GENERIC INVERTIBLE SHEAVES OF 2-TORSION AND GENERIC INVERTIBLE THETACHARACTERISTICS ON NODAL PLANE CURVES

Frabrizio Catanese*

1. Definitions and statements of the results

Definition 1. Let D be a reduced plane curve of degree n, and L an invertible sheaf on D such that $L^2 \cong \mathcal{O}_D$. L is said to be generic if the following condition holds:

$$H^o(D, L([\tfrac{n-3}{2}])) = 0 \tag{2}$$

(here, as usual, square brackets denote the integral part of a real number). Moreover L is said to be good if the pull back of L to the normalization of D is not trivial.

We remark that, if n is at least 3, and L is generic as in def. 1, then L is not trivial on D.

Definition 3. Let C,D be nodal curves of degree n in $\mathbf{P}^2$ (i.e. C,D are reduced and their only singularities are nodes).

A degeneration of C to D is the datum of a proper flat family $f: S \longrightarrow T$, such that

i) T is a smooth curve, S is a reduced divisor in $\mathbf{P}^2 \times T$, f is induced by the projection of $\mathbf{P}^2 \times T$ onto T

ii) there exist t_o, t_1 in T such that $f^{-1}(t_o) \cong C$, $f^{-1}(t_1) \cong D$

---

* Research partly supported by M.P.I.

iii) the fibration f is locally analytically isomomorphic to a product except exactly at a finite set of nodes $p_1,\dots,p_d$ of D (here we are using the isomorphism provided by ii)), which are smooth points of S.

The nodes $p_1,\dots,p_d$ are called the virtual nodes of the degeneration (according to Severi [Se]).

Definition 4. A nodal curve C is said to be of the main stream (according to the terminology of B. Moishezon, cf. [Mo]) if there exists a degeneration of C to a nodal curve D consisting of n different lines (D is often referred to as "the union of n lines in general position").

Definition 5. A thetacharacteristic F on a reduced plane curve C of degree n is a rank-1 torsion free sheaf on C such that $F \cong Hom_{\mathcal{O}_C}(F,\omega_C)$, where $\omega_C \cong \mathcal{O}_C(n-3)$ is the dualizing sheaf on C (cf. [Ba], [Be]).

A thetacharacteristic F is said to be invertible if F is an invertible sheaf: in this case the condition of being a thetacharacteristic reads out more directly as $F^2 \cong \mathcal{O}_C(n-3)$. A thetacharacteristic is said to be generic even if $H^o(C,F) = 0$, generic odd if $H^o(C,F) = 1$.

Definition 6. A nodal curve C is said to be of even multidegree if each irreducible component D of C has even degree.

Before stating our results, we remark that, for each pair of integers n,d, with $2d \leq (n-1)(n-2)$, the family of irreducible nodal curves of the main stream $V_{n,d}$ is a smooth irreducible non empty locally closed subvariety of $\mathbf{P}^N$ (here $N = n/2(n+3)$) (cf. [Se],[Wa],[Ta]). Moreover, if C_1 and C_2 are nodal curves with transversal intersections, then the union C of C_1 and C_2 is of the main stream if and only if C_1 and C_2 are of the main stream.

It makes threfore sense to talk about a "generic" nodal curve of the main stream. We can now state the main results of this paper.

Theorem 7. *For a generic nodal curve of the main stream* C, *there exists a generic invertible sheaf of 2-torsion* L. *If not all the components of* C *are rational, one can also assume that* L *is good.*

We observe here that, when n = degree of C is odd, and L is as in thm. 7, then $L((n-3)/2)$ is a generic even thetacharacteristic. Of course a nodal curve C of even degree admits invertible thetacharacteristics if and only if C has even multidegree (cf. def. 6).

Theorem 8. *A generic nodal curve of the main stream with even multidegree has a generic even invertible thetacharacteristic.*

We notice that an entirely similar method can be used to handle generic odd invertible thetacharacteristics, and that the above results have corollaries regarding the possibility of writing the equation of C as a determinant of a symmetric matrix of linear (resp.: quadratic) forms (cf. [Ba], [Ca]). We refer to [C-O] for a general discussion and for more complete results about (not necessarily invertible) thetacharacteristics on plane curves, and also to [Ha], for a nice tretament of the parity of invertible thetacharacteristics. As a final remark, we work over the field C of the complex numbers, though with minor changes everything works over an algebraically closed field of char $\neq 2$.

2. Two lemmas in linear algebra

In this section we shall work with a vector space over a field K of char $\neq 2$.

