(\mathcal{O}_C)$. Given a graded module $M=\bigoplus_{m\in\mathbb{Z}} M_m$, M(r), for $r\in\mathbb{Z}$, is M with a shift of degrees given by $(M(r))_m=M_{r+m}$. A shall be throughout the paper the polynomial ring $\mathbb{C}[Y_0,Y_1,Y_2,Y_3]$ (the symbol $\mathbb{C}\{\ldots\}$ denoting the ring of converging power series), endowed with a grading $A=\bigoplus_{k=0}^{\infty} A_k$ and, if $g\in A_k$, $A_{(g)}=\{a/g^n \mid a\in A_{nk}\}$.

Standard notations are: G.C.D. for the greatest common divisor of a set of positive integers, $\equiv$ for congruence, δ_{ij} for the Kronecker symbol ($\delta_{ij}=1$ for $i=j$, $\delta_{ij}=0$ for $i\neq j$).

$\mathbb{F}_n$ is the Segre-Hirzebruch rational ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(n))$, and σ_∞ its section corresponding to the projection of $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(n)$ onto $\mathcal{O}_{\mathbb{P}^1}$: σ_∞ has self-intersection $-n$ and is the unique irreducible curve on $\mathbb{F}_n$ with negative self-intersection.

If S is a smooth surface and p is a point on S, $B\ell_p(S)$ denotes the blow-up of S at p, and $B\ell_{p_1,\ldots,p_r}(S)$ is defined inductively as follows: p_r is a point in the surface $B\ell_{p_1,\ldots,p_{r-1}}(S)$ and $B\ell_{p_1,\ldots,p_r}(S)=B\ell_{p_r}(B\ell_{p_1,\ldots,p_{r-1}}(S))$; moreover, if $j>i$, p_j is said to be infinitely near to p_i if the surjective morphism $\Pi_{j,i}$ of $B\ell_{p_1,\ldots,p_{j-1}}(S)$ onto $B\ell_{p_1,\ldots,p_{i-1}}(S)$ maps p_j to p_i.

Further notations shall be introduced in the course of the paper.

Standard abbreviations are: "s.t."("such that"), "i.e." ("id est"), "resp."("respectively"), "cf."("compare"), "e.g."("exempli gratia").

1. REGULAR SURFACES OF GENERAL OR OF ANTIGENERAL TYPE.

Throughout the paper S will be a complete smooth algebraic surface such that either:

(1.1) $\exists$ an integer $m>0$ s.t. $|m\, K_S|$ defines a morphism ϕ_m onto a surface Σ

or

(1.2) $\exists$ an integer $m>0$ s.t. $|-m\, K_S|$ defines a morphism ϕ_m onto a surface Σ.

The two cases are clearly mutually exclusive, and hypothesis (1.1) forces S to be a minimal model, since then $K_S \cdot C \geq 0$ for each irreducible curve C; conversely, (cf. [Bo]) if S is the minimal model of a surface of general type, (1.1) holds. We shall furthermore make the assumption that S be a regular surface, i.e. $q(S)=h^1(\mathcal{O}_S)=h^1(K_S)=0$.
This last condition is automatically verified in case (1.2) by virtue of the following lemma.

Lemma 1.3. *If* S *is a regular surface satisfying* (1.1) *or* (1.2), *then* $H^1(n\,K)=0$ *for each* $n\in\mathbb{Z}$. *Moreover a surface satisfying* (1.2) *is regular.*

Proof. *By Serre duality* $h^1(r\,K)=h^1((1-r)K)$.
Moreover, by the vanishing theorem of Mumford-Ramanujam ([MU 2], [Ra]) $H^1(r\,K)=0$ *for* $r<0$ *in case* (1.1), *for* $r>0$ *in case* (1.2).
Therefore, in case (1.2) $H^1(r\,K)=0$ *for all* $r\in\mathbb{Z}$, *in particular* $H^1(\mathcal{O}_S)=H^1(K)=0$ *and* S *is regular.*
In case (1.1) $H^1(r\,K)=0$ *for* $r<0$, $r\geq 2$ *always, and for* $r=0,1$ *under the assumption that* S *be regular.*

Q.E.D.

We digress now briefly on the case of a surface S satisfying (1.2), usually called a weak Del Pezzo surface.
Clearly S is then a rational surface, $K^2>0$, $K\cdot C\leq 0$ for each irreducible curve C. By the Index Theorem $(K^2)\ (C^2) \leq (K\cdot C)^2$, equality holding iff $C\sim\lambda K$ with $\lambda\in\mathbb{Q}$, and by the genus formula $2p(C)-2=C^2+CK$; hence for an irreducible curve C one has $C^2 \geq -2$, and $C^2 \geq 1$ if C is not isomorphic

to $\mathbb{P}^1$ (since then $C^2+CK\geq 0$, so $C^2\geq -CK \geq 0$: but if $C^2=CK=0$ the Index Theorem implies $C\sim 0$, a contradiction since C is an effective divisor), moreover $C\cdot K=0$ implies $C^2=-2$, $C\cong\mathbb{P}^1$. Now S dominates a minimal model S' and, a fortiori, $D^2\geq -2$ for each irreducible curve D on S', therefore S' can only be $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$, $\mathbb{F}_2$.
In fact, if S is not minimal, we can assume $S'=\mathbb{P}^2$, since if $p\in\mathbb{P}^1\times\mathbb{P}^1$, $B\ell_p(\mathbb{P}^1\times\mathbb{P}^1)\cong B\ell_{p_1,p_2}(\mathbb{P}^2)$ where p_2 is not infinitely near to p_1, whereas, if $q\in\mathbb{F}_2-\sigma_\infty$, $B_q(\mathbb{F}_2)\cong B\ell_{q_1,q_2}(\mathbb{P}^2)$, where q_2 is infinitely near to q_1. Therefore if S is not minimal, there exists a sequence $p_1,\ldots,p_r$ of (possibly infinitely near) points on $\mathbb{P}^2$, with $r\leq 8$ (since $K_S^2>0$!), s.t. $S\cong B\ell_{p_1,\ldots,p_r}(\mathbb{P}^2)$. Since $C^2\geq -2$ for every irreducible curve C on S, one must have

(1.4) no more than 3 p_i's lie on a line

(1.5) no more than 6 p_i's lie on a conic

(1.6) the set $\{p_1,\ldots,p_r\}$ can be partitioned into subsets $\{p_{i_1},\ldots,p_{i_k}\}$ with $p_{i_1}\in\mathbb{P}^2$, $p_{i_{(j+1)}}$ infinitely near to p_{i_j} but not lying on the proper transform of $p_{i_{(j-1)}}$.

Since clearly for $\mathbb{P}^1\times\mathbb{P}^1$ and $\mathbb{F}_2$ there exists a divisor H with $2H\equiv -K$, and s.t. $|H|$ defines a birational morphism onto a quadric in $\mathbb{P}^3$, one asks whether, given $r\leq 8$, $p_1,\ldots,p_r$ satisfying (1.4), (1.5), (1.6), $S=B\ell_{p_1,\ldots,p_r}(\mathbb{P}^2)$ is a weak Del Pezzo surface.
Let E_i be the total transform of p_i in S, and let H be the total transform of a line in $\mathbb{P}^2$: since $-K_S\equiv 3H-\sum_{i=1}^{r}E_i$ and thus $K_S^2=9-r\geq 1$, (1.2) is verified if $K_S\cdot C\leq 0$ for every irreducible curve C on S. If C is one of the exceptional divisors, then by (1.6) either $C^2=-1$ or $C^2=-2$, hence $K_S\cdot C\leq 0$.
By the Riemann-Roch Theorem $h^\circ(-K_S)=10-r$, therefore $|-K_S|$ is at least a pencil; if $K_S\cdot C>0$, C is not exceptional and is a fixed part of $|-K_S|$: hence C is the proper transform either of a line, or of a conic, but then $K_S\cdot C>0$ contradicts (1.4), (1.5).
We can summarize the foregoing discussion in the following

Proposition 1.7. *A weak Del Pezzo surface is either* $\mathbb{P}^1 \times \mathbb{P}^1$ *, or* $\mathbb{F}_2$ *, or is the blow up* $B\ell_{p_1,\dots,p_r}(\mathbb{P}^2)$ *of* $\mathbb{P}^2$ *at* $r\leq 8$ *points, possibly infinitely near, but satisfying* (1.4),(1.5),(1.6).

To our surface S is naturally attached a graded ring

the canonical ring $R = \bigoplus_{m=0}^{\infty} H^{\circ}(m\,K)$ in case (1.1),

the anticanonical ring $R = \bigoplus_{m=0}^{\infty} H^{\circ}(-m\,K)$ in case (1.2).

By the results of [Za],[Mu 1] these are finitely generated and one calls $X = \mathrm{Proj}(R)$ the canonical (resp.: anticanonical) model of S.
X can also be obtained as follows: by the Index theorem, if D is a divisor with $K\cdot D=0$, then $D^2<0$ unless $D\sim 0$. Therefore the irreducible curves C with $K\cdot C=0$ are independent in $\mathrm{Pic}(S)\oplus\mathbb{Q}$ (cf. [Bo], prop.1), and, as we already noticed before, they are smooth rational curves with $C^2=-2$. X is obtained by contracting these curves and has only Rational Double Points as singularities (cf. [Ar 1], [Ar 2], [Bo]): in particular X is normal, the dualizing sheaf ω_X is invertible, and, $\Pi: S \to X$ being the contraction morphism, one has $\mathcal{O}_S(K_S)\cong\Pi^*(\omega_X)$.
Every morphism $\phi_m: S \to \Sigma$ as in (1.1), (1.2) factors through Π and $\Psi_m : X \to \Sigma$.

Remark 1.8. Ψ_m *is a finite morphism, being proper and quasi-finite, moreover* ω_X *(resp. :* ω_X^{-1}*) is ample on* X.

We refer to reader to [Bo] and [Sa] for more precise results on the structure of these morphisms.

§ 2. GOOD (WEIGHTED) CANONICAL PROJECTIONS.

The set up being as in the preceding section, we denote, for $m\geq 0$, by R_m the homogenous part of degree m of the graded ring R (i.e. $R_m=H^{\circ}(mK)$ in case (1.1), $R_m=H^{\circ}(-mK)$ in case (1.2)).
We choose then y_0,y_1,y_2,y_3 homogeneous elements of R such that

(2.1) $y_i\in R_{ee_i}$, where $e,e_0,\dots e_3$ are positive integers and the greatest common divisor of e_0, e_1, e_2, e_3 is 1

(2.2) the four divisors $C_i = \mathrm{div}(y_i)$ have empty intersection (we can easily assume then that three of them have an empty intersection).

It will also be convenient in the sequel to assume that each y_i does not belong to the subring of R generated by the other y_j's.
Let A be the ring $\mathbb{C}[Y_0, Y_1, Y_2, Y_3]$ graded so that Y_i is an homogeneous element of degree e_i.
Then (cf. [Do], [De]) $\mathbb{P}(e_0, e_1, e_2, e_3) = \mathrm{Proj}(A)$, (which in the sequel we shall simply denote by $\mathbb{P}$), is a 3-dimensional weighted projective space and the four sections $y_0, \dots, y_3$ define a morphism $\phi: S \to \mathbb{P}$ which we shall call a good canonical projection. By (2.2), since $K_S^2 > 0$, the image of ϕ is a surface $\Sigma \subset \mathbb{P}$ and ϕ factors as

(2.3)
$$\begin{array}{ccc} S & \xrightarrow{\ \phi\ } & \Sigma \subset \mathbb{P} \\ & \Pi \searrow \quad \nearrow \Psi & \\ & X & \end{array}$$
where Ψ is a finite morphism

Σ being an hypersurface in $\mathbb{P}$, the projecting weighted cone is defined by an homogeneous element f in A of degree n, hence $\Sigma = \mathrm{Proj}(A_{/(f)})$.
R, being via ϕ an overring of $A_{/(f)}$, can be naturally viewed as an A-module.
From now on, since $H^\circ(X, \mathcal{O}_X(m\ K_X)) \cong H^\circ(S, \mathcal{O}_S(m\ K_S))$, we shall restrict our considerations to the finite morphism $\Psi: X \to \Sigma$.

