# MODULI OF ALGEBRAIC SURFACES

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## Contents of the Paper

## Introduction

This paper reproduces with few changes the lectures I actually delivered at the
C. I. M. E. Session in Montecatini, with the exception of most part of one lecture
where I talked at length about the geography of surfaces of general type: the reason
for not including this material is that it is rather broadly covered in some survey
papers which will be published shortly ([Pe], [Ca 3], [Ca 2]).

Concerning my original (too ambitious) intentions, conceived when I accepted
Eduardo Sernesi's kind invitation to lecture about moduli of surfaces, one may
notice some changes from the preliminary program: the topics "Existence of
moduli spaces for algebraic varieties" and "Moduli via periods" were not treated.
The first because of its broadness and complexity (I realized it might require a
course on its own, while I mainly wanted to arrive to talk about surfaces of general
type), the second too because of its vastity and also for fear of overlapping with the
course by Donagi (which eventually did not treat period maps and variation of

Hodge structures). Anyhow the first topic is exhaustively treated in Popp's lecture notes ([Po]) and in the appendices to the second edition of Mumford's book on Geometric Invariant Theory ([Mu 2]), whereas the nicest applications of the theory of variation of Hodge structures to moduli of surfaces are amply covered in the book by Barth-Peters-Van de Ven ([B-P-V]).

Also, I mainly treated moduli of surfaces of general type, and fortunately Seiler lectured on the results of his thesis ([Sei 1,2,3]) about the moduli of (polarized) elliptic surfaces: I hope his lecture notes are appearing in this volume.

Instead, the part on Kodaira-Spencer's theory of deformations and its connections with the classical theory of continuous systems started to gain a dominant role after I gave a series of lectures at the Institute for Scientific Interchange (I. S. I.) in Torino on this subject. In fact, after Zappa ( cf. [Zp], [Mu 3]) discovered the first example of obstructed deformations, a smooth curve in an algebraic surface, it was hard to justify most of the classical statements about moduli (and in fact, cf. lecture four, some classical problems about completeness of the characteristic system have a negative answer).

Interest in moduli was revived only through the pioneering work of Kodaira-Spencer and later through Mumford's theory of geometric invariants. Mumford's theory is more algebraic and deals mostly with the problem of determining whether a moduli space exists as an algebraic or projective variety, whereas the trans-cendental theory of Kodaira and Spencer (in fact applied in an algebraic context by Grothendieck and Artin) applies to the more general category of com plex manifolds (or spaces), at the cost of producing only a local theory. In both issues, it is clear that it is not possible to have a good theory of moduli without imposing some restriction on complex manifolds or algebraic varieties.

Surfaces of general type are a case when things work out well, and one would like first to investigate properties and structure of this moduli spaces, then to draw from these results useful geometric consequences. It is my impression that for these purposes (e.g. to count number of moduli) the Kodaira-Spencer theory is by far more useful, and not difficult to apply in many concrete cases. In fact, it seems that in most applications only elementary deformation theory is needed, and that's one reason why these lecture notes cover very little of the more sophisti-cated theory (cf. §10 for more details). The other reason is that the author is not an expert in modern deformation theory and realized rather late about the existence or importance of some literature on the subject: in particular we would like to recommend the beautiful survey paper ([Pa]) by Palamodov on deformation of

complex spaces, whose historical introduction contains rather complete information regarding the material treated in the first three lectures.

Since the style of the paper is already rather informal, we don't attempt any discussion of the main ideas here in the introduction, and, before describing with more detail the contents, we remark that the paper (according to the C. I. M. E. goals) is directed to and ought to be accessible to non specialists and to beginning graduate students. Of course, reasons of space have obliged us to assume some familiarity with the language of algebraic geometry, especially sheaves and linear systems.

Finally, in many points references are omitted for reasons of economy and the lack of a quotation of some author's name (or paper) should not be interpreted as any claim of originality on my side, or as an underestimation of some scientific work.

§1-5 summarizes the essentials of the Kodaira-Spencer-Kuranishi results needed in later sections, following existing treatments of the topic ([K-M], [Ku 3]), whereas §6 is devoted to a single but enlightening example. §7 deals with deformations of automorphisms, whereas §8-9 are devoted to Horikawa's theory of deformations of holomorphic maps, with more emphasis to applications, such as deformation of surfaces in 3-space, or of complete intersections, and include some examples of everywhere obstructed deformations, due to Mumford and Kodaira. §10 is a "mea culpa" of the author for the topics he did not treat, §11-13 try to compare Horikawa's and Schlessinger-Wahl's theory of embedded deformations, whereas §12 consists of a rewriting, with some simplifications of notation, of Kodaira's paper ([Ko 3]) treating embedded deformations of surfaces with ordinary singularities. §14-17 give a basic resumé on classification of surfaces and §18-19 are devoted to basic properties of surfaces of general type and a sketchy discussion of Gieseker's theorem on their moduli spaces. §20-23 include a rough outline of recent work of the author and a result of I. Reider: §20 deals with the number of moduli of surfaces of general type, §22 outlines the deformation theory of $(\mathbb{Z}/2)^2$ covers, §21 and 23 exhibit examples of moduli spaces with arbitrarily many connected components having different dimensions, and discuss also the problem whether the topological or the differentiable structure should be fixed.

# LECTURE ONE: ALMOST COMPLEX STRUCTURES and the
## KURANISHI FAMILY

In this lecture I will review the construction, due to Kuranishi, of the complex structures, on a compact complex manifold $M$, sufficiently close to the given one. To do this, one has to use the notion of almost complex structures, of integrable ones: in a sense one of the main theorems, due to Newlander and Nirenberg, is a direct extension of a basic theorem of differential geometry, the theorem of Frobenius.

## §1. Almost complex structures

Let $M$ be a differentiable (or $C^\omega$, i.e. real analytic) manifold of dimension equal to $2n$, $T_M$ its real tangent bundle.

**Definition 1.1.** An almost complex structure on $M$ is the datum of a splitting $T_M \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$, with $T^{1,0} = \overline{T^{0,1}}$.

Naturally, the splitting of $T_M \otimes \mathbb{C}$ induces a splitting for the complexified cotangent bundle $T_M^\vee \otimes \mathbb{C} = (T^{1,0})^\vee \oplus (T^{0,1})^\vee$ ($(T^{1,0})^\vee$ is the annihilator of $T^{0,1}$), and for all the other tensors. In particular for the $r^{th}$ exterior power of the cotangent bundle, one has the decomposition $\wedge^r(T_M^\vee \otimes \mathbb{C}) = \bigoplus_{p+q=r} \wedge^p(T^{1,0})^\vee \otimes \wedge^q(T^{0,1})^\vee$.

We shall denote by $\mathcal{E}^{p,q}$ the sheaf of $C^\infty$ sections of $\wedge^p(T^{1,0})^\vee \otimes \wedge^q(T^{0,1})^\vee$ (resp. by $\mathcal{C}^{p,q}$ the sheaf of $C^\omega$ sections), by $\mathcal{E}^r$ the sheaf of $C^\infty$ sections of $\wedge^r(T_M^\vee \otimes \mathbb{C})$.

The De Rham algebra is the differential graded algebra $(\mathcal{E}^*, d)$, where $\mathcal{E}^* = \bigoplus_{r=0}^{2n} \mathcal{E}^r$, and $d$ is the operator of exterior differentiation. For a function $f$, $df \in \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$ and one can write accordingly $df = \partial f + \bar{\partial} f$; the problem is whether for all forms $\varphi$ one can write $d = \partial + \bar{\partial}$, with $\partial: \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q}$, $\bar{\partial}: \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}$ (then one has $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$, since $d^2 = 0$). Hence one poses the following

**Definition 1.2.** The given almost complex structure is integrable if

$$d(\mathcal{E}^{p,q}) \subset \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1} .$$

As a matter of fact, it is enough to verify this condition only for $p = 1$, $q = 0$.

**Lemma 1.3.** The almost complex structure is integrable $\iff d(\mathcal{E}^{1,0}) \subset \mathcal{E}^{2,0} \oplus \mathcal{E}^{1,1}$. [Hence another equivalent condition is: $\mathcal{E}^{1,0}$ generates a differential ideal.]

Proof. The question being local, we can take a local frame for $\mathcal{E}^{1,0}$, i.e. sections $\omega_1, \ldots, \omega_n$ of $\mathcal{E}^{1,0}$ whose values are linearly independent at each point (locally, $\mathcal{E}^{1,0}$ is a free module of rank $n$ over $\mathcal{E}^0$, and $\{\omega_1, \ldots, \omega_n\}$ is a basis). Our weaker condition is thus that

(1.4) $$d\omega_\alpha = \sum_{\beta < \gamma} \varphi_{\alpha\beta\gamma}\, \omega_\beta \wedge \omega_\gamma + \sum_{\beta, \gamma} \psi_{\alpha\beta\bar\gamma}\, \omega_\beta \wedge \overline{\omega_\gamma}$$

(where $\varphi_{\alpha\gamma\beta}$ and $\psi_{\alpha\beta\bar\gamma}$ are functions) since every $\omega \in \mathcal{E}^{1,0}$ can be written as $\sum_{\alpha=1}^n f_\alpha \omega_\alpha$, and $\{\omega_\beta \wedge \omega_\gamma \mid 1 \le \beta < \gamma \le n\}$ is a local frame for $\mathcal{E}^{2,0}$, $\{\omega_\beta \wedge \bar\omega_\gamma \mid 1 \le \beta, \gamma \le n\}$ is a local frame for $\mathcal{E}^{1,1}$. Now $\mathcal{E}^{0,1} = \overline{\mathcal{E}^{1,0}}$ hence $d(\mathcal{E}^{0,1}) \subset \mathcal{E}^{1,1} \oplus \mathcal{E}^{0,2}$ and one verifies $d(\mathcal{E}^{p,q}) \subset \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1}$ by induction on $p, q$, since locally any $\eta \in \mathcal{E}^{p,q}$ can be written as $\sum_{\alpha=1}^n \eta_\alpha \wedge \omega_\alpha + \sum_{\alpha=1}^n \vartheta_\alpha \wedge \bar\omega_\alpha$, with $\eta_\alpha \in \mathcal{E}^{p-1,q}$, $\vartheta_\alpha \in \mathcal{E}^{p,q-1}$. Q.E.D.

At this stage, one has to observe that if $M$ is a complex manifold, then $(T^\vee)^{1,0} = (T^{1,0})^\vee$ is generated (by definition !) by the differentials $df$ of holomorphic functions (at least locally, if one has a chart $(z_1, \ldots, z_n): U \to \mathbb{C}^n$, $dz_1, \ldots, dz_n$ give a frame for $(T^\vee)^{1,0}$). Conversely, one defines, given an almost complex structure, a function $f$ to be holomorphic if $\bar\partial f = 0$ (i.e., $df \in \mathcal{E}^{1,0}$); one sees easily, by the local inversion theorem of U. Dini, that the almost complex structure comes from a complex structure on $M$ if and only if for each $p$ in $M$ there do exist holomorphic functions $F_1, \ldots, F_n$ defined in a neighborhood $U$ of $p$ and giving a frame of $\mathcal{E}^{1,0}$ over $U$. This occurs exactly if and only if the almost complex structure is integrable: we have thus the following (cf. [N-N], [Hör] for a proof).

Theorem 1.4 (Newlander-Nirenberg). An almost complex structure on a $C^\infty$ manifold comes from a (unique) complex structure if and only if it is integrable.

Following Weil ([We], p. 36-37) we shall give a proof in the case where everything is real-analytic, because then we see why this is an extension of the theorem of Frobenius that we now recall (see [Spiv I] for more details, or [Hi]).

Theorem 1.5. Let $\varphi_1, \ldots, \varphi_r$ be 1-forms defined in an open set $\Omega$ in $\mathbb{R}^n$ and linearly independent at any point of $\Omega$. Then for each point $p$ in $\Omega$ there do exist local coordinates $x_1, \ldots, x_n$ such that the span of $\varphi_1, \ldots, \varphi_r$ equals the span of $dx_1, \ldots, dx_r$, $\Leftrightarrow$ $\varphi_1, \ldots, \varphi_r$ span a differential ideal (i.e., $\forall\, i = 1, \ldots, r$ $\exists$ forms $\vartheta_{ij}$ ($j = 1, \ldots, r$), s.t. $d\varphi_i = \sum_{j=1}^r \varphi_j \wedge \vartheta_{ij}$).

<u>Proof</u>. The usual way to prove the theorem is to consider, $\forall$ $p'$ in $\Omega$ the space $V_p$, of tangent vectors killed by $\varphi_1, \ldots, \varphi_r$ : then in a neighborhood $U$ of $p$ there exist vector fields $X_{r+1}, \ldots, X_n$ spanning $V_p$, for any $p'$ in $U$. Since

$$\varphi_i([X_j, X_k]) = X_j(\varphi_i(X_k)) - X_k(\varphi_i(X_j)) - d\varphi_i(X_j, X_k)$$

we see that the vector field $[X_j, X_k]$ at each $p'$ in $U$ lies in $V_p$,. One looks then for coordinates $x_1, \ldots, x_n$ s.t. $V_p$, is spanned by $\partial/\partial x_{r+1}, \ldots, \partial/\partial x_n$, and these coordinates are obtained by induction on $(n-r)$. In fact, by taking integral curves of the vector field $X_n$, one can assume $X_n = \partial/\partial x_n$, and replaces $X_i$ by $Y_i = X_i -$ $(X_i x_n) X_n$, which span the subspace $W_p$, of vectors in $V_p$, killing $x_n$, and so also the vector field $[Y_i, Y_j]$ at each point $p'$ in $U$ lies in $W_p$, (if $X(x_n) = 0$, $Y(x_n) = 0 \Rightarrow [X,Y](x_n) = 0!$). By induction there are coordinates $(y_1, \ldots, y_n)$ with $W_p$, spanned by $\partial/\partial y_{r+1}, \ldots, \partial/\partial y_{n-1}$. We can replace $X_n = \sum_{j=1}^n a_j(y)(\partial/\partial y_j)$ by $Y_n = \sum_{j=1}^r a_j(y)(\partial/\partial y_j) + a_n(y)(\partial/\partial y_n)$; since $[(\partial/\partial y_i), Y_n]$ $(i = r+1, \ldots, n-1)$ equals

$$\sum_{j \neq n+1, \ldots, n-1} \frac{\partial a_j(y)}{\partial y_i} \frac{\partial}{\partial y_j}$$

but on the other hand, this vector field is in $V_p'$, thus it is a multiple of $Y_n$ by a function $f$. But then, on the one hand, $[(\partial/\partial y_i), Y_n](x_n) = 0$ (since $Y_n(x_n) = X_n(x_n) = 1!$), on the other hand this quantity must equal $fY_n(x_n) = f$. Hence the functions $a_j(y)$ $(j = 1, \ldots, r, n)$ depend only upon the variables $y_1, \ldots, y_r, y_n$, so, by taking integral curves of the vector field $Y_n$, we can assume $Y_n = \partial/\partial y_n$ also.

<div align="right">Q.E.D.</div>

We have given a proof of the well known theorem of Frobenius just to notice that the only fact that is repeatedly used is the following: if $X$ is a non zero vector field, then there exist coordinates $(x_1, \ldots, x_n)$ s.t. $X = \partial/\partial x_n$. This follows from the theorem of existence and unicity for ordinary differential equations and from Dini's theorem. Both these results hold for holomorphic functions (they are even simpler, then), therefore, given a non zero holomorphic vector field $Z = \sum_{i=1}^n a_i(w) \partial/\partial w_i$ on an open set in $\mathbb{C}^n$ (i.e., the $a_i$'s are holomorphic functions), there exist local holomorphic coordinates $z_1, \ldots, z_n$ around each point such that $Z = \partial/\partial z_n$.

The conclusion is that the theorem of Frobenius holds verbatim if we replace $\mathbb{R}^n$ by $\mathbb{C}^n$, we consider holomorphic $(1,0)$ forms $\varphi_1, \ldots, \varphi_r$, and we require local holomorphic coordinates $z_1, \ldots, z_n$ s.t. the $\mathbb{C}$-span of $\varphi_1, \ldots, \varphi_r$ be the $\mathbb{C}$-span of $dz_1, \ldots, dz_r$. The proof of the Newlander-Nirenberg theorem in the real analytic case follows then from the following.

<u>Lemma 1.6</u>. Let $\Omega$ be an open set in $\mathbb{R}^{2n}$, let $\omega_1, \ldots, \omega_n$ be real analytic complex valued 1-forms defining an integrable almost complex structure (i.e., 1.4 holds). Then, around each point $p \in \Omega$, there are complex valued functions $F_1, \ldots, F_n$ s.t. the span of $dF_1, \ldots, dF_n$ equals the span of $\omega_1, \ldots, \omega_n$.

<u>Proof</u>. Take local coordinates $x_1, \ldots, x_n$ around $p$ s.t. each $\omega_\alpha$ is expressed by a power series $\sum_{j=1}^{2n} \sum_K f_{\alpha j, K} \, x^K dx_j$, where $K = (k_1, \ldots, k_{2n})$ denotes a multi-index. Then $\overline{\omega}_\alpha = \sum_{j, K} \overline{f}_{\alpha j, K} \, x^K dx_j$ and, if we consider $\mathbb{R}^{2n}$ as contained in $\mathbb{C}^{2n}$, upon replacing the monomial $x^K$ by the monomial $z^K$ and $x_j$ by $dz_j$ (here $x_j$ is the real part of $z_j$!), $\omega_\alpha$ and $\overline{\omega}_\alpha$ extend to holomorphic 1-forms $\omega_\alpha$, $\eta_\alpha$ in a neighborhood of $p$ in $\mathbb{C}^{2n}$. Since $\omega_1, \ldots, \omega_n, \overline{\omega}_1, \ldots, \overline{\omega}_n$ are a local frame for $\mathcal{E}^1$, the $\omega_\alpha$'s, $\eta_\alpha$'s give a basis for the module of holomorphic 1-forms, therefore one can write

$$d\omega_\alpha = \sum_{\beta < \gamma} \varphi_{\alpha\beta\gamma} \, \omega_\beta \wedge \omega_\gamma + \sum_{\beta, \gamma} \psi_{\alpha\beta\gamma} \, \omega_\beta \wedge \eta_\gamma + \sum_{\beta < \gamma} \xi_{\alpha\beta\gamma} \, \eta_\beta \wedge \eta_\gamma \ .$$

By restriction to $\mathbb{R}^{2n}$, using (1.4) we see that $\xi_{\alpha\beta\gamma} \equiv 0$, hence $\omega_1, \ldots, \omega_n$ span a differential ideal, hence Frobenius applies and there exist new holomorphic coordinates in $\mathbb{C}^{2n}$, $w_1, \ldots, w_{2n}$ s.t. the span of $dw_1, \ldots, dw_n$ equals the span of $\omega_1, \ldots, \omega_n$. We simply take $F_i$ to be the restriction of $w_i$ to $\mathbb{R}^{2n}$.     Q.E.D.

<u>Remark 1.7</u>. Assume that for $t = (t_1, \ldots, t_m)$ in a neighborhood of the origin in $\mathbb{C}^m$ one is given real analytic 1-forms $\omega_{t, 1}, \ldots, \omega_{t, n}$ as in lemma 1.6 which are expressed by convergent power series in $t_1, \ldots, t_m$, and define an integrable almost complex structure when $t$ belongs to a complex analytic subspace $B$ containing the origin. Then, for $t$ in $B$, the conclusions of lemma 1.6 hold with $F_{t, 1}, \ldots, F_{t, n}$ expressed as convergent power series in $(t_1, \ldots, t_m)$. In fact, if a vector field $X_t$ is given by a convergent power series in $t_1, \ldots, t_m$ also the solutions of the associated differential equation are power series in $t_1, \ldots, t_m$: moreover, by the local inversion theorem for holomorphic functions, if $f(x, t): \Omega \to \Omega$ is locally invertible, real analytic in $x$ and complex analytic in $t$, then the local inverse is also complex analytic in $t$.

## §2.   Small deformations of a complex structure

If $U$ is a vector subspace of a vector space $V$, and $W$ is a supplementary subspace of $U$ in $V$ (thus we identify $V$ with $U \oplus W$), then all the subspaces $U'$, of the same dimension, sufficiently close to $U$, can be viewed as graphs of a linear

map from $U$ to $W$: we apply this principle pointwise to define a small variation of an almost complex structure (hence also of a complex structure).

Definition 2.1. A small variation of an almost complex structure is a section $\varphi$ of $T^{1,0} \otimes (T^{0,1})^\vee$ (the variation is said to be of class $C^r$ if $\varphi$ is of class $C^r$).

Remark 2.2. To a small variation $\varphi$ we associate the new almost complex structure s.t. $T_\varphi^{0,1} = \{(u, v) \in T^{1,0} \oplus T^{0,1} \mid u = \varphi(v)\}$, since there is a canonical isomorphism of $T^{1,0} \otimes (T^{0,1})^\vee$ with $\mathrm{Hom}(T^{0,1}, T^{1,0})$.

We assume from now on that $M$ is a complex manifold: then, in terms of local holomorphic coordinates $(z_1, \ldots, z_n)$ one can write $\varphi$ as

$$(2.3) \qquad \varphi = \sum_{\alpha, \beta} \varphi_\alpha^{\bar\beta}(z) \, d\bar{z}_\beta \otimes \frac{\partial}{\partial z_\alpha}$$

so that

$$T_\varphi^{0,1} = \left\{ \left( \sum_\alpha u_\alpha \frac{\partial}{\partial z_\alpha}, \ \sum_\beta v_\beta \frac{\partial}{\partial z_\beta} \right) \ \middle| \ u_\alpha = \sum_\beta \varphi_\alpha^{\bar\beta} v_\beta \right\}$$

and is annihilated by $(T_\varphi^{1,0})^\vee$, the span of $\{\omega_\alpha = dz_\alpha - \sum_\beta \varphi_\alpha^{\bar\beta} d\bar{z}_\beta\}$. On the other hand, by what we've seen $T_\varphi^{0,1}$ is spanned by the $\xi_\gamma$'s, where

$$\xi_\gamma = \frac{\partial}{\partial \bar{z}_\gamma} + \sum_\alpha \varphi_\alpha^{\bar\gamma} \frac{\partial}{\partial z_\alpha} \quad .$$

Since $d\omega_\alpha = -\sum_\beta d\varphi_\alpha^{\bar\beta} \wedge d\bar{z}_\beta$, we are going to write down the integrability condition (1.4), which can be interpreted as

$$(2.4) \qquad d\omega_\alpha(\xi_\gamma, \xi_\delta) = 0 \quad \forall \ \alpha, \gamma, \delta \ (\gamma < \delta) .$$

We have

$$-d\omega_\alpha = \sum_{\beta, \epsilon} \left( \frac{\partial \varphi_\alpha^{\bar\beta}}{\partial z_\epsilon} \, dz_\epsilon \wedge d\bar{z}_\beta + \frac{\partial \varphi_\alpha^{\bar\beta}}{\partial \bar{z}_\epsilon} \, d\bar{z}_\epsilon \wedge d\bar{z}_\beta \right) \quad ,$$

which belongs to $\mathcal{E}^{1,1} \oplus \mathcal{E}^{0,2}$, hence kills pairs of vectors of type $(1,0)$. We get thus the condition

$$d\omega_\alpha \left( \frac{\partial}{\partial \bar{z}_\gamma}, \ \frac{\partial}{\partial \bar{z}_\delta} \right) + d\omega_\alpha \left( \sum_{\alpha'} \varphi_{\alpha'}^{\bar\gamma} \frac{\partial}{\partial z_{\alpha'}}, \ \frac{\partial}{\partial \bar{z}_\delta} \right)$$

$$+ d\omega_\alpha \left( \frac{\partial}{\partial \bar{z}_\gamma}, \ \sum_{\alpha''} \varphi_{\alpha''}^{\bar\delta} \frac{\partial}{\partial z_{\alpha''}} \right) = 0 \quad ,$$

boiling down to

$$(2.5') \qquad \frac{\partial \varphi_\alpha^{\bar{\delta}}}{\partial \bar{z}_\gamma} - \frac{\partial \varphi_\alpha^{\bar{\gamma}}}{\partial \bar{z}_\delta} + \sum_\epsilon \frac{\partial \varphi_\alpha^{\bar{\delta}}}{\partial z_\epsilon} \varphi_\epsilon^{\bar{\gamma}} - \frac{\partial \varphi_\alpha^{\bar{\gamma}}}{\partial z_\epsilon} \varphi_\epsilon^{\bar{\delta}} = 0 .$$

The condition that $(2.5')$ holds for each $\alpha$, and $\gamma < \delta$, can be written more simply as

$$(2.5) \qquad \bar{\partial} \varphi = \frac{1}{2} [\varphi, \varphi] \quad ,$$

where

$$\bar{\partial} \varphi = \sum_\alpha \left( \sum_{\gamma < \delta} \left( \frac{\partial \varphi_\alpha^{\bar{\gamma}}}{\partial \bar{z}_\delta} - \frac{\partial \varphi_\alpha^{\bar{\delta}}}{\partial \bar{z}_\gamma} \right) d\bar{z}_\gamma \wedge d\bar{z}_\delta \right) \otimes \frac{\partial}{\partial z_\alpha} \quad ,$$

$$[\varphi, \varphi] = 2 \sum_\alpha \sum_{\gamma < \delta} \left( \sum_\epsilon \frac{\partial \varphi_\alpha^{\bar{\delta}}}{\partial z_\epsilon} \varphi_\epsilon^{\bar{\gamma}} - \frac{\partial \varphi_\alpha^{\bar{\gamma}}}{\partial z_\epsilon} \varphi_\epsilon^{\bar{\delta}} \right) d\bar{z}_\gamma \wedge d\bar{z}_\delta \otimes \frac{\partial}{\partial z_\alpha}$$

$$= \sum_{\alpha, \epsilon, \gamma, \delta} d\bar{z}_\gamma \varphi_\epsilon^{\bar{\gamma}} \left( \frac{\partial \varphi_\alpha^{\bar{\delta}}}{\partial z_\epsilon} \right) \wedge d\bar{z}_\delta \otimes \frac{\partial}{\partial z_\alpha} - d\bar{z}_\delta \varphi_\alpha^{\bar{\delta}} \left( \frac{\partial \varphi_\epsilon^{\bar{\gamma}}}{\partial z_\alpha} \right) \wedge d\bar{z}_\gamma \otimes \frac{\partial}{\partial z_\epsilon} .$$

We shall explain these definitions while recalling some standard facts on Dolbeault cohomology and Hodge theory (harmonic forms).

So, let $V$ be a holomorphic vector bundle, and let $(U_\alpha)$ be a cover of $M$ by open sets where one has a trivialization $V_{|U_\alpha} \cong U_\alpha \times \mathbb{C}^r$, hence fibre vector co-ordinates $v_\alpha$, related by $v_\alpha = g_{\alpha\beta} v_\beta$ where $g_{\alpha\beta}$ is an invertible $r \times r$ matrix of holomorphic functions. We let $\mathcal{E}^{0, p}(V)$ be the space of ($\mathbb{C}^\infty$) sections of $V \otimes \wedge^p(T^{0,1})^\vee$: since $\bar{\partial} g_{\alpha\beta} = 0$, it makes sense to take $\bar{\partial}$ of $(0, p)$ forms with values in $V$ (i.e., elements of $\mathcal{E}^{0, p}(V)$), and we have the Dolbeault exact sequence of sheaves

$$0 \to \mathcal{O}(V) \to \mathcal{E}(V) \xrightarrow{\bar{\partial}_1} \mathcal{E}^{0,1}(V) \xrightarrow{\bar{\partial}_2} \cdots \xrightarrow{\bar{\partial}_n} \mathcal{E}^{0,n}(V) \to 0 \quad ,$$

where $\mathcal{O}(V)$ is the sheaf of holomorphic sections of $V$. We have the theorem of Dolbeault (the $\mathcal{E}^{0, k}(V)$ are soft sheaves).