Lemma 9. *Let* $v_1,\dots,v_k$ *be independent vectors in a vector space* V , *and let* $w_1,\dots,w_k$ *be any vectors in* V . *The one can choose numbers* $e_1=+1$ *or* $=-1$, *for* $i=1,\dots,k$, *such that, setting* $u_i = v_i + e_i w_i$ *the* k *vec-- tors* $u_1,\dots,u_k$ *are independent.*

Proof. $v_1 \wedge \dots \wedge v_k \neq 0$, but, since $2v_i = (v_i+w_i)+(v_i-w_i)$, we have:

$$0 \neq 2^k v_1 \wedge \dots \wedge v_k = \sum_{e_i=\pm 1} (v_1+e_1w_1)\wedge \dots \wedge (v_k+e_kw_k)$$

Since the sum is not zero, there exists a non zero summand, i.e. there exist $e_1,\dots,e_k$ such that $u_1 = v_1+e_iw_i,\dots,u_k = v_k+w_k$ are linearly independent vectors.

Q.E.D.

Lemma 10. *Let* V *and* W *be vector spaces of the same dimension* $=k$. *Assume that there are given*

$$\begin{cases} c_1,\dots,c_k & \text{independent linear forms from } V \text{ to } K \\ a_1,\dots,a_k & \text{independent linear forms from } V \text{ to } K \\ d_1,\dots,d_k & \text{independent linear forms from } W \text{ to } K \\ b_1,\dots,b_k & \text{linear forms from } W \text{ to } K. \end{cases}$$

Then one can choose numbers $e_i = \pm 1$, *for* $i=1,..,k$, *such that the linear map* $\alpha : V \oplus W \longrightarrow K^{2k}$, *given by the* $2k$ *linear forms* $(c_i \oplus d_i)$, $(a_i \oplus e_ib_i)$, *is an isomorphism.*

Proof. Take on V the basis dual to $\{c_1,\dots,c_k\}$, and on W the basis dual to $\{d_1,\dots,d_k\}$. Let A be the matrix associated, in the given basis, to the linear map of V to K^k determined by the linear forms $a_1,\dots,a_k$, and associate in analogous way a matrix B to the linear forms $b_1,\dots,b_k$. Let further E be the matrix $\mathrm{diag}\{e_1,\dots,e_k\}$, and I be the identity $(k \times k)$ matrix.

Then the matrix associated to α is in block form

$$\begin{pmatrix} I & I \\ A & EB \end{pmatrix} \quad :$$

therefore α is an isomorphism if and only if the matrix $A-EB$ is invertible.

Since A is invertible, we can apply lemma 9 setting $v_i = i^{th}$ row of A, $w_j = j^{th}$ row of B.

Q.E.D.

3. Auxiliary results

According to def. 1, let D be a reduced plane curve of degree n, let L be a generic invertible sheaf of 2-torsion, and let R be a line transversal (in fact this hypothesis is not needed) to D: let then $p_1,\dots,p_n$ be the points of intersection of D with R.

We have therefore the exact sequence

$$(11) \qquad 0 \longrightarrow L([(n-3)/2]) \longrightarrow L([(n-3)/2]+1) \longrightarrow \bigoplus_{i=1}^{n} \mathbb{C}_{p_i} \longrightarrow 0$$

Remark 12. The exact cohomology sequence associated to (11) gives:

a) for n odd an isomorphism

$$r: \ H^o(L(n-1)/2) \longrightarrow \bigoplus_{i=1}^{n} \mathbb{C}_{p_i}$$

b) for n even an exact sequence

$$0 \longrightarrow H^o(L(n-2)/2) \xrightarrow{\ r\ } \bigoplus_{i=1}^{n} \mathbb{C}_{p_i} \xrightarrow{\ \partial\ } H^1(L(n-4)/2) \longrightarrow 0$$

where the first and the last vector space are dual to each other by Serre duality.

Proof. By Serre duality there is a non degenerate pairing

$$H^o(L(i)) \times H^1(L(n-3-i)) \longrightarrow H^1(\mathcal{O}_D(n-3)) \cong \mathbb{C} .$$

Therefore, for n odd, $H^1(L(n-3)/2) = H^o(L(n-3)/2) = 0$, by (2), and for the same reason when n is even $H^1(L(n-2)/2) = H^o(L(n-4)/2) = 0$.

Q.E.D

Proposition 13. *Let* D *be a reduced plane curve,* R *a line transversal to* D, *and let* C *be the union of* D *and* R. *For every* $L \in Pic_2(D)$ *which is generic, there exists* $F \in Pic_2(C)$ *which is generic and such that* $F \otimes \mathcal{O}_D \cong L$.