Definition 2.4. *We shall say that* (e_0, e_1, e_2, e_3) *are normalized if the greatest common divisor of every three of them is equal to* 1. *We remark that in fact* $\mathbb{P} = \mathbb{P}(e_0, e_1, e_2, e_3)$ *is isomorphic to a weighted projective space* $\mathbb{P}(d_0, d_1, d_2, d_3)$ *with* (d_0, d_1, d_2, d_3) *normalized. For example,* $\mathbb{P} = \mathbb{P}(1,2,2,2)$ *is isomorphic to* $\mathbb{P}^3$, *but this isomorphism does not give an isomorphism of* $\mathcal{O}_{\mathbb{P}}(1)$ *with* $\mathcal{O}_{\mathbb{P}^3}(1)$ *(this last is isomorphic to* $\mathcal{O}_{\mathbb{P}}(2)$, *while* $\mathcal{O}_{\mathbb{P}} \cong \tilde{\mathcal{O}}_{\mathbb{P}}(1)$). *Also, in general the sheaves* $\mathcal{O}_{\mathbb{P}}(n)$ *are not necessarily invertible (cf.* [Do], 1.3.1, 1.3.2*).*
In the e_i*'s are normalized, then to* $\mathcal{O}_{\mathbb{P}}(1)$ *corresponds a Weil divisor a multiple of which is a very ample Cartier divisor.*

If the e_i's are normalized, we have in case (1.1) $\Psi^*(\mathcal{O}_{\mathbb{P}}(1)) = \mathcal{O}_X(e\ K_X)$, $\phi^*(\mathcal{O}_{\mathbb{P}}(1)) = \mathcal{O}_S(e\ K_S)$, (e has to be replaced by -e in case (1.2)).
Via the Serre correspondence (cf. [De], prop. 4.2) to every finitely generated graded A-module M is naturally associated a coherent sheaf of $\mathcal{O}_{\mathbb{P}}$-modules $\tilde{M}$ such that

(2.5) $M_n = \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(-n), \tilde{M})$.

In the case when $e>1$, we can split R as a direct sum of graded A-modules

(2.6) $R = \bigoplus_{i=0}^{e-1} R^{(i)}$, where $R_m^{(i)} = R_{me+i}$.

<u>Proposition 2.7.</u> $\tilde{R}^{(i)} \cong \Psi_*(\mathcal{O}_X(i\ K_X))$ *(resp.* $\Psi_*(\mathcal{O}_X(-i\ K_X))$ *in case* (1.2)).

<u>Proof.</u> *It suffices to consider case* (1.1) (S *of general type) and we have to show then the following:*

$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(-n), \Psi_*\ \mathcal{O}_X(i\ K)) = H^\circ(\mathcal{O}_X((ne+i)K))$.

Clearly the term on the left equals $H^\circ(\mathbb{P}, \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(-n), \Psi_*\mathcal{O}_X(i\ K)))$.
Now, for every element $g \in A$*, homogeneous of degree* d*, and with* $f \nmid g$*, we can consider the principal affine open set* $D_+(g) = \mathrm{Spec}(A_{(g)})$.
Since Ψ *is finite onto* Σ, $\Psi^{-1}(D_+(g))$ *is also affine. The assumption that* G.C.D.$(e_0,e_1,e_2,e_3)=1$ *guarantees the existence, for each* n*, of an integer* $k>0$ *and of a monomial* $\lambda(Y) = \lambda(Y_0,\dots,Y_3)$ *of degree* $kd+n$.
We observe the following facts:

$H^\circ(D_+(g), \mathcal{O}_{\mathbb{P}}(-n)) = \{P\cdot g^{-h} \mid P \in A_{hd-n}\}$

$H^\circ(D_+(g), \mathcal{O}_{\mathbb{P}}) = \{Q\ g^{-h} \mid Q \in A_{hd}\}$

$H^\circ(D_+(g), \Psi_*\mathcal{O}_X(iK)) = H^\circ(\Psi^{-1}(D_+(g)), \mathcal{O}_X(iK))$

Setting $y = (y_0,\dots,y_3)$, *we have* $\Psi^*(Q\cdot g^{-h}) = Q(y)\cdot g^{-h}(y)$.
To $u \in H^\circ(D_+(g), \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(-n), \Psi_*\mathcal{O}_X(iK))$ *we associate a section* u *of* $\mathcal{O}_X((ne+i)K)$ *on* $\Psi^{-1}(D_+(g))$ *defined as follows:*
for every $P \in A_{hd-n}$*, with* $f \nmid P$*, set* $\hat{u} = u(P\cdot g^{-h})\cdot g^h(y)\cdot P(y)^{-1}$.
We claim that $\hat{u}$ *is well-defined.*
In fact, assume P_1, P_2 *are such that* $f \nmid P_i$, $P_i \in A_{h_i d-n}$*, and let* $\lambda(Y)$ *be as above:*

then $u\,(P_1\;P_2\lambda\cdot g^{-(k+h_1+h_2)}) = u(P_1\;g^{-h_1})\cdot P_2(y)\lambda(y)g^{-(k+h_2)}(y) =$

$= u(P_2\;g^{-h_2})P_1(y)\lambda(y)\;g^{-(k+h_1)}(y).$

Since $\lambda(y)$ *is non zero, and* X *is irreducible, the definition is immediately seen to be well posed.*
Moreover the rational section $\hat{u}$ *is regular on* $\Psi^{-1}(D_+(g))$, *since its polar locus is contained in* $\bigcap_{P,f\not| P} \mathrm{div}(P)$.
Conversely, any such $\hat{u}$ *defines* u *via:*

$$u(P\;g^{-h}) = \hat{u}\;P(y)\;g^{-h}(y).$$

Hence $Hom_{O_{\mathbb{P}}}(O_{\mathbb{P}}(-n),\Psi_* O_X(iK)) = \Psi_*(O_X((ne+i)K))$
and we are through.

Q.E.D.

Proposition 2.8. *R is a Cohen-Macaulay A-module.*

Proof. *Again we treat only case* (1.1), *and we can assume, by* 2.2, *that* $C_0\cap C_1\cap C_2=\emptyset$ $(C_i=\mathrm{div}(y_i))$: *since dim* $R=3$, *it suffices to prove that* y_0, y_1, y_2 *form a regular sequence.*
Note first of all that R is an integral domain: then also, by lemma 1.3, $R/y_0\,R$ *is isomorphic to* $R' = \bigoplus_{m=0}^{\infty} H^\circ(C_0, O_{C_0}(m\,K_S))$. *Since* y_1 *vanishes only at a finite number of points of* C_0, *we get the exact sequences*

$$0 \longrightarrow R' \xrightarrow{\;y_1\;} R' \longrightarrow R'/y_1R' \to 0$$

$$0 \longrightarrow R'/y_1R' \longrightarrow \bigoplus_{m=0}^{\infty} H^\circ(C_0\cap C_1, O_{C_0\cap C_1}(mK_S)) \longrightarrow$$

$$\longrightarrow \bigoplus_{m=0}^{\infty} H^1(C_0, O_{C_0}((m-e_1)K_S))$$

where, by the Riemann-Roch theorem, since $\omega_{C_0} \overset{\sim}{=} O_{C_0}((1+e_0)K_S)$, *in the last direct sum the terms with* $m>1+e_0+e_1$ *are zero.*
Since y_2 *does not vanish on* $C_0\cap C_1$, y_2 *is not a zero-divisor of* R'/y_1R'.

Q.E.D.

Since *A* is a polynomial ring, we can choose a minimal set of homogeneous generators of *R* as a graded *A*-module, $v_1=1, v_2, \ldots v_h$, homogeneous of

respective degrees $\ell_1=0 < \ell_2 \leq \ell_3 \dots \leq \ell_h$.
By Hilbert's syzygy theorem (cf. e.g. [Z-S] p. 240, [A-N] pp. 575-588) and by prop. 2.8 the v_i's determine a free homogeneous minimal resolution of the graded A-module R. In case e=1, the resolution reads out as

$$(2.9)\quad 0 \longrightarrow \bigoplus_{j=1}^{h} A(-r_j) \xrightarrow{\alpha} \bigoplus_{i=1}^{h} A(-\ell_i) \longrightarrow R \to 0$$

(where α is a square matrix with α_{ij} a homogeneous polynomial in A of degree $r_j - e_i$ if $r_j>e_i$, $\alpha_{ij}=0$ otherwise).
In fact the kernel of the surjection of $\bigoplus_{i=1}^{h} A(-\ell_i)$ onto R is locally free by prop. 2.8, must be of rank $\leq$ h by the injectivity of α, and of rank at least h since dim R =3; moreover clearly $\alpha_{ij} = 0$ if $r_j<e_i$, but also when $r_j=e_i$ since $\{v_1,\dots,v_h\}$ is a minimal set of generators.
In case e>0, instead of (2.9) we have a direct sum on k=0,...,e-1 of the following exact sequences

$$(2.10)\quad 0 \longrightarrow \bigoplus_{j=1}^{h_k} A(-r_j^{(k)}) \xrightarrow{\alpha^{(k)}} \bigoplus_{i=1}^{h_k} A(-\ell_i^{(k)}) \longrightarrow R^{(k)} \to 0$$

<u>Lemma 2.11</u>. det $\alpha=(f)^{\deg \Psi}$ *(resp.:* $\det(\alpha^{(k)})=(f)^{\deg \Psi}$ *in case* e>0).

<u>Proof.</u> *Localization being flat, we get an exact sequence of sheaves on* $\mathbb{P}$, *by proposition* 2.7:

$$0 \to \bigoplus_{j=1}^{h_k} \mathcal{O}_{\mathbb{P}}(-r_j^{(k)}) \xrightarrow{\alpha^{(k)}} \bigoplus_{i=1}^{h_k} \mathcal{O}_{\mathbb{P}}(-\ell_i^{(k)}) \to \Psi_*(\mathcal{O}_X(kK)) \to 0$$

Now, $\mathbb{P}$ *being non singular in codimension* 1, $\mathcal{O}_{\mathbb{P},\Sigma}$ *is a discrete valuation ring, and the stalk of* $\Psi_*(\mathcal{O}_X(kK))$ *at the generic point of* Σ *is a* $\mathbb{C}(\Sigma)$*-vector space of dimension equal to* deg(Ψ), *and we get the desired result from the structure theorem of modules over principal ideal rings (cf.* [Ja I], *th.* 3.8, *p.* 176*).*

<u>Q.E.D.</u>

We can derive a very easy, but useful corollary of (2.11), which we state now for e=1, but holds also in case e>1 upon replacing h by h_k, ℓ_j by $\ell_j^{(k)}$, r_i by $r_i^{(k)}$.
We have already assumed $\ell_1\leq\ell_2\leq \dots \leq \ell_h$, and we can assume

$r_1 \geq r_2 \geq \ldots \geq r_h$; we stress again the fact that if $\deg(\alpha_{ij}) = r_j - \ell_i \leq 0$, then α_{ij} must be zero.

Corollary 2.12. $\ell_j < r_{h-j+1}$, $\forall j=1,\ldots,h$.
If Ψ *is birational* ($\deg \Psi = 1$), *then also* $\ell_j < r_{h-j+2}$.

Proof. *If* $\ell_t \geq r_{h-t+1}$, *then* $\alpha_{ij} = 0$ *for* $i \geq t, j \geq h-t+1$; *therefore, for each permutation* σ *of* $\{1,\ldots,h\}$ *we must have an element* $\alpha_{i\sigma(i)}$ *which equals zero, hence* $\det(\alpha) = 0$, *a contradiction.*
In case $e_t \geq r_{h-t+2}$, *expanding* $\det(\alpha)$ *as* $\sum_\sigma \varepsilon(\sigma) \prod_{i=1}^{h} \alpha_{i\sigma(i)}$ *the only* σ*'s contributing a non zero term are those for which* $\sigma(\{t,\ldots,h\}) = \{1,\ldots,h-t+1\}$, *therefore* $\det(\alpha) = \det(\tilde{\alpha}) \det(\hat{\alpha})$, *where* $\tilde{\alpha}$ *is the minor formed by the rows* $(t,\ldots,h)$ *and columns* $(1,\ldots,h-t+1)$ $\hat{\alpha}$ *is the minor formed by the rows* $(1,\ldots,t-1)$ *and columns* $(h-t+2,\ldots,h)$.
We conclude by lemma 2.11.

Q.E.D.