Theorem 2.6.

$$H^i(M, \mathcal{O}(V)) \cong \frac{\ker H^0(\bar{\partial}_{i+1})}{\operatorname{Im} H^0(\bar{\partial}_i)} .$$

So $\bar{\partial}$ is well defined for our $\varphi \in \mathcal{E}^{0,1}(T^{1,0})$. For further use, we shall use the notation $\Theta = \mathcal{O}(T^{1,0})$. To explain the bracket operation, we notice that this is a bilinear operation

$$[\ ,\ ]:\ \mathcal{E}^{o,\,p}(T^{1,\,0}) \times \mathcal{E}^{o,\,q}(T^{1,\,0}) \to \mathcal{E}^{o,\,p+q}(T^{1,\,0})$$

which in local coordinates $(z_1, \ldots, z_n)$, if

$$\varphi = \sum_{I = \{i_1 < \cdots < i_p\}} \sum_{\alpha} f_{\alpha}^{\overline{I}}(d\overline{z}_{i_1} \wedge \ldots \wedge d\overline{z}_{i_p}) \otimes \frac{\partial}{\partial z_{\alpha}} = \sum_{I,\,\alpha} f_{\alpha}^{\overline{I}}\, d\overline{z}^{\wedge I} \otimes \frac{\partial}{\partial z_{\alpha}}$$

and

$$\psi = \sum_{J,\,\epsilon} g_{\epsilon}^{\overline{J}}\, d\overline{z}^{\wedge \overline{I}} \otimes \frac{\partial}{\partial z_{\epsilon}} \quad,$$

is such that

$$[\varphi,\psi] = \sum_{I,\,J,\,\alpha,\,\epsilon} d\overline{z}^{\wedge I} \wedge d\overline{z}^{\wedge J} \otimes \left[ f_{\alpha}^{\overline{I}} \frac{\partial g_{\epsilon}^{\overline{J}}}{\partial z_{\alpha}} \frac{\partial}{\partial z_{\epsilon}} - g_{\epsilon}^{\overline{J}} \frac{\partial f_{\alpha}^{\overline{I}}}{\partial z_{\epsilon}} \frac{\partial}{\partial z_{\alpha}} \right]$$

The bracket operation enjoys the following properties

i) $[\psi,\varphi] = (-1)^{pq+1}[\varphi,\psi]$

(2.7)  ii) $\overline{\partial}[\varphi,\psi] = [\overline{\partial}\varphi,\psi] + (-1)^{p}[\varphi,\overline{\partial}\psi]$

iii) if $\xi$ is in $\mathcal{E}^{o,\,r}(T^{1,\,0})$, then the Jacobi identity holds, i.e.,

$$(-1)^{pr}[\varphi,[\psi,\xi]] + (-1)^{qp}[\psi,[\xi,\varphi]] + (-1)^{rq}[\xi,[\varphi,\psi]] = 0\ .$$

Before recalling the Hodge theory of harmonic forms, we remark that, if we have a small variation $\varphi(t)$ of complex structure depending on a parameter $t = (t_1, \ldots, t_m)$, setting $B = \{t \mid \overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]\}$, $B$ is precisely the set of points $t$ for which $\varphi(t)$ defines a complex structure: but in order that the complex charts depend holomorphically upon $t$ for $t$ in $B$ (we assume, of course, that $\varphi(t)$ be a power series in $t_1, \ldots, t_m$), we want (cf. remark 1.6) $B$ to be a complex subspace. The Kuranishi family, as will be explained in the second lecture, is a natural choice to embody all the small variations of complex structures with the smallest number of parameters.

Now, let $V$ be again a holomorphic vector bundle on $M$, and assume that we choose Hermitian metrics for $V$ and $T^{1,\,0}$, so that for all the bundles $V \otimes (T^{o,\,p})^{\vee}$ is determined a Hermitian metric (if $M$ is $C^{\omega}$, we can assume the metric to be $C^{\omega}$). Thus a volume form $d\mu$ is given also on $M$, and thus, for $\varphi, \psi \in \mathcal{E}^{o,\,p}(V)$ a Hermitian scalar product is defined by $(\varphi,\psi) = \int_{M} \langle \varphi,\psi \rangle_{x}\, d\mu$. ($\langle \varphi,\psi \rangle_{x}$ is the value which the Hermitian product, given for the fibre of $V \otimes (T^{o,\,p})^{\vee}$ at the point x, takes on the values of $\varphi$ and $\psi$ at x).

It is therefore defined the adjoint operator $\bar{\partial}^*: \; \mathcal{E}^{\,0,\,p+1}(V) \to \mathcal{E}^{\,0,\,p}(V)$ by the usual formula $(\bar{\partial}\varphi, \xi) = (\varphi, \bar{\partial}^* \xi)$, and one forms the $\underline{\text{Laplace operator}}$

$$\Box = \bar{\partial}\,\bar{\partial}^* + \bar{\partial}^*\,\bar{\partial} \; .$$

We have $\Box : \; \mathcal{E}^{\,0,\,p}(V) \to \mathcal{E}^{\,0,\,p}(V)$ and the space of harmonic forms is

(2.8) $$\mathcal{K}^p(V) = \{\varphi \in \mathcal{E}^{\,0,\,p}(V) \mid \Box\varphi = 0\} = \{\varphi \mid \partial\varphi = \bar{\partial}^*\varphi = 0\}$$

The main result is that one has an orthogonal direct sum decomposition (where we simply write $\mathcal{E}^{\,p}$ for $\mathcal{E}^{\,0,\,p}(V)$)

(2.9) $$\mathcal{E}^p = \mathcal{K}^p \overset{\perp}{\oplus} \bar{\partial}\mathcal{E}^{p-1} \overset{\perp}{\oplus} \bar{\partial}^* \, \mathcal{E}^{p+1}$$

$\underline{\text{Remark 2.10.}}$ $\mathcal{K}^p \overset{\perp}{\oplus} \bar{\partial}\mathcal{E}^{p-1}$ consists of the space $\Gamma(\ker \bar{\partial})$ of all the $\bar{\partial}$ closed p-forms: in fact if $\bar{\partial}\,\bar{\partial}^*\varphi = 0$, then $0 = (\bar{\partial}\,\bar{\partial}^*\varphi, \varphi) = \|\bar{\partial}^*\varphi\|^2 \Rightarrow \bar{\partial}^*\varphi = 0$. Therefore, in view of Dolbeault's theorem one has the following

$\underline{\text{Theorem 2.11}}$ (Hodge). $\mathcal{K}^p(V)$ is naturally isomorphic to $H^p(M, \Theta(V))$. Moreover for each $\varphi \in \mathcal{E}^{\,0,\,p}(V)$ there is a unique decomposition

$$\varphi = \eta + \Box\,\psi \;, \quad \text{with} \quad \eta = H(\varphi) \in \mathcal{K}^p, \;\; \psi = G(\varphi) \in (\mathcal{K}^p)^\perp \;.$$

H is obviously a projector (the "harmonic projector") onto the finite dimensional space $\mathcal{K}^p$, whereas G is called the Green operator. We refer to [K-M] again for the proof of the following

$\underline{\text{Proposition 2.12.}}$ $\bar{\partial}$, $\bar{\partial}^*$ commute with G, and the product of $\bar{\partial}$, $\bar{\partial}^*$ or G with H on both sides gives zero.

## §3. Kuranishi's equation and the Kuranishi family

Fix once for all an Hermitian metric on $T^{1,0}$ and let $\mathcal{K}^p$ be $\mathcal{K}^p(T^{1,0})$: we can therefore identify, by Hodge's theorem, harmonic forms in $\mathcal{K}^p$ with cohomology classes in $H^p(M, \Theta)$. Recall also that, by (2.7.ii), a bracket operation is defined

$$[ \; ]: H^p(M, \Theta) \times H^q(M, \Theta) \to H^{p+q}(M, \Theta).$$

Let $\eta_1, \ldots, \eta_m$ be a basis for $\mathcal{K}^1$, so that we can identify a point $t \in \mathbb{C}^m$ with the harmonic form $\sum_{i=1}^m t_i \eta_i$. Consider the following equation

(3.1) $$\varphi(t) = \sum t_i \eta_i + \frac{1}{2} \bar{\partial}^* G[\varphi(t), \varphi(t)] \;.$$

It is easy to see that one has a formal power series solution $\varphi = \sum_{m=1}^\infty \varphi_m(t)$,

where $\varphi_m(t)$ is homogeneous of degree $m$ in $t$: in fact by linearity on $t$ of $\bar{\partial}^*, G$

$$\varphi_1(t) = \sum_i t_i \eta_i \ , \qquad \varphi_2(t) = \frac{1}{2} \bar{\partial}^* G[\varphi_1(t), \varphi_1(t)] , \qquad \varphi_3 = \bar{\partial}^* G[\varphi_1, \varphi_2] , \ \ldots$$

The power series converges in a neighborhood of the origin because $G$ is a regularizing operator of order $2$ (with respect to Hölder or Sobolev norms).

We want to show that $B = \{ t \mid \varphi(t)$ converges, and defines a complex structure on $M \}$ is a complex subspace around the origin in $\mathbb{C}^m$. We know that $B = \{ t \mid \bar{\partial}\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0 \}$ and we claim that the following holds

Lemma 3.2. $\bar{\partial}\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0$ if and only if $H[\varphi(t), \varphi(t)] = 0$ .

Proof. The "only if" part is clear, since $H\bar{\partial} = 0$. Conversely, we want to show that $\psi = \bar{\partial}\varphi - \frac{1}{2}[\varphi, \varphi]$ equals zero. Now $\varphi = \varphi_1 + \frac{1}{2}\bar{\partial}^* G[\varphi, \varphi]$ by Kuranishi's equation, and $\varphi_1$ is harmonic: hence

$$\psi = \frac{1}{2}\bar{\partial}\bar{\partial}^* G[\varphi, \varphi] - \frac{1}{2}[\varphi, \varphi] = \frac{1}{2}(\bar{\partial}\bar{\partial}^* G - \mathrm{id})([\varphi, \varphi]) .$$

But the identity $\mathrm{id}$ equals $H + \Box G = H + \bar{\partial}\bar{\partial}^* G + \bar{\partial}^* \bar{\partial}G$, thus (since $H[\varphi, \varphi] = 0$ by assumption) $-2\psi = \bar{\partial}^* \bar{\partial}G[\varphi, \varphi] = $ (since $\bar{\partial}$, $G$ commute) $= \bar{\partial}^* G \bar{\partial}[\varphi, \varphi] = $ (by 2.7) $= 2\bar{\partial}^* G[\bar{\partial}\varphi, \varphi] = $ (since $[[\varphi, \varphi], \varphi] = 0$ by Jacobi's identity) $= 2\bar{\partial}^* G[\psi, \varphi]$. We have therefore reached the conclusion that $\psi(t) = -\bar{\partial}^* G[\psi(t), \varphi(t)]$, in particular for any Sobolev norm $\| \ \|$, $\|\psi(t)\| \le \text{cost.} \ \|\psi(t)\| \ \|\varphi(t)\|$. But since $\|\varphi(t)\|$ is infinitesimal as $t \to 0$, we get that for $t$ small $\|\psi(t)\| < \|\psi(t)\|$, hence $\|\psi(t)\| = 0$ and $\psi(t) = 0$ as we want to show. $\qquad$ Q.E.D.

We use now the standard notation $h^i(\mathfrak{F}) = \dim_{\mathbb{C}} H^i(M, \mathfrak{F})$, for a coherent sheaf on $M$, and we state an immediate consequence of (3.2).

Corollary 3.3. If $m = h^1(\Theta)$ as before, $k = h^2(\Theta)$, then $B$ is defined by $k$ holomorphic functions $g_1, \ldots, g_k$ of $t = (t_1, \ldots, t_m)$ which have multiplicity at least $2$ at the origin. Moreover, if we identify $\mathbb{C}^m$ with $H^1(\Theta)$, $\mathbb{C}^k$ with $H^2(\Theta)$, the function $g^{(2)}: \mathbb{C}^m \to \mathbb{C}^k$ given by the quadratic terms of $g_1, \ldots, g_k$ corresponds to the quadratic function associated to the symmetric bilinear function

$$[ \ , \ ]: H^1(\Theta) \times H^1(\Theta) \to H^2(\Theta) .$$

Proof. Let $\xi_1, \ldots, \xi_k$ be an orthonormal basis for $\mathcal{H}^2$. If $\omega$ is in $\mathcal{E}^{0,2}(T^{1,0})$, $H\omega = 0$ is equivalent to $(\omega, \xi_i) = 0$ for $i = 1, \ldots, k$. Therefore, by lemma 3.2, $B$ is defined by the $k$ functions $g_i(t) = ([\varphi(t), \varphi(t)], \xi_i) = 0$. Since

$\varphi(t) = \sum t_i \, \eta_i + o(t)$, $g_i(t)$, which is clearly a convergent power series in $t$, has a McLaurin expansion

$$g_i(t) = \sum_{j,k=1}^{m} ([\eta_j, \eta_k], \, \xi_i) \, t_j \, t_k + o(t^2) \, . \qquad\qquad \underline{Q.E.D.}$$

The final step is to observe that the varying complex structures on $M$, parametrized by $t \in B$, can be put together to give a structure of complex space to the product $M \times B$, we have

<u>Theorem 3.4.</u>  On $M \times B$ there exists a structure of complex space $\mathcal{X}$ such that the projection on the second factor induces a holomorphic map $\Pi : \mathcal{X} \to B$ such that

    i) each fiber $X_t = \Pi^{-1}(t)$ is the complex manifold obtained by endowing $M$ with the complex structure defined by $\varphi(t)$;

    ii) for each point $p \in M = X_0$ there exists a neighborhood $U$ in $M$ and a neighborhood $V$ in $\mathcal{X}$ such that $V$ is biholomorphic to $U \times B$ under a map $\psi : U \times B \to V$ s.t. $\pi \circ \psi$ is projection on the second factor.

<u>Sketch of Proof.</u>  By remark 1.7, for each point $p$ in $M$ there is a neighborhood $U$ and functions $F_{t,i}(x)$ $(i = 1, \ldots, n)$ s.t. for any $t$ in $B$ they give a local chart for the complex structure defined by $\varphi(t)$. Let $F(t,x) = (F_{t,1}(x), \ldots, F_{t,n}(x))$: $U \to \mathbb{C}^n$; we use $(F, t)$: $U \times B \to \mathbb{C}^n \times B$ to give the local charts for the complex structure $\mathcal{X}$. The inversion theorem of U. Dini ensures then that the complex structure on $\mathcal{X}$ is globally well defined, and that ii) holds. $\qquad$ Q.E.D.

# LECTURE TWO: DEFORMATIONS OF COMPLEX STRUCTURES AND KURANISHI'S THEOREM

In this lecture I will review the notion of deformation of complex structure introduced by Kodaira and Spencer, the notion of pull-back, versal family,..., define the Kodaira-Spencer map, and state the theorem about the semi-universality of the Kuranishi family.

## §4. Deformations of complex structure

Let $M$ be as usual a compact complex manifold.

**Definition 4.1.** A deformation of $M$ consists of the following data: a morphism of complex spaces $\Pi: \mathcal{X} \to B$, a point $0 \in B$, an isomorphism of the fiber $X_0 = \Pi^{-1}(0)$ with $M$ s.t. $\Pi$ is proper and flat.

**Remark 4.2.** $\Pi$ is said to be smooth if $\forall\, b \in B$ the fibre $X_b = \Pi^{-1}(b)$ is smooth (and reduced, of course!). A deformation $\Pi$ is smooth (at least if one shrinks $B$) by virtue of the following

**Lemma 4.3.** Let $\Pi^*: \mathcal{O}_{B,o} = A \to \mathcal{O}_{\mathcal{X},p} = R$ be a homomorphism of local rings of complex spaces, and assume that $\Pi^*$ makes $R$ a flat $A$-module, and that moreover $\Pi^*$ has a smooth fibre, i.e. $R/\Pi^* \mathcal{m}_A \cong \mathbb{C}\{\bar{x}_1,\ldots,\bar{x}_n\}$, $\mathcal{m}_A$ being the maximal ideal of $A$. Then $R \cong A\{x_1,\ldots,x_n\}$.

**Proof.** Let $x_1,\ldots,x_n$ be such that $x_i$ maps to $\bar{x}_i$ through the surjection $R \twoheadrightarrow R/\Pi^* \mathcal{m}_A$. Thus $\Pi^*$ defines a homomorphism $f: A\{x_1,\ldots,x_n\}$ to $R$. $f$ is surjective by Nakayama's lemma, and we claim that flatness implies the injectivity of $f$. Let $K = \ker f$, so that we have an exact sequence

$$0 \to K \to A\{x_1,\ldots,x_n\} \to R \to 0 .$$

Tensoring with the $A$ module $A/\mathcal{m}_A \cong \mathbb{C}$ we get, since $\mathrm{Tor}_1^*(R, A/\mathcal{m}_A) = 0$ (see [Dou 2], proposition 3) an exact sequence

$$0 \to K \oplus A/\mathcal{m}_A \to \mathbb{C}\{\bar{x}_1,\ldots,\bar{x}_n\} \overset{\Pi^*}{\to} R/\Pi^* \mathcal{m}_A \to 0 .$$

Since $\Pi^*$ is, by assumption, an isomorphism, $K \otimes A/\mathcal{m}_A = 0$, hence $K = 0$, again by Nakayama's lemma (applied to $K$ as an $R$ module!).

<div align="right">Q.E.D.</div>

Remark 4.4. A deformation is said to be smooth if $B$ is smooth: in this case $\Pi$ is just a proper map with surjective differential at each point. Lemma 4.3 shows that property ii) of theorem 3.4 holds for every deformation. In the case when $B$ is smooth, a classical theorem of Ehresmann ([Eh]) asserts that $\Pi$ is a differentiable fibre bundle. This is a local result, and to give an idea of the proof (especially to stress the importance of vector fields!), we can assume $B$ to be a k cube in $\mathbb{R}^k$. Then one wants to show that $\mathcal{X}$ is diffeomorphic to $M \times B$ via a map compatible with the two projections on $B$: one assumes $\mathcal{X}$ to have a Riemannian metric, so that, if $x_1, \ldots, x_k$ are coordinates in $\mathbb{R}^k$, one can lift the vector field $\partial / \partial x_k$ to $\mathcal{X}$ to a unique vector field $\xi$ that is orthogonal to the fibres of $\Pi$. Then one proves the result by induction on k: if $B' = B \cap \mathbb{R}^{k-1}$, one takes the integral curves of $\xi$ to construct a diffeomorphism of $\Pi^{-1}(B') \times (-1, 1)$ with $\mathcal{X}$, and then applies induction $(\Pi^{-1}(B') \cong M \times B')$ to infer that $\mathcal{X} \cong M \times B$. If everything (the Riemannian metric included) is $C^\omega$, the above proof yields a $C^\omega$ diffeomorphism. An analogous result holds in the general case.

Theorem 4.5. Given a deformation $\Pi: \mathcal{X} \to B$ (shrinking $B$ if necessary) there exists a real analytic $(C^\omega)$ diffeomorphism $\gamma: M \times B \to \mathcal{X}$ with $\Pi \circ \gamma = $ projection of $M \times B \to B$, and such that $\gamma$ is holomorphic in the second set of variables.

Idea of proofs. In the $C^\infty$ case (cf. [Ku 3], p. 19-23) the proof is easier: there exists a finite cover $V_\alpha$ of $\mathcal{X}$ such that, if $U_\alpha = M \cap V_\alpha$ ($M$ is identified with $X_o$), $V_\alpha \cong U_\alpha \times B$ under a biholomorphism $\varphi_\alpha$. We can assume $\mathcal{X} \subset \mathbb{R}^N$: using these $\varphi_\alpha$'s and a partition of unity subordinate to the cover $V_\alpha$ we can define a $C^\infty$ morphism of $\mathcal{X}$ to a tubular neighborhood $T$ of $M$ in $\mathbb{R}^N$, and then we compose with a retraction of $T$ to $M$ to get $\varphi: \mathcal{X} \to M$ such that $\varphi_{|M} = $ identity. Then $\varphi \times \Pi$ gives the required diffeomorphism.

In the real analytic case, one can use the fact that, if $T_M$ is the real tangent bundle of $M$, $H^1(M, T_M) = 0$: then the power series method of [K-M], pp. 45-55 gives the desired result.

Using the diffeomorphism $\gamma: M \times B \to \mathcal{X}$, for each $b \in B$ one gets a small variation of complex structure $\varphi(b) \in A^{0,1}(T^{1,0})$ which depends holomorphically upon b. If $(B, o) \subset (\mathbb{C}^r, o)$ and $t_1, \ldots, t_r$ are coordinates on $\mathbb{C}^r$ ($r = \dim \mathfrak{m}_{B,o} / \mathfrak{m}^2_{B,o}$) one can write $\varphi(b) = \Sigma t_i \eta_i + o(t)$, and the linear map $\rho$ from $\mathbb{C}^r = T_{B,o}$ (Zariski tangent space to B at 0) to $H^1(\Theta)$ such that $-\rho(\partial / \partial t_i) = $ class of $\eta_i$ in $H^1(\Theta_M)$, is called the Kodaira-Spencer map, and we shall soon give an easier way to define and compute it.

<u>Definition 4.6</u>. Let $\Pi: \mathcal{X} \to B$ be a deformation of $M$, and let $V_\alpha$ be a (finite) cover of $\mathcal{X}$ which locally trivializes $\Pi$, i.e. such that there exists a biholomorphism $\varphi_\alpha: V_\alpha \to U_\alpha \times B$ ($U_\alpha = V_\alpha \cap M$, and we tacitly assume $\Pi \circ \varphi_\alpha^{-1}$ to be the projection of $U_\alpha \times B \to B$). Let $\xi$ be a tangent vector to $B$ at $0$, and let $\vartheta_\alpha$ be the unique lifting of $\xi$ (viewed as a constant vector field in $\mathbb{C}^r \supset B$) to $V_\alpha$ given by the trivialization $\varphi_\alpha$. Then $\vartheta_\alpha - \vartheta_\beta$, restricted to $U_\alpha \cap U_\beta$, is a vertical vector field, thus $(\vartheta_\alpha - \vartheta_\beta) \in H^1(M_\alpha, \Theta)$. We define $\rho: T_{B,o} \to H^1(\Theta)$ to be the linear map such that $\rho(\xi) = \vartheta_\alpha - \vartheta_\beta$ and it is easy to see that $\rho$ is well defined, independently of the choice of the cover and of the trivializations (for instance, changing trivializations, $\vartheta$ is replaced by $\vartheta_\alpha'$ such that $\vartheta_\alpha - \vartheta_\alpha'$ is a vertical vector field, therefore $\vartheta_\alpha - \vartheta_\beta$ is cohomologous to $\vartheta_\alpha' - \vartheta_\beta'$). $\rho$ is called the <u>Kodaira-Spencer</u> map.

The Kodaira-Spencer map gives the first order obstruction to the global liftability of a vector field, and its importance lies in its functorial nature, that we are now going to explain.

<u>Definition 4.7</u>. Let $\Pi: \mathcal{X} \to B$ be a deformation of $M$, and let $f: B' \to B$ be a morphism of complex spaces, $O'$ a point of $B'$ with $f(O') = 0$. Then the pull back $f^*(\mathcal{X})$ is given by $\mathcal{X}' = \{(x, b') \mid x \in \mathcal{X}, b' \in B' \text{ s.t. } \Pi(x) = f(b')\} \subset \mathcal{X} \times B'$, with $\Pi'$ induced by projection on the second factor.

Let $f_*: T_{B',o'} \to T_{B,o}$ be the differential of $f$ at $O'$, and let $\rho, \rho'$ be the respective Kodaira-Spencer maps of $\mathcal{X}, \mathcal{X}'$: we have

$$(4.8) \qquad \rho' = \rho \circ f_*$$

as it is immediately verified.

Grothendieck's point of view was, in particular, that in order to compute $\rho(\xi)$ it suffices to choose $B' = \{t \in \mathbb{C} \mid t^2 = 0\}$, and $f$ the unique morphism of $B' \to B$ s.t. $f_*(\partial/\partial t) = \xi$; also, in the context of pull-back, the meaning of $\rho = 0$ is that, if $B^I$ is the subspace of $B$ defined by $\mathfrak{m}_{B,o}^2$, $i: B^I \to B$ is the inclusion, then $i^*(\mathcal{X}) \cong B^I \times M$ if and only if $\rho = 0$.

We can thus verify that the two definitions we have given of $\rho$ do, in fact, coincide, limiting ourselves to 1-parameter deformations.

<u>Lemma 4.9</u>. Let $\Pi: \mathcal{X} \to B$ be a deformation of $M$ with base $B = \{t \in \mathbb{C} \mid t^2 = 0\}$, whose associated small variation of complex structure is given by the form $\varphi(t) = t\eta$, with $\eta \in A^{0,1}(T^{1,0})$: then, using the Dolbeault isomorphism, $\rho(\partial/\partial t)$ is the class of $\eta$ in $H^1(\Theta)$.

**Proof.** We choose trivializing charts on $U_\alpha \times B$, with $z_1^\alpha, \ldots, z_n^\alpha$ coordinates in $U_\alpha$, given by $\zeta_j^\alpha(z,t) = z_j^\alpha + t\, w_j^\alpha$. $\varphi(t)$ on $U_\alpha$ is expressed by

$$t \sum_{i,j} \eta_i^{\alpha \bar{j}} \, d\bar{z}_j^\alpha \otimes \frac{\partial}{\partial z_i^\alpha}$$

and the condition that $\zeta_i^\alpha$ be holomorphic is that $(\bar\partial + \varphi)(\zeta_i^\alpha) = 0$, i.e.

$$\bar\partial\, w_i^\alpha + \sum_j \eta_i^{\alpha \bar{j}} \, d\bar{z}_j^\alpha = 0 \; ,$$

hence locally $\eta$ can be expressed as

$$- \sum_i \bar\partial\, w_i^\alpha \otimes \frac{\partial}{\partial z_i^\alpha} \qquad .$$

In view of the way the Dolbeault isomorphism is gotten, it suffices to verify that, if we set

$$- \sum_i w_i^\alpha \otimes \frac{\partial}{\partial z_i^\alpha} = \Lambda_\alpha$$

then $\Lambda_\alpha - \Lambda_\beta$ is cohomologous to $\vartheta_\alpha - \vartheta_\beta$. Now $\vartheta_\alpha = \partial/\partial t$ in the $\alpha$-chart, therefore, expressing $\vartheta_\alpha - \vartheta_\beta$ in the $\alpha$-chart, we get, if $\zeta_j^\alpha = f_j^{\alpha\beta}(\zeta^\beta, t)$ is the change of coordinates,

$$- \sum_i \left.\frac{\partial f_i^{\alpha\beta}(\zeta^\beta, t)}{\partial t}\right|_{t=0} \frac{\partial}{\partial z_i^\alpha} = - \sum_i g_i^{\alpha\beta}(z^\beta) \frac{\partial}{\partial z_i^\alpha}$$

if we set

$$f_j^{\alpha\beta}(\zeta^\beta, t) = h_j^{\alpha\beta}(\zeta^\beta) + t\, g_j^{\alpha\beta}(\zeta^\beta) \; .$$

But this last expression equals $\zeta_j^\alpha = z_j^\alpha + t\, w_j^\alpha$, hence, by the chain rule,

$$w_i^\alpha = g_i^{\alpha\beta}(z^\beta) + \sum_k \frac{\partial h_i^{\alpha\beta}}{\partial z_k^\beta} \cdot w_k^\beta \; ,$$

and we are done, since

$$\frac{\partial}{\partial z_k^\beta} = \sum_i \frac{\partial h_i^{\alpha\beta}}{\partial z_k^\beta} \frac{\partial}{\partial z_i^\alpha} \qquad . \qquad\qquad \text{Q.E.D.}$$

§5. Kuranishi's theorem

Definition 5.1. A deformation $\Pi: \mathcal{X} \to B$ of M is said to be complete if for any other deformation $\Pi': \mathcal{X}' \to B'$, there exists a neighborhood $B''$ of $O'$ in $B'$ and f: $B'' \to B$ such that $\mathcal{X}'' = \mathcal{X}'|B''$ is isomorphic to the pull-back $f^*(\mathcal{X})$. The deformation $\Pi: \mathcal{X} \to B$ is said to be universal if it is complete, and moreover f is (locally) unique (respectively: semi-universal if it is complete and $f_*: T_{B',o'} \to T_{B,o}$ is unique).

