

## CONNECTED COMPONENTS OF MODULI SPACES

F. CATANESE

### 0. Introduction

Let  $S$  be a minimal surface of general type (complete and smooth over  $\mathbb{C}$ ), and let  $\mathcal{M} = \mathcal{M}(S)$  (resp.,  $\mathcal{M}^{\text{diff}}$ ) be the coarse moduli space of complex structures on the oriented topological (resp., differential) 4-manifold underlying  $S$ .

By Gieseker's theorem [5],  $\mathcal{M}(S)$  is a quasiprojective variety, and the number  $\nu(S)$  of its irreducible components is bounded by a function  $\nu_0(K^2, \chi)$  of the two (topological) invariants  $K^2 = K_S^2$ ,  $\chi = \chi(\mathcal{O}_S)$ .

Let  $\lambda(S)$  be the number of connected components of  $\mathcal{M}(S)$ : this short note answers a question raised in a previous paper [1], showing that the above number  $\lambda(S)$  can be arbitrarily large.

As in [1], to which we shall constantly refer, again we restrict our attention to bidouble (i.e., Galois with group  $(\mathbb{Z}/2)^2$ ) covers of  $Q = \mathbb{P}^1 \times \mathbb{P}^1$ : indeed, (cf. [2]) we conjecture a stronger result to hold true, namely that many of the different irreducible components of  $\mathcal{M}$  we thus obtain are in fact connected components of  $\mathcal{M}$ .

The idea of proof is rather simple: if  $S$  and  $S'$  are deformations of each other, then there exists a diffeomorphism  $f: S \rightarrow S'$  such that  $f^*(K_{S'}) = K_S \in H^2(S, \mathbb{Z})$ , and, in particular, if  $r(S) = \max\{r \in \mathbb{N} \mid (1/r)K_S \in H^2(S, \mathbb{Z})\}$ , then  $r(S) = r(S')$ .

In view of Donaldson's recent result [3], it is possible that the integer  $r(S)$  could be an invariant of the differentiable structure for these surfaces; it is not clear at the moment whether nicer properties are enjoyed by the moduli spaces  $\mathcal{M}^{\text{diff}}(S)$ . Nevertheless, when the complex dimension is at least 3, it seems (cf. [6], [7]) that similar phenomena of high disconnectedness should appear also for  $\mathcal{M}^{\text{diff}}$ .

**1. Statement and proof of the main result**

**Theorem.** *For each natural number  $k$  there exist minimal models  $S_1, \dots, S_k$  of surfaces of general type such that*

- (a)  $S_i$  is simply-connected ( $i = 1, \dots, k$ ),
- (b) for  $i \neq j$ ,  $S_i$  and  $S_j$  are (orientedly) homeomorphic but not a deformation of each other.

**Remark 1.** From the proof it shall also follow that the number of moduli of  $S_i$  (cf. [1, p. 484]) differs from the number of moduli of  $S_j$  for  $i \neq j$ .

Let us recall an arithmetical result, proved by E. Bombieri in the Appendix to [1].

**Lemma 2.** *For each positive integer  $k$ , there exist integers  $m, T$ , and  $k$  distinct factorizations of  $6^m$ ,*

$$u'_i v'_i = 6^m \quad (i = 1, \dots, k),$$

together with integers  $w_i, z_i$  ( $i = 1, \dots, k$ ) such that, setting  $u_i = Tu'_i$  and  $v_i = Tv'_i$ , the following system of equalities and inequalities is satisfied for  $i = 1, \dots, k$ :

$$\begin{aligned} u_i v_i &= T6^m = M, & w_i z_i - 2(u_i + v_i) &= N, \\ (u_i + 2)/3 < w_i < u_i - 4, & (v_i + 2)/3 < z_i < v_i - 4. \end{aligned}$$

**Corollary 3.** *In the notations of Lemma 2, the greatest common divisors  $(u_i, v_i)$  assume at least  $k/2$  distinct values.*

*Proof.* Set  $u'_i = 2^{x_i} 3^{y_i}$ . We can clearly assume  $x_i \leq m - x_i$ , hence  $(u_i, v_i) = T2^{x_i} 3^{\min(y_i, m - y_i)}$ . Since the factorizations are distinct,  $(u_i, v_i) = (u_j, v_j)$  for  $i \neq j$  if and only if  $x_i = x_j, y_i = m - y_j$ . q.e.d.

Given a smooth projective variety  $X$  we denote by  $NS(X)$  the Neron-Severi group of divisors modulo numerical equivalence (which we shall denote by  $\sim$ , leaving the symbol  $\equiv$  for linear equivalence). Note that, more generally on a compact complex manifold  $X$ ,

$$NS(X) = (\ker(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X)))/\text{torsion}.$$

**Lemma 4.** *Let  $X, Y$  be smooth projective varieties and let  $\pi: X \rightarrow Y$  be a finite Galois cover with group  $G$ . Then*

- (i)  $\pi^*: NS(Y) \rightarrow NS(X)$  is injective, maps to  $L = (\ker \pi_*)^\perp = NS(X)^G$ , and  $L/\text{im } \pi^*$  is a torsion subgroup of exponent at most the order of  $G$ .