In order to prove a weak symmetry statement about resolutions (2.9) and (2.10), we adopt for the time being the notations cf. [G-D] III, 2.1, pp. 95 and following.
Let U be the open affine cover of X given by the sets $U_i = X - \{y_i = 0\}$ and let F be the sheaf $\mathcal{O}_X(k\,K)$ (in case (1.2) replace k by -k).
We set $F(*) = \bigoplus_{n \in \mathbb{Z}} F \otimes \mathcal{O}_X(n\, e\, K)$ (analogously in case (1.2)): obviously we have $H^i(X, F(*)) = H^i(U, F(*))$.
Then $H^\circ(X, F(*)) = R^{(k)}$,
$H^1(X, F(*)) = 0$.
We denote by (y) the maximal ideal (y_0, y_1, y_2, y_3) in A and by $H^i((y),\ldots)$ the local cohomology group (Koszul cohomology).
From (2.10), since $H^i((y), A(r)) = 0$ for $i \leq 3$, and since, by prop. 2.1.3 of [G-D], we have

$$H^i((y), R^{(k)}) \cong \begin{cases} 0 & \text{for } i \leq 2 \\ H^2(X, F(*)) & \text{for } i=3, \end{cases}$$

we obtain, by the long exact sequence of local cohomology

$$(2.13)\quad 0 \longrightarrow H^3((y),\mathcal{R}^{(k)}) \longrightarrow H^4((y),\mathcal{L}^1) \xrightarrow{\alpha^{(k)}} H^4((y),\mathcal{L}^\circ) \to 0,$$

where we have set for convenience (when e>1, the case e=1 being a special case,

$$(2.14)\quad \mathcal{L}^1 = \bigoplus_{j=1}^{h_k} A(-r_j^{(k)}), \quad \mathcal{L}^\circ = \bigoplus_{i=1}^{h_k} A(-\ell_i^{(k)}).$$

Now, given a graded module M, according to [Se], p. 263, we define the dual $M^\vee$ to be the graded module s.t. $(M^\vee_m)$ is the dual vector space of M_{-m}, and we recall that the functor $M \to M^\vee$ is exact and contravariant.

By the duality theorem for $\mathbb{P}$ ([De]), thm. 5.2), if $s = \sum_{i=0}^{3} e_i$

$$H^4(A(-r))^\vee = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(\mathcal{O}_{\mathbb{P}}(-n-r), \mathcal{O}_{\mathbb{P}}(-s)) = A(-s+r).$$

Since $H^3((y),\mathcal{R}^{(k)})^\vee \cong H^2(X,\mathcal{F}(*))^\vee \cong$

$$\Big(\bigoplus_{n\in\mathbb{Z}} H^2(X,\mathcal{O}_X(\pm(en+k)K))\Big)^\vee \cong \bigoplus_{n\in\mathbb{Z}} H^2(X,\mathcal{O}_X(\pm(k-en)K))^\vee \cong \text{(by Serre duality on X)}$$

$$\cong \bigoplus_{n\in\mathbb{Z}} H^\circ(X,\mathcal{O}_X((\pm(en-k)+1)K)) \cong \begin{cases} \mathcal{R}^{(1-k)} & \text{in case (1.1)} \\ \mathcal{R}^{(-1-k)} & \text{in case (1.2)} \end{cases}$$

we obtain, dualizing (2.13), another minimal resolution of $\mathcal{R}$, $\mathcal{R}^{(k)}$, namely

(2.15) e=1 (where, as usual, $\mp$ means: "-" in case (1.1), "+" in case (1.2))

$$0 \longrightarrow \bigoplus_{i=1}^{h} A(\ell_i - s \mp 1) \xrightarrow{t_\alpha} \bigoplus_{j=1}^{h} A(r_j - s \mp 1) \longrightarrow \mathcal{R} \longrightarrow 0$$

(2.16) e>1, case (1.1) k=0,1

$$0 \longrightarrow \bigoplus_{i=1}^{h_k} A(\ell_i^{(k)} - s) \xrightarrow{t_\alpha(k)} \bigoplus_{j=1}^{h_k} A(r_j^{(k)} - s) \longrightarrow \mathcal{R}^{(1-k)} \longrightarrow 0$$

(2.17) e>1, case (1.1) $k \geq 2$, or case (1.2)

$$0 \longrightarrow \bigoplus_{i=1}^{h_k} A(\ell_i^{(k)} - s+1) \xrightarrow{t_\alpha(k)} \bigoplus_{j=1}^{h_k} A(r_j^{(k)} - s+1) \longrightarrow \mathcal{R}^{(e-k\pm 1)} \longrightarrow 0.$$

At this point we remark that for each of the graded modules under consideration we have obtained two minimal free resolutions, which must be necessarily isomorphic, and therefore we obtain a bunch of equalities between the $\ell_i^{(k)}$'s and $r_j^{(k)}$'s.

For the applications we are going to illustrate in the sequel of this section, it will suffice to notice that (2.9) and (2.15) give, when e=1:

(2.18) $\ell_i = s \pm 1 - r_i$.

The following results are an easy application of (2.18), but they are already contained in a recent preprint of Ciliberto ([Ci 3]), where more general results are given.

Corollary 2.19. *If* $p_g \geq 4$ *and* $|K|$ *is free from base points, then* R *is generated by elements of degree* ≤ 3. *If* $p_g=4$, R *is generated by elements of degree* ≤ 2 *if the image of the canonical map is not a quadric.*

Proof. *In fact then one can choose 4 independent sections* $y_0, \ldots, y_3$ *of* $H^\circ(K)$ *giving a good projection* $\phi : S \to \mathbb{P}^3$.

We have then $e_0 = \ldots = e_3 = 1$, *so* $\ell_i = 5 - r_i$ *by* (2.18). *Since we claim that* R *is generated as a ring by the elements of degree* ≤ 3, *this will hold if* R *is generated as an* A*-module by such elements, i.e. if* $r_h \geq 2$.
Assume $r_h \leq 1$: *since by* (2.12) $r_h > \ell_1 = 0$, $r_h = 1$. *But then, since* $\ell_2 \geq 1$, $\det(\alpha)$ *would be divisible by a linear form, absurd. Finally, if* $p_g = 4$, *then* $\ell_2 \geq 2$ *and, if* $r_h = 2$, *it follows that* $\det(\alpha)$ *is divisible by a quadratic form, and we conclude by lemma* 2.11.
(Moreover, in this last case, one would have $r_{h-1} = 3$*)*

Q.E.D.

We recall now a recent result of P. Francia ([Fr]):

Theorem. *If* S *is a minimal surface of general type, then* $|2K|$ *is free from base points if* $p_g \geq 2$ *or if* $p_g = 1$ *and* $q=0$.

Hence, if S is also regular, and $p_g \geq 1$, one can choose a good projection ϕ with $(e_0, e_1, e_2, e_3) =$

a) (1,1,1,2) if $p_g \geq 3$ and $|K|$ has no fixed component

b) (1,1,2,2) if $p_g \geq 2$

c) (1,2,2,2) if $p_g = 1$, $K^2 \geq 2$

In all these cases, except if $p_g=2$, $K^2=1$, we can assume that $\det(\alpha)$ has no factor of degree less than, respectively, 1, 2, 3 (notice that we can assume, since $h^\circ(2K)=1+p_g+K^2$, that the y_j's of degree 2 are linearly independent modulo quadratic monomials in the y_i's of degree 1).

Remark. *Surfaces with* $K^2=p_g=1$ *are studied in* [Ca 2,3], *those with* $p_g=2, K^2=1$ *are studied in* [Ho], *and for them* R *is generated by elements of degree less than* 3, *resp.* 5.

With an entirely analogous argument to the one used in the proof of corollary 2.19 we obtain

Corollary 2.20. *R is generated by elements of degree less than* 4 *in case* a) *or case* b) *with* $p_g=2$, *and of degree less than* 5 *in the remaining cases.*

Remark. *To obtain more precise results in this direction, a less rough analysis is needed, taking e.g. into account the numerical relation* $K^2 \prod_i e_i = \sum_j (s+1-2\ell_j)$ *(obtained by looking at the Hilbert polynomials of the modules occurring in* 2.9*), and also the ring structure of* R.

§ 3. RING STRUCTURE AND SYMMETRY

In this section we shall make stronger assumptions on the good projection ϕ considered in § 2.
In terms of the associated finite morphism Ψ of (2.3) we shall assume either

(3.1) $\Psi: X \to \Sigma$ is the normalization map (Ψ being finite, this condition is equivalent to the requirement that ϕ be birational)

or

(3.2) $\Psi: X \to \Sigma$ is of degree 2 and Σ is a normal surface.

Definition 3.3. *ϕ is said to be a quasi-generic canonical projection if one of the conditions* (3.1), (3.2) *is satisfied.*

Also, for simplicity of notations, we shall assume e=1: we want to show that if ϕ is quasi-generic, then the matrix α giving the resolution

(2.9) can be chosen to be a symmetric matrix.

We already remarked that Σ is defined by a homogenous polynomial f, whose degree we shall denote by n (recalling that $s=\sum_{i=0}^{3} e_i$, we have $\sum_{j=1}^{h}(s+1-2\ell_j)=n$ in case (3.1), $=2n$ in case (3.2)).

By [De] prop. 5.9 the sheaf $\mathcal{O}_\Sigma(n-s)$ is the dualizing sheaf ω_Σ for Σ.

Let's first discuss case (3.1), when ϕ is birational.

Let $\mathcal{I}'\subset\mathcal{O}_\Sigma$ be the conductor ideal of Ψ, i.e. $\mathcal{I}'=\mathcal{H}om_{\mathcal{O}_\Sigma}(\Psi_*\mathcal{O}_X,\mathcal{O}_\Sigma)$ and let $\mathcal{I}$ be the inverse image of $\mathcal{I}'$ under the surjective homomorphism of $\mathcal{O}_{\mathbb{P}}$ onto $\mathcal{O}_\Sigma$.

<u>Definition 3.4.</u> *$\mathcal{I}$ is called the adjoint ideal to Σ.*

Now (cf. [Ha] 7.2, page 249, to whose notation we adhere) $\Psi^!\omega_\Sigma$ is a dualizing sheaf for X, hence $\Psi^!\omega_\Sigma\cong\omega_X$, hence

$$\Psi_*\omega_X\cong\Psi_*(\Psi^!\omega_\Sigma)\cong\mathcal{H}om_{\mathcal{O}_\Sigma}(\Psi_*\mathcal{O}_X,\omega_\Sigma)\cong\mathcal{H}om_{\mathcal{O}_\Sigma}(\Psi_*\mathcal{O}_X,\mathcal{O}_\Sigma(n-s)).$$

Since $\omega_\Sigma=\mathcal{O}_\Sigma(n-s)$ is torsion-free, we clearly have (since $\mathcal{O}_\Sigma\hookrightarrow\Psi_*\mathcal{O}_X$, taking $\mathcal{H}om_{\mathcal{O}_\Sigma}$) an injective homomorphism ρ:

$$(3.5)\qquad 0\longrightarrow\Psi_*\omega_X\xrightarrow{\ \rho\ }\omega_\Sigma\cong\mathcal{O}_\Sigma(n-s)$$

and, in fact, in view of the characterization of ω_Σ in terms of residues (cf. [Re] pp. 284-5),

(3.6) $\mathcal{I}'\omega_\Sigma\subset\rho(\Psi_*\omega_X)$, equality of sheaves holding at the points of Σ where ω_Σ is invertible.

For further simplification we shall consider only case 1.1, from now on.

<u>Definition 3.7.</u> *Let* v *be a homogeneous element of* R *of degree* d; *noting that* $R(1)$ *is (by* 2.7) *the module associated to the sheaf* $\Psi_*\omega_X$, *we have that* v, *by* 2.5, *determines a homomorphism* $\tilde{v}:\mathcal{O}_{\mathbb{P}}(-(d-1))\to\Psi_*\omega_X$, *and therefore* $\rho\circ\tilde{v}$ *is represented by an element* $\tilde{\tilde{v}}$, *homogeneous of degree* n+d-s-1, *in* $A/(f)$. $B\in A$ *is said to be a <u>lifting</u> of* v *if*

the residue class of B *in* $A/(f)$ *is just* $\tilde{\tilde{v}}$ *(note that a lifting* B *of* v *therefore always exists, and is unique if* $d<s+1$*).*

In terms of the foregoing definition, we can formulate the main theorem of this section in the following way.

Theorem 3.8. *Let* Ψ *be a quasi-generic birational projection, and let* $v_1=1,v_2,\ldots,v_h$ *be a minimal set of homogeneous generators for the A-module R. Let moreover* β *be a symmetric matrix whose entries* β_{ij} $(i,j=1,\ldots,h)$ *belong to A and are a lifting of* $(v_iv_j)\in R$.
One can then choose a minimal resolution of R, corresponding to the choice of $v_1,\ldots,v_h$*, such that (cf. 2.9) the matrix* α *is symmetric and such that* $\beta=\Lambda^{h-1}(\alpha)$ *(and therefore* $\alpha=(f)^{-(h-1)}\Lambda^{h-1}(\beta)$*, since* $f=\det(\alpha)$ *in case* (3.1)*).*

Remark 3.9. *The proof would be immediate if one knew a priori that* $\det(\beta)\neq 0$ *since then, setting* α *as* $f^{-(h-1)}\Lambda^{h-1}(\beta)$*, we would have* $\alpha\beta=f\,I_h$ *(*I_h *being the identity* $h\times h$ *matrix), therefore* α *would be a matrix giving all the relations among the* v_j*'s (in fact, since R is an integral domain,* $\sum_j \alpha_{ij}v_jv_h=0$ *in R implies* $\sum_j \alpha_{ij}v_j=0$ *in R, conversely if* $\sum_j \gamma_jv_j=0$ *in R,* $\forall h$ $\sum_j \gamma_j\beta_{jh}$ *is divisible by* f*, hence* $\exists\,\lambda_h$ *with* $f\lambda_h=\sum_j \gamma_j\beta_{jh}$*, and then, since* $\alpha\beta=f\,I_h$*, and A is an integral domain,* $\gamma_j=\sum_h \alpha_{jh}\lambda_h$*).*

Proof of theorem 3.8. *We shall compare the two minimal free resolutions of R given, respectively, by* (2.9) *and* (2.15)*, and we use the notations introduced in* (2.14)*, i.e.* $L^1=\bigoplus_{j=1}^{h} A(-r_j)$*,* $L^\circ=\bigoplus_{i=1}^{h} A(-\ell_i)$.