Remark 5.2. In view of (4.8), a complete deformation is semi-universal if the associated Kodaira-Spencer map $\rho$ is injective. Let us see that $\rho$ is surjective for a complete family; in fact, by lemma (4.9) and the Kuranishi equation (3.1), the Kuranishi family has a bijective Kodaira-Spencer map: hence, if a complete family exists, it must have a surjective Kodaira-Spencer map.

Proposition 5.3. A semi-universal family is unique up to isomorphism.

Proof. By completeness, if $\Pi: \mathcal{X} \to B$, $\Pi': \mathcal{X}' \to B'$ are semi-universal, there exist $f': B \to B'$, $f: B' \to B$ with $\mathcal{X}' = f^*(\mathcal{X})$, $\mathcal{X} = f'^*(\mathcal{X}')$. Hence $\mathcal{X} = (f'f)^*(\mathcal{X})$, and by semi-universality $(f'f)_* = $ identity, but also $(ff')_* = $ id, therefore, by the local inversion theorem, f, f' are isomorphisms. Q.E.D.

We can now state the theorem of Kuranishi, referring the reader, for a complete proof, to [Dou 1], [Ku 3], [Ku 2], and to [K-M] and [Ku 1] for weaker versions.

Theorem 5.4 (Kuranishi)

    i) The Kuranishi family is semi-universal, and $f_*$ coincides (up to sign) with the Kodaira-Spencer map $\rho$.

    ii) The Kuranishi family is complete for $b \in B - \{o\}$, when viewed as a deformation of $X_b$.

    iii) If $H^0(\Theta_M) = 0$, the Kuranishi family is universal.

Let's draw some corollaries of the above theorem, noting that by proposition 5.3, the Kuranishi family is "the" semi-universal family of deformations, and that, by (3.3), the Kuranishi family is smooth if $H^2(\Theta) = 0$.

Corollary 5.5. If a deformation $\Pi': \mathcal{X}' \to B'$ has a smooth base $B'$, and surjective Kodaira-Spencer map $\rho'$, then it is complete and moreover the Kuranishi family of deformations is smooth.

**Proof.** Let $f : B' \to B$ be such that $\mathcal{X}' = f^*(\mathcal{X})$. By taking a smooth submanifold of $B'$, we can assume $f_*$ to be bijective. But then $B'$ is a neighborhood of $O \in \mathbb{C}^r$, and we have $f : \mathbb{C}^r \to B \subset \mathbb{C}^r$ with $f_*$ invertible: by the local inversion theorem $f(\mathbb{C}^r)$ contains a neighbourhood of $O \in \mathbb{C}^r$, hence gives a local isomorphism $f : B' \cong \mathbb{C}^r \overset{\cong}{\to} B \cong \mathbb{C}^r$.          Q.E.D.

Finally, we mention a refinement of part iii) of the theorem ([Wav]).

**Theorem 5.6** (Wavrick). If the base $B$ of the Kuranishi family is reduced, and $h^o(X_t, \Theta_t)$ is constant, then the Kuranishi family is __universal__.

In the next paragraph we shall discuss the example of the Segre-Hirzebruch surfaces, which illustrates how certain statements in the above theorems cannot be improved. We only remark here that if $M$ is a curve ($n = 1$), the $H^2(\Theta) = 0$ and the Kuranishi family is smooth of dimension $h^1(\Theta) = 3g - 3 + a$, where $g$ is the genus of the curve and $a = h^o(\Theta)$ is the dimension of the group of automorphisms of $M$.

## §6. The example of Segre-Hirzebruch surfaces

The Segre-Hirzebruch surface $\mathbb{F}_n$ (where $n \in \mathbb{N}$) is, in fancy language, the $\mathbb{P}^1$ bundle $\mathbb{P}(V_n)$ associated to the rank 2 vector bundle $V_n$ such that $\mathfrak{G}(V_n) \cong \cong \mathfrak{G}_{\mathbb{P}^1} \oplus \mathfrak{G}_{\mathbb{P}^1}(n)$. By abuse of language we shall identify $V$ with $\mathfrak{G}(V)$, therefore we get a split exact sequence

(6.1)  $$O \to \mathfrak{G}_{\mathbb{P}^1} \to V_n \to \mathfrak{G}_{\mathbb{P}^1}(n) \to 0 \ .$$

We consider all the rank 2 vector bundles $V$ which fit into an exact squence like (6.1), which are classified by $H^1(\mathfrak{G}_{\mathbb{P}^1}(-n))$, a vector space of dimension $(n-1)$, and we consider the family of ruled surfaces $\mathbb{P}(V)$, thus obtained, as a deformation of $\mathbb{F}_n$. In concrete terms, we take $B = \mathbb{C}^{n-1}$, with coordinates $t_1, \ldots, t_{n-1}$, and we obtain $\mathcal{X}$ glueing $\mathbb{P}^1 \times \mathbb{C} \times B$ ($= \mathbb{P}^1 \times (\mathbb{P}^1 - \{\infty\}) \times B$) with $\mathbb{P}^1 \times \mathbb{C} \times B$ ($= \mathbb{P}^1 \times (\mathbb{P}^1 - \{o\}) \times B$) by the identification of $(y_0, y_1, z, t_1, \ldots, t_{n-1})$ with $(y_0', y_1', z', t_1, \ldots, t_{n-1})$ if

(6.2)  $$z' = \frac{1}{z} \ , \quad y_1' = y_1 z^{-n} \ , \quad y_0' = y_0 + y_1 \sum_{i=1}^{n-1} t_i z^{-i}$$

(note then that $y_0 = y_0' + y_1' \cdot \sum_i t_i z^{n-i}$).

Now we shall compute the Kodaira-Spencer map of the family $\Pi : \mathcal{X} \to B$ we have just constructed.

$$\rho\left(\frac{\partial}{\partial t_i}\right) = \left(\frac{\partial}{\partial t_i}\right) - \left(\frac{\partial}{\partial t_i}\right)' \quad,$$

where the prime means we are writing a vector field using the second chart. So, expressing $(\partial/\partial t_i)$ using the first chart, we get

$$\rho\left(\frac{\partial}{\partial t_i}\right) = -(y_1' z^{n-i}) \frac{\partial}{\partial y_0} = (-y_1 z^{-i}) \frac{\partial}{\partial y_0} \quad.$$

We shall now show that these vector fields generate the Čech cohomology group $H^1(\{U, U'\}, \Theta_{\mathbb{F}_n})$, where $U = \mathbb{P}^1 \times \mathbb{C}$ of the second chart, and we notice that, since $H^1(\Theta_{\mathbb{P}^1 \times \mathbb{C}}) = 0$, the Čech cohomology group we are going to compute is indeed $H^1(\mathbb{F}_n, \Theta_{\mathbb{F}_n})$, hence the Kodaira-Spencer map will be bijective, and the constructed family will be the Kuranishi family.

Let's work on $\mathbb{F}_n$, where $z' = 1/z$, $y_1' = y_1 z^{-n}$, $y_0' = y_0$: since on $\mathbb{P}^m$ we have the Euler exact sequence

$$(6.3) \qquad 0 \to \mathcal{O}_{\mathbb{P}^m} \xrightarrow{\begin{pmatrix} x_0 \\ \vdots \\ x_m \end{pmatrix}} \mathcal{O}_{\mathbb{P}^m}(1)^{m+1} \xrightarrow{\left(\frac{\partial}{\partial x_0}, \ldots, \frac{\partial}{\partial x_m}\right)} \Theta_{\mathbb{P}^m} \to 0$$

vector fields in $\mathbb{P}^1$ have as basis $x_0 \frac{\partial}{\partial x_0}$, $x_0 \frac{\partial}{\partial x_1}$, $x_1 \frac{\partial}{\partial x_0}$. Therefore the holomorphic sections of $\Theta_{\mathbb{F}_n}$ on $U \cap U'$ can be written uniquely as

$$(6.4) \qquad \sum_{i \in \mathbb{Z}} z^i \left( a_{i00} y_0 \frac{\partial}{\partial y_0} + a_{i01} y_0 \frac{\partial}{\partial y_1} + a_{i10} y_1 \frac{\partial}{\partial y_0} \right) + \sum_{j \in \mathbb{Z}} b_j z^j \frac{\partial}{\partial z} \quad.$$

where $a_{i00}$, $a_{i01}$, $a_{i10}$, $b_j \in \mathbb{C}$. These sections are holomorphic also on $U$ if only non zero terms occur with $i \in \mathbb{N}$, $j \in \mathbb{N}$. Since

$$\frac{\partial}{\partial y_0'} = \frac{\partial}{\partial y_0} \quad, \quad \frac{\partial}{\partial z'} = -z^2 \frac{\partial}{\partial z} + nz y_0 \frac{\partial}{\partial y_0} \quad, \quad \frac{\partial}{\partial y_1'} = \frac{\partial}{\partial y_1} z^n \quad,$$

we write a regular section on $U'$ in terms of the first coordinate, and we have

$$\sum_{i \in \mathbb{N}} (z')^i \left( a_{i00}' y_0' \frac{\partial}{\partial y_0'} + a_{i01}' y_0' \frac{\partial}{\partial y_1'} + a_{i10}' y_1' \frac{\partial}{\partial y_0'} \right) + \sum_{j \in \mathbb{N}} b_j' (z')^j \frac{\partial}{\partial z'}$$

$$= \sum_{i \in \mathbb{N}} \left( a_{i00}' y_0 \frac{\partial}{\partial y_0} \right) z^{-i} + \left( a_{i01}' y_0 \frac{\partial}{\partial y_1} \right) z^{n-i} + \left( a_{i10}' y_1 \frac{\partial}{\partial y_0} \right) z^{-n-i} - \sum_{j \in \mathbb{N}} \left( b_j' \frac{\partial}{\partial z} \right) z^{-j+2}$$

$$+ \sum_{j \in \mathbb{N}} \left( nb_j' y_0 \frac{\partial}{\partial y_0} \right) z^{-j+1} \quad.$$

Since $H^1(\Theta_{\mathbb{F}_n}) = H^0(U \cap U', \Theta_{\mathbb{F}_n})/H^0(U, \Theta_{\mathbb{F}_n}) + H^0(U', \Theta_{\mathbb{F}_n})$, we see that

(6.5) $\qquad \left\{ y_1 z^{-1} \dfrac{\partial}{\partial y_0}, \ldots, y_1 z^{-(n-1)} \dfrac{\partial}{\partial y_0} \right\}$ is a basis of $H^1(\Theta_{\mathbb{F}_n})$ .

Furthermore, since $H^0(\mathbb{F}_n, \Theta_{\mathbb{F}_n}) = H^0(U, \Theta_{\mathbb{F}_n}) \cap H^0(U', \Theta_{\mathbb{F}_n})$, we have that

(6.6) $\qquad y_0 \dfrac{\partial}{\partial y_0}, \ y_0 \dfrac{\partial}{\partial y_1}, \ldots, z^n y_0 \dfrac{\partial}{\partial y_1}, \dfrac{\partial}{\partial z}, z \dfrac{\partial}{\partial z}, z^2 \dfrac{\partial}{\partial z} - n y_0 z \dfrac{\partial}{\partial y_0}$

(and $y_1 (\partial/\partial y_0)$ if $n = 0$) are a basis of $H^0(\mathbb{F}_n, \Theta_{\mathbb{F}_n})$.

Corollary 6.7. $h^1(\Theta_{\mathbb{F}_n}) = n-1$, $h^0(\Theta_{\mathbb{F}_n}) = n+5$ if $n > 0$, $6$ if $n = 0$, hence $\mathbb{F}_n \cong \mathbb{F}_m$ if and only if $n = m$. Furthermore, the family defined through the glueing (6.2) is the Kuranishi family of $\mathbb{F}_n$.

Let now $T_k \subset B$ be the determinantal locus

(6.8) $\qquad T_k = \left\{ t \mid \mathrm{rank} \begin{pmatrix} t_1 & \cdots & t_{k+1} \\ t_2 & & t_{k+2} \\ \vdots & & \vdots \\ t_{n-k-1} & & t_{n-1} \end{pmatrix} \leq k \right\}$

We refer to ([Ca 1], §1) for the proof of the following

Proposition 6.9. $T_k$ is an algebraic cone of dimension $\min(2k, n-1)$, and if $t \in T_k - T_{k-1}$, then $X_t \cong \mathbb{F}_{n-2k}$.

Remark 6.10. This example illustrates how the Kuranishi family can be semi-universal only for $t = 0$, and complete for $t \neq 0$. In this case $h^1(\Theta_{X_t})$ has a strict maximum for $t = 0$, and we notice that

(6.11) $\qquad h^1(\Theta_{X_t})$ is an uppersemicontinuous function in $t$, in general, and if $h^1(\Theta_{X_t})$ is constant on $B$ for the Kuranishi family, then the Kuranishi family (cf. 5.2) is also semi-universal for $t \neq 0$.

In this case, as we have seen, $h^0(\Theta_{X_t})$ is not constant: this was, in Wavrik's theorem (5.6), a sufficient condition for the universality of the Kuranishi family. We are going to show that for the above surfaces the Kuranishi family is not universal.

Example 6.12. Take $n = 2$ and our given family $\mathcal{X}$, obtained by the glueing $z' = 1/z$, $y_1' = y_1 z^{-2}$, $y_0' = y_0 + y_1 t_1 z^{-1}$. Consider the local biholomorphism of B to B sending $t_1$ to $t_1 f(t_1)$, where $f(t_1)$ is a holomorphic function with $f(0) = 1$. Then the pull-back is the family $\mathcal{X}'$ given by the glueing $\zeta' = 1/\zeta$, $\eta_1' = \eta_1 \zeta^{-2}$, $\eta_0' = \eta_0 + \eta_1 t_1 f(t_1) \zeta^{-1}$. But we obtain an isomorphism of $\mathcal{X}'$ with $\mathcal{X}$, compatible with the projections $\Pi$, $\Pi'$ on B, if we set $\zeta = z$, $\zeta' = z'$, $y_0 = \eta_0$, $y_0' = \eta_0'$, $y_1 = \eta_1 f(t_1)$, $y_1' = \eta_1' f(t_1)$. The condition $f(0) = 1$ ensures that the given isomorphism of $\mathbb{F}_n$ with the central fibre $X_0$ has not been changed.

# LECTURE THREE: VARIATIONS ON THE THEME OF DEFORMATIONS

## §7. Deformation of automorphisms

As an application of the theorem of Kuranishi, let's assume that $G$ is a finite (or compact) group of biholomorphisms of $M$. Then we clearly have a natural action of $G$ on $M \times B$, where $G$ acts trivially on the second factor, the base of the Kuranishi family.

If $\sigma \in G$, $t \in B$, clearly $\sigma$ is holomorphic on $X_t$ if and only if $\sigma_* T^{0,1}_t = T^{0,1}$, i.e. $\Leftrightarrow \sigma_* \varphi(t) = \varphi(t)$. Therefore $G \subset \mathrm{Aut}(X_t)$ for the set $\tilde{B}^G = \{ t \,|\, \sigma_* \varphi(t) = \varphi(t) \}$ (note that it makes sense to talk of $\sigma_*$ since $\sigma$ is an automorphism of $M = X_o$). This set is not so weird in general, since $\sigma_* \varphi(t) - \varphi(t)$ is a power series in $t$, we can only say that it is a complex subspace if $G$ is compact, since, by integration with respect to the invariant measure of $G$, we can assume $T^{1,0}_M$ to be endowed with a $G$-invariant Hermitian metric. For $G \ni \sigma$, $\sigma$ being holomorphic on $M$, $\sigma_*$ commutes with $\bar{\partial}$, but now the $G$-invariance of the metric implies that $\sigma_*$ also commutes with $\bar{\partial}^*$, $\square$, $G$, $H$.

Now $G$ acts naturally on the cohomology groups $H^q(\Theta)$, and we shall write, as customary,

$$H^q(\Theta)^G = \{ \eta \,|\, \in H^q(\Theta) \,|\, \sigma_* \eta = \eta \;\; \forall \; \sigma \in G \} \ .$$

If we identify $B$ with a complex subspace of $H^1(\Theta)$, we have

(7.1) $\quad \{ t \in B \,|\, \sigma \,|_{X_t} \text{ is holomorphic } \forall \; \sigma \in G \} = \tilde{B}^G = B \cap H^1(\Theta)^G = B^G$

In fact, since $\varphi(t) = t + \frac{1}{2} \bar{\partial}^* G [\varphi(t), \varphi(t)]$,

$$\sigma_* \varphi(t) = \sigma_* t + \frac{1}{2} \bar{\partial}^* G [\sigma_* \varphi(t), \sigma_* \varphi(t)] \ ,$$

therefore $\sigma_* \varphi(t)$ solves the Kuranishi equation for $\sigma_* t$, and $\sigma_* \varphi(t) = \varphi(\sigma_* t)$. Thus $\sigma_* \varphi(t) = \varphi(t)$ if and only if $t = \sigma_* t$, as we had to show.

But we can be more precise, because we have that

$$B^G = \{ t \in H^1(\Theta) \,|\, \forall \; \sigma \in G, \; t = \sigma_* t, \; H[\varphi(t), \varphi(t)] = 0 \}$$

but

$$\sigma_* H[\varphi(t), \varphi(t)] = H[\varphi(\sigma_* t), \varphi(\sigma_* t)] = (\text{if } t = \sigma_* t) = H[\varphi(t), \varphi(t)] \ ,$$

therefore we get that $B^G$ is a complex subspace of $H^1(\Theta)^G$ defined by $h^2(\Theta)^G$ equations of multiplicity at least 2 and such that their quadratic parts are associated to the symmetric bilinear mapping

$$[\ ,\ ]\colon\ H^1(\Theta)^G \times H^1(\Theta)^G\ \to\ H^2(\Theta)^G\ .$$

We get thus a lower bound for $\dim B^G$, and we observe that the family $\mathcal{X}_{|B^G} = \Pi^{-1}(B^G)$ has an action of $G$ which is holomorphic, fibre preserving, and such that the diffeomorphism type of the action is constant.

## §8. Deformations of non-degenerate holomorphic maps

Assume we are in the following situation: we are given a family of deformations of $M$, $\Pi\colon \mathcal{X} \to B$, and let's assume that, $W$ being a fixed complex manifold, one is given a holomorphic map $F\colon \mathcal{X} \to W \times B$, such that $\Pi = p_2 \circ F$, $p_2\colon W \to B$ being the projection on the second factor of the product. This general situation has been considered by Horikawa ([Hor 0], [Hor 1], [Hor 2]), here we shall limit ourselves to the case when

(8.1)    $f_t = p_1 \circ F\big|_{X_t}\colon X_t \hookrightarrow W$ is generically with injective differential $(f_t)_*$

(here $p_1\colon W \times B \to W$ is the first projection).

(8.1) is equivalent to saying that $\forall\, t\ (f_t)_*\ \Theta_{X_t} \to (f_t)^*\,\Theta_W$ is an injective homomorphism of sheaves, where $(f_t)^*$ denotes the analytic pull-back of a coherent sheaf. In particular, for $t = 0$, we have an exact sequence

(8.2)    $$0\ \to\ \Theta_{X_o}\ \xrightarrow{(f_o)_*}\ (f_o)^*\,\Theta_W\ \to\ N_{f_o}\ \to\ 0\ .$$

Definition 8.3. The cokernel $N_{f_o}$ of the homomorphism $(f_o)_*$ in (8.2) is called the normal sheaf of the holomorphic map, and $H^o(N_{f_o})$ is called the <u>characteristic system</u> of the map.

Proposition 8.4. There is a linear map $\rho_F\colon T_{B,o} \to H^o(N_{f_o})$ such that, if $\rho$ is the Kodaira-Spencer map of the given deformation, and $\partial$ is the coboundary map $\partial\colon H^o(N_{f_o}) \to H^1(\Theta_{X_o})$ of the long exact cohomology sequence of (8.2), then one has a <u>factorization</u> $\rho = \partial \circ \rho_F$. Such a $\rho_F$ is called the <u>characteristic map</u> of the family.

Proof. Let's take a finite cover $(V_\alpha)$ of $\mathcal{X}$ such that $V_\alpha \cong U_\alpha \times B$ as usual, so that, locally on $V_\alpha$, if $z^\alpha$ are local coordinates in $U_\alpha \subset X_o$, we can write

(8.5)    $$F(z^\alpha,\, t) = (f(z^\alpha,\, t),\, t)\ .$$

Let $\xi$ be a tangent vector in $T_{B,o}$ which can be extended as a vector field on $B$

(we always work with $(B, o)$ as a germ of complex space), and let $\vartheta_\alpha$ be the lift of $\xi$ to $V_\alpha$ determined by our choice of coordinates $(z^\alpha, t)$ on $V_\alpha$. We know that $\rho(\xi) = (\vartheta_\alpha - \vartheta_\beta)\big|_{X_o}$, and that in fact (since one can change coordinates), $\vartheta_\alpha$ is well defined only up to adding a vertical vector field. The differential $F_*$ sends $\vartheta_\alpha$ in a pair $(\psi_\alpha, \xi)$ where $\psi_\alpha$ is a section of $f^*(\Theta_W)$.

Restricting $\psi_\alpha$ to $U_\alpha \times \{0\}$, I get a global section $\psi$ of $N_{f_o} = f_o^*(\Theta_W)/\Theta_{X_o}$ and by definition, since $(f_o)_*(\vartheta_\alpha\big|_{U_\alpha \times \{0\}}) = \psi_\alpha$, we get

$$\partial(\psi) = (\vartheta_\alpha - \vartheta_\beta)\big|_{X_o} = \rho(\xi) \ .$$

So it suffices to set $\rho_F(\xi) = \psi$, and $\rho_F$ is well-defined and linear. $\hspace{1cm}$ Q. E. D.

The situation considered up to here embodies the classical theory of deformations of plane curves with nodes and cusps, and of surfaces with ordinary singularities, therefore we shall now give a definition which is consistent with the classical one in the second case, but is not a generalization of definition 5.1 (also we shall denote by $\varphi$ the map $\varphi \colon M \to W$ corresponding to $f_o$ via the isomorphism $M \cong X$).

Definition 8.6. The characteristic system $H^o(N_\varphi)$ of the map $\varphi \colon M \to W$ is complete if there exists a smooth deformation of the holomorphic map (i.e. $B$ is smooth) such that $\rho_F \colon T_{B, o} \to H^o(N_\varphi)$ is surjective.

Remark 8.7. $H^o(N_\varphi)$ is the exact analogue of $H^1(\Theta_M)$ in the case of deformations of a manifold. In general, a sufficient condition in order that the Kuranishi family be smooth is the sharp assumption $H^2(\Theta_M) = 0$: in a similar fashion Horikawa proves the following generalization of a previous theorem of Kodaira:

Theorem 8.8. The characteristic system of a map is complete if $H^1(N_\varphi) = 0$ .

In the next paragraph we shall discuss some particular example in the special case when $\varphi$ is an embedding: before doing so, we just make the following observation (in view of the long exact cohomology sequence associated to (8.2)).

(8.9) $\hspace{1cm}$ a necessary condition in order to deform $\varphi$ on a complete family $\mathcal{X}$ of deformations of $M$ is that $H^1(\varphi^*(\Theta_W)) \to H^1(N_\varphi)$ be injective.

## §9. Examples of embedded deformations and obstructed moduli

Assume now that $M$ is a subvariety of $W$, that $\dim W = r$, $\dim M = n$, and that $M$ is the locus of zeros of a section of a rank $(r-n)$ vector bundle $V$. We shall write $V_{|M}$ for $\mathcal{O}_W(V) \otimes \mathcal{O}_M$, and it is an elementary computation to see that, if the $N_{M|W}$ denotes the normal sheaf for the embedding of $M$ into $W$, then

$$(9.1) \qquad\qquad N_{M|W} = V_{|M} \quad ,$$

hence we have an exact sequence

$$0 \to \Theta_M \to \Theta_W \otimes \mathcal{O}_M \to V_{|M} \to 0$$

and clearly the characteristic system is complete if there is a surjection

$$H^O(W, V) \twoheadrightarrow H^O(M, V_{|M}) \ .$$

### Example 9.2.

$$W = \mathbb{P}^r \ , \quad V = \overset{r-n}{\underset{i=1}{\oplus}} \mathcal{O}_{\mathbb{P}^r}(m_i) \quad (m_i \geq 2) \ .$$

The ideal $\mathcal{J}_M$ admits a Koszul resolution

$$(9.3) \qquad 0 \to \wedge^{r-n}(V^{\vee}) \to \wedge^{r-n-1}(V^{\vee}) \to \cdots \to \wedge^2(V^{\vee}) \to V^{\vee} \to \mathcal{J}_M \to 0$$

$$\wedge \begin{pmatrix} f_1 \\ \vdots \\ \vdots \\ f_{n-r} \end{pmatrix}$$

dual to $\wedge^k V \xrightarrow{\qquad} \wedge^{k+1} V$, the $f_i$'s being the sections

of $\mathcal{O}_{\mathbb{P}^r}(m_i)$ s.t. $M = \{f_1 = \cdots = f_{n-r} = 0\}$

From (9.3), by induction, follows that

$$(9.4) \qquad\qquad H^i(\mathcal{J}_M(k)) = 0 \ \forall \ k \in \mathbb{Z} \text{ and for } i \leq n \quad .$$

In particular, $H^1(\mathcal{J}_M \cdot V) = 0$, hence, by the sequence

$$0 \to \mathcal{J}_M V \to V \to V_{|M} \to 0 \ ,$$

we infer that $H^O(\mathbb{P}^r, V)$ goes onto $H^O(M, V_{|M})$. Also, from the sequence

$$0 \to \mathcal{J}_M(k) \to \mathcal{O}_{\mathbb{P}^r}(k) \to \mathcal{O}_M(k) \to 0$$

follows that

(9.5) $\qquad H^i(\mathcal{O}_M(k)) = 0 \quad \forall\ k \in \mathbb{Z} \text{ if } i \le n-1 \quad (\text{e.g. } H^1(V_{|M}) = 0)$

It will be true that the family given by $H^o(V)$ is <u>complete</u> if and only if, in view of the long exact cohomology sequence associated to (9.1), $H^1(\mathcal{O}_{\mathbb{P}^r} \otimes \mathcal{O}_M) = 0$. Now, in view of the Euler sequence (6.3), and of (9.5), we have an exact cohomology sequence giving

(9.6) $\qquad H^1(\mathcal{O}_{\mathbb{P}^r} \otimes \mathcal{O}_M) = 0 \quad \text{if } n \ge 3, \text{ or if } n = 2 \quad \text{and}$

$$H^2(\mathcal{O}_M) \to H^2(\mathcal{O}_M)^{r+1} \text{ is injective.}$$

But in this last case (we use the standard notation $\Omega^i_M$ for $\mathcal{O}((\wedge^i T^{1,0}_M)^\vee))$, by Serre duality an equivalent condition is that the following map be surjective

$$H^o(\Omega^2_M(-1))^{r+1} \to H^o(\Omega^2_M).$$

But, by adjunction, $\Omega^2_M \cong \mathcal{O}_M(\Sigma\, m_i - r - 1)$, and since $H^o(\mathcal{O}_{\mathbb{P}}(k)) \to H^o(\mathcal{O}_M(k))$ is onto $\forall\ k$, we get that

(9.7) $\qquad H^1(\mathcal{O}_{\mathbb{P}^r} \otimes \mathcal{O}_M) = 0 \quad \text{unless } \Omega^2_M \cong \mathcal{O}_M \quad (\text{i.e. } \sum_1^{r-2} m_i = r+1\,,$

an equation which has only the following solutions for the $m_i$'s: $(4), (3,2), (2,2,2))$. In this last case $H^1(\mathcal{O}_M) = H^o(\Omega^1_M) = 0$ (M is a Kähler manifold via the Fubini-Study metric), but, since $\Omega^2_M \cong \mathcal{O}_S$, $H^2(\mathcal{O}_M)$ is dual by Serre duality to

$$H^o((\Omega^1_M) \otimes \Omega^2_M) = H^o(\Omega^1_M) = 0\ .$$

The conclusion is the following well known fact (cf. [Ser]).

(9.8) $\qquad$ If M is a (smooth) complete intersection in $\mathbb{P}^r$ of dimension $n \ge 2$, the characteristic system is complete, and the embedded deformations give a complete deformation, except if $n = 2$ and $\Omega^2_M \cong \mathcal{O}_M$: M is called then a K3 surface, and embedded deformations give a 19-dimensional subvariety of the Kuranishi family, which is smooth of dimension 20.