Proof. Let n be the degree of D, and $p_1,\dots,p_n$ be the points of intersection of D with R. Then we have the following exact sequence

$$(14) \qquad 0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_D \oplus \mathcal{O}_R \longrightarrow \bigoplus_{i=1}^{n} \mathbb{C}_{p_i} \longrightarrow 0$$

from which one deduces the following exact sequence

$$(15) \qquad 1 \longrightarrow \mathcal{O}_C^* \longrightarrow \mathcal{O}_D^* \oplus \mathcal{O}_R^* \longrightarrow \bigoplus_{i=1}^{n} \mathbb{C}_{p_i}^* \longrightarrow 1$$

whose associated long cohomology sequence yields

$$(16) \qquad 1 \longrightarrow \bigoplus_{i=1}^{n} \mathbb{C}_{p_i}^* / \mathbb{C}^* \longrightarrow Pic(C) \longrightarrow Pic(D) \oplus Pic(R) \longrightarrow 1$$

where C^* is embedded diagonally in $\bigoplus_{i=1}^{n} \mathbb{C}_{p_i}^*$.

Since the kernel of the exact sequence is 2-divisible, we have a corresponding exact sequence for the elements of 2-torsion, namely

$$(17) \qquad 1 \longrightarrow \bigoplus_{i=1}^{n} (\mu_2)_{p_i} / (\mu_2) \longrightarrow Pic_2(C) \longrightarrow Pic_2(D) \longrightarrow 1$$

where (μ_2) is the group of square roots of 1 in $\mathbb{C}^*$.

Therefore, up to isomorphism, we have 2^{n-1} possible extensions F of L. Moreover, if F is any of such extension, then $F \otimes \mathcal{O}_D \cong L$, $F \otimes \mathcal{O}_R \cong \mathcal{O}_R$, and every other extension is obtained by choosing $e = (e_1,\dots,e_n)$, with $e_1 = +1$ or -1, and modifying the glueing of the stalk $\mathcal{O}_{R,p_i}$ with L_{p_i} by the automorphism of L_{p_i} obtained by multiplication by e_i.

In this way for each $e = (e_1,\dots,e_n)$ we obtain another extension F_e, and the meaning of (16) is that $F_e = F_{e'}$ if and only if $e' = -e$. Now F is generic if and only if $H^o(C,F([(n-2)/2])) = 0$.

We have clearly the exact sequence

$$(18) \qquad 0 \longrightarrow H^o(C,F([(n-2)/2])) \longrightarrow H^o(R,\mathcal{O}_R([(n-2)/2]))\oplus H^o(D,L([(n-2)/2]))$$

$$\xrightarrow{\ \alpha\ } \bigoplus_{i=1}^{n} \mathbb{C}_{p_i} \longrightarrow \dots$$

We consider two cases separately:

i) n is odd, hence it suffices to show that ker (α) = 0; in this case $H^o(D,L(n-3)/2)) = 0$, therefore $\mathrm{Ker}(\alpha) = \{ s \mid s \in H^o(R,\mathcal{O}_R((n-3)/2))$ and s vanishes at $p_1,\dots,p_n\}$ and clearly $\ker(\alpha) = 0$, as we wanted.

ii) n is even = 2k, therefore, by (12) b), both summands in the middle term of (18) are k-dimensional vector spaces.

We are now able to apply lemma 10.

Set in fact $W = H^o(D,L((n-2)/2))$: by (12) b) again r has rank equal to k, therefore we can select $p_1,\dots,p_k$ such that the k linear forms $d_1,\dots,d_k$ obtained by evaluating sections of W on $p_1,\dots,p_k$ are linearly independent (in fact these linear forms are defined only up to non-zero scalar multiples, but this does not matter for our purposes).

We set further b_i = evaluation of sections of W at p_{k+i}, $V = H^o(R,\mathcal{O}_R(k-1))$, c_i = evaluation of sections of V at p_i, a_i = evaluation at p_{k+i} and the hypotheses of lemma 10 are clearly satisfied since $R \cong \mathbf{P}^1$. The statement of lemma 10 ensures now the existence of $e_1,\ldots,e_k$ such that, if you replace F by F_e, with $e = (e_1,\ldots,e_k, 1,\ldots.,1)$, then the corresponding α is an isomorphism. In particular then $H^o(C,F(n-2)/2)) = 0$.

Q.E.D.

Corollary 19. Let C be the union of n lines in general position. Then, if n is at least 3, there exist $L \in \mathrm{Pic}_2(C)$ which is generic.