Since two minimal resolutions are isomorphic, using the canonical isomorphisms of $H^4((y),L^\circ)^\vee$ *with* $L^1(1)$ *(resp.: of* $H^4((y),L^1)^\vee$ *with* $L^\circ(1)$*), and the isomorphism* τ *of* $R(1)$ *with* $H^3((y),R)^\vee$ *used to prove* (2.18)*, we infer the existence of isomorphisms* τ_1,τ_0 *which make the following diagram commute:*

$$
\begin{array}{ccccccccc}
0 \to & L^1 & \xrightarrow{\alpha} & L^\circ & \longrightarrow & R & \to 0 \\
& \downarrow \tau_1 & & \downarrow \tau_0 & & \| & \\
(3.10)\quad 0 \to & L^1 & \xrightarrow{t_\alpha} & L^\circ & \longrightarrow & R & \to 0 \\
& \wr\| & & \wr\| & & \downarrow \tau & \\
0 \to & H^4((y),L^\circ)^\vee(-1) & \xrightarrow{t_\alpha} & H^4((y),L^1)^\vee(-1) & \longrightarrow & H^3((y),R)^\vee(-1) & \to 0
\end{array}
$$

Let $\{E_1,\ldots,E_h\}$ *be the canonical basis of* L°, $\{E'_1,\ldots,E'_h\}$ *be the canonical basis of* L^1, *so that* E_i *maps to the element* v_i *of* R.

We recall now (cf. [Se], *p.* 253 *and foll.,* [Do] *p.*40*) how the local cohomology groups with respect to the maximal ideal* (y) *of* A *are effectively computed.*

Let V *be the free* A*-module having as basis the differentials* $dY_0,\ldots,dY_3$, *and graded in such a way that* $\deg(dY_i)=-me_i$, *and let* λ_m *be the homogeneous element of* V *(of degree* 0*) given by*

$$(3.11)\qquad \lambda_m = \sum_{j=0}^{3} Y_j^m(dY_j)$$

Then, for every A*-module* R, *the cohomology groups* $H^*((y),R)$ *are the limit, for* $m>>0$, *of the cohomology of the Koszul complex*

$$\ldots \to \Lambda^i V \otimes_A R \xrightarrow{\wedge \lambda_m} \Lambda^{i+1} V \otimes_A R \longrightarrow \ldots$$

We have therefore the following commutative diagram with exact rows, and with columns giving the Koszul complexes:

$$
\begin{array}{ccccccccc}
0 \longrightarrow & \Lambda^2 V \otimes_A L^1 & \xrightarrow{\alpha} & \Lambda^2 V \otimes_A L^\circ & \longrightarrow & \Lambda^2 V \otimes_A R & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
(3.12)\quad 0 \longrightarrow & \Lambda^3 V \otimes_A L^1 & \xrightarrow{\alpha} & \Lambda^3 V \otimes_A L^\circ & \longrightarrow & \Lambda^3 V \otimes_A R & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Lambda^4 V \otimes_A L^1 & \xrightarrow{\alpha} & \Lambda^4 V \otimes_A L^\circ & \longrightarrow & \Lambda^4 V \otimes_A R & \longrightarrow 0
\end{array}
$$

Let η *be any element in* $H^3((y),R)$: *then* η *is represented by a cocycle*

$$\eta' = \sum_{i=1}^{h} \eta_i\, v_i \quad \text{in} \quad \Lambda^3 V \otimes_A R$$

(hence $\eta_1,\ldots,\eta_h \in \Lambda^3 V$*).*

Letting d *be the coboundary map of the long cohomology sequence associa*-*ted to* (3.12), *we have that*

$$d(\eta) \textit{ is represented by } (\alpha)^{-1}(\lambda_m \wedge \sum_{i=1}^{h} \eta_i E_i).$$

Let r *be an integer* $\leq h$; *to compute* $\tau_\circ(E_r)$, *we use* (3.10): *it must hold*

$$(3.13) \quad < \tau_\circ(E_r), d\,\eta > = << \tau(v_r), \eta >> \qquad \forall \eta \in H^3((y), R)$$

$<,>$, $<<,>>$ *denoting the two pairings given by Serre duality on* $\mathbb{P}$ *and* X *respectively.*

To compare the above two pairings, let F *be the subsheaf of* $O_{\mathbb{P}}(n-s)$ *which is the inverse image of* $\rho(\Psi_* \omega_X)$ *(cf.* (3.5)*) under the surjective homomorphism of* $O_{\mathbb{P}}(n-s)$ *onto* $O_\Sigma(n-s)$: *we have the following exact sequences (where* $F \overset{\sim}{=} IO_{\mathbb{P}}(n-s)$ *except possibly at the non Gorenstein points of* Σ*):*

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & O_{\mathbb{P}}(-s) & \longrightarrow & F & \longrightarrow & \psi_*\omega_X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & O_{\mathbb{P}}(-s) & \longrightarrow & O_{\mathbb{P}}(n-s) & \longrightarrow & O_\Sigma(n-s) & \longrightarrow & 0 \end{array}$$

which give an isomorphism $\partial : H^2(\psi_* \omega_X) \to H^3(O_{\mathbb{P}}(-s)) \overset{\sim}{=} \mathbb{C}$, *as it is easy to check.*

Under the natural identifications described before (2.13), $\partial :\ H^3((y), R) \xrightarrow{\cong} H^4((y), A(-s))$ *is obtained by lifting a cocycle with values in* R *to a cocycle with values in* $A(n-s)$, *taking its coboundary, and then dividing by* f. *Hence*

$$<\tau_\circ(E_r), d\eta> = <\tau_\circ(E_r), \alpha^{-1}(\lambda_m \wedge \sum_{i=1}^{h} \eta_i E_i)> =$$

$$= <({}^t\alpha)^{-1}\tau_\circ(E_r),\ \lambda_m \wedge \sum_{i=1}^{h} \eta_i E_i>, \quad \textit{while}$$

$$<<\tau(v_r), \eta>> = <<\tau(v_r), \sum_{i=1}^{h} \eta_i v_i>> = \partial(\sum_{i=1}^{h} \eta_i (v_r v_i)) =$$

$$= \frac{1}{f}(\lambda_m \wedge \sum_{i=1}^{h} \eta_i \beta_{ri}).$$

Therefore, setting $\tau_\circ(E_r) = \frac{1}{f}({}^t\alpha) \sum_{j=1}^{h} \beta_{jr} E_j$, *we have a lifting of* τ, *and we conclude that*

$$(3.14) \quad \tau_\circ = \frac{1}{f}\ {}^t\alpha\ \beta, \qquad \tau_1 = \frac{1}{f}\ \beta\ \alpha.$$

To finish our proof, it suffices to change the given minimal free resolution in order to obtain, instead of α, $\tau_\circ\alpha = \frac{1}{f}\,{}^t\alpha\,\beta\,\alpha$, *which is now a symmetric matrix.*
Also (cf. remark 3.9), it is easily seen that $\beta = \Lambda^{h-1}\alpha$ *gives a lifting of the products* $v_i v_j$, *and* $\alpha\beta = \beta\alpha = f\ I_h$.

Q.E.D.

To end this section, we indicate how a similar theorem holds in case Ψ is a quasi-generic projection of degree 2 (cf. [Ca 1]). In this case, since Σ is normal, Σ is the quotient of X by a biregular involution σ: X→X, and we have a splitting of the functions on X into σ^*-invariant, resp. σ^*-antiinvariant ones: in other terms

(3.15) $\quad \Psi_* \mathcal{O}_X = \mathcal{O}_\Sigma \oplus \mathcal{F}$.

Accordingly, we have a splitting of R as a direct sum of A-submodules

(3.16) $\quad R = R'' \oplus R'$

We have, though, another splitting of R as a vector space: in fact σ induces an automorphism σ^* of R, and if we denote by R^+ (resp.: R^-) the (+1) (resp.:(-1)) eigenspace for σ^*, we also have

(3.17) $\quad R = R^+ \oplus R^-$ (note that in fact R^+ is a subring).

We observe, concerning this last splitting, that σ induces the trivial projectivity on $\mathbb{P}$, hence first of all the y_i's are eigenvectors for σ^*, moreover, σ being an involution, $\sigma^*(y_i) = \pm\, y_i$, so finally, since σ^* induces the identity on $\mathbb{P}$, either

(3.2.a) $\quad \sigma^*(y_i) = y_i \qquad (i=0,\ldots,3)$

or

(3.2.b) $\quad \sigma^*(y_i) = (-1)^{e_i}\, y_i \qquad (i=0,\ldots,3)$

Notice that $R''=R^+$, $R'= R^-$ if and only if we are in case a).
We also remark that, by the same argument of proposition 2.7 $\Psi_*\,\omega_X = \tilde{R}(\pm 1)$, hence

(3.18) $\quad \Psi_*\,\omega_X = \mathcal{O}_\Sigma(\pm 1) \oplus \mathcal{F}(\pm 1)$.

Now, in case b), since σ^* acts as multiplication by $(-1)^m$ on A_m, it is easy to see that $\mathcal{F}(\pm 1)$ corresponds to the sheaf of σ^*-invariant sections of ω_X (resp. : $\mathcal{O}_\Sigma(\pm 1)$ to the sheaf of antiinvariant ones).
The fundamental fact that we shall use here is that σ^* acts as the identity

on $H^2(S,\Omega^2_S)\cong H^2(X,\omega_X)$. This fact can easily be checked working over $\mathbb{C}$, as we do, since $H^2(S,\Omega^2_S)\cong H^4(S,\mathbb{C})\cong H^4(S,\mathbb{Z})\otimes\mathbb{C}$ and σ^* acts as the identity on $H^4(S,\mathbb{Z})$ (σ gives in fact an orientation preserving homeomorphism of S).

Therefore, while we apply Serre duality on X, we have $(R^+_m)^\vee = R^+_{\pm 1-m}$, and, in particular, in case b) we have

$$R^+_{2m} = R''_{2m}\ ,\quad R^+_{2m+1} = R'_{2m+1}$$

(and analogously for R^-).

We choose now a resolution of R given as a direct sum of a resolution of R'' and of a resolution of R', and let α be the matrix giving such a resolution (hence α is now in the form $\begin{pmatrix} f & 0 \\ 0 & \alpha' \end{pmatrix}$, since $R'' \cong A/_{(f)}$).

Applying the weak symmetry statement proved in §2, we see that $({}^t\alpha)$ gives a resolution for a module isomorphic to $R(\pm 1)$; but further, applying the above remarks on duality and σ^*-variance and looking at the computation given before (2.15), we see that ${}^t\alpha$ gives a resolution for

$$(3.19)\qquad \begin{cases} R'(\pm 1) \oplus R''(\pm 1) & \text{in case b)} \\ R''(\pm 1) \oplus R'(\pm 1) & \text{in case a).} \end{cases}$$

In case b), we conclude that, up to a shift of grades, also R' is a cyclic module $\cong A/_{(f)}$.

So in case b) R is generated by 1, v_2 and, if the matrix α is normalized as we did in §2 (with decreasing degrees as the row and column indeces increase), then α has the form $\begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix}$, and $\deg f = s \pm 1-\ell_2$

<u>Definition 3.20.</u> *Case* (3.2.b) *is called the standard case (of a quasi generic projection of degree* 2). *The canonical ring R is then generated by* y_0,y_1,y_2,y_3,v_2 *with the relations* $f(y_0,\dots,y_3)=0$, $v_2^2=G(y_0,\dots,y_3)$, *where* G *is a homogeneous polynomial of degree equal to* $2\ell_2=2(s\pm 1-n)$.

We pass now to the non standard case (3.2.a), which is the only one we are going to consider in the next paragraphs.

In case (3.2.a), first of all, by (3.19), $\deg(f)=s\pm 1$.

We choose as a minimal system of generators for the A-module R, $v_1=1$ as a generator for R^+, and $v_2,\dots,v_h$ a minimal (homogeneous) system of generators for R^-.