So, for instance, every small deformation of a smooth surface S of degree $m$ in $\mathbb{P}^3$ is still a surface in $\mathbb{P}^3$: but what happens in the large, according to the following definition?

Definition 9.9. Two manifolds $M, M'$ are said to be a deformation of each other (in the large) if they lie in the same class for the equivalence relation generated by:

$M \overset{\text{Def}}{\sim} M' \iff$ there exists a deformation: $\Pi : \mathcal{X} \to B$ of $M$ with irreducible $B$ and such that $\exists\, b \in B$ with $X_b$ isomorphic to $M'$. If $M \overset{\text{Def}}{\sim} M'$, we shall also say that $M'$ is a direct deformation of $M$. The upshot ([Hor 3]) is that already for degree $n = 5$ the surfaces which are deformations of quintic surfaces need not be surfaces in $\mathbb{P}^3$ any more!

The rough idea is as follows: Horikawa considers all the smooth surfaces such that $p_g = h^o(\Omega_S^2) = 4$, and such that $K^2 = 5$ (K, here and in the following, is a divisor of a (regular here, rational in other cases) section of $\Omega_S^2$): these numerical conditions, as we shall see, are invariant under any deformation, and indeed any complex structure on a surface orientedly homeomorphic to a smooth quintic surface in $\mathbb{P}^3$ must satisfy these conditions.

Studying the behaviour of the rational map $\varphi : S \to \mathbb{P}^3$ associated to the sections of $\Omega_S^2$, Horikawa shows that either

    I) $\varphi$ gives a birational morphism onto a 5-ic or

    IIa) $\varphi$ gives a rational map 2:1 onto a smooth quadric or

    IIb) $\varphi$ is 2:1 to a quadric cone.

Note that for a smooth 5-ic $\Omega_S^2 \cong \mathcal{O}_S(1)$, hence $\varphi$ is just inclusion in $\mathbb{P}^3$ and one is in case I. Surfaces of type I belong to one family of deformations, and their Kuranishi family is smooth of dimension 40: the same holds for surfaces of type IIa). For surfaces of type IIb), instead, $h^1(\Theta_S) = 41$, $h^2(\Theta_S) = 1$ therefore we know that the Kuranishi family gives a hypersurface in $\mathbb{C}^{41}$: Horikawa computes

$$[ \ , \ ] \ H^1(\Theta_S) \times H^1(\Theta_S) \to H^2(\Theta_S) \ ,$$

finding that, in suitable coordinates, the associated quadratic polynomial is $z_1 z_2$. Now, by the Morse lemma, there do exist new coordinates on $H^1(\Theta_S)$ s.t. the equation $g$ of $B$ is of the form $g(t) = t_1 t_2 + \psi(t_3, \dots, t_{41})$, where $\psi(t) = o(t^2)$. Since $H^o(\Theta_S) = 0$, the Kuranishi family is universal and surfaces of type IIb) form, as it is easy to show, a 39-dimensional variety which is contained in the singular locus of $B$ by Horikawa's computation of $[ \ , \ ]$. Now the singular locus of $B$ is given by $t_1 = t_2 = \partial\psi/\partial t_3 = \dots = \partial\psi/\partial t_{41} = 0$, hence it has dimension 39 iff $\psi \equiv 0$. Thus $g = g_1 g_2$, with $g_i(z) = z_i + o(z)$. The conclusion is now easy: when $g_1 = 0$, $g_2 \neq 0$ we have a surface of type I (a 5-ic), when $g_2 = 0$, $g_1 \neq 0$ we have a surface of type IIa, when $g_1 = g_2 = 0$, we have a surface of type IIb.

We end this section showing a nice example due to Mumford ([Mu 1] , cf. also appendix to Chapter V of [Za]) of varieties for which the Kuranishi family is everywhere non reduced. We notice that the terminology most frequently used adopts the following

Definition 9.10. A manifold M is said to have obstructed moduli if dim $B < h^1(\mathbb{O})$ (if and only if B is not smooth), i.e. if "not all infinitesimal deformations are integrable."

It frequently occurs that B may be singular at 0, but the phenomenon pointed out by Mumford is not so common (at least for the time being).

Example 9.11. ([Mu 1]). Let F be a smooth cubic surface in $\mathbb{P}^3$, E a straight line contained in F (hence $E^2 = -1$, $KE = -1$), and let H be a hyperplane section of F. The linear system $|4H + 2E|$ has no base points (this is clear outside E, on the other hand by the exact sequence $0 \to \mathbb{O}_F(4H) \to \mathbb{O}_F(4H+E) \to \mathbb{O}_E(3) \to 0$ since $H^1(\mathbb{O}_F(4H)) = 0$, we get that $|4H + E|$ has no base points and $H^1(\mathbb{O}_F(4H+E)) = 0$, then we conclude by the exact sequence $0 \to \mathbb{O}_F(4H+E) \to \mathbb{O}_F(4H+2E) \to \mathbb{O}_E(2) \to 0$, where $\mathbb{O}_E(i)$ is the sheaf of degree i on $E \cong \mathbb{P}^1$), so that we can pick a smooth curve C inside $|4H+2E|$. Since the canonical sheaf of C is $\mathbb{O}_C(3H+2E)$, we easily find that

(9.12)        $C \subset F$ is a smooth curve of genus g = 24 and degree 14.

Since the normal sheaf of C in F, $N_{C|F}$ is $\mathbb{O}_C(4H + 2E)$, which is non special, we get an exact sequence of normal sheaves

(9.13)                $0 \to H^o(N_{C|F}) \to H^o(N_{C|\mathbb{P}^3}) \to H^o(\mathbb{O}_C(3H)) \to 0$

and also $H^1(N_{C|\mathbb{P}^3}) \cong H^1(\mathbb{O}_C(3H))$, a dual vector space to $H^o(\mathbb{O}_C(2E))$, which has dimension 1 by virtue of the exact sequence

$$0 \to H^o(\mathbb{O}_F(2E)) \to H^o(\mathbb{O}_C(2E)) \to H^1(\mathbb{O}_F(-4H)) = 0 .$$

We see that the hypothesis in theorem 8.8 is not verified, and in fact the character-istic system is not complete, as we shall see. (9.13) gives $\dim_{\mathbb{C}} H^o(N_{C|\mathbb{P}^3}) = 57$, moreover F moves in a 19-dimensional linear system in $\mathbb{P}^3$, C varies in a 37-dimensional linear system on F, hence C belongs to a 56-dimensional family. Mumford shows that C cannot belong to an algebraic family of dimension 57 by the following arguments:

1) if $C'$ is a smooth curve of genus 24, degree 14, $\Theta_{C'}(4H)$ is non special and has 33 independent sections, so that $C'$ belongs to 2 independent quartic surfaces $G$, $G'$. Clearly, $C'$ is not a plane curve, and it is easy to check that $C$ is not contained in any quadric surface.

2) Thus, either

    a) $C'$ is not contained in any cubic surface or

    b) $C'$ belongs to a (unique) smooth cubic surface or

    b') $C'$ belongs to a singular cubic surface.

3) Assume now that $C$ belongs to an irreducible family of dimension $\geq 57$: since condition b') is a closed condition on the base of the family, there would exist a family of curves $C'$ of type a), with dimension $\geq 57$. But in case a), $G \cdot G' = C' + \Gamma$, where $\Gamma$ is a conic. Thus the complete intersection of $G$, $G'$, being Cohen-Macauley, is reduced and has at most triple points as singularities, so that $G$ and $G'$ have no singular points in common. Since $G$ and $G'$ intersect transversally at the points of $C' - \Gamma$, we can assume $G$ to be non-singular around $C'$.

4) By Noether's theorem, not all surfaces of degree 4 contain a conic, hence $G$ belongs to a family of dimension at most 33 (in fact, much less, see [G-H]), moreover it is easily verified that the characteristic system of $C'$ in $G$ has dimension 24, so that the dimension of such pairs $(C' \subset G)$ is at most 57, and since $C'$ belongs to a 1-dimensional system of quartic surfaces, we are done.

<u>Example 9.14</u> ([Ko 1], [Mu 1]). If $M$ is the blow-up of $\mathbb{P}^3$ with centre a curve $C$ as 9.11, then the base $B$ of the Kuranishi family of deformations is non reduced.

Before even setting up the notations, let's give a useful

<u>Definition 9.15.</u> Let $Y$ be a subvariety of a smooth variety $X$. We define $\Theta_X$ (-log $Y$) to be the sheaf of tangent vectors on $X$ which are tangent to $Y$ (i.e., $\xi \in \Theta_X$ (-log $Y$) if and only if, $\mathcal{J}_Y$ being the ideal sheaf of $Y$, $\forall g \in \mathcal{J}_Y$, $\xi(g) \in \mathcal{J}_Y$).

<u>Remark 9.16.</u> Clearly, $\mathcal{J}_Y \Theta_X \subset \Theta_X$(-log $Y$) and, by the definition, $\Theta_X/\Theta_X(\text{-log } Y)$ is the equisingular normal sheaf $N'_{Y|X}$ of $Y$ in $X$. ($N'_{Y|X}$ is the usual normal sheaf when $Y$ is smooth, cf. §11.) We have thus the exact sequences

(9.17) $$0 \to \Theta_X \text{(-log } Y) \to \Theta_X \to N'_{Y|X} \to 0 \quad .$$

Moreover the tangent sheaf of $Y$, $\Theta_Y$ is by definition the quotient

$$\Theta_X(-\log\ Y)/\mathcal{J}_Y\,\Theta_X \ \ .$$

Let's now set up the notation: we have $\Pi: M \to \mathbb{P}^3$ the blow-up map, $E$ is the exceptional divisor $\Pi^{-1}(C)$.

We have therefore two exact sequences

$$0 \to \Theta_{\mathbb{P}^3}(-\log\ C) \to (\Theta_{\mathbb{P}^3}) \to N_{C/\mathbb{P}^3} \to 0$$

$$0 \to \Theta_M(-\log\ E) \to \Theta_M \to N_{E|M} \to 0 \ \ .$$

First of all, by the exact sequence

$$0 \to \mathcal{J}_C\,\Theta_{\mathbb{P}^3} \to \Theta_{\mathbb{P}^3}(-\log\ C) \to \Theta_C \to 0 \ \ .$$

since $C$ has genus bigger than 2, $H^o(\Theta_C) = 0$, thus also $H^o(\Theta_{\mathbb{P}^3}(-\log\ C)) = 0$, since $H^o(\mathcal{J}_C\,\Theta_{\mathbb{P}^3}) = 0$ (this is the Lie algebra of the group of projectivities leaving $C$ fixed, which is the trivial group since $C$ contains 5 independent points).

Let's look at the exceptional divisor $E$: $E$ is the projectivized normal bundle of $C$ in $\mathbb{P}^3$, $E = \mathbb{P}(N_{C|\mathbb{P}^3})$, i.e. points in $E$ are lines in the normal bundle of $C$. In particular, if $\Theta_E(1)$ is the dual of the tautological invertible sheaf $\Theta_E(-1) \subset \Pi^*(N_{C|\mathbb{P}^3})$, then $\Pi_*\Theta_E(1) = N^{\vee}_{C|\mathbb{P}^3}$ ($\vee$ denoting the dual sheaf).

It is easy to verify that

$$(9.18) \qquad\qquad N_{E|M} \cong \Theta_E(-1) \ \ ,$$

hence we have

$$(9.19) \qquad\qquad H^o(N_{E|M}) = H^1(N_{E|M}) = 0 \ \ .$$

__Proof__ of (9.19). Since a sheaf of degree $(-1)$ on $\mathbb{P}^1$ has 0 cohomology in all degree, $\Pi_* N_{E|M} = \mathcal{R}^1\Pi_* N_{E|M} = 0.$ \hfill Q. E. D.

__Corollary 9.20.__ $\qquad H^i(\Theta_M) \cong H^i(\Theta_M(-\log\ E))$ for $i = 0, 1, 2.$

__Proposition 9.21.__ $\quad \Pi_*\Theta_M(-\log\ E) = \Theta_{\mathbb{P}^3}(-\log\ C)$, $\mathcal{R}^1\Pi_*\Theta_M(-\log\ E) = 0$, hence

$$H^i(\Theta_M(-\log\ E)) \cong H^i(\Theta_{\mathbb{P}^3}(-\log\ E)) \quad \text{for } i = 0, 1, 2$$

and $H^o(\Theta_M(-\log\ E)) = 0$ .

The proof of proposition 9.21 follows immediately from the following.

<u>Lemma 9.22.</u> Let $\Pi: X \to \mathbb{C}^2$ be the blow up of the origin $O$, $\mathfrak{M}$ the maximal ideal of the point $O$, $E$ the exceptional divisor $\Pi^{-1}(0)$. Then

$$\Pi_* \Theta_X(-\log E) = \mathfrak{M} \otimes_{\mathbb{C}^2} , \quad \mathcal{R}^1 \Pi_* \Theta_X(-\log E) = 0 .$$

<u>Proof.</u> $X$ is covered by two affine pieces $A$, $A'$ with coordinates $(u,v)$, resp. $(u',v')$ and $\Pi$ is given by $x = u$, $y = uv$ (resp. $x = u'v'$, $y = u'$). Thus

$$\begin{cases} \dfrac{\partial}{\partial u} = \dfrac{\partial}{\partial x} + v \dfrac{\partial}{\partial y} \\[2mm] \dfrac{\partial}{\partial v} = u \dfrac{\partial}{\partial y} \end{cases} \qquad \begin{cases} \dfrac{\partial}{\partial u'} = \dfrac{\partial}{\partial y} + v' \dfrac{\partial}{\partial x} \\[2mm] \dfrac{\partial}{\partial v'} = u' \dfrac{\partial}{\partial x} \end{cases}$$

Since $E$ is defined by the equation $u = 0$ $(u' = 0)$, $\Theta(-\log E)$ is generated by

$$u \frac{\partial}{\partial u} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad \text{and by} \quad \frac{\partial}{\partial v} = x \frac{\partial}{\partial y} \quad \text{on } A, \text{ and by}$$

$$u' \frac{\partial}{\partial u'} = y \frac{\partial}{\partial y} + x \frac{\partial}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial v'} = y \frac{\partial}{\partial x} \quad \text{on } A' .$$

Let $\xi$ be a section of $\Theta_X(-\log E)$ defined in a neighborhood of $E$: then we can express $\xi$ as

$$\xi = a(u,v) u \frac{\partial}{\partial u} + b(u,v) \frac{\partial}{\partial v} \qquad \text{and also}$$

$$\xi = \alpha(u',v') u' \frac{\partial}{\partial u'} + \beta(u',v') \frac{\partial}{\partial v'} , \quad \text{where}$$

$$a(u,v) = \sum_{i,j \geq 0} a_{ij} u^i v^j$$

and similarly for $b$, $\alpha$, $\beta$. We must have

$$ax \frac{\partial}{\partial x} + (ay + bx) \frac{\partial}{\partial y} = (\alpha x + \beta y) \frac{\partial}{\partial x} + (\alpha y) \frac{\partial}{\partial y} .$$

Hence, expressing $\xi$ as a rational section of $\Theta_{\mathbb{C}^2}$, we see that the coefficient of $\partial/\partial x$ can be any function $f$ of the type

$$x \cdot \sum_{i,j \geq 0} a_{ij} x^{i-j} y^j ,$$

such that it can also be expressed in the form

$$x \cdot \sum_{h,\ell \geq 0} \alpha_{h\ell} x^h \cdot y^{\ell-h} + y \sum_{h,\ell \geq 0} \beta_{h,\ell} x^h y^{\ell-h} .$$

It is immediate to see that the coefficient of $\partial/\partial x$ is a power series in x, y, vanishing at the origin, and, by symmetry, the same holds for the coefficient g of $\partial/\partial y$. It is also easy to verify that two such $f, g \in \mathcal{M} \Theta_{\mathbb{C}^2, 0}$ can be chosen arbitrarily. Instead, to show that $R^1 \Pi_*(\Theta_X(-\log E)) = 0$, it suffices to show that any section $\Lambda$ of $\Theta_X(-\log E)$ on $A \cap A'$ can be written as a sum $\Lambda^+ + \Lambda'$, where $\Lambda^+$ is regular on A, $\Lambda'$ is regular on A'. Now, $\Lambda$ can be written as

$$a(u,v) \, u \frac{\partial}{\partial u} + b(u,v) \frac{\partial}{\partial v} \quad ,$$

where

$$a(u,v) = \sum_{\substack{i \geq 0 \\ j \in \mathbb{Z}}} a_{ij} \, u^i \, v^j$$

can be written as

$$a(u,v) = a^+(u,v) + a'(u,v) \quad a^+ = \sum_{i,j \geq 0} a_{ij} \, u^i \, v^i \quad ,$$

and similarly $b(u,v) = b^+(u,v) + b'(u,v)$. Since

$$\Lambda^+ = a^+ \, u \frac{\partial}{\partial u} + b^+ \frac{\partial}{\partial v}$$

is regular on A, it suffices to verify (we omit this) that $\Lambda'$ is regular on A'.

$$\underline{Q.E.D.}$$

From 9.20 and 9.21 we infer that

$$H^0(\Theta_M) = 0, \quad H^2(\Theta_M) \cong H^1(N_{C|\mathbb{P}^3}), \quad H^1(\Theta_M) \cong H^0(N_{C|\mathbb{P}^3})/H^0(\Theta_{\mathbb{P}^3}) \quad .$$

These isomorphisms are natural, in fact one can verify that for each deformation of the embedding $C \hookrightarrow \mathbb{P}^3$ the Kodaira-Spencer map for the family of blown up 3-folds is the composition of the characteristic map of the deformation with the surjection of $H^0(N_{C|\mathbb{P}^3}) \twoheadrightarrow H^1(\Theta_M)$ (the kernel $H^0(\Theta_{\mathbb{P}^3})$ is due to the fact that blowing up projectively equivalent curves one obtains isomorphic 3-folds). Now, Kodaira ([Ko 1], thm. 6) proves that every small deformation of M is the blow-up of $\mathbb{P}^3$ with center a curve which is a deformation of C in $\mathbb{P}^3$, thereby showing that the Kuranishi family of M has dimension equal to 56-15 = 41, whereas, by what we saw, $h^1(\Theta_M) = 42$ for each blow-up M of a curve C as in 9.11. Thus the Kuranishi family B of M is singular at each point ($h^1(\Theta_{M_t})$ being constant for $t \in B$, B is the Kuranishi family for each $M_t$).

## §10. Further variations and further results

We have seen in §9 a 3-dimensional variety such that its Kuranishi family is universal at each point, but its base  B  is everywhere non reduced. We remarked in §5 that the base B of the Kuranishi family of curves is smooth: for surfaces Kas ([Kas]) found, using Kodaira's theory of elliptic surfaces, an example of a family of elliptic surfaces such that the generic dimension of  $H^1(\Theta_{S_t})$  would be strictly bigger than  dim B.  The family is constructed by deforming a certain class of algebraic surfaces. We suspect that this should not happen for surfaces of general type with  $H^1(S,\mathbb{R}) = 0$ and K ample (cf. lecture seven); it is anyhow clarified by Burns and Wahl ([B-W]) how the fact that  $K_S$  is not ample, in particular the existence of many curves  E  such that  $\Theta_E(K_S) \cong \Theta_E$ , forces the dimension of  $H^1(\Theta)$  to be bigger than  dim B : in particular, using classical results of Segre on the existence of surfaces  Σ  in  $\mathbb{P}^3$  with many nodes (also called conical double points, i.e. with local equation  $x^2 + y^2 + z^2 = 0$ ), they show that the blow-up of  Σ  at the nodes is a surface  S  with obstructed deformations (the rough idea being that nodes contribute by 1 to  $h^1(\Theta)$ , but all the small deformations are still surfaces in  $\mathbb{P}^3$ ). We already noted in the introduction that Zappa ([Zp]) was the first to show that the characteristic system of a submanifold does not need to be complete. His example is as follows (we follow, though, the description of [Mu 3]):

Example 10.1.  Let  E  be an elliptic curve and let  V  be the rank 2 bundle which occurs as a non trivial extension

$$(10.2) \qquad\qquad 0 \to \Theta_E \to V \to \Theta_E \to 0$$

(in fact, these extensions are classified by  $\mathbb{C}^*$  orbits in  $H^1(\Theta_E) \cong \mathbb{C}$ , so there is "only" one non trivial extension). The subbundle  $\Theta_E$  defines a section  C  of the  $\mathbb{P}^1$  bundle  $S = \mathbb{P}(V)$  over  E, and, since  $V/\Theta_E \cong \Theta_E$ ,  $N_{C|S} \cong \Theta_C$ . Nevertheless, there is no embedded deformation of  C  in  S, since  $H^1(\Theta_S) \cong H^1(\Theta_E)$ , and, if  C′  is algebraically equivalent to  C, then there is a divisor  L  of degree 0 on E  such that  (Π: S → E  being the bundle map)  $C' \equiv C + \Pi^*(L)$ .  But then  $\Pi_*(\Theta_S(C')) = V \otimes \Theta_E(L)$ , and, tensoring 10.2 with  $\Theta_E(L)$ , we infer that there are no sections if  $L \not\equiv 0$ , whereas, for  $L \equiv 0$ , the condition that the extension splits ensures that 10.2 is not exact on global sections (or geometrically, if

$$h^0(\Theta_S(C)) \geq 2 , \quad S \cong E \times \mathbb{P}^1 !) \quad .$$

Q.E.D.

As far as deformation theory is concerned, the Kodaira-Spencer-Kuranishi results were extended first in the direction of the deformations of isolated singularities (cf. [Po], [Gr 1]), and then the result of Kuranishi was extended to the case of compact complex spaces ([Gr 3], [Dou 3], [Pa]). On the other hand, Grothendieck ([Gro 1], [Gro 2]) contributed significantly to extension of the deformation theory, especially through the construction of the Hilbert schemes, parametrizing projective subschemes with fixed Hilbert polynomials (cf. §19 for a vague idea): his results were extended to the case of (compact subspaces of) complex spaces in Douady's thesis ([Dou 4]). Since the variations in the theme of deformations can be arbitrary, Schlessinger ([Sch]) approached the problem abstractly developing a general theory giving necessary and sufficient conditions for finding "power series solutions", i.e. finding a formal versal deformation space for a deformation functor: this theory is usually coupled with a deep theorem of Artin ([Ar]), giving criteria of convergence for the power series solutions. We don't try to sketch any detail, nor to mention further very interesting work, but we defer the reader to the very interesting article ([Pa]) of Palamodov already quoted in the introduction (and plead guilty for ignoring the post '76 period). We simply remark the importance of Palamodov's theorem 5.6 giving an algebraic description of the higher order terms in the Koranishi equations.

As far as I know, this result has not yet been applied in concrete geometric cases, but its validity should be tested in some example.

## LECTURE FOUR: THE CLASSICAL CASE

### §11. Deformations of a map and equisingular deformations of the image (infinitesimal theory)

We let, as in §9, $\varphi: X \to W$ be a non-degenerate holomorphic map, and we set $\Sigma = \varphi(X)$. Since $W$ is a smooth variety, we have in general, for every sub-variety $\Sigma$, the exact sequences

(11.1)
$$0 \to \Theta_W(-\log \Sigma) \to \Theta_W \to N'_{\Sigma \,|\, W} \to 0$$
$$0 \to \Theta_W(-\Sigma) \to \Theta_W(-\log \Sigma) \to \Theta_\Sigma \to 0 \ .$$

On the other hand, by dualizing (i.e., taking $\mathrm{Hom}_{\Theta_\Sigma}(\ \cdot\ , \ \Theta_\Sigma)$) the exact sequence

$$0 \to N^{\vee}_{\Sigma \,|\, W} \to \Omega^1_W \otimes \Theta_\Sigma \to \Omega^1_\Sigma \to 0$$

where $N^{\vee}_{\Sigma \,|\, W}$ is the conormal sheaf of $\Sigma$ in $W$, we get the long exact sequence

$$0 \to \Theta_\Sigma \to \Theta_W \otimes \Theta_\Sigma \to (N^{\vee}_{\Sigma \,|\, W})^* = N_{\Sigma \,|\, W} \to \mathrm{Ext}^1(\Omega^1_\Sigma, \Theta_\Sigma) \to 0 \ .$$

which splits into the short exact sequences

(11.2)
$$0 \to \Theta_\Sigma \to \Theta_W \otimes \Theta_\Sigma \to N'_{\Sigma \,|\, W} \to 0$$
$$0 \to N'_{\Sigma \,|\, W} \to N_{\Sigma \,|\, W} \to \mathrm{Ext}^1(\Omega^1_\Sigma, \Theta_\Sigma) \to 0 \quad .$$

**Example 11.3.** Assume $\Sigma$ is a hypersurface in $W$, locally defined by the equation $f(x_1, \ldots, x_n) = 0$. Then $N_{\Sigma \,|\, W} \cong \Theta_\Sigma(\Sigma)$, and $N'_\Sigma$ is the subsheaf defined by the ideal sheaf $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$. Thus if $g$ is a section of $N'_{\Sigma \,|\, W}$, then $f_t = f(x) + t g(x) = 0$ gives an infinitesimal deformation of $\Sigma$ which is "equisingular," i.e., modulo $(t^2)$, the locus of zero has not changed. In fact, if

$$g(x) = \sum_i \frac{\partial f}{\partial x_i} \cdot u_i(x) \quad ,$$

then, setting $u(x) = (u_1(x), \ldots, u_n(x))$, we have that $f_t(x) \equiv f(x + t u(x)) \pmod{t^2}$ by Taylor expansion.

**Definition 11.4.** The morphism $\varphi$ is said to be <u>stable</u> if the direct image sheaf $\varphi_*(N_\varphi)$ is isomorphic to the equisingular sheaf $N'_{\Sigma \,|\, W}$.

**Remark 11.5.** By looking at the stalks of $N'_{\Sigma \,|\, W}$ and of $\varphi_*(N_\varphi)$ at $p$, a smooth point of $\Sigma$, such that $\varphi$ has differential of maximal rank at the points in $\varphi^{-1}(p)$,

we see immediately that if $\varphi$ is stable, then $\varphi$ is birational onto its image (otherwise one should equip the locus $\Sigma$ with a scheme structure with nilpotent elements). We shall assume from now on $\varphi$ to be birational onto its image.

Proposition 11.6. Assume $\dim X = 1$, and that $p$ is a singular point of $\Sigma$: then if $\varphi$ is stable, then $p$ is an ordinary double point (node) and $\dim W = 2$.

Proof. In fact, the rank of $N'_{\Sigma|W}$ at $p$ ($\operatorname{rank}_p \mathcal{F} = \dim_{\mathbb{C}} \mathcal{F}/\mathfrak{m}_p \mathcal{F}$, $\mathfrak{m}_p$ being the maximal ideal of $p$) is $n = \dim W$, since $\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n$ generate $N'_{\Sigma|W}$ locally, while vector fields in $\Theta_\Sigma$ vanish at $p$ (cf. [Ro], thm. 3.2); whereas the rank of $\varphi_*(N_\varphi)$ at $p$ is just the sum

$$\sum_{\varphi(q) = p} \dim_{\mathbb{C}} \frac{(N_\varphi)_q}{\varphi^* \mathfrak{m}_p (N_\varphi)_q}$$

If $\varphi(q) = p$, then we can take local holomorphic coordinates $t$ at $q$, $(x_1, \ldots, x_n)$ at $p$, such that

$$\varphi(t) = \left( t^{a_1}, t^{a_1 + a_2} + \ldots, \ldots, t^{a_1 + a_2 + \ldots + a_r} + \ldots, 0, 0 \right)$$

where $a_i > 0$, and $r$ is the smallest dimension of a local smooth subvariety containing the branch of $\Sigma$ corresponding to $q$ (clearly, $a_1 = 1 \Longleftrightarrow r = 1$, and $r \geq a_1$). It is easy then to see that

$$\frac{\dim(N_\varphi)_q}{\varphi^* \mathfrak{m}_p (N_\varphi)_q} \geq n a_1 - 1 \quad .$$

Hence if this dimension has to be less than $n$, we get already $a_1 = 1$; moreover, since the sum over all these points $q$ has to be less than $n$, we infer that $n = 2$ and there are exactly two smooth branches. Either the two branches are transversal, and we have a node ($x_1 x_2 = 0$ in suitable coordinates on $W$), or we have a double point of type (we set $x = x_1$, $y = x_2$) $y^2 = x^{2k+2}$ ($k \geq 1$). In this case

$$\frac{\varphi_*(N_\varphi)}{\mathfrak{m}_p^2 \, \varphi_*(N_\varphi)} \quad ,$$

though, has dimension 4 whereas

$$\frac{N_{\Sigma, p}}{\mathfrak{m}_p^2 N_{\Sigma, p}} = \frac{(y, x^{2k+1})}{(y, x^{2k+1})(y^2, xy, x^2) + (y^2 - x^{2k+2})}$$

has dimension 5 (since $y, xy, y^2, x^{2k+1}, x^{2k+1} \cdot y$ are a $\mathbb{C}$-basis). Q. E. D.