*If  $B_1, \dots, B_k$  are the irreducible components of the branch divisor  $B$  of  $\pi$ , let (for  $i = 1, \dots, k$ )  $e_i$  be the order of the inertia group of any divisor in  $\pi^{-1}(B_i)$ , let  $d_i$  be the order of divisibility of the class of  $B_i$  in  $NS(Y)$  ( $d_i = \max\{d \mid \exists \Gamma_i \text{ s.t. } d\Gamma_i \sim B_i\}$ ), and set  $m_i = \text{g.c.d.}(e_i, d_i), a_i = e_i/m_i$ .*

Assume furthermore  $H_1(X, \mathbb{Z}) = 0$  and  $H^2(G, \mathbb{C}^*) = 0$  (e.g., if  $G$  is cyclic). Then

(ii) the exponent  $\beta$  of  $L/\text{Im } \pi^*$  is the least common multiple  $\alpha$  of the numbers  $a_1, \dots, a_k$ .

*Proof.* If  $m$  is the order of the group  $G$ , we have  $\pi_*\pi^* = m$  (Identity), hence  $\pi^*$  is injective, and  $\text{im } \pi^* \subset (\ker \pi_*)^\perp$  by the projection formula  $\pi^*x \cdot y = x \cdot \pi_*y$ . Moreover, since  $\pi^*\pi_* = \sum_{g \in G} g^*$ , tensoring over  $\mathbb{Q}$ ,  $\ker \pi_*$  is the kernel of the projector onto the subspace of invariants, and  $(\ker \pi_*)^\perp = \text{NS}(X)^G$ . If  $x \in L$ , then  $g^*x = x \ \forall g \in G$ ; hence  $mx = \pi^*(\pi_*x)$  and the first assertion is proven.

Since  $H_1(X, \mathbb{Z}) = 0$ , any element  $x$  in  $L$  is represented by a divisor  $D$  s.t.  $g^*D \equiv D \ \forall g \in G$ .

Consider the sheaf  $\mathcal{L} = \mathcal{O}_X(D)$  of rational functions  $f$  with  $\text{div}(f) - D \geq 0$ : by assumption,  $\forall g \in G$  there exists an isomorphism between  $\mathcal{L}$  and  $g^*\mathcal{L}$ , hence, defining  $G(\mathcal{L}) = \{(g, \tilde{g}) \mid g \in G \text{ and } \tilde{g} \text{ is an isomorphism from } g^*\mathcal{L} \text{ to } \mathcal{L}\}$ , we have a central extension

$$(5) \quad 0 \rightarrow \mathbb{C}^* \rightarrow G(\mathcal{L}) \rightarrow G \rightarrow 0.$$

We notice that

**Sublemma 6.** (5) splits if and only if  $D$  is linearly equivalent to a  $G$ -invariant divisor  $D'$  (i.e.,  $g(D') = D' \ \forall g \in G$ ).

*Proof.* The “if” part is obvious, since then  $\mathcal{L} \cong \mathcal{O}(D')$  and the condition  $\text{div}(f) - D' \geq 0$  is clearly  $G$ -invariant, hence there is an action of  $G$  on  $\mathcal{L}$  which makes (5) split. Conversely, if (5) splits there is an action of  $G$  on  $\mathcal{L}$ , and the sheaf  $(\pi_*\mathcal{L})^G$  is nonzero.

If  $H$  is a very ample divisor on  $Y = X/G$ , for  $m \gg 0$  the sheaf  $(\pi_*\mathcal{L})^G(mH)$  has a section, hence  $D + m\pi^*H$  is linearly equivalent to an effective divisor  $C$ , which is  $G$ -invariant. q.e.d.

Now the extensions of  $G$  by  $\mathbb{C}^*$  are classified by  $H^2(G, \mathbb{C}^*)$ ; hence, if  $H^2(G, \mathbb{C}^*) = 0$ ,  $D$  is linearly equivalent to a  $G$ -invariant divisor, and we can only consider the case of an effective  $G$ -invariant divisor  $C$ . In this case, if  $R_i = \pi^{-1}(B_i)_{\text{red}}$ , we can write  $C$  as  $C = C_R + C'$ , where  $C_R, C'$  are effective, no component of the ramification divisor  $R$  appears in  $C'$ , and  $C_R = \sum_{i=1}^k b_i R_i$  (since  $g(C) = C \ \forall g \in G$ , this is possible).

We have

$$\pi_*(C) = m\Gamma' + \sum_i (b_i m/e_i) B_i \sim m\Gamma' + \sum_i (b_i d_i m/e_i) \Gamma_i$$

since  $B_i$  is exactly  $d_i$ -divisible.

Now

$$mC \equiv \pi^*\pi_*(C) \equiv m\pi^*(\Gamma') + \sum_i (b_i d_i m/e_i) \pi^*(\Gamma_i),$$

hence

$$\alpha C \equiv \pi^*(\alpha\Gamma') + \sum_i (\alpha/a_i)b_i(d_i/m_i)\pi^*(\Gamma_i),$$

thus  $\alpha C$  belongs to  $\text{im } \pi^*$ .