The resolution for R^+ is clearly given by

$$0 \longrightarrow A(-n) \xrightarrow{f} A \longrightarrow R^+ \longrightarrow 0 \quad (n=\text{degree of } \Sigma)$$

while the resolution for R^- will be of the form

$$(3.21) \qquad 0 \longrightarrow L''^1 \xrightarrow{\alpha''} L''^0 \longrightarrow R^- \longrightarrow 0$$

The situation is technically simpler here, than in the birational case, since the product of 2 antiinvariant functions is invariant, so we have an A-bilinear map of $R^- \times R^- \to R^+$, hence it is evident how to "lift" $v_i v_j$, for $i,j \geq 2$, to an element B_{ij} of A. The role of the adjoint ideal is taken here by the branching ideal B, the inverse image in A of $R^- \cdot R^- \subset R^+$, and which is spanned by the B_{ij}'s $(i,j \geq 2)$.
The analogous theorem to 3.8 is, in this situation, the following

Theorem 3.22. *Given a non standard quasi-generic projection of degree 2, and a minimal system of homogeneous generators* $v_2, \ldots, v_h$ *of* R^- *as an A-module, let* β'' *be a symmetric matrix of size* $(h-1)\times(h-1)$ *whose entries* B_{ij} $(i,j=2,\ldots,h)$ *give a lifting of* $v_i v_j$. *Then one can choose a minimal free resolution of* R^-, *associated to the given choice of generators, such that* (cf. 3.21) *the matrix* α'' *is symmetric and* $\alpha''\beta'' = \beta''\alpha'' = f\, I_{h-1}$.

§ 4. DETERMINANTAL EQUATION, ADJOINTS AND THE CANONICAL RING.

Let's assume again that Ψ gives a quasi-generic projection, and let $1=v_1, v_2, \dots, v_h$ be a minimal set of homogeneous generators for R.
Then, since R is a ring, $v_i v_j \in R$, hence, for $i,j=2,\dots,h$, there exist , for $k=1,\dots,h$, homogeneous polynomials $\ell_{ij}^k = \ell_{ji}^k$ with the property that

$$(4.1) \qquad v_i v_j = \sum_{k=1}^{h} \ell_{ij}^k v_k,$$

and the ℓ_{ij}^k's are defined uniquely modulo the relations

$$(4.2) \qquad \sum_{j=1}^{h} \alpha_{ij} v_j = 0 \qquad (i=1,\dots,h)$$

Theorem 4.3. *R is generated by* 1, $y_0,\dots,y_3$, $v_2,\dots,v_h$, *and the relations among these generators are generated by the* $h(h+1)/2$ *relations* (4.1) *and* (4.2).

Proof. *Let D be the ring $\mathbb{C}[y_0,\dots,y_3,v_2,\dots,v_h]$ (graded in an obvious way), and let J be the ideal in D generated by the relations* (4.1) *and* (4.2).
Since the relations (4.1) *hold, we have that D/J is generated by $v_1=1, v_2,\dots,v_h$ as an A-module. Since the natural homomorphism of D in R is onto, R is a quotient of D/J, but on the other hand, since the relations* (4.2) *hold in D/J, D/J is a quotient of R as an A-module. Hence $R \cong D/J$.*

Q.E.D.

Remark 4.4. *Case* (3.2) *occurs exactly when one has* $\ell_{ij}^k=0$ *for* $k=2,\dots,h$, *(and then* $\ell_{ij} = B_{ij}$ *in case* 3.2.a*)*.

Let's consider now the case (3.1) when ψ is birational. Since X is the normalization of Σ, it must be true that the ring R be completely determined by the symmetric matrix α. In fact we know that $\beta = \Lambda^{h-1}(\alpha)$ is a matrix such that β_{ij} is a lifting of $v_i v_j$, hence (4.1) implies that

$$(4.5) \qquad \beta_{ij} = \sum_{k=1}^{h} \ell_{ij}^k \beta_{1k}$$

(the equality should a priori only hold modulo f, but, since $f = \sum_{k=1}^{h} \alpha_{1k} \beta_{1k}$, we can always modify the ℓ_{ij}^k's in order that (4.5)hold).

It is immediate to see that (4.5) is equivalent to the following Rank Condition mentioned in the introduction:

(R.C.) let α' be the matrix obtained by erasing the first row of the symmetric matrix α: then the ideal in A generated by the entries of $\beta=\Lambda^{h-1}(\alpha)$ equals the ideal I generated by the entries of $\Lambda^{h-1}(\alpha'),\beta_{11},\dots,\beta_{1h}$.

Remark 4.6 : *in the following we shall see that the* (R.C.) *is directly related to the theorem of Rouché-Capelli.*

Proposition 4.7. *Let α be a symmetric matrix of homogeneous polynomials of degrees as above, and assume that α satisfies* (R.C.), *that* $\text{depth}_I(A)\geq 2$, *and that* $f=\det(\alpha)\neq 0$.
Let $\hat{R}$ be the A-module generated by $v_1=1,v_2,\dots,v_h$ subject to relations (4.2), *and define an A-bilinear product on $\hat{R}$ according to* (4.1) *and the request that $v_1=1$ be the identity. Then this product is well defined, is commutative and associative, hence makes $\hat{R}$ into a commutative A-algebra R.*

Remark 4.8. *The condition* $\text{depth}_I A\geq 2$ *holds iff the β_{ij}'s have no common factor, and, a fortiori, if* $f=\det(\alpha)$ *is an irreducible polynomial.*

Proof. of prop. 4.7. *The ideal* I *has the following minimal free resolution*

$$(4.9)\qquad 0 \longrightarrow L'' \xrightarrow{\ \alpha'\ } L' \xrightarrow{\ \Lambda^{h-1}(\alpha')\ } I \longrightarrow 0$$

(L", L' being free A-modules of respective ranks equal h-1,h*), by virtue of Hilbert's theorem (cf. e.g.* [Ei] *, thm. 2,p. 122). In particular, if* $\sum_{t=1}^{h} g_t\,\beta_{1t}=0$ *in A, then there do exist, for* $i=2,\dots,h$, *elements u_i in A s.t.*

$$(4.10)\qquad g_t=\sum_{i=2}^{h} u_i\,a_{it},\quad \textit{for each}\quad t=1,\dots,h.$$

Again since $f=\sum_{t=1}^{h}\alpha_{it}\beta_{it}$, *we can obtain that the same conclusion hold if only* $\sum_{t=1}^{h} g_t\,\beta_{1t}\equiv 0 \pmod{f}$.

To check that the definition of the product is well posed (we have up to now only specified the product of a pair of generators), we must verify that

(4.11) $\quad (\sum_{j=1}^{h} \alpha_{ij} v_j) v_k = 0$ *in* $\hat{R}$, *i.e. that*

(4.12) $\quad \sum_{t,j=1}^{h} \alpha_{ij}\, \ell_{jk}^{t}\, v_t = 0.$

(4.12) *holds if and only if there do exist* $u_1, \dots, u_h \in A$ *s.t.*

$$\sum_{j=1}^{h} \alpha_{ij}\, \ell_{jk}^{t} = \sum_{s=1}^{h} u_s\, \alpha_{st}.$$

By our previous considerations, it suffices to show that $\sum_{t,j=1}^{h} \alpha_{ij}\, \ell_{jk}^{t}\, \beta_{1t} \equiv 0(f)$, *but this is true, since, by* (4.5), *this expression equals* $\sum_{j=1}^{h} \alpha_{ij}\, \beta_{jk} = f\, \delta_{ik}$.

The commutativity of the product is obvious since α *is a symmetric matrix, hence* β *is also symmetric and* $\ell_{ij}^{k} = \ell_{ji}^{k}$; *on the other hand, showing that the product is associative amounts to proving that, in* R,

(4.13) $\quad (v_i \cdot v_j) v_k = v_i (v_j \cdot v_k)$ *for each* $i,j,k=2,\dots,h$,

i.e. $(\sum_{t=1}^{h} \ell_{ij}^{t}\, v_t) \cdot v_k = v_i (\sum_{s=1}^{h} \ell_{jk}^{s}\, v_s)$, *or still more concretely*,

$$\sum_{r=1}^{h} \left[(\sum_{t=1}^{h} \ell_{ij}^{t}\, \ell_{tk}^{r}) - (\sum_{s=1}^{h} \ell_{jk}^{s}\, \ell_{is}^{r}) \right] v_r = 0 \quad \textit{in } \hat{R}.$$

By the same remark we used above, it suffices to show that

$$\sum_{r=1}^{h} \left[(\sum_{t=1}^{h} \ell_{ij}^{t}\, \ell_{tk}^{r}) - (\sum_{s=1}^{h} \ell_{jk}^{s}\, \ell_{is}^{r}) \right] \beta_{1r} \equiv 0 \pmod f,$$ *or, using* (4.5), *that*

(4.14) $\quad (\sum_{t=1}^{h} \ell_{ij}^{t}\, \beta_{tk}) - (\sum_{s=1}^{h} \ell_{jk}^{s}\, \beta_{is}) \equiv 0 \pmod f.$

We notice further that, since $f=\det(\alpha), \beta = \Lambda^{h-1}(\alpha)$, $\beta_{ij}\, \beta_{kt} \equiv \beta_{jk}\, \beta_{it} \pmod f$ (*cf.* [Ca 1], 1.2 *p.* 437).

Hence $\beta_{1u}(\sum_{t=1}^{h} \ell_{ij}^{t} \beta_{tk}) \equiv \sum_{t=1}^{h} \ell_{ij}^{t} \beta_{1t} \beta_{uk} =$

$$= \beta_{ij} \beta_{uk} \equiv \beta_{jk} \beta_{iu} = \sum_{s=1}^{h} \ell_{jk}^{s} \beta_{1s} \beta_{iu} \equiv$$

$$\equiv \beta_{1u} (\sum_{s=1}^{h} \ell_{jk}^{s} \beta_{is}).$$

We have thus obtained that the expression g *in* (4.14) *is such that* β_{1u} $g \in (f)$ $\forall$ $u=1,\dots,h$. *Since the* β_{1u}*'s have no common factor, it follows that* f *divides* g, *as we wanted.*

Q.E.D.

Assuming α to be a matrix with all the good properties we requested so far, we define X to be Proj(R) (cf. 4.7), so that the inclusion of $A/(f)$ into R defines a morphism $\Psi: X \to \Sigma$.
Since our aim is to recreate the situation we started with, we first notice that Ψ is a finite morphism : in fact R is a finite $A/(f)$-module generated by $v_1=1,\dots,v_h$, hence, over any affine piece of Σ, $\Psi_* \mathcal{O}_X$ is a finite $\mathcal{O}_\Sigma$-module. With the notations we have thus set we have

Proposition 4.15. *Assume the ring R is given as in* 4.7, *and assume that* $f=\det(\alpha)$ *is an irreducible polynomial. Then* $\Psi: X=\text{Proj}(R) \longrightarrow \Sigma =$ $= \text{Proj}(A/(f))$ *is a finite birational morphism.*

Proof. *By the remarks made previously, it suffices to exhibit a rational inverse for* Ψ.
We remark that, R being generated as a ring by $y_0,\dots,y_3$, $v_2,\dots,v_h$, *we have a natural embedding*

$$(4.16) \qquad X \subset \mathbb{P} = \tilde{\mathbb{P}} \; (e_0,\dots,e_3, \ell_2,\dots,\ell_h).$$

In this respect, we can view (since $v_1=1$*) equations* (4.2) *as a set of inhomogeneous linear equations in the unknows* $v_2,\dots,v_h$ *(therefore the rank condition is nothing else than the condition in the Rouché-Capelli theorem):*

$$\sum_{j=2}^{h} \alpha_{ij} v_j = - \alpha_{i1}.$$

The system of equations has a unique solution for the points y *in* $\mathbb{P}$ *where* α' *has rank* = h-1, *and then the solution is given, by Cramer's rule, for* j = 2,...,h, *by*

$$(4.17) \qquad v_j = \beta_{1j}/\beta_{11}.$$

Since, by the irreducibility of f, *the locus defined by* $I=(\beta_{11},\ldots,\beta_{1h})$ *has codimension* 1 *in* Σ, *we conclude that* Ψ^{-1} *is given by the restriction to* Σ *of the following rational map* Ψ^{-1}: $\mathbb{P} \dashrightarrow \tilde{\mathbb{P}}$ *s.t.*

$$(4.18) \qquad \Psi^{-1}(y) = (y_0\,\beta_{11}, y_1\,\beta_{11}, y_2\,\beta_{11}, y_3\,\beta_{11},\ \beta_{12},\ldots,\beta_{1h}).$$

Q.E.D.

Corollary 4.19. X *is the blow-up of* Σ *in the ideal* $\hat{I}$ *generated by* $(\beta_{11},\ldots,\beta_{1h})$.

Proof. *Obvious by* (4.18).

Q.E.D.