Proof. If $n=3$, then $\mathrm{Pic}_2(C) = \mu_2$, and $H^o(C,L) = 0$ if L is the non trivial element in $\mathrm{Pic}_2(C)$. By prop. 13 we can proceed by increasing induction on n.

Q.E.D.

We have two entirely analogous results.

Proposition 20. *Let* D *be a reduced plane curve of even multidegree, and* F *a generic even invertible thetacharacteristic on* D. *Let* Q *be a conic transversal to* D *and let* C *be the union of* D *and* Q. *Then there exists a generic even invertible thetacharacteristic* G *on* C *such that* $G \otimes \mathcal{O}_D \cong F(1)$.

Proof. Let n be the degree of D and let $p_1,\ldots,p_{2n}$ be the points of intersection of D with Q.

We only sketch the argument, since it parallels verbatim the one used in the proof of prop. 13.

We have the exact sequence

$$(21) \qquad 0 \longrightarrow H^o(C,G) \longrightarrow H^o(Q,\mathcal{O}_Q(n-1)) \oplus H^o(D,F(1)) \longrightarrow \bigoplus_{i=1}^{2n} \mathbb{C}_{p_i} \longrightarrow \cdots$$

where by $\mathcal{O}_Q(n-1)$ we mean the invertible sheaf on $Q \cong \mathbf{P}^1$ of degree (n-1) Again, by the exact sequence

$$(22) \qquad 0 = H^o(D,F(-1)) \longrightarrow H^o(D,F(1)) \longrightarrow \bigoplus_{i=1}^{2n} \mathbb{C}_{p_i}$$

and since $H^o(D,F(1))$ has dimension equal to n, we can apply lemma 10 and twist G in order to obtain an invertible thetacharacteristic G_e on C with $H^o(C,G_e) = 0$

Q.E.D.

Corollary 23. Let C be the union of n conics in general position, $n \geq 2$. Then C has an invertible generic even thetacharacteristic.

Proof. By prop. 20, it suffices to prove the beginnning step of the induction, i.e. the statmenet when $n=2$, $C = Q_1 \cup Q_2$. In this case if F is an invertible thetacharacteristic, and $H^o(C,F) \ni s \neq 0$, then it is well known that $2\,\mathrm{div}(s)$ corresponds to a common tangent line of Q_1 and Q_2. Since Q_1 and Q_2 have at most 4 tangents in common, either F is generic even, or $\dim H^o(C,F) = 1$ and F corresponds to one of the common tangents of Q_1 and Q_2. We conclude the proof by observing that, by the exact sequence analogous to (17), G has exactly 8 invertible thetacharacteristics.

Q.E.D.

3. End of the proof

Remark 24. If D is a nodal curve, and $p_1,\dots,p_d$ is any set of nodes of D, there exists a degeneration $f\colon S \longrightarrow T$ for which $p_1,\dots,p_d$ are the virtual nodes (this follows from the previously mentioned theory of Severi-

Wahl, cf. [Se][Wa][Ta]). In this case, taking the normalization S' of S by blow-ups in $\mathbf{P}^2 \times T$ with center the singular curves of S, one obtains $f'\colon S' \longrightarrow T$ where the fibre $f'^{-1}(t_o)$ is the normalization of D at the "effective" nodes of D, i.e. at the nodes of D which are not virtual, whereas $f'^{-1}(t)$ is the normalization of $f^{-1}(t)$ for $t \neq t_o$.

Lemma 25. *For a generic nodal curve of the main stream* C, *there exists a generic* $L \in \mathrm{Pic}_2(C)$.

Proof. Let $f\colon S \longrightarrow T$ be a degeneration of C to D, where D is the union of n lines in general position. By corollary 19 there exists $L_o \in \mathrm{Pic}_2(D)$ which is generic.

Now L_o can be extended to an invertible sheaf defined in a neighbourhood of D in S, and such that $L^2 \cong \mathcal{O}_S$. In fact, if D is the union of $R_1,\dots,R_n$, using arguments similar to the ones used in the proof of prop. 13 one can easily show that, choosing an open cover $\mathcal{U} = \{U_1,\dots,U_n\}$ of D with $U_i \supset R_i$, $U_i \cap R_j \cap R_k = \emptyset$ for $i \neq j \neq k \neq i$, L_o is determined by a cocycle g_{ij} in $H^1(\mathcal{U}, \mathcal{O}_D^*)$ with $g_{ij} = +1$ or -1.

By shrinking T to a suitable Zariski open neighbourhood of t_o, one can assume that L is defined on the whole of S, and also, by uppersemicontinuity, that, if $C_t = f^{-1}(t)$, $L_t = L \otimes \mathcal{O}_{C_t}$, then $H^o(C_t, L_t([(n-3)/2])) = 0$.