Conversely, we claim that the class of  $R_i$  in  $L/\text{im } \pi^*$  has period exactly equal to  $a_i$ .

In fact  $a_i R_i \equiv (d_i/m_i)\pi^*(\Gamma_i)$ , as we have seen, and if there exists a divisor  $\Gamma$  and some integer  $c$  dividing  $a_i$  such that  $cR_i \equiv \pi^*(\Gamma)$ , applying  $\pi_*$  we get

$$m\Gamma \sim c\pi_*(R_i) \sim (cm/e_i)B_i.$$

Hence  $e_i\Gamma \sim cB_i \sim cd_i\Gamma_i$ , thus  $(e_i = a_i m_i!)$   $a_i m_i \Gamma \sim (cm_i d_i/m_i)\Gamma$  and  $(d_i/m_i)\Gamma_i \sim a_i/c\Gamma$ . Since  $d_i/m_i$  and  $a_i/c$  are relatively prime,  $\Gamma_i$  is  $a_i/c$  divisible, therefore  $a_i = c$ .

**Remark.** The above proof shows that, in general,  $\beta \geq \alpha$ .

**Corollary 7.** *Let  $\pi: X \rightarrow Y$  be a finite Galois cover with group  $G$  s.t.  $\pi$  is the composition of Galois covers as in (ii) of Lemma 4, each such that the corresponding integer  $\alpha$  equals 1. Then  $\text{NS}(X)^G = \pi^*(\text{NS}(Y))$ .*

*Proof.* The proof is by induction on the number of steps: in fact if  $N$  is a normal subgroup of  $G$  and  $Z = X/N$  is smooth, let  $p: X \rightarrow Z$ ,  $q: Z \rightarrow Y$  be the quotient morphisms. Since  $p^*$  and  $q^*$  are injective by Lemma 4, we can identify  $\text{NS}(Y)$  and  $\text{NS}(Z)$  to subgroups of the free abelian group  $\text{NS}(X)$ .

Let  $\Gamma = G/N$ : by induction  $\text{NS}(Y) = \text{NS}(Z)^\Gamma = (\text{NS}(X)^N)^\Gamma = \text{NS}(X)^G$ . q.e.d.

**Remark.** The result of Corollary 7 can be stated in a greater generality, in particular one does not need the intermediate quotients to be smooth.

*Proof of the theorem.* Recall that  $\pi: S \rightarrow Q = \mathbb{P}^1 \times \mathbb{P}^1$  is a smooth simple bidouble cover of type  $(2a, 2b)$ ,  $(2n, 2m)$  if  $\pi$  is a finite  $(\mathbb{Z}/2)^2$  Galois cover,  $S$  is a smooth surface, and the branch locus of  $\pi$  consists of two curves of respective bidegrees  $(2a, 2b)$ ,  $(2n, 2m)$ .

Apply Lemma 2 to the integer  $2k$ . Then, for  $i = 1, \dots, 2k$  set (in the notations of the lemma)

$$\begin{aligned} a_i &= (u_i + w_i)/2 + 1, & n_i &= (u_i - w_i)/2 + 1, \\ b_i &= (v_i - z_i)/2 + 1, & m_i &= (v_i + z_i)/2 + 1, \end{aligned}$$

and let, for  $i = 1, \dots, 2k$ ,  $\pi_i: S_i \rightarrow Q$  be a smooth simple bidouble cover of type  $(2a_i, 2b_i)$ ,  $(2n_i, 2m_i)$ . As in [1], p. 506] we see that

$$K_{S_i}^2 = 8M, \quad \chi(\mathcal{O}_{S_i}) = \frac{3}{2}u_i v_i + (u_i + v_i) + 2 - \frac{1}{2}w_i z_i = \frac{3}{2}M + 2 - \frac{1}{2}N.$$

Moreover,

$$(8) \quad K_{S_i} = \pi_i^*(\mathcal{O}_Q(u_i, v_i))$$

and,  $u_i, v_i$  being even,  $K_{S_i}$  is 2-divisible: hence, by Freedman's theorem [4] (cf. also [1, Theorem 4.6]), all the surfaces  $S_i$  are (orientedly) homeomorphic. Applying Corollary 6 to  $\pi_i: S_i \rightarrow Q$ , and using (8), we see that  $r(S_i) = \max\{r \in \mathbb{N} \mid (1/r)K_{S_i} \in H^2(S_i, \mathbb{Z})\}$  equals the greatest common divisor  $(u_i, v_i)$ .

By Corollary 3 there are at least  $k$  of the  $2k$  surfaces  $S_1, \dots, S_{2k}$ , which satisfy the requirements of the theorem ( $S_i$  is simply connected by [1, Proposition 2.7]), since  $r(S)$  is a deformation invariant, as is easy to show.

### References

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UNIVERSITÀ DI PISA, ITALY