Remark 4.20. *The ideal* $\hat{I}$, *generated by* $\beta_{11},\ldots,\beta_{1h}$ *in* $\mathcal{O}_\Sigma$, *is contained in the conductor ideal* $I' = Hom_{\mathcal{O}_\Sigma}(\Psi_* \mathcal{O}_X, \mathcal{O}_\Sigma)$.

Proof. *Since* $\hat{I}$ *is generated by the* β_{1j}*'s, it suffices to prove that, the isomorphism of* $\mathbb{C}(\Sigma)$ *with* $\mathbb{C}(X)$ *being given by* (4.17), $\beta_{1j}\,v_k$ *is, for every* j,k, *a regular section (of an invertible sheaf) on* X.
But in fact $\beta_{1j}v_k = \beta_{1j}\beta_{1k}/\beta_{11}$, *and* $\beta_{1j}\beta_{1k} - \beta_{11}\beta_{jk} \equiv 0 \pmod f$, *hence* $\beta_{1j}v_k = \beta_{jk}$.

Q.E.D.

Proposition 4.21. *Assume that the only singularities of* X *are* R.D.P.*'s: then* $\hat{I} = I'$ *(cf.* 4.20*).*

Proof. *If* X *has only* R.D.P.*'s as singularities,* X *is normal and the dualizing sheaf* ω_X *is invertible.*
Since the local rings of Σ *are* C-M *(Cohen-Macaulay), the conductor ideal* I' *is also* C-M, *and equals the ideal* $\hat{I} = (\beta_{11},\ldots,\beta_{1h})$, *which is also* C-M, *provided equality holds in codimension* 1.
In codimension 1 *we can use Wilson's argument ([Wi], cor. 1.4) to the*

effect that the normalization X *equals the blow-up of* Σ *in the conductor ideal* I'*: we conclude then by cor.* 4.19.

Q.E.D.

For simplicity, let's assume now that $\omega_\Sigma \cong \mathcal{O}_\Sigma(n-s)$ is an invertible sheaf: then, as we saw before (cf. (3.6)), $\Psi_* \,\omega_X \cong I'\mathcal{O}_\Sigma(n-s)$. Furthermore the ring R is such that the A-module $R(1)$ ($R(-1)$ in case (1.2)) is isomorphic to the A-module associated to the coherent $\mathcal{O}_{\mathbb{P}}$-sheaf $\Psi_* \,\omega_X$: in fact, if I is the ideal generated by $(\beta_{11},\dots,\beta_{1h})$ in $\mathcal{O}_{\mathbb{P}}$, we have the exact sequence (4.9) for the associated A-module I, and I/(f)I is easily seen to be isomorphic to R. On the other hand, by the embedding (4.16) of X in $\tilde{\mathbb{P}}$, one sees immediately that $R(\pm 1)$ is the module associated to the invertible sheaf $\mathcal{O}_X(\pm 1)$. We have thus proven

Theorem 4.22. *Let* $e_0,\dots,e_3$ *be positive integers with* G.C.D.$(e_0,\dots,e_3)=1$, *and let* A *be the graded ring* $\mathbb{C}[Y_0,\dots,Y_3]$ *where* $\deg(Y_i)=e_i$. *Let moreover be given positive integers* $\ell_1=0<\ell_2\le \dots \le \ell_h$, *and a symmetric matrix* α *of size* $(h \times h)$, *with entries* α_{ij} *homogeneous polynomials of degree* $s+1+\ell_i-\ell_j$ *(resp:* $s-1-\ell_i-\ell_j$*), where* $s=\sum_{i=0}^{3} e_i$*; assume further that* α *satisfies the rank condition* (R.C.), *that* $f=\det(\alpha)$ *is an irreducible polynomial defining* $\Sigma \subset \mathbb{P}=\mathrm{Proj}(A)$ *with* ω_Σ *invertible, and finally that,* R *being the ring associated with* α *as in prop.* 4.7, X=Proj(R) *has only* R.D.P.*'s as singularities. Then* X *is the canonical model of a regular surface of general type (resp: the anticanonical model of a weak Del Pezzo surface) and* $\Psi: X \to \Sigma$ *is a quasi-generic birational projection (as in* § 2*).*

We have now an analogous result describing the case (3.2.a) of a non standard quasi-generic projection of degree 2, but in this case there is a big restriction on the degrees $\ell_2,\dots,\ell_h$. As we saw in remark 4.4, case (3.2.a) is a special case where $\alpha_{11}=f$ (equation of Σ), $\alpha_{1j}=0$ for $j\ge 2$, and $\det(\alpha'')=$ $=f$, hence f has degree equal to $(s\pm 1)$, while $\sum_{j=2}^{h}(s\pm 1-2\ell_j)=s\pm 1$, moreover (2.12) says that $\ell_j+\ell_{h-j+1}<s\pm 1$ (in particular $\ell_h<s\pm 1$), hence $\sum_{j=2}^{h}(s\pm 1-2\ell_j)=\sum_{j=1}^{h}(s\pm 1-\ell_j-\ell_{h-j+1})-(s\pm 1)>h-(s\pm 1) \Rightarrow h<2s\pm 2$.

On the other hand here the rank condition (R.C.) is trivially satisfied (since, in the notation set up in 3.21, 3.22, $\beta_{11}=f$, $\beta_{1j}=0$ for $j\geq 2$, $\beta_{ij}=B_{ij}$ for $i,j\geq 2$). For further simplification we assume that, according to def. 2.4, the e_i's are normalized.
We remark further that Σ must be normal (with $\omega_\Sigma = \mathcal{O}_\Sigma(1)$, i.e. Σ is a canonical surface*) and then the ideal $\mathcal{B}$ generated by the B_{ij}'s, defining a locus contained in the singular locus of Σ, is such the support of $\mathcal{O}_{\mathbb{P}}/\mathcal{B}$ is a finite set of points. Conversely, to a matrix α'' as above we associate a ring R generated by $y_0,\dots,y_3$, $v_2,\dots,v_h$ with the relations

$$(4.23)\qquad \begin{cases} \sum_{j=2}^{h} \alpha_{ij}\, v_j = 0 & (i=2,\dots,h) \\ v_i v_j = B_{ij} & ((B_{ij})=\beta'' \text{ being the adjoint matrix of } \alpha'') \end{cases}$$

and we denote, as usual, Proj(R) by X.

Theorem 4.24. *Let A, s, be as in thm. 4.22, and let be given positive integers* $\ell_2,\dots,\ell_h$ *such that* $2\sum_{j=2}^{h} \ell_j = (h-2)(s+1)$ *(resp.* $:=(h-2)(s-1)$*), and let* α'' *be a symmetric matrix of size* $(h-1)\times(h-1)$ *whose entries* α_{ij} $(i,j=2,\dots,h)$ *are homogeneous polynomials of degree* $s+1-\ell_i-\ell_j$ *(resp.:* $s-1-\ell_i-\ell_j$*). Assume that* $f=\det(\alpha'')$ *is an irreducible polynomial defining a normal surface* $\Sigma\subset\mathbb{P}$ *with* $\omega_\Sigma \cong \mathcal{O}_\Sigma(1)$ *invertible (and that the* e_i*'s are normalized). Let the ring R be given as in (4.23), and assume that* $X=\mathrm{Proj}(R)$ *has only* R.D.P.*'s as singularities.*
Then X is the canonical model of a surface of general type (resp.: the anticanonical model of a weak Del Pezzo surface), and $\Psi: X\to\Sigma$ *is a quasi-generic projection of degree 2.*

Proof. *Ψ is finite since R is a finite A/(f) module. Moreover, when* $B_{ii}\neq 0$, *we have* $v_i^2 = B_{ii}$, $v_j=(v_j\, v_i)v_i\cdot v_i^{-2} = B_{ij}\cdot B_{ii}^{-1}\cdot v_i$, *hence Ψ is unramified at the points where the rank of* α'' *is* $(h-2)$. *Since* ω_X, ω_Σ, *are invertible and Ψ is unramified in codimension 1, we have* $\omega_X = \Psi^*\omega_\Sigma = \Psi^*\mathcal{O}_\Sigma(\pm 1)$, *as we wanted to show.*

Q.E.D.

Remark 4.25. *If Σ has only* R.D.P.*'s as singularities, then the same occurs also for* X.

* But, cf. 2.4, if $\mathcal{O}(1) \cong \mathcal{O}$, Σ is a K3 surface!

In fact this is trivial for the points where Ψ *is unramified, whereas, if* $B_{ij}(P)=0$ $\forall i,j=2,\dots,h$, *then the inverse image of* P *is a single point* Q.

Since the germ (Σ,P) *is the quotient of* $(\mathbb{C}^2,0)$ *by a finite group* $G\subset SL(2,\mathbb{C})$, *and* (Σ,P) *is a quotient of* X *by* $(\mathbb{Z}/2)$ *and* Ψ *is locally ramified only at* Q, *also* (X,Q) *is of the form* $(\mathbb{C}^2,0)/G'$, G' *of index* 2 *in* G, *hence* Q *is a* R.D.P..

Conversely, if the points of Σ *where the* B_{ij}*'s vanish are smooth points of* $\mathbb{P}$, *then* Σ *has only* R.D.P.*'s as singularities if* X *does.*

In fact, the quotient of a rational singularity is again rational, and R.D.P.*'s are precisely the rational singularities with the Zariski tangent space of dimension* 3 *(cf.*[Ar 2]*).*

5. THE CLASSICAL CASE AND EXAMPLES.

At the end of last section, we have given two existence theorems, (4.22) and (4.24), for regular surfaces of general type (the existence part for weak Del Pezzo surfaces is of no interest, in view of prop. (1.17)). But whereas the matrices α'' in theorem (4.24) are only subject to an open condition, the matrices α considered in theorem (4.22) are also subject to the closed condition (R.C.).

Thus we want analyze condition (R.C.), observing first that (R.C.) is a global condition, but in fact equivalent to local ones: in this respect we shall show how the local generic case fits in with the classical picture of the "ordinary" singularities occurring for a generic projection.

As a second goal, we want to show that the theory developed so far is non vacuous, and that it is in fact, under mild conditions, possible to construct smooth X's.

We denote still by I the ideal in A generated by $\beta_{11},\dots,\beta_{1h}$, by $\mathcal{I}$ the associated ideal sheaf, and by $\Gamma=\mathrm{Proj}(A/I)$: since I is C-M Γ is a scheme of dimension 1 without embedded points.

Also, by 4.9, $\beta_{ij}\in I$ if and only if its stalk belongs at $\mathcal{I}$ for every point P in $\mathbb{P}$.

X being the blow-up of Σ in Γ, we want first of all $\Sigma-\Gamma$ to have at worst

R.D.P.'s as singularities. To simplify our local computations we shall assume that the points of Γ are smooth points of $\mathbb{P}$, e.g. that $\mathbb{P}=\mathbb{P}^3$. We denote again by α'' the minor of α obtained by deleting the first row and column of α, so that $\det(\alpha'')=\beta_{11}$, and we shall denote by F the associated surface, called the adjoint surface.

Remark 5.1. *For later use, we observe the following.*
We have showed that the datum of the ring R with an A-module structure and with a minimal system of homogeneous generators $1=v_1,\dots,v_h$ *and the datum of a symmetric matrix* α *satisfying certain conditions are equivalent to each other. Now, we can pass from one system of generators to another by acting with the group* G *of transformations* g *s.t.* $g(v_j)=$ $=\sum_{i=1}^{h} g_{ij}\, v_i$ $(j=2,\dots,h)$, *with* $g_{ij}\in A_{\ell_j-\ell_i}$.

Correspondingly, the matrix α is transformed into the matrix

$$(5.2)\qquad g\ \alpha\ {}^t g.$$

On the other hand, the group H of graded automorphisms of *A* also acts on the given set of matrices, and G and H give a semi-direct product (with G being the normal subgroup) $G \rtimes H$ s.t. the orbits of the action of $G \rtimes H$ give the isomorphism classes of the morphisms $\Psi: X\to\Sigma$.
We notice finally that the action of G is obtained as a composition of elementary transformations, of the kind: adding a multiple of a row to a given row and then performing the same operation for the corresponding columns, or multiplying by the same constant a row and the corresponding column.
We are now in a position to study, locally at $P\in\mathbb{P}$, the closed and the open condition imposed upon the matrix α: to do this, we shall work with the local ring $\mathcal{O}_{P,\mathbb{P}}$ of germs of holomorphic functions around P, denoting $\mathcal{M}_{P,\mathbb{P}}$ its maximal ideal.