Therefore, for $t \neq t_o$, we obtain a nodal curve of the main stream C_t of the same type as C, and endowed with a generic $L_t \in \mathrm{Pic}_2(C_t)$.

Q.E.D.

Remark 26. Working in the analytic category over $\mathbb{C}$, we notice that, exponentiating the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-D) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

one sees immediately that the obstruction to the surjectivity of $\text{Pic}(S) \longrightarrow \text{Pic}(D)$ lies in $H^2(\mathcal{O}_S(-D))$, and this last group is zero if you shrink T.

Proof of theorem 7. In view of the preceding lemma 25, it suffices to consider the case when C has an irreducible non rational component B.

Step I: reduction to the case when C is irreducible and non rational. Assume in fact that B is a non rational component of C, and assume that B has a generic good $L' \in \text{Pic}_2(B)$. We remark that, obviously, our assertion is proven if we show the existence of $L \in \text{Pic}_2(C)$ which is generic and such that $L \otimes \mathcal{O}_B \cong L'$. To accomplish this, we write C as $B \cup D$, and we choose a degeneration of D to D' which is a union of lines in general position, and is transversal to B.

Prop. 13 ensures that on $C' = B \cup D'$ there exists a generic $L'' \in \text{Pic}(C')$ such that $L'' \otimes \mathcal{O}_B \cong L'$. Now we conclude considering the degeneration of C to C' obtained keeping B fixed and degenerating D to D', and arguing as in lemma 25.

Step II: the case when C is irreducible and non rational.

In this case, let D be an irreducible rational curve of the same degree of C, and let L_o be a generic invertible sheaf in $\text{Pic}_2(D)$.

Since the normalization of D is P^1, L_o cannot be good, nevertheless there exists a node p of D such that, if D' is the normalization of D at the other nodes, then the pull-back of L_o to D' is not trivial. This assertion follows immediately from the isomorphism

$$(27) \qquad \text{Pic}_2(D) \cong \bigoplus_{i=1}^{m} (\mu_2)_{p_i}$$

where $p_1,\dots,p_m$ are the nodes of D.

By choosing a set of virtual nodes of D containing p, we can construct a degeneration (cf. remark 24) to D of an irreducible curve of the same genus of C.

Arguing as in lemma 25, (and using remark 26 if one does not want to repeat the argument which proves that L_o can be extended to L), we obtain an irreducible C_t with $L_t \in Pic_2(C_t)$ generic.

We claim that L_t is good: in fact, if $f: S \longrightarrow T$ is our degeneration, and $f': S' \longrightarrow T$ is the flat family obtained by taking the normalization of S, we have that the pull back of L to S' is non trivial, by our choice of the virtual nodes, when restricted to $f'^{-1}(t_o)$.

Hence, e.g. by semicontinuity, the pull back of L to S' is not trivial when restricted to $C'_t = f'^{-1}(t)$; but, by remark 24, C'_t is the normalization of C_t, and it is immediate to see that the above sheaf is nothing else than the pull-back of L_t to C'_t. This shows that L_t is good.

Q.E.D.

Proof of theorem 8.

Again by the Severi-Wahl theory of virtual nodes it follows that a nodal curve of the main stream with even multidegree can be degenerated to the union of conics in general position. The proof, using corollary 23, remark 26, is entirely similar to the one of lemma 25.

Q.E.D.

References

[Ba] Barth, W.
Moduli of vector bundles on the projective plane,
Inv. Math. **42**, 63-91 (1977)

[Be] Beauville, A.
Prym varieties and the Schottky problem
Inv. Math. **41**, 149-196 (1977)

[Ca] Catanese, F.
Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications
Inv. Math. **63**, 433-465 (1981)

[C-O] Catanese, F.-Oort, F.
Thetacharacteristics on plane curves,
in preparation.

[Ha] Harris, J.
Thetacharacteristics on singular curves
Preprint (1980)

[Mo] Moishezon, B.
Stable branch curves and braid monodromies,
Springer LNM **862**, 107-192 (1981)

[Se] Severi, F.
Vorlesungen über algebraische geometrie (Anhang F),
Teubner, Leipzig (1921)

[Tj] Tjurin, A.
On intersection of quadrics
Russian Math. Surveys, **30**, 6, 51-105 (1975)

[Ta] Tannenbaum, A.
Families of algebraic curves with nodes,
Comp. Math. **41**, 107-126 (1980)

[Wa] Wahl, J.
Deformations of plane curves with nodes and cusps
Am. J. of Math. **96**, 529-577 (1974)