Example 11.7. The morphism $t \mapsto (t^2, t^3)$, giving an ordinary cusp, is not stable since in fact the deformation $t \mapsto (a_0 + t^2, b_0 + b_1 t + t^3)$ gives a node: in fact $\partial x_1 / \partial t = 2t$, $\partial x_2 / \partial t = b_1 + t^2$, and $t = \pm \sqrt{-b_1}$ are two points of $X$ mapping to the same point.

Remark 11.8. We refer to [Math 1, 2] for a thorough and general discussion about stable map germs: here we shall limit ourselves in the sequel to discuss ordinary singularities when $\dim X = 2$. Before dealing with this special case, let's see what is true in general.

Theorem 11.9. There is a natural injective homomorphism of $N'_{\Sigma \mid W}$ to $\varphi_* N_\varphi$, if you assume $\varphi$ to be finite and birational onto $\Sigma$.

Proof. We have the two following exact sequences, with the homomorphism $\psi$ induced by pull-back $\varphi^* : \Theta_\Sigma \to \varphi_* \Theta_X$ .

$$(11.10) \qquad 0 \to \Theta_\Sigma \to \Theta_W \otimes \Theta_\Sigma \to N'_{\Sigma \mid W} \to 0$$

$$\downarrow \psi$$

$$0 \to \varphi_* \Theta_X \to \Theta_W \otimes \varphi_* \Theta_W \to \varphi_* N_\varphi \to R^1 \varphi_* \Theta_X = 0$$

and we have to verify that $\psi(\Theta_\Sigma) \subset \varphi_* \Theta_X$ . I.e., this is what we need to verify: if $\mathcal{J}_\Sigma$ is the ideal sheaf of $\Sigma$, and $(x_1, \ldots, x_n)$ are coordinates in $W$, whenever $a_1(x), \ldots, a_n(x)$ are functions such that $\forall f \in \mathcal{J}_\Sigma$,

$$\sum_{i=1}^{n} a_i(x) \frac{\partial f}{\partial x_i} \in \mathcal{J}_\Sigma \quad ,$$

then there do exist, for each point $y$ s.t. $\varphi(y) = x$, functions $\beta_1(y), \ldots, \beta_m(y)$ ($m = \dim X$, $y = (y_1, \ldots, y_m)$ are coordinates on $X$) such that

$$a_i(x) = \sum_{j=1}^{m} \beta_j(y) \left( \frac{\partial x_i}{\partial y_j} \right) \quad .$$

Since $X$ is smooth, by Hartog's theorem it will suffice to show the existence of such functions outside a subvariety $\Gamma$ of codimension at least 2 in $X$.

We first remark that, since we are assuming $\varphi$ to be birational onto $\Sigma$, the subvariety $Z \subset X$ where $\varphi$ is not of maximal rank has image $\varphi(Z) \subset \text{Sing}(\Sigma)$ (if $\varphi : \mathbb{C}^n_{,0} \to \mathbb{C}^n_{,0}$ has local degree 1, then it is a local biholomorphism). If

$x \in \Sigma$ - Sing($\Sigma$), there is nothing to prove, otherwise there exists a codimension 2 subvariety $\Delta$ of $\Sigma$ with

i) $\Delta \subset$ Sing($\Sigma$) ,

ii) if $x \in$ Sing($\Sigma$) - $\Delta$ , $\Sigma$ is locally biholomorphic to $\Sigma^1 \times \mathbb{C}^{m-1}$ where $\Sigma^1$ is a curve in $\mathbb{C}^{n-m+1}$ ,

iii) $\varphi^{-1}(\Delta) = \Gamma$ has codimension at least 2 in $X$ (this follows from the assumption that $\varphi$ be finite,

iv) if $x \in$ Sing($\Sigma$) - $\Delta$ , $y \in \varphi^{-1}(x)$, then there are coordinates $(y_1, \ldots, y_m)$ around $y$ such that, using the local biholomorphism $\Sigma \cong \Sigma^1 \times \mathbb{C}^{m-1}$ ,

$$\varphi(y) = (\varphi^1(y), y_2, \ldots, y_m).$$

By our previous remark and iv) above, it suffices to prove our result in the case when dim $X$ = dim $\Sigma$ = 1. In this case, we denote by $t$ a local coordinate at a point $y$ of $X$, according to tradition, and we may assume that, $\forall \; 1 \le i < j \le n$, projection on the $(i, j)$ coordinates maps $\Sigma$ birationally to a plane curve of equation $F_{ij}(x_i, x_j) = 0$. Since $F_{ij} \in \mathcal{I}_\Sigma$ , we have

$$a_i(x(t)) \frac{\partial F}{\partial x_i}(x_i(t), x_j(t)) + a_j(x(t)) \frac{\partial F}{\partial x_j}(x_i(t), x_j(t)) \equiv 0 \; .$$

Assume we show that there exists $\forall \; i, j$ , a meromorphic function $v(t)$ with

$$a_i(x(t)) = v(t) \frac{dx_i(t)}{dt} \; , \qquad a_j(x(t)) = v(t) \frac{dx_j(t)}{dt}$$

then all the $(2 \times 2)$ minors of the matrix

$$\begin{pmatrix} \dfrac{dx_1(t)}{dt} & \cdots & \dfrac{dx_n(t)}{dt} \\ a_1(x(t)) & \cdots & a_n(x(t)) \end{pmatrix}$$

vanish, and we can conclude that there is a $v(t)$ with $a_i(x(t)) = v(t) (dx_i(t)/dt)$ for each $i$, provided the following holds true.

Lemma 1.12. Let $\Sigma$ be a germ of plane curve singularity, with equation $f(x, y) = 0$ and let $x = \varphi_1(t)$, $y = \varphi_2(t)$ be a parametrization of a branch of $\Sigma$. Then, if $a_1(t)$, $a_2(t)$ are functions such that

$$a_1(t) \frac{\partial f}{\partial x}(\varphi_1(t), \varphi_2(t)) + a_2^{(t)} \frac{\partial f}{\partial y}(\varphi_1(t), \varphi_2(t)) \equiv 0 \; ,$$

there does exist a meromorphic function $v(t)$ with

$$a_i(t) = v(t) \, \frac{d\varphi_i(t)}{dt} \qquad (i = 1, 2) \ .$$

<u>Proof.</u> Write $f = f_1 f_2$ where $f_1 = 0$ is the local equation of the given branch. Clearly

$$\frac{\partial f}{\partial x}(\varphi_1(t), \varphi_2(t)) = \frac{\partial f_1}{\partial x}(\varphi_1(t), \varphi_2(t)) \cdot f_2(\varphi_1(t), \varphi_2(t))$$

and analogously for $\partial f / \partial y$: since $f_2(\varphi_1(t), \varphi_2(t) \neq 0$, we can indeed assume $\Sigma$ to have only one branch. Without loss of generality we may assume

$$\varphi_1(t) = t^m \ , \quad \varphi_2(t) = g(t) = t^{m+c} + \cdots$$

($m$ is the multiplicity, $c$ the local class). As classical, we use a base change $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ sending $(t, y)$ to $(x = t^m, y)$: then $\varphi^{-1}(\Sigma)$ consists of $m$ smooth branches, of equation $y - g(t \epsilon^i) = 0$, with $\epsilon = \exp(2\pi \sqrt{-1}/m)$, $i = 1, \ldots, m$ (the first of these branches coincides with the given parametrization). Clearly the pull-backs $\varphi^*(\partial f / \partial y)$ and $\varphi^*(\partial f / \partial x)$ coincide, respectively, with

$$\frac{\partial f(t, y)}{\partial y} \quad \text{and} \quad \frac{1}{m \, t^{m-1}} \, \frac{\partial f(t, y)}{\partial t}$$

Since

$$f(t, y) = \prod_{i=1}^{m} (y - g(t \epsilon^i)) \ ,$$

by assumption

$$a_1(t) \, \frac{\partial f(t, y)}{\partial t} + a_2(t) \, m \, t^{m-1} \, \frac{\partial f(t, y)}{\partial y}$$

vanishes identically after plugging in $y = g(t)$. We get

$$-a_1(t) \sum_{i=1}^{m} \prod_{j \neq i} (y - g(t \epsilon^j)) \, \epsilon^i \frac{\partial g}{\partial t} (t \epsilon^i) + a_2(t) \, m \, t^{m-1} \sum_{i=1}^{m} \prod_{j \neq i} (y - g(t \epsilon^j))$$

and plugging in $y = g(t)$, we obtain

$$-a_1(t) \, \frac{\partial g(t)}{\partial t} + a_2(t) \, mt^{m-1} \equiv 0 \ .$$

<div align="right">Q. E. D. for the lemma.</div>

Now

$$a_i(x(t)) = v(t) \, \frac{dx_i(t)}{dt} \ ;$$

let $m$ be the multiplicity of the branch (i.e. $m = \min \operatorname{ord}_t x_i(t)$) and assume $x_1(t) = t^m$: then $\operatorname{ord}_t a_i(x(t)) \geq m$, hence $\operatorname{ord}_t v(t) \geq 0$ and $g$ is holomorphic. It remains to be proved that the given homomorphism of $N_{\Sigma \mid W}$ into $\varphi_* N_\varphi$ is

injective. In view of (11.10) we have to show that if a section $\xi$ of $\Theta_W \otimes \mathfrak{G}_\Sigma$ lies in the image of $\varphi_* \Theta_X$, then its image in $N_{\Sigma|W}$ equals zero.

By 11.2 it suffices to show that its image in $N_{\Sigma|W} = \mathrm{Hom}(N_{\Sigma|W}^\vee, \mathfrak{G}_\Sigma)$ is zero. Let $\nu$ be a section of $N_{\Sigma|W}^\vee$ : since $\xi$ is tangent to $\Sigma$ at the smooth points of $\Sigma$, $\langle \xi, \nu \rangle$ vanishes on an open dense set, thus $\langle \xi, \nu \rangle \equiv 0$, $\Sigma$ being reduced. $\qquad$ Q. E. D.

We shall not pursue here the analogue of Kuranishi's theory for these deformation theories (cf. [Sch], [Wah], [B-W], [Pa]), in fact, as we have shown already and will see in the sequel, it is very hard to compute the obstructions in almost all the examples, whereas geometry can help to find a complete family of deformations.

## §12. Surfaces with ordinary singularities

Here $X$ is a smooth surface and will hence be denoted by $S$, $\varphi : S \to \Sigma \subset W$, where $W$ is a smooth 3-fold is a finite map, birational onto its image $\Sigma$, which possesses only the following type of singularities:

    i) nodal curve ($xy = 0$ in local holomorphic coordinates)

    ii) triple points ($xyz = 0$ in local coordinates)

    iii) pinch points ($x^2 - zy^2 = 0$ in local coordinates).

$\Delta$ will be the double curve ($= \mathrm{Sing}(\Sigma)$) of $\Sigma$, smooth at points of type i), iii), with local equations $x = y = 0$, and with a triple transversal point at each triple point. We let $D = \varphi^{-1}(\Delta) \subset S$, and notice that a pinch point $p'$ has just one inverse image point $p$, where we can choose local coordinates $(u,v)$ such that

$$(12.1) \qquad\qquad \varphi(u,v) = (uv, v, u^2)$$

hence in particular $D = \{(u,v) \mid v = 0\}$.

**Proposition 12.2.** If $\Sigma$ has ordinary singularities, the morphism $\varphi$ is stable (i.e., ch. 11.4, $\varphi_*(N_\varphi) \cong N'_{\Sigma|W}$).

**Proof.** In view of 11.6 and 11.9, it suffices to consider the case of triple and pinch points, and to prove that $\eta'_{\Sigma|W}$ goes onto $\varphi_* N_\varphi$. To do this, we shall explicitly compute these two sheaves.

**Lemma 12.3.** $N'_{\Sigma|W} \subset N_{\Sigma|W} \cong \mathfrak{G}_\Sigma(\Sigma)$ is the subsheaf of sections $g$ vanishing on $\Delta$ and satisfying the further linear condition: $\partial g / \partial y = 0$ at the pinch points.

Proof. $g \in N'_{\Sigma \mid W}$ iff it belongs locally to the Jacobian ideal of $\Sigma$, i.e. $(x, y)$ for the nodal points, $(xy, yz, xz)$ for the triple points, $(x, y^2, yz)$ for the pinch points. At the pinch points, $g \in (x, y^2, yz) \iff g = x g_1 + y g_2$ with $g_2(0, 0, 0) = 0 \iff \partial g / \partial y (0, 0, 0) = 0$.

Q.E.D.

Remark 12.4. Since $g$ vanishes on $\Delta$, clearly $\partial g / \partial z = 0$ at a pinch point $p'$. Hence the condition $\partial g / \partial y = 0$ can be formulated also as: $\xi(g) = 0$ for each tangent vector at $p'$ lying in the tangent cone to $\Sigma$, $\{x^2 = 0\}$. Clearly this last formulation is independent of the choice of coordinates.

Lemma 12.5. Let $p_1, \ldots, p_k$ be the points of $S$ mapping to the pinch points $p'_1, \ldots, p'_k$ of $\Sigma$: then $N_\varphi \cong \Pi^k_{i=1} \, \mathcal{m}_{p_i} \, \mathcal{O}_S (\varphi^* \Sigma - D)$.

Proof. By 11.9 and 12.3 we know that $N_\varphi$ coincides with $\varphi^* (N'_{\Sigma \mid W})$ except at a finite number of points, and that $\varphi^* (N'_{\Sigma \mid W})$ equals to $\mathcal{J}' \mathcal{O}_S (\varphi^* \Sigma - D)$, where $\mathcal{J}'$ is an ideal sheaf of a 0-dimensional scheme. Hence $N_\varphi$ is also of the form $N_\varphi = \mathcal{J} \mathcal{O}_S (\varphi^* \Sigma - D)$, with $\dim(\text{supp}(\mathcal{O}_{S/\mathcal{J}})) = 0$. To determine the ideal $\mathcal{J}$, we first notice that $\text{supp}(\mathcal{O}_{S/\mathcal{J}}) = \{p_1, \ldots, p_k\}$, then that, at $p_i$, $N_\varphi \cong \text{coker} \, \mathcal{O}^2 \xrightarrow{\Phi} \mathcal{O}^3$ where $\Phi =$ differential of $\varphi$, sends a pair $(g_1, g_2)$ to a triple $f_1 = v g_1 + u g_2$, $f_2 = g_2$, $f_3 = 2u g_1$. The homomorphism of $\mathcal{O}^3 \to \mathcal{m}_{p_i}$ sending $(f_1, f_2, f_3)$ to $(2u f_1 - v f_3)$ clearly gives an isomorphism of $N_\varphi$ with $\mathcal{m}_{p_i}$.

Q.E.D.

We can now finish the proof that $\varphi_* (N_\varphi) = N'_{\Sigma \mid W}$: in fact $\varphi_* \mathcal{O}_S (-D) = \mathcal{O}_\Sigma (-\Delta)$, as it is easy to see, whereas at the pinch points $\mathcal{m}_{p_i} \mathcal{O}_S (-D) = (uv, v^2)$, whereas $g \in N'_{\Sigma \mid W}$ iff $g = x g_1 + y g_2$ with $g_2 \in \mathcal{m}_{p'_i}$; as we have already seen, $\varphi^* (g) = uv \, \varphi^* (g_1) + v \, \varphi^* g_2 \in \mathcal{m}_{p_i} \mathcal{O}_S (-D)$, and we can conclude since both sheaves $\varphi_* (N_\varphi) \supset N_{\Sigma \mid W}$ have codimension 1 in $\mathcal{O}_\Sigma (\Sigma - \Delta)$.

Q.E.D. for Proposition 12.2.

Corollary 12.6. If $\Sigma$ has ordinary singularities only in the smooth 3-fold $W$, then there exists an exact sequence

$$0 \to H^0 (\mathcal{O}_S) \to H^0 (\varphi^* \mathcal{O}_W) \to H^0 (N'_{\Sigma \mid W}) \to H^1 (\mathcal{O}_S) \to H^1 (\varphi^* \mathcal{O}_W)$$

$$\to H^1 (N_{\Sigma \mid W}) \to H^2 (\mathcal{O}_S) \to H^2 (\varphi^* \mathcal{O}_W) \to H^2 (N'_{\Sigma \mid W}) \to 0$$

Proof. Obvious from the Leray spectral sequence for the finite map $\varphi$.

Q.E.D.

<u>Definition 12.7.</u>  $\mathfrak{O}_W(\Sigma)(-\Delta-c')$, where  $c'$  stands for "cuspidal conditions," is defined to be the inverse image of  $N'_{\Sigma}|_W$  under the surjective homomorphism  $\mathfrak{O}_W(\Sigma) \to \mathfrak{O}_{\Sigma}(\Sigma) = N_{\Sigma}$.

The heuristic explanation for  $\mathfrak{O}_W(\Sigma)(-\Delta-c')$  (cf. [Ko 2]) is as follows: assume that you deform the singular locus of  $\Sigma$  by deforming with a parameter  $t$  the local coordinates  x, y, z; then if  $X(t) = x + t\xi + \cdots$ ,  $Y(t) = y + t\eta + \ldots$ ,  $Z(t) = z + t\zeta + \ldots$ , the local equation of  $\Sigma$  changes as follows:

$$\begin{cases} XY = xy + t(\xi y + \eta x) + \cdots \\ XYZ = xyz + t(\xi yz + \eta xz + \zeta xy) + \cdots \\ X^2 - Y^2 Z = x^2 - y^2 z + t(2\xi x - 2zy\eta - \zeta y^2) + \cdots \end{cases}$$

Hence, if  $f = 0$  was the old equation, the new one is of the form  $f + tg + \ldots$ , where  g  is a section of  $\mathfrak{O}_W(\Sigma)$  vanishing on  $\Delta$  and satisfying the cuspidal conditions.

We clearly have an exact sequence ($f$  is a section with  $\mathrm{div}(f) = \Sigma$ ) .

(12.8) $$0 \to \mathfrak{O}_W \xrightarrow{\ f\ } \mathfrak{O}_W(\Sigma)(-\Delta-c') \to N'_{\Sigma}|_W \to 0 ,$$

and Kodaira, after Severi, gives the following (cf. [Ko 2]).

<u>Definition 12.9.</u>  $\Sigma$  is said to be <u>regular</u> if  $H^1(\mathfrak{O}_W(\Sigma)(-\Delta-c')) = 0$, and <u>semi-regular</u> if  $H^1(\mathfrak{O}_W(\Sigma)(-\Delta-c')) \to H^1(N_{\Sigma}|_W)$  is the zero map.

<u>Remark 12.10.</u>  The two definitions coincide if  $H^1(\mathfrak{O}_W) = 0$, e.g. for  $W = \mathbb{P}^3$ .

We have the following.

<u>Theorem 12.11</u> (Kodaira, [Ko 2]).  If  $\Sigma$  is semi-regular the characteristic system of the map  $\varphi : S \to \Sigma$  is complete; moreover, there is a smooth semi-universal family  $\{\varphi_t\}$  of deformations of  $\varphi : S \to \Sigma$  such that the characteristic system is complete also for  $t \neq 0$.

Unfortunately, the condition of semi-regularity is a very strong assumption upon  $\Sigma \subset W$: we shall, following Kodaira ([Ko 2] , [Ko 3]), consider from now on only the classical case where  $W = \mathbb{P}^3$ , and regularity coincides with semi-regularity.

<u>Theorem 12.12.</u> If $\Sigma$ is a surface in $\mathbb{P}^3$ of degree $n$ with ordinary singularities, $\Sigma$ is (semi)-regular if and only if the cuspidal conditions are independent on the space of polynomials of degree $n$ vanishing on the double curve $\Delta$ of $\Sigma$ (i.e.,

$$0 \to H^o(\mathcal{O}_{\mathbb{P}^3}(nH)(-\Delta - c')) \to H^o(\mathcal{O}_{\mathbb{P}^3}(nH)(-\Delta)) \xrightarrow{\partial g/\partial y_i} \underset{i}{\oplus} \mathbb{C}_{p_i'} \to 0$$

is exact, where $H$ is the hyperplane divisor on $\mathbb{P}^3$).

<u>Proof.</u> By assumption

$$H^1(\mathcal{O}_{\mathbb{P}^3}(nH)(-\Delta - c')) = H^1(\mathcal{O}_{\mathbb{P}^3}(nH)(-\Delta)) \quad .$$

By the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(nH)(-\Delta) \to \mathcal{O}_{\Sigma}(nH)-\Delta) \to 0$$

we have thus to show that $H^1(\mathcal{O}_{\Sigma}(nH-\Delta)) = 0$. Denoting still by $H$ the pull-back of a hyperplane, we have $H^1(\mathcal{O}_{\Sigma}(nH - \Delta)) = H^1(\mathcal{O}_S(nH - D))$. Since, by adjunction, the canonical divisor on $S$ is $(n-4)H - D$, by Serre duality, our space is dual to $H^1(\mathcal{O}_S(-4H))$, which is zero since $H$ is ample (e.g. by Kodaira's vanishing theorem).

<div align="right">Q. E. D.</div>

The preceding criterion of regularity is not so easy to apply directly, thus the usual method is to relate the equisingular deformations of $\Sigma$ to the (equisingular) deformations of $\Delta$ (observing that sections of $\mathcal{O}_{\mathbb{P}^3}(nH)$ vanishing on $\Delta$ of order 2 give trivial infinitesimal deformations of $\Delta$).

We have the usual exact sequences (cf. 9.17)

(12.13.i) $$0 \to \Theta_{\mathbb{P}^3}(-\log \Delta) \to \Theta_{\mathbb{P}^3} \to N'_{\Delta|\mathbb{P}^3} \to 0$$

(12.13.ii) $$0 \to \Theta_{\Delta} \to \Theta_{\mathbb{P}^3} \otimes \mathcal{O}_{\Delta} \to N'_{\Delta|\mathbb{P}^3} \to 0$$

and moreover ([Ko 3], thm. 4).

<u>Theorem 12.14.</u> There exists an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(nH)(-2\Delta) \to \mathcal{O}_{\mathbb{P}^3}(nH)(-\Delta - c') \to N'_{\Delta|\mathbb{P}^3} \to 0 \quad .$$

<u>Idea of Proof</u> (see loc. cit. for details). Let $\tilde{\Delta}$ be $\Delta-\{$triple points and pinch points of $\Sigma\}$, and let $\tilde{N}$ be the normal bundle of $\tilde{\Delta}$ in $\mathbb{P}^3$. Since the conormal sheaf $N^{\vee}$ of $\Delta$ is just $\mathcal{O}_{\mathbb{P}^3}(-\Delta)/\bar{\mathcal{O}}_{\mathbb{P}^3}(-2\Delta)$, the basic claim is that there exists an isomorphism of $\tilde{N}$ into $\tilde{N}^{\vee}|_{\tilde{\Delta}} \otimes \mathcal{O}_{\mathbb{P}^3}(nH)$: and after that one has to check that this isomorphism extends at the cuspidal points onto the subsheaf defined by the cuspidal

conditions, and at the triple points there is a similar verification. Since $\tilde{N}$ and $N^{\vee}\big|_{\tilde{\Delta}}$ are dual bundles, the key point is that the equation $f$ of $\Sigma$, locally of the form $xy = 0$, induces locally two sections of $N^{\vee}\big|_{\tilde{\Delta}}$, and globally a non vanishing section of $\wedge^2 N^{\vee}\big|_{\tilde{\Delta}} \otimes \mathcal{O}_{\mathbb{P}^3}(nH)$, thereby inducing a non degenerate pairing $\tilde{N} \times \tilde{N} \to \mathcal{O}_{\tilde{\Delta}}(nH)$, hence the desired isomorphism. Q. E. D.

The important feature of 12. 14 is that the left term of the exact sequence depends only upon the double curve $\Delta$ and the degree $n$ of $\Sigma$, but not upon $\Sigma$. Moreover, given any curve $\Delta$ in $\mathbb{P}^3$, by Serre's theorem ([Se ]), there is an integer

$$(12.15) \qquad n_o(\Delta) = \min \{ n \mid H^i(\mathcal{O}_{\mathbb{P}^3}(kH)(-2\Delta) = 0 \quad \forall \ i = 1, 2, \ k \geq n \} \ .$$

Theorem 12. 16. Let $\Sigma$ be a surface of degree $n$ with ordinary singularities in $\mathbb{P}^3$ having $\Delta$ as double curve; if $n \geq n_o(\Delta)$, $\Sigma$ is regular if and only if $H^1(N'_{\Delta | \mathbb{P}^3}) = 0$. In particular $\Sigma$ is regular if $H^1(\mathcal{O}_{\mathbb{P}^3} \otimes \mathcal{O}_{\Delta}) = 0$.

Proof. By the cohomology sequence attached to 12.14, and by 12.15,

$$H^1(\mathcal{O}_{\mathbb{P}^3}(nH)(-\Delta - c')) \cong H^1(N'_{\Delta | \mathbb{P}^3}) \ .$$

The other assertion follows from (12.13.ii). Q. E. D.

Theorem 12.17. Let $\Sigma$ be a surface of degree $n$ with ordinary singularities in $\mathbb{P}^3$ having $\Delta$ as double curve, and let $\hat{\Delta}$ be the normalization of $\Delta$.

  i)  If $T$ is the divisor on $\hat{\Delta}$ given by the sum of the triple points, and $H$ is the hyperplane divisor, then if $n \geq n_o(\Delta)$ and $\mathcal{O}_{\hat{\Delta}}(H-T)$ is non-special on (every component of) $\hat{\Delta}$, then $\Sigma$ is regular.

  ii) If there exists a surface $\Sigma'$ of degree $n'$ containing $\Delta$, and such that the divisor $\varphi^*(\Sigma')$ on $S$ has no multiple components, then $n_o(\Delta) \leq n + n' - 3$.

Proof. i) by 12.16 it suffices to show $H^1(\mathcal{O}_{\mathbb{P}^3} \otimes \mathcal{O}_{\Delta}) = 0$. By the Euler sequence (6.3) tensored with $\mathcal{O}_{\Delta}$, it suffices to show that $H^1(\mathcal{O}_{\Delta}(1)) = 0$. Now, if $\psi : \hat{\Delta} \to \Delta$ is the normalization map, $\psi_*(\mathcal{O}_{\hat{\Delta}}(H-T)) = \mathcal{M}_T \mathcal{O}_{\Delta}(1)$, where $\mathcal{M}_T$ is the ideal sheaf of the triple points. Hence $H^1(\mathcal{M}_T \mathcal{O}_{\Delta}(1)) = 0$ and we are done by the exact sequence

$$0 \to \mathcal{M}_T \mathcal{O}_{\Delta}(1) \to \mathcal{O}_{\Delta}(1) \to \tau \to 0$$

where $\tau$ is a skyscraper sheaf with stalk $\cong \mathbb{C}$ at each triple point.

ii) let $k$ be an integer $\geq n + n' - 3$ and consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(kH - \Sigma) \to \mathcal{O}_{\mathbb{P}^3}(kH)(-2\Delta) \to \mathcal{O}_{\Sigma}(kH - 2\Delta) \to 0$$

Since $H^i(\mathcal{O}_{\mathbb{P}^3}(kH - \Sigma)) = H^i(\mathcal{O}_{\mathbb{P}^3}((k-n)H)) = 0$ for $i = 1, 2$, it suffices to show the vanishing of $H^1(\mathcal{O}_{\Sigma}(kH - 2\Delta))$. Since $\mathcal{O}_{\Sigma}(-2\Delta) = \varphi_* \mathcal{O}_S(-2D)$, we want the vanishing of $H^1(\mathcal{O}_S(kH - 2D))$. As in 12.12, since $K_S = (n-4)H - D$, the dual vector space is, by Serre duality, $H^1(\mathcal{O}_S(-(k-n+4)H + D))$. By assumption, $n'H \equiv \varphi^*(\Sigma') = D + \Gamma$, hence we want the vanishing of $H^1(\mathcal{O}_S(-aH - \Gamma))$, where $a = k - n + 4 - n' \geq 1$. But $|aH + \Gamma|$ maps to a surface and $|aH + \Gamma|$ contains a reduced connected divisor, hence one can apply the Ramanujam vanishing theorem (cf. e.g. [Bo], [Ram]).