Remark 5.3. *Since* $\mathcal{O}_{P,\mathbb{P}}^{\,*}=(\mathcal{O}_{P,\mathbb{P}_n}^{\,*})^2$, *by the Babylonian theorem, quadratic forms on* $(\mathcal{O}_{P,\mathbb{P}})^k$ *split as* $q_1\oplus q_2$, *where the associated matrices* A_1, A_2 *are s.t.* A_1 = *identity matrix,* A_2 *has entries in* $\mathcal{M}_{P,\mathbb{P}}$. *Also, since we want the ideal spanned by* $\Lambda^{h-1}(\alpha')$ *to be invariant, we shall only allow to change* α *to* $g\alpha\,{}^t g$ *where* g *belongs to the subgroup* G *of*

$GL(h,\mathcal{O}_{P,\mathbb{P}})$ *of the matrices whose first column is* $= {}^t(1,0,\dots,0)$.

We observe now that, if $P\in\Gamma$, then $\alpha''(P)$ is not invertible, hence rank $(\alpha''(P)) \leq h-2$.

Case I: rank $(\alpha''(P)) = h-2$ ($P\in\Gamma$, also).

We can assume then, acting with G, that $\alpha = \begin{pmatrix} x & z & 0 \\ z & y & \\ 0 & & I_{h-2} \end{pmatrix}$

where $y, z\in \mathcal{M}_{P,\mathbb{P}}$, since P is a point of Γ.

$I=(y,z)$, so the (R.C.) implies that $\exists\ a,b\in\mathcal{O}_{P,\mathbb{P}}$ s.t. $x=az+by$. Therefore a local equation for Σ is given by

$$(5.4)\qquad f = azy + by^2 - z^2 = \Delta y^2 - \zeta^2$$

where $\zeta = z - \frac{a}{2}y$, $\Delta = (b+\frac{a^2}{4})$, hence $(y,z)=(y,\zeta)= I$.

We impose now the condition that X, the blow-up of Σ in I, have only R.D.P.'s as singularities, and in particular we ask when is X smooth. An easy computation shows that X is wholly contained in one of the two charts of the blow-up of $\mathbb{P}$ along Γ, hence there are given local coordinates (x_1,x_2,x_3,t), s.t. $\mathcal{O}_{P,\mathbb{P}} \cong \mathbb{C}\{x_1,x_2,x_3\}$, where X is defined by

$$(5.5)\quad \begin{cases} \Delta - t^2=0 \\ \zeta - ty \end{cases}, \text{ where } \Delta,\zeta,y\in\mathbb{C}\{x_1,x_2,x_3\}.$$

Two further possibilities occur: if $\Delta(P)\neq 0$, then P has two inverse images P_1,P_2 in X, and the local form of the upper left corner of α can be reduced to $\begin{pmatrix}\eta & \zeta' \\ \zeta' & \eta\end{pmatrix}$ with $\eta = \Delta y$, $\zeta' = \Delta^{1/2}\zeta$. Then P_1 and P_2 are R.D.P.'s (resp.: smooth points) of X $\Leftrightarrow$ the surfaces $\eta+\zeta'$, $\eta-\zeta'$ have a R.D.P. in P (resp.: a smooth point in P).

If $\Delta(P)=0$, then P has only one inverse image in X, which is a smooth point of X if and only if $\{\Delta=\zeta=0\}$ is a smooth curve at P. The local form of the upper part of α is $\begin{pmatrix}\Delta y & \zeta \\ \zeta & y\end{pmatrix}$ and we have a R.D.P. if and only if

i) Δ is a local parameter at P, say $\Delta=x_1$, and then $\zeta(t^2,x_2,x_3) - t\, y(t^2,x_2,x_3)$ defines a R.D.P.

ii) ζ is a local parameter at P, (say $\zeta= x_3$) and $x_3=ty, \Delta=t^2$ define a R.D.P.

In particular, if P is a smooth point of Γ, then one can choose coordinates x_1,x_2,x_3 such that the local equation of Σ is either

$$(5.6)\quad \begin{cases} x_1x_2=0:(\mathrm{I}) = (\Gamma \text{ double curve of } \Sigma) & \text{or} \\ x_3^2-x_1x_2^2=0:(\mathrm{II})=(P \text{ pinch point of the double curve}) & \text{or} \\ x_3^2-\Delta x_2^2=0:(\mathrm{III}) \text{ with } t^2-\Delta(x_1,x_2,tx_2) \text{ defining a R.D.P.} \end{cases}$$

(in particular in the first two cases we have smooth points of X, in the third ony if $\Delta=bx_2+c\,x_3+\phi$ with $b,c\in\mathbb{C}$, $\phi\in M^2_{P,\mathbb{P}}$, and $b\neq 0$).

Case II: rank $(\alpha''(P)) = h-3$ (hence automatically $P\in\Gamma$).

We can assume, up to G equivalence, that α has the form

$$(5.7)\quad \begin{vmatrix} \gamma & u & v & \\ u & x & z & 0 \\ v & z & y & \\ & 0 & & I_{h-3} \end{vmatrix} \quad \text{where } x,y,z\in M_{P,\mathbb{P}}.$$

Proposition 5.8. *rank* $(\alpha''(P)) = h-3 \Rightarrow$ *rank* $(\alpha(P)) = h-3$.

Proof. *If* $u(P)\neq 0$, *then* $\beta_{33}(P)= u^2(P)\neq 0$, *against* (R.C.).
An entirely analogous argument yields $v(P)=0$.
If $\gamma(P)\neq 0$, *we may assume* $\gamma\equiv 1$ *(then* γ *is a square* $\gamma=w^2$, *and we can divide first row and column by* w*). I being the ideal* $(xy-z^2,uz-xv,uy-zv)$, *we shall derive a contradiction if we show that* $I\subset(u,v)^m$ *for each* m.
It suffices then to show that $I\subset(u,v)^3$*: in fact, by induction, if* $m\geq 3$, *and* $I\subset(u,v)^m$, *since (*γ *being* $\equiv 1$, *and* (R.C.) *holding true)* $x\equiv u^2$ (mod *I*), $z\equiv uv$ (mod *I*), $y\equiv v^2$ (mod *I*), *we have* $I\subset(u,v)\cdot I\subset(u,v)^{m+1}$.
We shall show that $x,y,z\in(u,v)^2$, *so* $I\subset(u,v)^3$ *will follow a fortiori.*
We have, by (R.C.), $x=u^2+\lambda(xy-z^2)+\mu\cdot(uz-xv)+\nu(uy-zv)$, *for suitable* $\lambda,\mu,\nu\in O_{P,\mathbb{P}}$, *hence* $x(1-\lambda y+\mu v)\in(u,z,y,v)^2$. *Hence* $x\in(u,z,y,v)^2$, *and analogously we infer that* $y\in(u,z,x,v)^2$, *hence* $y,x\in(u,z,v)^2$.
Finally $z\in(u,v,x,y)^2\subset(u,v,z)^2\Rightarrow z\in(u,v)^2$, *thus our claim follows at once.*

Q.E.D.

Question 5.9. *We believe that* (R.C.) *may imply that* $rank(\alpha) = rank(\alpha') = rank(\alpha'')$ *for each point of* Γ*, but we did not check it.*

Remark 5.10. *A little bit more of computation shows that, if* F *is the ideal spanned by* (x,y,z) *in* $\mathcal{O}_{P,\mathbb{P}}$, *then* u,v, *belong to* F*, while* $\gamma x, \gamma y, \gamma z$ *belong to* F^2*: then, e.g. if* (x,y,z) *form a regular sequence, also* γ *belongs to* F*. In this case also, since* I *has no embedded primes, we see that, given* $u,v \in F$*,* γ *is uniquely determined modulo* I *(if* γ_1, γ_2 *are two solutions, then, if* $g=\gamma_1-\gamma_2$, gx, gy, $gz \in I$, $\Rightarrow g \in I$*).*

Let's assume now that (x,y,z) form a regular sequence in $\mathcal{O}_{P,\mathbb{P}}$: we can, by remark 5.10, assume, acting with a suitable element of G, that the upper part of the matrix α has the form
$$\begin{pmatrix} c_1x+c_2y+c_2z & gy & d_1x+d_2y+d_3z \\ gy & x & z \\ d_1x+d_2y+d_3z & z & y \end{pmatrix}$$
and an elementary computation shows that (R.C.) is satisfied (cf. remark 5.10) if

(5.11) $c_1=d_1d_2$, $c_2=d_3g+d_2^2$, $c_3=d_2d_3+gd_1$,

i.e. $\gamma = d_2v + g(d_3y+d_1z)$.

We are going to describe the nature of the singularity of Σ when g,d_1,d_2,d_3 are general and to make our discussion clearer let's assume that the d_i's, g, are indeterminates, as well as x,y,z.

Let's work thus in the ring $R[x,y,z]$, where $R=\mathbb{C}[d_1,d_2,d_3,g]$.

Then the 2x2 minors of the matrix
$$\tilde{\alpha} = \begin{pmatrix} d_2v + g(d_3y+d_1z) & gy & v \\ gy & x & z \\ v & z & y \end{pmatrix}$$
define conics in $\mathbb{P}^2_R$, and they belong to the R-module of conics generated by $xy-z^2$, $gyz-xv$, gy^2-zv.

Since these last three are independent, they span a net with a base scheme of length 3 (if we work now on K = alg. closure of R), which consists in fact of 3 distinct points. Therefore $f=\det(\tilde{\alpha})$ defines a cubic $\tilde{\Sigma}$ in $\mathbb{P}^2_K$ which has 3 double points: in fact if $\tilde{B}=\Lambda^2\tilde{\alpha}$, $\tilde{B}$ vanishes at these 3 points by (R.C.), and $f^2=\det \tilde{B}$, so that f^2 vanishes of order

at least 3 at these points.
Hence $\tilde{\Sigma}$ consists of 3 lines in $\mathbb{P}^2_K$, and therefore f splits into the product of 3 linear forms in $R'[x,y,z]$, where R' is an algebraic extension of a localization of R.
In fact, even in the special case $g\equiv 0$, we have $f=v(x^2d_1+z^2d_2+xzd_3)$, hence the singularity of Σ is that of a triple point, i.e.

(5.12) $f=x_1x_2x_3$ in suitable holomorphic coordinates (x_1,x_2,x_3).

Definition 5.13. *We shall say that the matrix* α *is semiordinary if the following conditions hold (we assume here that* $\mathbb{P}$ *is smooth*

i) rank $\alpha''(P)\geq h-3$ *for each point* $P\in\mathbb{P}$ *and, at the points where* $\alpha''(P)$ *has rank* $=h-3$, $\Lambda^{h-2}(\alpha'')$ *generates the maximal ideal* $M_{P,\mathbb{P}}$ *(hence* P *is a conical double point for the adjoint surface* F),

ii) F *is smooth at the points where rank* $(\alpha'')=h-2$,

iii) Γ *is smooth at these points,*

iv) the entries of $\Lambda^{h-2}(\alpha')$ *vanish at the points* P *where* $\Lambda^{h-2}(\alpha'')$ *is zero,*

v) at the points where rank $(\alpha'')=h-3$, Γ *consists of* 3 *smooth transversal branches*

vi) α *satisfies* (R.C.)

α *is said to be ordinary if moreover*

vii) Σ *has ordinary singularities (i.e.* $\Sigma-\Gamma$ *is smooth, at the smooth points of* Γ Σ *has singularities of type* (5.6) (I),(II), *and at the triple points of* Γ Σ *has a singular point of type* (5.12)*).*

Now let's consider, after that integers $\ell_2,\dots,\ell_h$ as in thm. 4.24 have been fixed, the vector space T of matrices $\alpha = {}^t\alpha$ s.t. $\alpha_{ij}\in A_{s+1-\ell_i-\ell_j}$.
We have two natural fibrations $T \xrightarrow{\pi'} T' \xrightarrow{\pi''} T''$ of vector spaces, s.t. $\pi'(\alpha)=\alpha'$, $\pi''(\alpha')=\alpha''$, with our usual notations.
Let $S\subset T$ be the set of semiordinary matrices α, $O\subset S\subset T$ the set of ordinary matrices: clearly O is open in S. We have that S maps into $S''=\{\alpha''|$ s.t. i), ii) of (5.13) hold$\}$, respectively into $S'=\{\alpha' \mid$ i)-v) of (5.13) hold$\}$.
Now S'' is a Zariski open set in T" (cf. e.g. [Ca 1] thm. 2.8, or [Ba]),

and, in order to study the fibration $\pi'': S' \to S''$, we define $\hat{S}'=\{\alpha' \mid i),ii),iv) \text{ hold}\}$. $\pi''|_{\hat{S}'}: \hat{S}' \to S''$ is a vector space fibration and S' is open in $\hat{S}'$, as we are going to see.

<u>Proposition 5.14.</u> *If* $\alpha''\in S''$, *then* $\pi''^{-1}(\alpha'')\cap\hat{S}'$ *has codimension in* $\pi''^{-1}(\alpha'')$ *equal to* 2 t, *where* t *is the number of singular points of* $F=\{\det(\alpha'')=0\}$.

<u>Proof.</u> *Let* $P_1,\dots,P_t$ *be the singular points of* F, *and let* $b:\tilde{F}\to F$ *be the blow-up of* F *at* $P_1,\dots,P_t$. *Denote by* A_i $(i=1,\dots,t)$ *the exceptional curve* $b^{-1}(P_i)$: $A_i \cong \mathbb{P}^1$ *and* $A_i^2 = -2$. *Let* $\tilde{\Gamma}$ *be the proper transform of* Γ in $\tilde{F}$, *and let* H *be a divisor s.t.* $\mathcal{O}_{\tilde{F}}(H)\equiv b^*(\mathcal{O}_F(1))$. *The symmetric matrix* α'' *determines a sheaf* $\mathcal{F}$ *on* F *which is the cokernel of (cf.*[Ca 1], *§ 2)*

$$(5.15)\qquad 0 \longrightarrow \bigoplus_{j=2}^{h} \mathcal{O}_{\mathbb{P}}(\ell_j-s-1) \xrightarrow{\alpha''} \bigoplus_{i=2}^{h} \mathcal{O}_{\mathbb{P}}(-\ell_i) \longrightarrow \mathcal{F} \longrightarrow 0:$$

we have $\mathcal{F}=b_*\,\mathcal{O}_{\tilde{F}}(L)$, *where* L *is a divisor with* $2L \equiv \delta H + \sum_{i=1}^{t} A_i$, *for a suitable integer* δ. *There exists then a positive integer* m *s.t.*

$$(5.16)\qquad \tilde{\Gamma} \equiv L + mH - 2\sum_{i=1}^{t} A_i \quad :$$

in fact by (iv) $2\tilde{\Gamma} + 3\sum_{i=1}^{t} A_i$ *is linearly equivalent to a multiple of* H *(notice that* $\mathrm{div}_F(\Sigma)=2\Gamma$, *ibidem).*

Since $H^1(\tilde{F},\mathcal{O}_{\tilde{F}}(L+mH))=0 \quad \forall m\in\mathbb{Z}$, *we have*

$$0 \longrightarrow H^\circ(\mathcal{O}_{\tilde{F}}(\tilde{\Gamma})) \longrightarrow H^\circ(\mathcal{O}_{\tilde{F}}(L+mH)) \longrightarrow \bigoplus_{i=1}^{t} H^\circ(\mathcal{O}_{2A_i}(L)) \longrightarrow 0,$$

and, since $A_i\cdot L= -1$, $H^\circ(\mathcal{O}_{A_i}(L))=0$, *and then* $H^\circ(\mathcal{O}_{2A_i}(L)) \cong$
$\cong H^\circ(\mathcal{O}_{A_i}(-A_i+L)) \cong H^\circ(\mathcal{O}_{\mathbb{P}^1}(1)) = \mathbb{C}^2$.

<u>Q.E.D.</u>

We remark now that $\alpha'\in S'$ iff $\tilde{\Gamma}$ is a smooth curve in the linear system $|L+mH-2\sum_{i=1}^{t} A_i|$, which is transversal to the exceptional curves

$A_1,\dots,A_t$, and such a $\tilde{\Gamma}$ can be found, by Bertini's theorem, if the given linear system has no fixed points in $\tilde{F}$.
We can conclude our discussion of S concerning $\pi' : S \to S'$; by remark 5.10 α_{11} is uniquely determined modulo I locally, hence also globally modulo $I(s+1)$: in particular if the degree of F is bigger than (s+1) (what holds true in all but a finite number of cases), then α_{11} is uniquely determined by α'. We can summarize our remarks as follows:

Theorem 5.17. *Assume that* $\pi' : S \to S'$ *is dominant: then* S *is irreducible.*

Hence in some cases there is a natural way to define a natural irreduducible unirational component of the moduli space, as we mentioned already in the introduction.
It is now time to pass to a few examples.

Example I : We saw in (2.19) that if S is a minimal surface of general type with $q=0$, $p_g = 4$, $|K|$ free from base points and with image Σ which is not a quadric, then $\ell_2 = \dots \ell_h = 2$, and $K^2 = 5+(h-1)$.
Let's consider the first non trivial case, i.e. when $K^2=6$. In this situation, assume first $\deg \phi = 2$.
Then Σ must be normal, since otherwise, $\pi : \tilde{\Sigma} \to \Sigma$ being the normalization map, one would have $\phi_* \mathcal{O}_S = \pi_* \mathcal{O}_{\tilde{\Sigma}} \oplus F$ and the number of a minimal system of generators of R would be at least 3, since then $\pi_* \mathcal{O}_{\tilde{\Sigma}} \supsetneq \mathcal{O}_\Sigma$.
Since the degrees of α are $\begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix}$, we see that we have a standard double cover (i.e. case 3b), i.e. the canonical ring R is generated by $1, v_2$ with relations (α being $= \begin{pmatrix} 0 & \alpha_{12} \\ \alpha_{12} & 0 \end{pmatrix}$)

$$\begin{cases} \alpha_{12}(y)=0 \\ v_2^2=G(y) \end{cases}$$, where G is a quartic form.

When $\Psi : X \to \Sigma$ is birational, we have a matrix $\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix}$

with deg $(\alpha_{11})=5$, deg $(\alpha_{12})=3$, deg $(\alpha_{11}) = 1$.

Clearly then the α_{ij}'s are $\not\equiv 0$, $f=\det(\alpha)$ being irreducible.

(R.C.) here gives the simple condition that one can write α_{11} as $G\alpha_{22} + Q\alpha_{12}$, where G is a quartic and Q a quadratic form. The curve Γ is here a plane cubic, generally there are no triple points and 12 pinch-points.

The canonical ring R is generated by $1, v_2$ with relations

$$\begin{cases} \alpha_{12} + \alpha_{22} v_2 = 0 \\ G\alpha_{22} + \alpha_{12} (Q + v_2) = 0 \\ v_2^2 = G + Q v_2 \end{cases}$$

and we see that we obtain the double covers of a cubic exactly in the special case when $\{\alpha_{22} \equiv 0\}$.

We have then

Theorem 5.18. *Surfaces with* $q = 0$, $p_g = 4$, $K^2 = 6$, *such that* $|K|$ is *free from base points and does not map onto a quadric form an irreducible unirational open set of their moduli space of dimension* 38.

Proof. *The other assertion being clear, let's compute the dimension. Given* $\alpha_{12}, \alpha_{22}, \alpha_{11}$ *belongs to a vector space of dimension* $(35+10-4)=41$, *hence we have a family depending on* $(4+20+41-1) = 64$ *parameters: since* $\dim PGL(4) = 15$, $\dim G = 11$ *(cf.* (5.1)*) we reach the desired conclusion.*

Q.E.D.

Example II: we keep on assuming $p_g = 4$ $(q=0)$, but set $K^2 = 7$.
If $|K|$ is free from base points, then clearly $\Psi : X \to \Sigma$ is birational.
We have a matrix

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix}$$

where $\alpha_{22}, \alpha_{23}, \alpha_{33}$ are linear forms, $\deg(\alpha_{11}) = 5$, $\deg(\alpha_{12}) = \deg(\alpha_{13}) = 3$.
Our first remark is that the three linear forms $\alpha_{22}, \alpha_{23}, \alpha_{33}$ cannot be proportional, because then Γ would not have dimension $=1$, and by the same argument it cannot be $\alpha_{22} \equiv \alpha_{33} \equiv 0$.

Therefore, either the linear forms are linearly independent, or else the datum of α'' corresponds to the datum of a pencil of quadratic forms on $\mathbb{P}^1$, hence we can assume to have one of the following cases, (up to acting with G):

a) $\alpha_{33} \equiv 0$

b) $\alpha_{23} \equiv 0$

c) $\alpha_{22}, \alpha_{23}, \alpha_{33}$ linearly independent.

Theorem 5.19. *Regular surfaces with* $p_g=4$, $K^2=7$, $|K|$ *free from base points form an irreducible unirational open set of their moduli space M.*

Proof. *We recall the well known fact that each irreducible component of M has dimension at least* $10\chi - 2K^2 = 36$ *(cf. [Ca 4,5]), and we shall show first that surfaces of type* a), b) *form a nowhere dense constructible set.*
We also remark that I *has a resolution of the form*

$$0 \longrightarrow A(-5)^2 \longrightarrow A(-2) \oplus A(-4)^2 \longrightarrow I \longrightarrow 0,$$

therefore $\dim_{\mathbb{C}} I_5 = 26$ *(hence* α_{11} *depends upon 26 parameters, once* α' *is fixed). In case* a) *we set* $x = \alpha_{22}$, $y = \alpha_{23}$ *: by* (R.C.) $\alpha_{13}^2 \in I \subset (x,y)$, *hence* $\alpha_{13} \in (x,y)$ *and also then* $\alpha_{12} \in (x,y)$ $(\alpha_{11}x - \alpha_{12}^2 \in I)$. *Acting with*

G *we can achieve that* $\alpha_{12} \equiv 0$, *then.*
We obtain thus a family depending on 42 *parameters, which is left invariant by the subgroup of projectivities for which the linear forms* x,y *are eigenvectors, which has dimension* 9, *hence surfaces of type* a) *belong to a constructible set of dimension* ≤ 33.
In case b), *we set* $\alpha_{22}=x$, $\alpha_{33}=y$; *again by* (R.C.) $\alpha_{12}, \alpha_{13} \in (x,y)$ *and, acting with* G, *and assuming to have chosen projective coordinates* (x,y,z,w), *we can get that* α_{12} *does not contain the monomial* x *(resp:* α_{13} *does not contain* y*).*
Then $\alpha_{12}= y\, q(y,z,w)$, $\alpha_{13}=x\, q'(x,z,w)$ *and we have a family depending on* 38 *parameters, and we conclude by exactly the same argument as above.*
For surfaces of type c) *we use the same fibrations we have considered to prove theorem* 5.17.
We set $x=\alpha_{22}$, $y=\alpha_{33}$, $z=\alpha_{23}$ *and recall that by* 5.8 *all the* α_{ij}*'s belong to the ideal* (x,y,z).
We denote by V *the space of matrices* α *with* $\alpha'' = \begin{pmatrix} x & z \\ z & y \end{pmatrix}$ *fixed.* V *contains the locus* $Y=\{\alpha \in V \mid \alpha$ *satisfies* (R.C.) *and* α' *defines a* 1*-dimensional scheme* $\Gamma\}$, *and* V *fibres onto an open set* $U=\{\alpha_{12},\alpha_{13} \in (x,y,z) \mid \Gamma$ *is* 1*-dimensional*$\} \subset \mathbb{C}^{38}$, *with fibres either empty or affine spaces of dimension* 26.
Y *contains an open set (cf.* [Elk]*)* Y' *such that, for* $\alpha_{11}, \alpha_{12}, \alpha_{13}$ *in* Y' *the corresponding surface* X *has only* R.D.P.*'s as singularities.*
Now, for α_{12}, α_{13} *general, we have* Γ *with only one triple point as singularity and of degree* 7 *and to prove that* Y *is irreducible it suffices to show the existence of* α_{11} *s.t.* (R.C.) *holds, what is an immediate consequence of formulas* (5.11), *which indeed, if* d_1,d_2,d_3,g *are quadratic forms, give an irreducible rational family an open set of which parametrizes all surfaces of type* c) *(it is easy to see that for general choice of these forms we have* Σ *with ordinary singularities, and hence* X *smooth).*

<u>Q.E.D.</u>

<u>Example III</u>: Let S be a weak Del Pezzo surface of degree 7.
Therefore S is obtained by blowing up $\mathbb{P}^2$ twice, as we noticed before.
$|-K|$ gives an embedding of the anticanonical model X of S(X=S if p_2 is

not infinitely near to p_1) into $\mathbb{P}^7$. Taking a projection with centre a $\mathbb{P}^3$ not intersecting X, we obtain a surface Σ of degree 7 in $\mathbb{P}^3$ which has a determinantal equation $f=\det(\alpha)=0$, with $\alpha = \begin{pmatrix} G & {}^tq \\ q & \alpha'' \end{pmatrix}$ with (α'') a 4x4 symmetric matrix of linear forms, q a column of quadratic forms, G a cubic form. If the projection is generic, Σ has ordinary singularities (and an extra node if $S \not\cong X$) and a double curve Γ of degree 14 with 10 triple points at the 10 nodes of the quartic symmetroid $F=\{\det(\alpha'')=0\}$.

Taking ℓ to be a general linear form, the surface $\ell f-\det^2(\alpha'')=0$ is a surface with ordinary singularities whose normalization is a simply connected minimal surface with $K^2=2$, $p_g=1$, $q=0$ (cf. [E] pp. 316-320, [C-D] § 5).

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Address of the Author:

Dipartimento di Matematica
Università di Pisa
Via Buonarroti, 2
56100 - P I S A (ITALIA)