Q. E. D.

By 12.14, if $n \geq n_o(\Delta)$, then $H^o(\mathcal{O}_{\mathbb{P}^3}(nH)(-\Delta - c'))$ goes onto $H^o(N'_{\Delta|\mathbb{P}^3})$: on the other hand, by (12.8) this surjective homomorphism factors through the one onto $H^o(N'_{\Sigma|\mathbb{P}^3})$, which has the subspace $\mathbb{C}f$ as its kernel ($f = 0$ being the equation of $\Sigma$). Assume now $\Delta$ to be smooth (thus $\Sigma$ has no triple points): then if the characteristic system of $\Sigma$ is complete, and $n \geq n_o(\Delta)$, then also the characteristic system of $\Delta$ is complete; moreover, Kodaira (loc. cit., p. 246) proves the converse.

Theorem 12.18. Let $\Sigma$ be a surface of degree $n$ in $\mathbb{P}^3$ with ordinary singularities and smooth double curve $\Delta$. Assume $n \geq n_o(\Delta)$: then the characteristic system of $\Delta$ is complete if and only if the characteristic system of $\Sigma$ is complete.

This theorem, combined with Mumford's example 9.11 of a family of space curves $\Delta$ for which the characteristic system is never complete for each $\Delta$, shows the existence of many surfaces $\Sigma$ such that all their equisingular deformations do not have a complete characteristic system: in fact, given an $n$ such that $\mathcal{O}_{\mathbb{P}^3}(n)(-2\Delta)$ is generated by global sections, it follows by Bertini's theorem that the general section $f \in H^o(\mathcal{O}_{\mathbb{P}^3}(n)(-2\Delta))$ defines a surface $\Sigma$ smooth away from $\Delta$, and with ordinary singularities only.

This result, obtained 20 years ago, culminated a very long history of attempts to show that the characteristic system of a surface $\Sigma$ with ordinary singularities should always be complete (we defer the reader to [En], [Za], especially Mumford's appendix to chapter V for a more thorough discussion).

We simply want to remark again that the fact that the characteristic system is not com plete does not imply the singularity of the base B of the Kuranishi family: in fact, for $t \in B$ one can have a deformation $\varphi_t$ of the holomorphic map $\varphi : S \to \mathbb{P}^3$ if and only if the cohomology class of H $(= \varphi^*$ (hyperplane)) remains of type $(1,1)$ on $S_t$.

Example 12.19. A classical case where Kodaira's theorem 12.17 applies is the case of Enriques' surfaces $\Sigma$, with equation

$$f(x_0, x_1, x_2, x_3) = x_0^2 x_1^2 x_2^2 x_3^2 \left( \sum_{i=0}^{3} \frac{1}{x_i^2} \right) + x_0 x_1 x_2 x_3\, q(x) \ ,$$

where $q(x)$ is a general quadratic form. Here $\Delta$ consists of the six edges of the coordinate tetrahedron $\{x_0 x_1 x_2 x_3 = 0\}$ , $n = 6$. The normalization $\hat{\Delta}$ of $\Delta$ consists of 6 copies of $\mathbb{P}^1$ , and $\mathbb{O}_{\hat{\Delta}}(H-T)$ has degree $(-1)$ on each component, hence is non special $(H^1(\mathbb{O}_{\mathbb{P}^1}(-1)) = 0\,!)$. The surface $\Sigma'$ to be taken is a general cubic surface of equation

$$x_0 x_1 x_2 x_3 \left( \sum_{i=0}^{3} a_i \frac{1}{x_i} \right) = 0 \ ,$$

hence $n_o(\Delta) \geq 6$, and $\Sigma$ is regular.

The characteristic system has dimension 25 and it is easy to see that a smooth complete family of deformations of $\Sigma$ is obtained by taking images under projectivities of surfaces in the above 10-dimensional family. Working out the exact sequence 12.6, we see that the above 10-dimensional family has bijective Kodaira-Spencer map, so that the Kuranishi family of S is smooth, 10-dimensional. This last result can also be gotten in a simpler way: since $K_S = 2H - D$, and $\varphi^*(x_0 x_1 x_2 x_3) = 2D$, we get $2K_S \equiv 0$. On the other hand, if $K_S \equiv 0$, there would be a quadric containing $\Delta$, what is easily seen not to occur. Hence $K_S \not\equiv 0$, $2K_S \equiv 0$. Moreover $\chi(\mathbb{O}_S) = 1$. Taking the square root $w$ of $x_0 x_1 x_2 x_3$ and then normalizing the surface $\Sigma' = \{(w, x_0, x_1, x_2, x_3)\,|\,w^2 = x_0 x_1 x_2 x_3,\ f(x) = 0\}$, we get a smooth surface $S'$ (called a K3 surface) possessing an unramified double cover $\Pi : S' \to S$. It is easy to see that $K_{S'} \equiv 0$, and, since $\chi(\mathbb{O}_{S'}) = 2$, $H^1(\mathbb{O}_{S'}) = H^0(\Omega^1_{S'}) = 0$. Now $H^2(\mathbb{O}_S)$ is the Serre dual of $H^0(\Omega^1_S \otimes \Omega^2_{S'}) = H^0(\Omega^1_{S'}) = 0$, hence there are no obstructions for the Kuranishi family of S (of $S'$, too).

Example 12.20 ([Ko 3], [Hor 4], [Us  ]). Let $\Delta \subset \mathbb{P}^3$ be a smooth curve, the complete intersection of two surfaces, $\Delta = \{F = G = 0\}$. Then one can consider the smooth family of surfaces of degree $n$ having $\Delta$ as double curve. By our assumption, it is easy to check that, if $\deg F = a$, $\deg G = b$, the equation of $\Sigma$ can be written in the form

$$(12.21) \qquad\qquad AF^2 + 2BFG + CG^2 \, ,$$

where $\deg A = n - 2a$, $\deg B = n - a - b$, $\deg C = n - 2b$.

The results of Kodaira-Horikawa and Usui can be summarized as follows: varying $A$, $B$, $C$ one gets a surjective characteristic map, so that

(12.22 i)   the characteristic system is complete.

Using the standard Euler sequence, it is possible to prove that, in the exact sequence

$$H^0(N'_{\Sigma \,|\, \mathbb{P}^3}) \;\to\; H^1(\Theta_S) \;\to\; H^1(\varphi^* \Theta_{\mathbb{P}^3}) \;\overset{\sigma}{\to}\; H^1(N'_{\Sigma \,|\, \mathbb{P}^3})$$

the homomorphism $\sigma$ is injective, hence in particular

(12.22 ii)   The Kuranishi family of $S$ is smooth.

Furthermore,

(12.22 iii)   the above surfaces are not (semi)-regular,

(12.22 iv)   the pairing $H^1(\Theta) \times H^0(\Omega^2_S) \to H^1(\Omega^1_S)$ is non degenerate in the first factor (this result is called Infinitesimal Torelli property, and actually Usui proves the above result, provided $n \geq n_0(\Delta)$, also in the more general case of theorem 12.17).

This example shows clearly how the condition of semi-regularity is much too restrictive (in fact, as we noticed, it is an analogue of the condition $H^2(\Theta_X) = 0$ in order to ensure smoothness of the Kuranishi family).

§13.  Generic multiple planes and equisingular deformations of plane curves
       with nodes and cusps

We consider again a smooth surface $S$ and a finite morphism $\varphi: S \to \mathbb{P}^2$, of degree $d$; we let, as usual, $H$ be the pull-back of a line, and we denote by $R$ the ramification divisor of $\varphi$, i.e., given the exact sequence

(13.1) $$0 \to \Theta_S \xrightarrow{\varphi_*} \varphi^* \Theta_{\mathbb{P}^2} \to N_\varphi \to 0$$

the divisor of zeros of $(\Lambda^2 \varphi_*) \in H^0((\Lambda^2 \varphi^* \Theta_{\mathbb{P}^2}) \otimes (\Lambda^2 \Theta_S)^\vee)$.

The following is a classical

Definition 13.2. $\varphi$ is said to be a generic multiple plane (or a stable morphism) if R is smooth, $\varphi(R) = B$ (the branch locus) has only nodes and ordinary cusps as singularities, $\varphi\big|_R : R \to B$ is generically 1-1.

At the points of R, the normal form of $\varphi$ (for suitable local holomorphic coordinates in the source and in the target) is as follows:

(13.3 i)   $\varphi(x,y) = (x^2, y)$ at the points $p \in R$ with $\varphi(p)$ not a cusp of B

(13.3 ii)   $\varphi(x,y) = (y, yx - x^3)$ at the points $p_i$ of R with $\varphi(p_i) = p_i'$ a
      cuspidal point of B.

(13.3 i) and ii) imply that $N_\varphi$ is an invertible $\Theta_R$-sheaf. It is easier though, to compute $\varphi_*(N_\varphi)$: we apply $\varphi_*$ to 13.1 to obtain

$$0 \to \varphi_* \Theta_S \xrightarrow{\hat\varphi} \Theta_{\mathbb{P}^2} \otimes \varphi_* \Theta_S \to \varphi_* N_\varphi \to 0 \quad .$$

Proposition 13.4. The trace map $t : \Theta_{\mathbb{P}^2} \otimes \varphi_* \Theta_S \to \Theta_{\mathbb{P}^2}$ induces an isomorphism of $\varphi_* N_\varphi$ with $\Theta_{\mathbb{P}^2}/\Theta_{\mathbb{P}^2}(-\log B)$; in particular, $\varphi_* N_\varphi = N'_{B|\mathbb{P}^2}$.

Proof. The statement is almost obvious for the points of $\mathbb{P}^2$ which are not cuspidal points of B. At a cuspidal point $p_i'$, if $p_i \in R$ is such that $\varphi(p_i) = p_i'$, we see immediately that

$$(\varphi_* N_\varphi)_{p_i'} \cong \Theta_{\mathbb{P}^2} \otimes \varphi_* \Theta_{S, p_i} / \hat\varphi(\varphi_* \Theta_{S, p_i}) \quad .$$

These two are $\Theta_{\mathbb{P}^2, p_i'}$ modules and, if we choose coordinates as in 13.3 ii), $(y, z)$ at $p_i$ such that $z = yx - x^3$, a basis for $\varphi_* \Theta_{S, p_i}$ is given by

$$\frac{\partial}{\partial x}, \ x\frac{\partial}{\partial x}, \ x^2\frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ x\frac{\partial}{\partial y}, \ x^2\frac{\partial}{\partial y} \ ,$$

while a basis for $\Theta_{\mathbb{P}^2} \otimes \varphi_* \Theta_{S, p_i}$ is given by

$$\frac{\partial}{\partial y}, \ x\frac{\partial}{\partial y}, \ x^2\frac{\partial}{\partial y}, \ \frac{\partial}{\partial z}, \ x\frac{\partial}{\partial z}, \ x^2\frac{\partial}{\partial z} \ .$$

Since the Jacobian matrix $\varphi_*$ is $\begin{pmatrix} 0 & 1 \\ (y-3x^2) & x \end{pmatrix}$ , we see that the image of $\hat{\varphi}$ is the submodule generated by

(13.5)
$$
\begin{cases}
\dfrac{\partial}{\partial y} + x\dfrac{\partial}{\partial z}, \; x\dfrac{\partial}{\partial y} + x^2\dfrac{\partial}{\partial z}, \; x^2\dfrac{\partial}{\partial y} + (yx-z)\dfrac{\partial}{\partial z}, \\[2mm]
(y-3x^2)\dfrac{\partial}{\partial z}, \; (3z-2xy)\dfrac{\partial}{\partial z}, \; (3xz-2yx^2)\dfrac{\partial}{\partial z}.
\end{cases}
$$

We use now the symbol $\equiv$ to denote congruence modulo the submodule image $(\hat{\varphi})$, and we deduce from (13.5) that

$$
x\frac{\partial}{\partial z} \equiv -\frac{\partial}{\partial y}, \; x^2\frac{\partial}{\partial z} \equiv \frac{1}{3}y\frac{\partial}{\partial z}, \; x\frac{\partial}{\partial y} \equiv x^2\frac{\partial}{\partial z} \equiv -\frac{1}{3}y\frac{\partial}{\partial z},
$$

$$
x^2\frac{\partial}{\partial y} \equiv z\frac{\partial}{\partial z} - yx\frac{\partial}{\partial z} \equiv z\frac{\partial}{\partial z} + y\frac{\partial}{\partial y}, \; 3z\frac{\partial}{\partial z} \equiv 2yx\frac{\partial}{\partial z} \equiv -2y\frac{\partial}{\partial y},
$$

$$
0 \equiv 3zx\frac{\partial}{\partial z} - 2yx^2\frac{\partial}{\partial z} \equiv -3z\frac{\partial}{\partial y} - \frac{2}{3}y^2\frac{\partial}{\partial z}.
$$

We readily infer that $\varphi_*(N_\varphi)$ is isomorphic to the quotient of $\Theta_{\mathbb{P}^2}$ by the submodule generated by $2y(\partial/\partial y) + 3z(\partial/\partial z)$ and by $9z(\partial/\partial y) + 2y^2(\partial/\partial z)$: it is immediate to check that this last submodule is indeed $\Theta_{\mathbb{P}^2}(-\log B)$, since the equation of $B = \varphi(R)$ is given by $4y^3 - 27z^2 = 0$.                                Q. E. D.

The previous proposition shows that infinitesimal deformations of the stable map $\varphi$ correspond to infinitesimal equisingular deformations of B. On the other hand, if $\{\varphi_t : S_t \to \mathbb{P}^2\}_{t \in T}$ is a deformation of $\varphi$, it is easy to verify that the condition that $\varphi_t$ be stable is an open one, and $\{B_t\} = \{\varphi_t(R_t)\}$ is an equisingular family of plane curves with nodes and cusps. Conversely, if $\{B_t\}_{t \in T}$ is an equisingular family of curves with nodes and cusps which is a deformation of $B = B_{t_o}$, we see that for t near to $t_o$ the pairs $(\mathbb{P}^2, B)$ and $(\mathbb{P}^2, B_t)$ are diffeomorphic, and in particular $\Pi_1(\mathbb{P}^2 - B_t) \cong \Pi_1(\mathbb{P}^2 - B)$. Thus, the associated subgroup of the covering $\varphi : S - \varphi^{-1}(B) \to \mathbb{P}^2 - B$ determines another smooth surface $S_t$ with a stable morphism $\varphi_t : S_t \to \mathbb{P}^2$ and it is not difficult to verify that in this way we get a deformation of $\varphi$ with base T. We have thus

Theorem 13.6.   There is a natural isomorphism between the characteristic system $H^0(N_\varphi)$ of a generic multiple plane $\varphi$ and the equisingular characteristic system $H^0(N'_{B|\mathbb{P}^2})$ of its branch curve B. Also, the characteristic system of $\varphi$ is complete if and only if the equisingular characteristic system of B is complete.

Again 9.11 and 12.18 imply the existence of plane curves with nodes and cusps whose equisingular characteristic system is obstructed: in fact, if $\varphi' : S \to \Sigma \subset \mathbb{P}^3$ is a map to a surface with ordinary singularities, it is well known that there exists a point $p$ in $\mathbb{P}^3$ such that the projection $\Pi : \mathbb{P}^3 - \{p\} \to \mathbb{P}^2$ with centre $p$ makes $\varphi = \Pi \circ \varphi'$ into a generic multiple plane. It is then clear that if $\{\varphi'_t\}$ is a deformation of $\varphi'$, then $\{\varphi_t = \Pi \circ \varphi'_t\}$ is a deformation of $\varphi$. Conversely, as remarked before example 12.19, if $\{\varphi_t\}$ is a deformation of $t$, then there is a deformation $\varphi'_t$ of $\varphi'$ if and only if setting $H_t = \varphi_t^*$ (hyperplane in $\mathbb{P}^2$), the four sections $x_0, \ldots, x_3$ of $H^o(\mathbb{G}_S(H))$ extend holomorphically in $t$ to 4 sections $x_{0t}, \ldots, x_{3t}$ of $H^o(\mathbb{G}_{S_t}(H_t))$.

This property holds in particular if $\dim H^1(\mathbb{G}_{S_t}(H_t))$ is independent of $t$: we defer the reader to [Wah] for more details, as well as for a very precise account of the theory of equisingular deformations of curves with nodes and cusps. Again here we have to remark that Enriques tried several times to show that curves with nodes and cusps were unobstructed, but this is not true, by the example of Mumford-Kodaira-Wahl.

# LECTURE FIVE: SURFACES AND THEIR INVARIANTS

## §14. Topological invariants of surfaces

In this section and the following ones, we shall very quickly review some basic facts about the topology of compact complex surfaces, and roughly outline the Enriques-Kodaira classification of (compact) complex surfaces. We defer the reader to [Bo-Hu], [Be 1], [B-P-V], and also to the survey papers [Ci], [Ca 2] for a thorough, update and exhaustive treatment. Given a (compact) complex surface S, we shall consider its underlying structure as an oriented topological 4-manifold, and also its differentiable manifold structure.

The main topological invariants of S are

$$
(14.1) \begin{cases}
\Pi_1(S): \text{ the fundamental group of } S \\
b_i(S) = \dim_{\mathbb{Q}} H_i(S, \mathbb{Q}), \text{ the Betti numbers of } S \\
e(S) = \displaystyle\sum_{i=0}^{4} (-1)^i b_i(S) = 2 - 2b_1 + b_2, \text{ the topological Euler-Poincaré} \\
\quad \text{characteristic of } S \\
T: \text{ the torsion subgroup in } H_1(S, \mathbb{Z}) \text{ (and in } H^2(S, \mathbb{Z})) \\
Q = H^2(S, \mathbb{Z}) \to \mathbb{Z}, \text{ the integral unimodular quadratic form given by} \\
\quad \text{cup product (composed with evaluation on the fundamental} \\
\quad \text{class of } S) \\
b^+, b^-: \text{ the indices of positivity, resp. negativity, of } Q \\
\tau = b^+ - b^-: \text{ the signature of the manifold (note that the rank of } Q \\
\quad \text{is } b_2 = b^+ + b^-)
\end{cases}
$$

The differentiable structure determines the real tangent bundle of S and its second Stiefel-Whitney class $w_2(S)$ (cf. [Mi-Sta]), by a theorem of Wu, determines whether $Q(x)$ is an even (i.e. $Q(x)$ even $\forall$ x in $H^2(S, \mathbb{Z})$) or odd form, since $Q(x) \equiv w_2(S) \cdot x \pmod 2$. Now, it is known (cf. [Se 2]) that all indefinite unimodular quadratic forms are determined by their rank, signature and party: if they are odd, then they are diagonalizable over $\mathbb{Z}$ (hence with $\pm 1$ entries on the diagonal), and if they are even, they can be brought to a block diagonal form, with building blocks $U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$
E_8 = \begin{pmatrix}
2 & 0 & -1 & 0 & & & \\
0 & 2 & 0 & -1 & & & \\
-1 & 0 & 2 & -1 & & & \\
0 & -1 & -1 & 2 & -1 & & \\
& & & -1 & 2 & -1 & \\
& & & & -1 & 2 & -1 \\
& & & & & -1 & 2
\end{pmatrix} \quad \text{or} \quad -E_8
$$

Notice that $\tau(U_2) = 0$, $\tau(E_8) = 8$, and, by a theorem of Rokhlin [Rk], if $w_2 = 0$, then $\tau \equiv 0 \pmod{16}$ ($w_2 = 0 \Rightarrow Q$ is even, but not conversely, cf. [Hab] or 17.6).

What can be said about $Q$ when $Q$ is definite? Donaldson [Do 1] recently established the following remarkable result.

Theorem 14.2. Let $M$ be a compact oriented 4-dimensional manifold with definite intersection form $Q$: then $Q$ is diagonalizable (i.e., its matrix is $\pm$ Identity in a suitable basis).

The importance of the intersection form $Q$ lies in the fact that it is the unique topological invariant when the 4-manifold $M$ is simply connected. We have in fact ([Fre])

Theorem 14.3 (M. Freedman). Let $M$, $M'$ be compact oriented topological 4-manifolds, and assume that they are simply-connected, and have the same intersection form $Q$. If $M$, $M'$ have a differentiable structure, they are topologically equivalent. More generally, given $Q$, there are at most two topological types of 4-manifolds $M$ with form $Q$ and $\Pi_1(M) = 0$: if there are two, they are distinguished by the property whether $M \times [0, 1]$ admits or doesn't admit a differentiable structure.

§15. Analytic invariants of surfaces

Before giving a list of invariants, it is convenient to clarify that in some cases we are talking about biholomorphic invariants, in others about bimeromorphic invariants. To explain the notion of a bimeromorphic map, we recall that two smooth algebraic varieties $X$ and $Y$ were classically said to be birational if their fields of rational functions $\mathbb{C}(X)$, $\mathbb{C}(Y)$ would be isomorphic. Such an isomorphism does not induce a biholomorphic map, but only a "generalized" graph, i.e., a closed subvariety $\Gamma$ of $X \times Y$ such that, $p_1$, $p_2$ being the projections on both factors:

(15.1)   i)   there exist closed subvarities $I_X$ of codimension at least 2 in $X$ (resp.: $I_Y$), such that the restriction of $p_1$ from

$$\Gamma - p_1^{-1}(I_X) \;\to\; X - I_X$$

is biholomorphic (same condition for $p_2$),

ii)   $\Gamma$ is irreducible (hence $\Gamma$ is the closure of $\Gamma - p_1^{-1}(I_X)$).

Replacing the word "subvariety" by the word "closed analytic subspace" (actually, in the sequel we shall make little distinction, since by Chow's theorem a closed analytic subspace of a compact algebraic variety is an algebraic subvariety), we obtain the definition of a bimeromorphic map.

**Definition 15.2.** A bimeromorphic map between compact complex manifolds $X$, $Y$ is a biholomorphic map $\varphi$ between open sets of $X$ and $Y$, $U_X$ and $U_Y$ , such that $X - U_X$ , $Y - U_Y$ are closed analytic subsets, and the closure $\Gamma$ of the graph of $\varphi$ is a closed analytic subset of $X \times Y$ satisfying properties 15.1.

In general, for a compact complex manifold $X$, we denote by $\mathbb{C}(X)$ its field of meromorphic functions and we recall the following famous result of Siegel [Sie 1].

**Theorem 15.3.** $\mathbb{C}(X)$ is a finitely generated extension field of $\mathbb{C}$ with transcendence degree $a(X)$ over $\mathbb{C}$ with $a(X) \le n = \dim_{\mathbb{C}} X$.

**Remark 15.4.** It is easy to see that a bimeromorphic map between $X$ and $Y$ induces an isomorphism between $\mathbb{C}(X)$ and $\mathbb{C}(Y)$. Moreover, for an algebraic variety $X$ $\mathbb{C}(X)$ coincides with the field of rational functions and, if $\dim Y = a(Y)$, $\mathbb{C}(Y) \cong \mathbb{C}(X)$, then $Y$ is bimeromorphic to $X$. Such a $Y$ does not need to be algebraic if $\dim \ge 3$, and is usually called a Moishezon manifold (cf. [Moi 1], [Moi 2]).

In dimension 2 the bimeromorphic maps are obtained as composition of certain elementary bimeromorphic maps which we are going now to describe.

**Example 15.5.** Let $X$ be a complex manifold, $p$ a point in $X$, $(z_1, \ldots, z_n)$ coordinates in a neighborhood $U$ of $p$, with $p$ corresponding to the origin. Let $\tilde{U}$ be the closure of the graph of the meromorphic map $U \to \mathbb{P}^{n-1}$ sending $(z_1, \ldots, z_n)$ to the line $\mathbb{C}(z_1, \ldots, z_n)$. Then, glueing $\tilde{U}$ with $X - \{p\}$ in an obvious way, we obtain a new manifold $\tilde{X}$, with a proper holomorphic and bimeromorphic map $\sigma : \tilde{X} \to X$ such that

  i) $\sigma^{-1}(p)$, which is called the exceptional subvariety and denoted by $E$, is isomorphic to $\mathbb{P}^{n-1}$ .

  ii) The normal bundle $N_{E|\tilde{X}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$.

  iii) $\sigma|_{\tilde{X}-E}$ is a biholomorphism.

$\sigma$ is called an <u>elementary modification</u> and $\sigma^{-1}$ is called the <u>blow-up</u> of the point $p$.

The following result is classical.

Theorem 15.6. Every bimeromorphic map of complex surfaces factors as a composition of blow-ups followed by a composition of elementary modifications.

Definition 15.7. A complex surface $S$ is said to be minimal if every holomorphic and bimeromorphic map $\sigma : S \to S'$ is a biholomorphism.

Remark 15.8. If $\tilde{S} \overset{\sigma}{\to} S$ is an elementary modification, then the lattice $H^2(\tilde{S}, \mathbb{Z})$, equipped with the quadratic form $\tilde{Q}$, is the orthogonal direct sum $H^2(S, \mathbb{Z}) \oplus \mathbb{Z}E$, where $E$ is the cohomology class of the exceptional curve $E$, with $\tilde{Q}(E) = -1$. In particular $\tilde{Q}$ is odd and $b_2(\tilde{S}) = b_2(S) + 1$.

Conversely, if a complex surface $\tilde{S}$ contains an exceptional curve $E$ of the I kind, i.e. $E \cong \mathbb{P}^1$, $N_{E|\tilde{S}} = \mathbb{O}_{\mathbb{P}^1}(-1)$ (or, equivalently $\tilde{Q}(E) = -1$) then there exists an elementary modification $\sigma : \tilde{S} \to S$ with $\sigma(E) = $ a point in $P$, and the second Betti number of $S$ is equal to $b_2(\tilde{S}) - 1$. This is the classical result of Castelnuovo and Enriques, extended by Kodaira in the non algebraic case, and generalized by Grauert in [Gr 2] ; combining this with another deep theorem of Castelnuovo and Kodaira, we obtain the following

Theorem 15.9. A complex surface $S$ is minimal if and only if it does not contain an exceptional curve of the I kind. Every complex surface $S'$ is a blow-up of a minimal surface $S$, and $S$ is unique up to biholomorphism except if $S'$ is ruled, i.e. $S'$ is bimeromorphic to a product $C \times \mathbb{P}^1$.

Another biproduct of the structure theorem 15.6 of bimeromorphic maps of surfaces is the following theorem of Chow and Kodaira.

Theorem 15.10. A (smooth) complex surface $S$ is projective (i.e., a submanifold of some projective space) if and only if $a(S) = 2$.

We should also remark that, by the result of Kodaira ([Ko 1], thm. 6) quoted at the end of §9, all small deformations of a non minimal complex surface are again non minimal, while we saw (use prop. 6.19, and the fact that $\mathbb{F}_1$ is the blow-up of a point in $\mathbb{P}^2$) that is is not true for deformations in the large.

We can now start to review some of the classical bimeromorphic invariants of surfaces. The following are numerical invariants.

(15.11)

$$\begin{cases} p_g \text{ , the } \underline{\text{geometric genus}}, \text{ is } h^2(\mathbb{O}_S) = \text{dimension}_{\mathbb{C}} H^2(\mathbb{O}_S) \\ q \text{ , the } \underline{\text{irregularity}}, \text{ is } h^1(\mathbb{O}_S) \\ P_m \text{ , the } m^{th} \underline{\text{plurigenus}}, \text{ is } h^0((\Omega_S^2)^{\otimes m}) = h^0(\mathbb{O}_S(mK)) \end{cases}$$

A more subtle invariant is the graded ring

$$(15.12) \qquad \mathbb{R}(S) = \bigoplus_{m=0}^{\infty} H^0(\mathbb{O}_S(mK)), \text{ the } \underline{\text{canonical ring}}.$$

**Definition 15.13.** Let $Q(S)$ be the field of fractions of homogeneous elements of the same positive degree in $\mathbb{R}(S)$: then, either $Q(S) = \emptyset$, or $Q(S)$ is algebraically closed in $\mathbb{C}(S)$, and the $\underline{\text{Kodaira dimension of }S}$, $\text{Kod}(S)$ is

$$\begin{cases} -\infty & \text{if } Q(S) = \emptyset \\ \text{tr deg}_{\mathbb{C}} Q(S), & \text{otherwise.} \end{cases}$$

The above definition is not the unique possible one: denoting by $\varphi_m$ the $m^{th}$ pluricanonical map, i.e. the rational map $\varphi_m : S \dashrightarrow \mathbb{P}^{P_m - 1}$ attached to the sections of $H^0(\mathbb{O}_S(mK))$, one can also define $\text{Kod}(S)$ to be $\max_m (\dim \varphi_m(S))$.

From the complex point of view, one can consider divisors $D$, $D'$ and consider the intersection number of two divisors, $D \cdot D'$, as classically defined through algebraic equivalence: since to a divisor $D$ one associates the invertible sheaf $\mathbb{O}_S(D)$, one sees that, denoting by $c_1$ (I Chern class) the homomorphism of $H^1(\mathbb{O}_S^*)$ (classifying isomorphism classes of invertible sheaves) to $H^2(S, \mathbb{Z})$ appearing in the cohomology sequence of the exponential sequence

$$0 \to 2\pi i \mathbb{Z} \to \mathbb{O} \xrightarrow{\exp} \mathbb{O}^* \to 0 \text{ ,}$$

the bilinear form of intersection on $H^2(S, \mathbb{Z})$ extends the classical intersection product. It is in fact true more generally that many of the analytical invariants, whose definition depends upon the complex structure of $S$, are in fact determined only by the topological structure of $S$.

Notice that, by 15.6 and 15.8, $\Pi_1(S)$, $T$, $b_1$, $b^+$ are bimeromorphic invariants.

Another classical invariant is

$$(15.14) \qquad p^{(1)} = K^2 + 1, \text{ the linear genus of } S .$$

$K^2$, as well as $p^{(1)}$, are not bimeromorphic invariants, but, if a surface $S'$ is not ruled, one can consider, e.g., the linear genus, and all the possible analytical

and topological invariants of the unique minimal surface $S$ bimeromorphic to $S'$ ($S$ is called the minimal model of $S'$).

If you perform a blow-up, $e = 2 - 2b_1 + b_2$ goes up by 1, $K^2$ drops by 1: hence $K^2 + e$ is a bimeromorphic invariant, and an extension to complex surfaces of a classical theorem of Noether identifies it with a combination of previously encountered invariants. We have (cf. [Hi 1]), the following

Theorem 15.15. $(K^2 + e) = 12(1 - q + p_g) = 12 \chi$ ($\chi = 1 - q + p_g$ is the Euler-Poincaré characteristic of the structure sheaf $\mathcal{O}_S$).

Another result of the same type, which represented a real breakthrough in the classification of complex manifolds, is the index theorem of Atiyah-Singer-Hirzebruch.

Theorem 15.16. $3\tau = 3(b^+ - b^-) = K^2 - 2e$ .

Let's observe now that, by Serre duality, $p_g$ is also the dimension of $H^o(\Omega^2_S)$; in general holomorphic 1 and 2 forms on a surface are d-closed, so that there is, using DeRham's theorem, an inclusion $H^o(\Omega^1_S) \subset H^1(S, \mathbb{C})$ , $H^o(\Omega^2_S) \subset H^2(S, \mathbb{C})$. Moreover, if $\eta \in H^o(\Omega^1_S)$ , $\bar{\eta}$ a $\bar{\partial}$-closed $(0,1)$ form and gives, by Dolbeault's theorem, a cohomology class in $H^1(\mathcal{O})$. In this way one sees that $b^+ \geq 2p_g$, $h^o(\Omega^1_S) \leq q$ , $(2q - b_1) \geq 0$, and the upshot is that, by a clever manipulation, the index theorem tells you that the sum (of positive integers!) $(b^+ - 2p_g) + (2q - b_1)$ equals 1: we thus have, (cf. [Ko 4]).

Theorem 15.17. If $b_1$ is even, $b_1 = 2q$, $b^+ = 2p_g + 1$, $h^o(\Omega^1_S) = q$; if $b_1$ is odd, $b_1 = 2q - 1$, $b^+ = 2p_g$, $h^o(\Omega^1_S) = q - 1$. In particular, if $b_1$ is even and $p_g = 0$, $S$ is a projective surface.

There are several other results pertaining to inequalities between numerical invariants, or limitations in their range, but it is more convenient to postpone these to the next section, as being part of the classification theory.

We just end this paragraph by mentioning a consequence of the previous theorem

Corollary 15.18. The intersection form of a complex surface $S$ is semi-negative definite if and only if $b_1$ is odd and $p_g = 0$ .

## LECTURE SIX: OUTLINE OF THE ENRIQUES-KODAIRA CLASSIFICATION

### §16. Definition of the main classes of the classification

First of all, the main purpose of the Enriques-Kodaira classification is to partition all surfaces, considered only up to bimeromorphic equivalence, into 7 classes, in such a way that the knowledge (by explicit calculations) of some numerical invariant (or some more refined invariant, as the triviality of the canonical divisor) may allow to draw several conclusions about the structure and the geometry of some surfaces taken into consideration.

Its analogue in dimension 1 is the rough subdivision of curves according to the Kodaira-dimension, $Kod = -\infty$ if the genus $g$ is 0, i.e. the curve is $\mathbb{P}^1$, $Kod = 0$ if $g = 1$, i.e. you have an elliptic curve, $Kod = 1$ if the genus is at least 2. The classification of curves according to their genus is more refined, but it is closely related to the knowledge of the topology of algebraic curves; in the surface case, a complete classification, less rough than the one given by Enriques and Kodaira, seems for the time being out of reach, due to the problem of classifying all the surfaces of general type.

**Definition 16.1.** A surface $S$ is said to be of general type if $Kod(S) = 2$, or, equivalently, if $Q(S) = \mathbb{C}(S)$ (cf. 15.13).

**Definition 16.2.** A surface $S$ is said to be rational if it is bimeromorphic to $\mathbb{P}^2$, in particular a rational surface is ruled.

**Definition 16.3.** A surface $S$ is said to be elliptic if it admits an elliptic fibration, i.e., a surjective morphism $f: S \to B$, with $B$ a curve, and with the smooth fibres of $f$ being elliptic curves.

**Remark 16.4.** If $S$ is elliptic, $a(S) \geq a(B) = 1$; conversely, if $a(S) = 1$, there exists a curve $B$ with $\mathbb{C}(S) \cong \mathbb{C}(B)$, and an elliptic fibration $f : S \to B$, such that all the curves of $S$ are components of the fibres of $f$. If $S$ is elliptic, then $Kod(S) \leq 1$, and $S$ is not of general type.

We can now pass to the list of the seven classes of surfaces, some of them being divided into subclasses.

Class 1):   Ruled surfaces (i.e. bimeromorphic to a product $C \times \mathbb{P}^1$): these are distinguished by the irregularity $q$ which equals the genus of $C$, and

they are rational if $q = 0$. Their minimal models are $\mathbb{P}^2$, and $\mathbb{P}^1$- bundles over curves C.

From now on, since the minimal model is unique, we shall only talk about the minimal models.

Class 2):  $\underline{\text{K3 surfaces}}$, defined by the condition $q = 0$, $K \equiv 0$.

Class 3):  $\underline{\text{Complex tori}}$, i.e. surfaces biholomorphic to a quotient $\mathbb{C}^2/\Omega$, where $\Omega$ is a subgroup generated by 4 vectors linearly independent over $\mathbb{R}$.

Class 4):  Elliptic surfaces with $b_1$ even, $P_{12} \neq 0$, $K \neq 0$, divided into two sub- classes distinguished by the Kodaira dimension.

Class 4), Kod = 0:  $\underline{\text{Enriques surfaces}}$, i.e. normalization of surfaces as in 12.19, or $\underline{\text{hyperelliptic surfaces}}$ (explicitly described in $[\text{B-DF}]$, cf. $[\text{Be 1}]$, pgs. 112-114), quotients of a product of two elliptic curves $E_1 \times E_2$ by the action of a subgroup G of $E_1$ acting on $E_1 \times E_2$ by sending $(x_1, x_2)$ to $(x_1 + g, g(x_2))$, for a suitable action of G on $E_2$ such that $E_2/G \cong \mathbb{P}^1$.

Class 4), Kod = 1:  Canonically elliptic surfaces with $b_1$ even: $\varphi_{12}$, the $12^{\text{th}}$ pluricanonical map, gives an elliptic fibration.

Class 5)  Surfaces of general type (cf. lecture seven).

Class 6)  Elliptic surfaces with $b_1$ odd, $P_{12} \neq 0$, with subclasses

Class 6), Kod = 0:  $\underline{\text{Kodaira surfaces}}$, i.e. surfaces of the form $\mathbb{C}^2/G$, where G is a group of affine transformations of the form $(z_1, z_2) \mapsto$ $(z_1 + a, z_2 + \bar{a} \; z_1 + b)$. They are distinguished into $\underline{\text{primary}}$ ones, with $b_1 = 3$, $K \equiv 0$, and $\underline{\text{secondary}}$ ones, with $b_1 = 1$, $K \neq 0$; the secondary ones admit an unramified cover of finite degree which is a primary Kodaira surface.

Class 6), Kod = 1:  Canonically elliptic surfaces with $b_1$ odd: $\varphi_{12}$ gives an elliptic fibration.

Class 7):  Surfaces with $b_1 = 1$, $P_{12} = 0$ (and Kod = $-\infty$ in fact): we shall not say much about these, since their classification has not been yet complete- ly accomplished.

§17.  Criteria of classification and features of some classes

The following result is the prototype of all criteria of classification

Theorem 17.1 (Castelnuovo's criterion).  A surface  S  is rational if and only if
$q = P_2 = 0$.

Ruled surfaces are also the ones which admit several characterizations.

Theorem 17.2.  A surface  S  with  $b_1$  even is ruled if and only if one of the follow-
ing equivalent conditions is satisfied:

  i)   $P_{12} = 0$ .
  ii)   These exists a curve  C, not exceptional of the I kind, with  $K \cdot C < 0$.
Moreover, if  S  is a minimal model and  $b_1$  is even,
  iii)   $K^2 < 0$  if and only if  S  is ruled and  $q \geq 2$ .
The following results hold instead when  S  is minimal, but without the assumption
that  $b_1$  be even.
  iv)   $K^2 > 0$,  $P_2 = 0$  if and only if  S  is rational
  v)   $K^2 > 0$,  $P_2 \neq 0$  if and only if  S  is of general type.
  vi)   $e < 0$  if and only if  S  is ruled and  $q \geq 2$ .

The definition of  K3  surfaces we gave in  §16  was one of the less explicit
ones : in fact, we could have chosen to define a K3 surface according to the follow-
ing beautiful theorem of Kodaira (conjectured earlier by Andreotti and Weil).

Theorem 17.3.  A minimal surface  S  is a K3 surface if and only if  S  is a direct
deformation, with non singular base, of a non singular surface of degree 4 in  $\mathbb{P}^3$ .

We notice that some of the K3 surfaces can be elliptic, as well as some com-
plex tori, and this fact justifies the condition  $K \not\equiv 0$  used to define Class 4).  We
defer the reader to  [B-P-V]  for an excellent survey about K-3 surfaces and their
moduli space.

The following theorem characterizes complex tori.

Theorem 17.4.  A minimal surface  S  is a complex torus if and only if  $b_1 = 4$,
$K \equiv 0$.  Moreover, a surface with  $K \equiv 0$  (necessarily minimal) has  $q = 0, 2$, and
thus  $b_1 = 0, 4, 3$  according to whether  S  is a K3 surface, a complex torus, or a
primary Kodaira surface.

As to class 4), we recall that we have already implicitly remarked that the two subclasses are distinguished by the value of $P_{12}$ being 1 or $\geq 2$, we have in fact the following.

Proposition 17.5.

$\qquad$ $Kod(S) = -\infty$ if and only if $P_{12} = 0$.

$\qquad$ $Kod(S) = 0$ if and only if $P_{12} = 1$.

$\qquad$ $Kod(S) = 1$ if and only if $P_{12} \geq 2$ and $K^2 = 0$ for the minimal

$\qquad\qquad$ model of S.

In particular, for the subclass of class 4), where $Kod = 0$, we have a rather nice characterization.

Theorem 17.6. A surface S is hyperelliptic if and only if $P_{12} = 1$, $b_1 = 2$. A surface with $P_{12} = 1$, $b_1 = 0$ is a K3 surface if $p_g = 1$ and an Enriques surface if $p_g = 0$ (then $P_2 = 1$).

For the subclass of class 4) with $Kod = 1$, we see from proposition 17.5 that a characterization of a minimal model S is

(17.7) $$K^2 = 0, \quad P_{12} \geq 2, \quad b_1 \text{ even}$$

but, if we are given a non minimal surface, then the conditions $b_1$ even, and $|12K|$ yielding a rational map with image a curve are easier to check. As a matter of fact, to detect the exceptional curves of the I kind on a surface with $Kod \geq 0$ (i.e. $P_{12} \neq 0$), the standard way is to look at the smallest $m$ such that $P_m \neq 0$, and then to look at the fixed part of $|mK|$, to check which components of this divisor are exceptional.

$\qquad$ Surfaces of general type being taken into account by theorem 17.2.v), we notice that the surfaces in classes 6) and 7) are the ones with $b_1$ odd and, in particular, they cannot be Kählerian. From surface classification and results of Kodaira, Miyaoka, Todorov and Siu follows also the remarkable

Theorem 17.8. A complex surface with $b_1$ even is a deformation of an algebraic surface and is Kählerian.

$\qquad$ The two classes 6) and 7) are distinguished by the value of $P_{12}$, which is $\neq 0$ for class 6), and 0 for class 7) (remark, though, that Kodaira's class VII is different from our class 7), being defined by the condition $b_1 = 1$, and thereby including

also secondary Kodaira surfaces and some canonically elliptic surfaces). The sub-class of Kodaira surfaces is again characterized by

(17.9)                                    $P_{12} = 1$, $b_1$ odd

and we defer the reader to [Ko 4] for a very detailed description of these surfaces.

For lack of time, we don't attempt to describe the known examples and the classification results for surfaces in class 7), referring to [Nak 1] for a nice and updated survey.

Let's just notice that, when $b_1 = 1$, then, by a result of Kodaira on elliptic surfaces, we have $p_g = 0$, and the intersection form $Q$ is semi-negative definite by 15.18. More precisely, by Noether's formula 15.15, since $\chi = 0$, $b_2^- = b_2 = e$ $= -K^2$. Hence, as in the case of ruled surfaces with $q \geq 2$, $K^2 < 0$ as soon as the Betti number $b_2$ is $\neq 0$. On the other hand, if $S$ is elliptic, by Kodaira's canonical divisor formula (cf. [B-P-V], pgs. 161-164), a multiple $mK$ of $K$ is linearly equivalent to a multiple $rF$ of a fibre $F$ of the elliptic fibration, hence in particular $K^2 = 0$ (the above canonical divisor formula shows also that a surface $S$ admits more than one elliptic fibration only if it is not canonically elliptic and in fact only if $Kod = 0$, since either $r = 0$ or $K$ determines the elliptic fibration).

We have thus the following

<u>Theorem 17.10.</u>  A minimal surface $S$ has $K^2 < 0$ if and only if either $S$ is ruled with $q \geq 2$ ($b_1$ odd) or $S$ has $b_1 = 1$, $b_2 > 0$.

Since $\chi = K^2 + e$, by Castelnuovo's theorem 17.2.vi) and the above one follows

<u>Theorem 17.11.</u>  A surface $S$ has $\chi < 0$ if and only if $S$ is ruled with $q \geq 2$.

# LECTURE SEVEN: SURFACES OF GENERAL TYPE AND THEIR MODULI

## §18. Surfaces of general type, their invariants and their geometry

Let's observe that Noether's theorem 15.15 and the index theorem 15.16 ensure that the analytically defined invariants $K^2$ and $\chi$ are determined by $e$ and $\tau$, therefore $K^2$ and $\chi$ are topological invariants and the advantage of dealing with them stems from the fact that, unlike $K^2$ and $e$, they don't have to satisfy any congruence relation. Also, by theorem 17.2, we have $K^2$, $\chi \geq 1$ for a minimal surface $S$ of general type (these are called Castelnuovo's inequalities), and two more inequalities are satisfied.

$$(18.1) \qquad K^2 \geq 2p_g - 4 \geq 2\chi - 6 \qquad \text{(Noether's inequality)}$$

$$K^2 \leq 9\chi \qquad \text{(Bogomolov-Miyaoka-Yau's inequality)}$$

In fact, by classification, the inequality is true for all surfaces, except for ruled surfaces of irregularity $q \geq 2$, which have $K^2 = 8(1-q)$, $\chi = 1 - q$. S. T. Yau [Ya] proved indeed a much stronger theorem, in particular it follows from his results the following

**Theorem 18.2.** If a surface of general type $S$ has $K^2 = 9\chi$, then the universal cover $\hat{S}$ of $S$ is biholomorphic to the unit ball in $\mathbb{C}^2$.

An easily proven but nice corollary concerns the surfaces $S$ for which the intersection form $Q$ is positive definite: in fact, if $b_2 = b^+ = \tau$, since $K^2 \leq 9\chi$, $4K^2 = 3(12\chi) = 3(K^2 + e)$, i.e., $K^2 - 2e \leq e$, which is in turn equivalent to $3\tau \leq e = 2 + b_2 - 2b_1$. Then $2b_2 \leq 2(1 - b_1)$ and thus $b_2 = 1$, $b_1 = 0$, $p_g = 0$ $(b^+ = 1 = 1 + 2p_g)$ hence $\chi = 1$, $K^2 = 12 - 3 = 9$.

**Corollary 18.3.** The only complex surfaces for which the intersection form $Q$ is positive definite have $b_2 = 1$ and $K^2 = 9$, $\chi = 1$: they are either $\mathbb{P}^2$ or a surface of general type with the unit ball as universal cover.

By a result of Kodaira, the plurigenera are completely determined by the invariants $K^2$, $\chi$; we have in fact

**Theorem 18.4.** If $S$ is a minimal surface of general type, and $m \geq 2$, $P_m = \chi + (1/2) K^2 m (m - 1)$.

As a consequence of the theory of pluricanonical mappings, that we are going now to explain, there follows the result that surfaces with given invariants $K^2$, $\chi$ belong to a finite number of deformation types.

Let's go back to the canonical ring $\mathfrak{R}(S)$, defined in 15.12; 18.4 tells us that its Hilbert polynomial is determined by $K^2$, $\chi$, and clearly two minimal surfaces of general type $S$, $S'$ are isomorphic if and only if $\mathfrak{R}(S)$ and $\mathfrak{R}(S')$ are isomorphic graded rings. Before mentioning directly how to recover $S$ from $\mathfrak{R}(S)$, let's remark that, $\mathbb{C}(S)$ being finitely generated, there exists an $m$ such that every function in $\mathbb{C}(S)$ can be written as a fraction whose numerator and denominator are sections of $H^o(\mathfrak{O}_S(mK))$. In other terms, there exists an $m$ such that the $m^{th}$ pluricanonical map $\varphi_m$ is birational onto its image $\Sigma_m$. Unfortunately, unlike the case of curves, one cannot expect $\varphi_m$ to be an embedding: in fact, though $K \cdot C \geq 0$ for each irreducible curve on $S$ (17.2.ii)), there can be a finite number of curves $C \cong P^1$ with $K \cdot C = 0$ $(C^2 = -2)$, and $K$ is ample iff these curves do not exist on $S$. Otherwise, since $\mathfrak{R}(S)$ is finitely generated, one can take $X =$ Proj $\mathfrak{R}(S)$ (its points correspond to maximal homogeneous ideals in $\mathfrak{R}(S)$), and there exists a holomorphic map $\Pi: S \to X$ satisfying the following properties

(18.5)　i)　$X$ is a normal surface

　　　　ii)　if $i: X^o \to X$ is the inclusion morphism of the non-singular part of $X$, then the sheaf of Zariski differentials $\omega_X = i_*(\Omega^2_{X^o})$ is invertible and $\Pi^*(\omega_X) = \mathfrak{O}_S(K_S)$ (i.e., $X$ has only Rational Double Points, R.D.P.'s, as singularities).

　　　　iii)　every pluricanonical map $\varphi_m: S \to \Sigma_m$ factors through $\Pi$ and $\tilde{\varphi}_m: X \to \Sigma_m$.

Definition 18.6.　$X = \text{Proj}(\mathfrak{R}(S))$ is called the canonical model of $S$.

We defer the reader to [Ca 3] for a survey of recent results on pluricanonical maps of surfaces of general type, and we content ourself with stating a by now classical result of Bombieri.

Theorem 18.7.　If $m \geq 5$, $\tilde{\varphi}_m: X \to \Sigma_m$ is an isomorphism.

## §19. Pluricanonical images and Gieseker's moduli variety

Recall that a projective variety $\Sigma \subset P^N$ is said to be projectively normal if the restriction homomorphisms

$$H^o(\mathfrak{G}_{P^N}(k)) \rightarrow H^o(\mathfrak{G}_\Sigma(k))$$

are surjective for each integer $k \geq 0$. In the case where $\Sigma = \Sigma_m$ is the pluricanonical image of a surface $S$ of general type we have (cf. [Ci 2]).

**Theorem 19.1.** If $m \geq 8$, $\tilde{\varphi}_m : X \rightarrow \Sigma_m$ is an isomorphism onto a projectively normal surface.

We are going now to discuss very loosely the main line of ideas which lead to Gieseker's theorem about the existence of moduli spaces of surfaces of general type. First of all, let's recall Mumford's definition (cf. [Mu 2]).

**Definition 19.2.** A variety $\mathfrak{M}_{K^2,\chi}$ is said to be a coarse moduli space for surfaces of general type $S$ with given invariants $K^2$, $\chi$ if there exists a bijection $\lambda$ between $\mathfrak{M}_{K^2,\chi}$ and the set of isomorphism classes $[S]$ of minimal surfaces as above, satisfying the following property: for each deformation $\mathfrak{S} \overset{p}{\rightarrow} B$ of such surfaces, there is given a unique morphism $\psi : B \rightarrow \mathfrak{M}_{K^2,\chi}$ such that $\psi(b) = \lambda^{-1}([S_b])$, and the correspondence $(\mathfrak{S} \overset{p}{\rightarrow} B) \rightarrow \psi$ is compatible with pull-backs (i.e., to the family $f^*\mathfrak{S} \rightarrow B'$ corresponds $\psi \circ f = \psi' : B' \rightarrow \mathfrak{M}_{K^2,\chi}$).

**Theorem 19.3** (Gieseker). There exists a coarse moduli space $\mathfrak{M}_{K^2,\chi}$ which is a quasi-projective variety. Moreover, two surfaces $S$, $S'$ correspond to points in the same connected component of $\mathfrak{M}_{K^2,\chi}$ if and only if $S$ is a deformation of $S'$.

The key point consists in taking all the $m^{th}$ canonical images $\Sigma_m$ of our surfaces (with $K^2$, $\chi$ fixed): they are, if $m \geq 8$, projectively normal surfaces of fixed degree (= $m^2 K^2$) in a fixed projective space $\mathbb{P}$ of dimension $P_m - 1 = \chi - 1 + m(m-1)/2 \ K^2$.

Now, $\Sigma_m$ and $\Sigma'_m$ are projectively equivalent if and only if the corresponding surfaces $S$, $S'$ are isomorphic, and one has to construct a quotient by the group $PGL(P_m)$ of a variety $\hat{\mathfrak{K}}$ parametrizing our surfaces $\Sigma_m$. To this purpose, one has first to use the Hilbert scheme (cf.[Gro2], [Mu 3], [Ser 2]) technique: since these notes are meant to be elementary, let's indicate the main idea.

Since $\Sigma = \Sigma_m$ is projectively normal, if we denote by $\mathcal{J}_\Sigma$ the ideal sheaf of $\Sigma$ in $\mathbb{P}$, the space $H^o(\mathcal{J}_\Sigma(n))$ has fixed dimension equal to

$$\binom{n + P_m}{P_m} - P_{nm}$$

Now, there exists an integer $n$, depending only on the Hilbert polynomial $P(n)$ of a variety (here $P(n) = \chi + K^2/2 \ nm(nm - 1)$, hence $P$ depends only upon $K^2, \chi, m$), such that the ideal sheaf $\mathcal{J}_\Sigma$ of $\Sigma$ is generated by the subspace $V_\Sigma = H^o(\mathcal{J}_\Sigma(n))$ of the fixed vector space $W = H^o(\mathcal{O}_\mathbb{P}(n))$. Let $r$ be the dimension of $V_\Sigma$, $t$ the dimension of $W$: then $r = t - P(n)$, and $r, t$ depend only upon $K^2, \chi, m$.

<u>Definition 19.4.</u> The $n^{th}$ Hilbert point of $\Sigma$ is the point $\mathbb{P}(\wedge^r V_\Sigma) \in \mathbb{P}(\wedge^r W)$, belonging to a Grassman manifold $G(r, W)$ of dimension $r \cdot P(n)$.

The Hilbert scheme of subschemes of $\mathbb{P}$ with Hilbert polynomial $\mathbb{P}(n)$ is the closed subscheme $\mathcal{K}$ of the Grassmann manifold $G(r, W) \subset \mathbb{P}(\wedge^r W)$ corresponding to the set of $r$-dimensional subspaces $V$ of $W$ such that the ideal sheaf $\mathcal{J} = V\mathcal{O}_\mathbb{P}$ generated by $V$ defines a subscheme $\Sigma(V)$ with Hilbert polynomial $P(n)$. $\mathcal{K}$ is the basis of a universal family, i.e., there is a subscheme $Z$ of $\mathcal{K} \times \mathbb{P}$ such that the fibre of $Z$ over $V \in \mathcal{K}$ is just the subscheme $\Sigma(V)$. This $\mathcal{K}$ is too big, and first of all one has to take the open set $\mathcal{K}_o \subset \mathcal{K}$

(19.5)    $\mathcal{K}_o = \{ V \in W \mid V\mathcal{O}_\mathbb{P}$ defines a connected surface $\Sigma(V)$ with only R. D. P.'s as singularities $\}$.

Over $\mathcal{K}_o$ lies the restriction $Z_o$ of the universal family $Z$, and inside $\mathcal{K}_o$ lies the closed subscheme

(19.6)    $\hat{\mathcal{K}} = \{ V \mid \Sigma = \Sigma(V)$ has $\omega_\Sigma(-1) \cong \mathcal{O}_\Sigma \}$.

The restriction $\hat{Z}$ of the universal family enjoys the following

<u>19.7. Universal property</u> of $\hat{Z} \subset \hat{\mathcal{K}} \times \mathbb{P}$: for each family $p: \mathcal{S} \to B$ of minimal surfaces of general type, with given invariants $K^2, \chi$, and for each choice of $P_m$ independent sections of the locally free sheaf $p_*(\omega_{\mathcal{S}|B}^{\otimes m})$, there does exist a unique pair of morphisms

$$\alpha: \mathcal{S} \to \hat{Z}, \quad f: B \to \hat{\mathcal{K}}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\ \alpha\ } & \hat{Z} \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \xrightarrow{\ f\ } & \hat{\mathcal{H}} \end{array}$$

and such that $\alpha$ gives fibrewise the $m^{th}$-canonical map

$$\varphi_m : S_b \to \mathbb{P} \ .$$

Now, all the surfaces of general type with those given invariants $K^2$, $\chi$ appear as (minimal resolutions of) fibres of the family $\hat{Z} \to \hat{\mathcal{H}}$ of canonical models; since the base $\hat{\mathcal{H}}$ is quasi-projective, hence it has a finite number of components, and, by a result of G. Tjurina [Tju], the canonical models $X$, $X'$ of two surfaces of general type $S$, $S'$ are a deformation of each other if and only if $S$, $S'$ are deformations of each other, it is thus proven (cf. 19.3).

(19.8)    The surfaces of general type with given $K^2$, $\chi$ belong to a finite number of deformation classes. In particular there is only a finite number of diffeomorphism types.

It is also clear that the group

(19.9)    $G' = PSL(P_m)$ acts on $\hat{\mathcal{H}}$, and there is a bijection between the set of orbits of $G$ and the set of isomorphism classes of surfaces of general type with invariants $K^2$, $\chi$.

Thus, the problem of the existence of the coarse moduli space $\mathcal{M}_{K^2, \chi}$ is reduced to the existence of a categorical quotient (cf. [Mu 2], chap. I) for the action of $G$ on $\hat{\mathcal{H}}$.    This is a problem in the realm of the so-called Geometric Invariant Theory: in general (cf. [Mu 2], Appendix) such quotients always exist as algebraic (or Moishezon) spaces (i.e., as complex spaces bimeromorphic to algebraic varieties).    In fact, we recall again, what Gieseker proves is that

(19.10)    The categorical quotient $\hat{\mathcal{H}}/G'$ exists as a quasi-projective variety.

The idea is as follows, the Hilbert point belongs to

$$\mathbb{P}(\wedge^r W) = \mathbb{P}(\wedge^r (Sym^n(\mathbb{C}^{P_m}))) \ ,$$

and one wants to show that the invariant homogeneous functions of degree $\ell$, i.e. the sections in

$$A = H^o( P (\wedge^r (Sym^n(\mathbb{C}^{P_m}))), \; \mathbb{O}(\ell))^{G'} ,$$

for m, n, $\ell$ sufficiently big separate the orbits, so that $\hat{\mathcal{K}}/G'$ sits inside a projecttive variety. We have here the typical situation of Geometric Invariant Theory: a vector space

$$U = \wedge^r (Sym^n (\mathbb{C}^{P_m})) ,$$

and an action of $SL(P_m) = G$ . Then

(19.11)    A point $u \in U$ is

        1) unstable if $\overline{Gu} \ni 0$

        2) semistable if $\overline{Gu} \not\ni 0$

        3) stable if it is semistable and the stabilizer of u is finite.

and then the G invariant polynomials define a morphism $\psi$ from $\mathbb{P}(U)^{ss} = \{$semi-stable points$\}$ to a projective variety Y in such a way that the restriction of $\psi$ to the stable points $\mathbb{P}(U)^{s}$ separates the orbits (in fact, the stable points have closed orbits, and two closed orbits are separated by some G invariant polynomials.

Remark 19.12. In our case the condition that the stabilizer of u be finite, when u corresponds to the Hilbert point of a pluricanonical image $\Sigma_m$ follows from a general result of Matsumura [Mat]. In fact, $\{ g \mid g(\Sigma_m) = \Sigma_m \} = Aut(\Sigma)$ is a linear algebraic group which, if not finite, would have a non-trivial Cartan subgroup, which is a rational variety: but then $\Sigma$ would be uniruled (X, with dim X = n is said to be uniruled if there exist Y, with dim Y = n-1, and a dominant rational map of $Y \times \mathbb{P}^1$ into X), and in particular all the plurigenera of $\Sigma$ would vanish.

The really difficult point is to prove that these Hilbert points are semistable, and this is done with hard combinatorial estimates using the Hilbert-Mumford stability criterion.

19.13. u is semi-stable if and only if for each 1-parameter subgroup of G, $t \in \mathbb{C}^* \to g(t)$, where

$$g(t) = A \begin{pmatrix} t^{a_o} & & 0 \\ & \ddots & \\ & & a_p \\ 0 & & t \end{pmatrix} A^{-1} \qquad (\text{with } \Sigma\, a_i = 0)$$

one has $\lim_{t \to 0} g(t)(u) \neq 0.$

From the fact that $\hat{\mathcal{K}}/G = \mathcal{M}_{K^2,\chi}$ is a coarse moduli space (cf. 19.2) it follows easily the following

Corollary 19.14. Let S be a minimal surface of general type with invariants $K^2$, $\chi$: then, if B is the base of the Kuranishi family of S, then, locally around the point [S] corresponding to the isomorphism class of S, $\mathcal{M}_{K^2,\chi}$ is analytically isomorphic to the quotient of B by the finite group Aut(S).

§20. The number of moduli M(S) of a surface

Definition 20.1. For a surface of general type S , we define M(S), the number of moduli of S, to be equal to the dimension of the base B of its Kuranishi family, (i.e. the maximum of the dimensions of the irreducible components of B.

Remark 20.2. By 19.14, M(S) is the dimension of $\mathcal{M}_{K^2,\chi}$ at the point [S] corresponding to S.

Remark 20.3. More generally, Kodaira and Spencer ([K-S], Chap. V, §11) define the number of moduli of a complex manifold X to be the maximal dimension of the base T of an effectively parametrized family $\mathcal{X} \to T$ of deformations of X, i.e. such that the Kodaira-Spencer map $\rho_t$ is injective for each t .

It was conjectured by Noether that M(S) would be $10\chi - 2K^2$ , whereas Enriques realized that $10\chi - 2K^2$ was only a lower bound for M(S). In fact, by the Hirzebruch-Riemann-Roch formula (cf. [Hi]), $-(10\chi - 2K^2)$ equals the Euler-Poincaré characteristic of $\Theta$, i.e. $h^o(\Theta) - h^1(\Theta) + h^2(\Theta)$. In general, $H^o(\Theta_X)$ is the Lie algebra of a real Lie group of biholomorphisms of X: since (cf. 19.12) Aut(S) is finite, $H^o(\Theta_S) = 0$, and, as we saw in 3.3.

(20.4) $\qquad 10\chi - 2K^2 = h^1(\Theta) - h^2(\Theta) \le \dim B = M(S)$.

On the other hand, $M(S) \le h^1(\Theta_S)$ , and we indeed conjecture that for the general surface S in each component of the moduli space $\mathcal{M}_{K^2,\chi}$ equality holds, provided $q(S) = 0 \sqrt{}$ and K is ample (for reasons stemming from [Ca 4] and [Ca 5]), so that $\mathcal{M}$ should be a reduced variety. In any case, finding the dimension $h^1(\Theta)$ is by no means easier (and in some cases even more difficult) than to compute M(S), therefore we just observe that $h^2(\Theta_S) = h^o(\Omega_S^1 \otimes \Omega_S^2)$, hence

(20.5) $$M(S) \le h^1(\Theta) = 10\chi - 2K^2 + h^o(\Omega^1 \otimes \Omega^2) \quad ,$$

and one can give an upper bound for $M(S)$ by giving an upper bound for $h^o(\Omega^1 \otimes \Omega^2)$ in terms of $\chi$, $K^2$, q. It is clear that, doing so, one does not obtain the best estimate, because one is giving an upper bound for the dimension of the Zariski tangent space of each point of the base $B$ of the Kuranishi family, and not simply an upper found for $\dim B$, as we already noticed.

A way to bound $h^o(\Omega^1_S \otimes \Omega^2_S) = h^o(\Omega^1_S(K))$ is to use the existence of a smooth curve $C$ in $|2K|$ as soon as $K^2 \ge 5$, or $p_g \ge 1$, except possibly if $K^2 = 3$, 4 (this follows from recent results of Francia [Fra] and Reider [Rei 2]). In fact then, by the exact sequences

$$0 \to \Omega^1_S(-K) \to \Omega^1_S(K) \to \Omega^1_S(K) \otimes \mathcal{O}_C \to 0$$

$$0 \to \mathcal{O}_C(-K) \to \Omega^1_S(K) \otimes \mathcal{O}_C \to \mathcal{O}_C(4K) \to 0$$

since $\Omega^1_S(-K) \cong \mathcal{O}_S$ , we get

$$h^o(\Omega^1_S(K)) \le h^o(\mathcal{O}_C(4K)) = 5K^2 \ .$$

Otherwise, one looks for a smooth curve in $|mK|$, with $m = 3$, and the result is (with an improvement only of the constant in [Ca 6], Thm. B).

Theorem 20.6. We have the inequalities

$$10\chi - 2K^2 \le M(S) \le 10\chi + 3K^2 + 18 \ .$$

If $|K|$ contains a smooth curve, then $M(S) \le 10\chi + q + 1$ .

In the case when the surface $S$ has $q \ne 0$, since $q = h^o(\Omega^1_S)$ , surely $h^o(\Omega^1_S \otimes \Omega^2_S) \ne 0$, since there is a bilinear map $H^o(\Omega^1_S) \times H^o(\Omega^2_S) \to H^o(\Omega^1_S \otimes \Omega^2_S)$ which is non-degenerate in each factor, nevertheless the existence of holomorphic 1-forms can be used for our desired bound.

For irregular $(q \ne 0)$ surfaces, a powerful tool is given by the analysis of the Albanese map

$$\alpha: S \to A = \text{Alb}(S) = H^o(\Omega^1_S)^\vee / H_1(S, \mathbb{Z}) \ ,$$

such that $\alpha(p) = \int_{p_o}^{p}$ (where $p_o$ is a fixed point and the linear functional on $H^o(\Omega^1)$ is clearly defined only up to $\int_\gamma$ , for $\gamma \in H_1(S, \mathbb{Z})$). The condition that

$\alpha$ should have differential of maximal rank = 2 except that at a finite set of points can also be phrased as

(20.7)     there do exist $\eta_1, \eta_2 \in H^0(\Omega_S^1)$ such that $C = \text{div}(\eta_1 \wedge \eta_2)$
           is a reduced and irreducible curve (in $|K|$ !).

Castelnuovo [Cas] tried to prove the inequality $M \leq p_g + 2q$ under the assumption that $\alpha(S)$ would be a surface, i.e., assuming that the differential of $\alpha$ would have rank 2 outside of a curve: we showed in [Ca 6] that there are infinitely many families of surfaces $S$ for which $\alpha(S)$ is a surface, and $M \geq 4p_g + o(p_g)$. In fact, we also proved that (20.7) implies $M \leq p_g + 3q - 3$, and conjectured that (20.7) would imply the Castelnuovo inequality. We shall now give Reider's simple proof [Rei 1] of this inequality.

<u>Theorem 20.8</u> (Reider). If $S$ satisfies 20.7, then $M \leq p_g + 2q$ .

<u>Proof.</u> Since $C$ is irreducible, $\eta_1$ and $\eta_2$ don't vanish simultaneously on a curve, hence we can assume $\eta = \eta_1$ to have only isolated zeros. Let $Z$ be the 0-dimensional scheme of zeros of $\eta$: we have then the Koszul complex

(20.9)          $$0 \to \mathcal{O}_S \xrightarrow{\ \eta\ } \Omega_S^1 \xrightarrow{\ \wedge \eta\ } \mathcal{J}_Z \Omega_S^2 \to 0$$

where $\mathcal{J}_Z$ is the ideal sheaf of $Z$. Tensoring with $\mathcal{O}_S(K)$, since $h^1(\Theta) = h^1(\Omega_S^1(K))$, we get

(20.10)          $$h^1(\Theta_S) \leq q + h^1(\mathcal{J}_Z \Omega_S^2) - 1 .$$

The basic fact is that the ideal sheaf of $C$ is contained in $\mathcal{J}_Z$, hence we have an exact sequence

(20.11)          $$0 \to \mathcal{O}_S(-C) \to \mathcal{J}_Z \to \mathcal{F} \to 0 ,$$

where $\mathcal{F}$ is a torsion free, rank 1 sheaf on $C$. Tensoring (20.11) with $\mathcal{O}_S(2K)$ yields

(20.12)          $$h^1(\mathcal{J}_Z(2_S)) \leq q + h^1(\mathcal{F}(2_S)) - 1 .$$

Since $\mathcal{O}_C(2K_S)$ is the dualizing sheaf of $C$, by Grothendieck duality

$$h^1(\mathcal{F}(2K_S)) = \dim \text{Hom}(\mathcal{F}(K_S), \mathcal{O}_C(K)) .$$

On the other hand, there is a bilinear map

$$H^o(\mathfrak{F}(K)) \times \text{Hom}(\mathfrak{F}(K), \mathfrak{G}_C(K)) \to H^o(\mathfrak{G}_C(K))$$

which is non-degenerate in each factor; since these are complex vector spaces, a well known application of the Segre product (projectived tensor product), gives the inequality

$$h^1(\mathfrak{F}(2K_S)) \leq h^o(\mathfrak{G}_C(K)) - h^o(\mathfrak{F}(K)) + 1$$

$$\leq p_g + q - 1 - h^o(\mathfrak{F}(K)) + 1 .$$

To give a lower bound for $h^o(\mathfrak{F}(K))$, we tensor (20.11) with $\mathfrak{G}_S(K)$, to get $h^o(\mathfrak{F}(K)) \geq h^o(\mathfrak{J}_Z(K)) - 1 \geq q - 2$ by the sequence (20.9). It suffices to put these inequalities together.

Q.E.D.

We defer the reader to [Rei 1] for other results of this type, and we note in fact that Reider simply uses the existence of a 1-form $\eta$ with isolated zeros, and the existence of a reduced irreducible curve $C$ in $|K|$ such that $\mathfrak{G}_S(-C) \subset \mathfrak{J}_Z$, so that the method can be generalized taking $C \in |mK|$ with such a property. It would be interesting to give an upper bound for $M(S)$ in the case of a surface $S$ fibred over a curve $B$ of genus $\geq 1$.

# LECTURE EIGHT: BIHYPERELLIPTIC SURFACES AND PROPERTIES OF THE MODULI SPACES

### §21. Moduli spaces of surfaces of general type and their properties

Let $S$ be a minimal surface of general type with invariants $K^2$, $\chi$ and consider the Gieseker moduli space $\mathcal{M}_{K^2,\chi}$, which has a finite number of connected components. We can define two moduli spaces $\mathcal{M}^{top}$ and $\mathcal{M}^{diff}$ which are contained in $\mathcal{M}_{K^2,\chi}$, and are indeed a union of connected components of $\mathcal{M}_{K^2,\chi}$, as follows ($\mathcal{M}^{diff} \subset \mathcal{M}^{top}$).

Definition 21.1. $\mathcal{M}^{top}(S)$ (resp.: $\mathcal{M}^{diff}(S)$) is $\{[S'] \in \mathcal{M}_{K^2,\chi} \mid$ there exists an orientation preserving homeomorphism (resp.: diffeomorphism) between $S$ and $S'\}$.

We note (cf. §14.17) that if $Q$ is even and $K^2 \geq 9$, then every complex structure on the topological manifold underlying $S$ corresponds to a minimal surface $S'$ of general type homeomorphic to $S$. Thus $\mathcal{M}^{diff}$, e.g., is then a coarse moduli space for all the (integrable almost) complex structures on the differentiable manifold underlying $S$.

Remark 21.2. Since the Stiefel-Whitney class $w_2$ (cf. 14) is the mod 2 reduction of $c_1(K) \in H^2(S, \mathbb{Z})$, we see that the intersection form $Q$ is even if and only if $c_1(K) \in 2H^2(S, \mathbb{Z})$. Therefore, Freedman's theorem 14.3 has as a corollary that two simply connected complex surfaces $S_1$, $S_2$ are homeomorphic if and only if they have the same invariants $K^2$, $\chi$, and for both of them the same answer holds true to the question: does $K_i \in 2H^1(S_i, \mathbb{Z})$ or does it not?

We note that in complex dimension $n = 1$, the notion of homeomorphic and diffeomorphic are the same, and that the moduli space $\mathcal{M}_g$ of curves of genus $g$ is connected, also irreducible, and of pure dimension $3g - 3$ (more of its properties are known, cf. [H-M], and Harer's notes in this volume).

In the next paragraphs, we shall consider some families of surfaces, somehow a generalization of hyperelliptic curves, by which we shall see that $\mathcal{M}^{top}$ does not share any of these three good properties with $\mathcal{M}_g$. We have in fact the following (cf. [Ca 6], [Ca 8]).

Theorem 21.3. For each natural number $k$, there exists a minimal surface of general type $S$ such that $\mathcal{M}^{top}(S)$ has at least $k$ irreducible components

$Y_1, \ldots, Y_k$ with

    i) $\dim(Y_i) \neq \dim(Y_j)$ for $i \neq j$

    ii) $Y_i$ and $Y_j$ lie, for $i \neq j$, in different connected components of $\mathcal{M}^{top}(S)$.

It is not clear at the time being how smaller than $\mathcal{M}^{top}$ is really $\mathcal{M}^{diff}$:
Donaldson [Do 2] has shown the existence of two homeomorphic, but not diffeomorphic, complex surfaces, so we should expect to have $\mathcal{M}^{diff} \neq \mathcal{M}^{top}$ in general, and $\mathcal{M}^{diff}$ could still have some nice properties.

## §22. Bidouble covers and their deformations

**Definition 22.1.** A smooth bidouble cover (it is a fourfold cover) is a Galois finite cover, $\varphi \colon S \to X$ with group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, between smooth varieties.

**Example 22.2.** Let $\varphi \colon \mathbb{P}^2 \to \mathbb{P}^2$ be the morphism defined by $(x_0, x_1, x_2) \to (x_0^2, x_1^2, x_2^2) = (z_0, z_1, z_2)$. The group $(\mathbb{Z}/2)^2$ acts with the three covering involutions $\sigma_0, \sigma_1, \sigma_2$ such that $\sigma_i^*(x_j) = x_j$ if $i \neq j$, $\sigma_i^*(x_i) = -x_i$. The fixed locus for $\sigma_i$ is the line $\{x_i = 0\} = R_i$ plus the point $R_j \cap R_k$, if $(i, j, k)$ is a permutation of $(1, 2, 3)$, and the branch locus $B = B_1 + B_2 + B_3$ consists of the 3 coordinate lines, respective images of $R_1$, $R_2$, $R_3$. This example, though easy, gives all the ingredients of the geometric picture in the general situation.

    As in the example one defines (cf. [Ca 6]) $\sigma_0$, $\sigma_1$, $\sigma_2$ to be the three non-trivial elements of the group, and denotes by $R_i$ the divisorial part of the fix locus of $\sigma_i$, by $B_i$ the set theoretic image of $R_i$.

    If $z_i = 0$ is a section of $\mathcal{O}_X(B_i)$ with $\operatorname{div}(z_i) = B_i$, we see from the example above that it is not possible to take directly only the square root $x_i$ of $z_i$, but that it is possible to take the square roots $w_i$ of $z_0 z_1 z_2 / z_i$ (these three double covers correspond to the three distinct subgroups of order 2 of $(\mathbb{Z}/2)^2$, giving each a factorization of $\varphi$ as a composition of two double covers). In general, thus, there are divisors $L_0$, $L_1$, $L_2$ on $X$ such that $2L_i \equiv B_0 + B_1 + B_2 - B_i$, $B_k + L_k \equiv L_0 + L_1 + L_2 - L_k$, and $S$ is the smooth surface, in the rank 3 bundle on $X$ which is the direct sum of the three line bundles corresponding to the $L_i$'s, defined by the equations

(22.3)
$$
\begin{cases}
w_i^2 = z_0 z_1 z_2 / z_i \\[2mm]
z_k w_k = w_0 w_1 w_2 / w_k
\end{cases}
$$

There is a natural way of deforming these equations, and, computing $N_\varphi$ and the characteristic map of the family, one can check whether the characteristic system of the morphism $\varphi$ is complete.

We refer to [Ca 6] for more details, and observe that equations 22.3 take a much simpler form, and the natural way of deforming is easier to see if one assumes one involution, say $\sigma_0$, to have only isolated fixed points, i.e., if one assumes $z_0 = 1$.

Definition 22.4. A simple bidouble cover is a smooth bidouble cover such that one of the three covering involutions has a fixed set of codimension at least 2.

The equations 22.3 simplify then (set $x_1 = w_2$, $x_2 = w_1$) to

$$(22.5) \qquad \begin{cases} x_1^2 = z_1 \\ x_2^2 = z_2 \ , \end{cases}$$

and a natural way to deforming them is to set

$$(22.6) \qquad \begin{cases} x_1^2 = z_1 + b_1 x_2 \\ x_2^2 = z_2 + b_2 x_1 \ , \end{cases}$$

for $b_i$ a section of $\mathcal{O}_X(B_i - L_i)$. In general, applying $\varphi_*$ to the exact sequence

$$0 \to \Theta_S \to \varphi^* \Theta_X \to N_\varphi \to 0 \ ,$$

one obtains

$$(22.7) \qquad 0 \to \varphi_* \Theta_S \to \Theta_X \otimes (\mathcal{O}_X \oplus \mathcal{O}_X(-L_1) \oplus \mathcal{O}_X(-L_2) \oplus \mathcal{O}_X(-L_3))$$

$$\to \bigoplus_{i=0}^{2} (\mathcal{O}_{B_i}(B_i) \oplus \mathcal{O}_{B_i}(B_i - L_i)) \to 0$$

Now, the parameter space for the natural deformations of $\varphi$ is the vector space

$$\bigoplus_{i=0}^{2} H^0(\mathcal{O}_X(B_i) \oplus \mathcal{O}_X(B_i - L_i))$$

and one obtains

Theorem 22.8. The characteristic system of the map $\varphi$ is complete if $H^0(\mathcal{O}_X(B_i))$ goes onto $H^0(\mathcal{O}_{B_i}(B_i))$, and the same holds for

$$H^O(\mathcal{O}_X(B_i - L_i)) \to H^O(\mathcal{O}_{B_i}(B_i - L_i)) \ .$$

If furthermore $H^1(\mathcal{O}_X) = H^1(\mathcal{O}_X(-L_i)) = 0$, then the Kuranishi family of S is smooth.

Actually the hypotheses in the above theorem are rather restrictive, and not strictly needed, anyhow they are sufficient for our application (cf. §23). Also, a similar result should hold true more generally for smooth Abelian covers.

§23. Bihyperelliptic surfaces

Hyperelliptic curves are double covers of $\mathbb{P}^1$, and if we multiply all the previous terms by two we get

Definition 23.1. A bihyperelliptic surface is a smooth bidouble cover of $\mathbb{P}^1 \times \mathbb{P}^1$.

In the following, we shall limit ourselves to consider simple bihyperelliptic surfaces. These are determined by the two branch curves $B_1$, $B_2$ of respective bidegrees $(2n, 2m)$, $(2a, 2b)$. One can allow also the two curves $B_1$, $B_2$ to acquire singularities, but in such a way that the bidouble cover defined by equations (22.5) have only R.D.P.'s as singularities: we shall call the resulting surfaces admissible.

(23.2)  Denote by $\eta_{(a,b)(n,m)}$ the subset of the moduli space corresponding to natural deformations of simple bihyperelliptic surfaces with branch loci of bidegrees $(2a, 2b)$, $(2n, 2m)$. Denote further by $\hat{\eta}_{(a,b)(n,m)}$ the subset corresponding to admissible surfaces (these are the surfaces whose canonical models are defined by equations (22.6), and occur precisely when those equations give surfaces with at most R.D.P.'s as singularities).

An easy application of theorem 22.8 shows

Theorem 23.3. $\eta_{(a,b)(n,m)}$ is a Zariski open irreducible subset of the moduli space. In particular, the closure $\overline{\eta}_{(a,b)(n,m)}$ is irreducible (and contains $\hat{\eta}_{(a,b)(n,m)}$).

Remark 23.4. i) Clearly

$$\eta_{(a,b)(n,m)} = \eta_{(b,a)(m,n)} = \eta_{(n,m)(a,b)} = \eta_{(m,n)(b,a)} :$$

apart from these trivial equalities, all the $\eta_{(a,b)(n,m)}$'s can be proven to be different (by the inflectionary behaviour of the canonical map of the general surface in the family), and hence they are disjoint by theorem 23.3.

ii) Also, since $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$ is a deformation of $\mathbb{F}_{2n}$, one can also show (cf. [Ca 1] ) that, enlarging the set $\eta_{(a,b)(n,m)}$ to the smooth bidouble covers of $\mathbb{F}_{2n}$, and doing the same for the admissible covers, a result similar to 23.3 holds true.

iii) If $a > 2n$, $m > 2b$, it follows easily from equations (22.6) that all the surfaces in $\hat{\eta}_{(a,b)(n,m)}$ are admissible (simple) bihyperelliptic surfaces.

We can now sketch the main arguments for the proof of theorem 21.3.

(23.5)   Bihyperelliptic surfaces are simply-connected, and their invariants $K^2$, $\chi$ are expressed by quadratic polynomials P, Q of $(a, b, n, m)$.

(23.6)   Also $\dim \eta_{(a,b)(n,m)}$ is given by an easy function of $(a, b, n, m)$.

(23.7)   Letting $r(S) = \max \{ m \mid c_1(K) \in mH^2(S, \mathbb{Z}) \}$, since for a family $\mathcal{S} \to B$, $\mathbb{O}_{S_t}(K_t) = \mathbb{O}_{S_t} \otimes \omega_{\mathcal{S}|B}$, we have that $r$ is a locally constant function on the moduli space. Moreover (cf. [Ca 7] for the proof, using easy arguments of group cohomology), if $[S] \in \eta_{(a,b)(n,m)}$, then $r(S) = $ G.C.D. $(a+n-2, b+m-2)$.

(23.8)   One has to show (this was done by Bombieri in the appendix to [Ca 6], that for each $k$, there exist $k$ 4-tuples $(a,b)(n,m)$ giving the same values for $K^2, \chi$, and $k$ distinct values for both $M(S)$ and $r(S)$, and one can further assume $r(S)$ to take even values. But when $w_2 = 0$, Q is even, the S's are simply connected, therefore (cf. 21.2) one gets $k$ distinct irreducible components, of different dimensions, belonging to the same moduli space $\mathfrak{m}^{top}(S)$, and lying in $k$ distinct connected components of $\mathfrak{m}^{top}(S)$. I conjecture the closures of the $\eta_{(a,b)(n,m)}$'s (at least if $a > 2n$, $m > 2b$) to be themselves connected

components of the moduli space. The following has been proven up to now ([Ca 7]), and it is an encouraging result, since one of the most difficult problems is, in general, to describe deformations in the large of complex manifolds.

<u>Theorem 23.9.</u> If $a > 2n$, $m > 2b$ the closure of $\eta_{(a,b)(n,m)}$ consists of admissible covers of some $\mathbb{F}_{2k}$, with

$$k \leq \max\left(\frac{n}{a-1}, \frac{n}{m-1}\right).$$

In particular $\hat{\eta}_{(a,b)(n,m)}$ is a closed subvariety of the moduli space if

$$a \geq \max(2n+1, b)$$
$$m \geq \max(2n+1, m).$$

<u>Idea of proof:</u> If $a > 2n$, $m > 2b$, then (cf. 23.4.iii) all the surfaces in $\eta_{(a,b)(n,m)}$ are simple bihyperelliptic and, given a 1-dimensional family $\mathsf{S} \xrightarrow{p'} T$, with $[S_t] \in \eta_{(a,b)(n,m)}$ for $t \neq t_o$, one wants to conclude that $S_{t_o}$ is still an admissible cover of some $\mathbb{F}_{2k}$. The key point is that $(\mathbb{Z}/2)^2$ acts birationally on $\mathsf{S}$, preserving the deformation morphism $p'$, but indeed it acts biregularly on the family $\mathcal{X} \xrightarrow{p} T$ of the canonical models of the previous family $p': \mathsf{S} \to T$. What we have to show is that, if $Z = \mathcal{X}/(\mathbb{Z}/2)^2$, (then $Z_t = \mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq t_o$), then $Z_{t_o} = \mathbb{F}_{2k}$. To achieve this goal it suffices to show that $q: Z \to T$ is a smooth fibration, since every deformation of a minimal rational ruled surface with $Q$ even is again a surface of the same type. Now, the singularities that $Z_{t_o}$ can have are quotients of R.D.P.'s by $\mathbb{Z}/2$ or $(\mathbb{Z}/2)^2$, and can be explicitly classified: but many of them can be shown not to occur since any smoothing of these singularities would contribute, through the vanishing cohomology of the Milnor fibre, a subspace of $H^2(Z_t, \mathbb{Z})$ of dimension $\geq 2$ over which $Q$ is negative definite. This is a contradiction, since $Z_t \cong \mathbb{P}^1 \times \mathbb{P}^1$, and other arguments again by contradiction, eliminate the other remaining possibilities.

I should finally remark that to prove that the closure of $\eta_{(a,b)(n,m)}$ is open in the moduli space, it would suffice (by the results of [Ca 6] and [Ca 7]) to prove also when the canonical model of $S$ is singular (i.e., the bidouble cover is admissible, but not smooth) that the base $B$ of the Kuranishi family of $S$ is locally irreducible.

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