FOOTNOTES TO A THEOREM OF I. REIDER

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Introduction.

_ After the suggestion of one of the editors of these Proceedings, we publish this article which essentially reproduces a letter I wrote to Igor Reider on november 1986, after giving a seminar at the Institute Mittag-Leffler on Reider's results which have appeared in [Rei].

§1 is devoted to giving a more general and precise version of a result stated in an article by Griffiths and Harris ([G-H] , [Rei]), applying a construction due to Serre to construct vector bundles on algebraic surfaces starting from 0-dimensional subschemes failing to impose independent conditions to certain linear systems ; this version has not been superseded by the results appearing in Tjurin's new article ([Tju]) .

§2 supplies instead details for the proof (proposition 3 of [Rei]) that m-canonical systems , if m is at least 3 , give (in characteristic 0 , and with a couple of exceptions) embeddings of the canonical models of surfaces of general type ; for these details Reider defers the reader to the above quoted letter , and this result is due to cooperation with Torsten Ekedahl who in fact devised the final trick to solve the combinatorial problem to which the proof had been reduced ([Ek]).

It is a pleasure to acknowledge the warm hospitality and stimulating atmosphere I found at the Mittag-Leffler Institute in september '86 , and to thank the organizers of the Conference for their kind invitation .

§1 ZERO CYCLES ON SURFACES AND RANK 2 BUNDLES.

In this section X shall be a projective normal Gorenstein surface over an

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algebraically closed field k , i.e. X is normal and Cohen-Macaulay and ω_X, the dualizing sheaf of X , is an invertible sheaf ; we shall denote by K a Cartier divisor associated to ω_X .

We let Z be a purely 0-dimensional subscheme of X ; Z shall also be called a 0-cycle.

If we assume further that Z is a local complete intersection (l.c.i. , for short) , then , denoting , for $p \in$ supp (Z) , by R_p the local ring $\mathcal{O}_{Z,p}$ of Z at p , and by R(Z) the direct sum of the R_p 's , we have that

(1.1) $R= R_p$ is a 0-dimensional Gorenstein ring , in other terms , there is a non-degenerate pairing $R \times R \longrightarrow k$ given by local duality .

(1.2) $R=R_p$ has a natural decreasing filtration , given by the powers of the maximal ideal of p , and the last non zero term of this filtration is called the <u>socle</u> of R , and shall be denoted by $S = S_p$. The condition that R be a Gorenstein ring implies that S is a 1-dimensional k-vector space.

(1.3) We recall moreover that the pairing (1.1) is compatible with the algebra structure on R , i.e. , for f,g $\in$ R , $< f, g > = < 1 , fg >$, and therefore the socle S is just the annihilator of the maximal ideal $\mathcal{M}_p$ of $R = R_p$.

In the sequel , given a k-vector space V , we shall denote by $V^{\vee}$ its dual.

<u>Theorem 1.4</u>

Let X be a Gorenstein surface and Z a 0-cycle on X ; let L be a Cartier divisor on X and [L] the invertible sheaf associated to the Cartier divisor .

If J_Z denotes the ideal sheaf of Z , we may consider the exact sequence

(*) $H^0 ([K+L]) \xrightarrow{r} H^0 ([K+L] |_Z) \longrightarrow H^1 (J_Z [K+L])$,

and consider an isomorphism of the middle term with R(Z) (given by some local trivialization of [K + L]) .

Then there is an isomorphism between

i) the group of extensions $0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow J_Z[L] \longrightarrow 0$,
modulo the subgroup of extensions $0 \longrightarrow \mathcal{O}_X \longrightarrow E' \longrightarrow [L] \longrightarrow 0$
(giving E as the subsheaf of E' defined as the preimage of $J_Z[L]$) .

ii) the group of linear forms $\alpha \varepsilon R(Z)^{\vee}$ vanishing on the image of $H^0([K+L])$.

Moreover , in the above isomorphism , E is locally free if and only if Z is a l.c.i. and ,writing α_p for the restriction of α to R_p , α_p does not vanish on the socle S_p of R_p .

Proof. Dualizing the exact sequence (*) , we obtain that the group of linear forms α as in ii) is isomorphic to the space $H^1(J_Z[K+L])^{\vee}$ modulo the subspace $H^1([K+L])^{\vee}$, and we conclude for the first assertion since these two vector spaces are naturally isomorphic to $Ext^1(J_Z[L], \mathcal{O}_X)$, resp. to $Ext^1([L], \mathcal{O}_X)$.

We denote by α^* an extension in $Ext^1(J_Z[L], \mathcal{O}_X)$ inducing α .

We have to see when does the extension α^* give a locally free sheaf E. First of all , since E has rank 2 , if E is locally free , then Z is locally defined by two equations , so Z must be a l.c.i. . Moreover , the local to global spectral sequence for Ext provides a natural map :

$$Ext^1(J_Z[L], \mathcal{O}_X) \longrightarrow H^0(\mathcal{E}xt^1(J_Z[L], \mathcal{O}_X) \cong H^0(\mathcal{E}xt^2(\mathcal{O}_Z[L], \mathcal{O}_X) \cong Ext^2(\mathcal{O}_Z[K+L],[K]) \cong H^0([K+L]|_Z)^{\vee} \cong R(Z)^{\vee}$$

(the last two isomorphisms being respectively given by Serre duality on X and by the chosen trivialization of [K + L] around Z).

The given extension α^* thus naturally maps to α , with α_p giving a local extension $0 \longrightarrow \mathcal{O}_{X,p} \longrightarrow E_p \longrightarrow J_{Z,p} \longrightarrow 0$ as follows .

Using local duality we can identify $R_p^{\vee}$ with R_p , hence we can pick a function g around p whose class in R_p represents α_p .

Moreover , since Z is a l.c.i. , the ideal J_Z is locally generated by two functions h_1, h_2 , and then E_p is given as the cokernel of the homomorphism of free sheaves associated to the transpose of the row

(g, h_1, h_2) so that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{X,p} \longrightarrow \mathcal{O}_{X,p}^3 \longrightarrow E_p \longrightarrow 0$$

and the embedding of $\mathcal{O}_{X,p}$ in E_p is induced by the isomorphism of $\mathcal{O}_{X,p}$ with the first factor of $\mathcal{O}_{X,p}^3$ (hence the quotient of E_p by $\mathcal{O}_{X,p}$, if h is the column with coefficients h_1 , h_2 , is isomorphic to

$\mathcal{O}_{X,p}{}^2 / h\mathcal{O}_{X,p}$, and thus to $J_{Z,p}$ as desired) .
It is now clear that E_p is locally free if and only if g does not vanish at p , i.e. its class does not annihilate the socle S_p of R_p .

Q.E.D.

Remark 1.5. If H^1 ([K + L]) = 0 , then for each $\alpha \in R(Z)^v$ there is a unique extension α^* inducing α .

Example 1.6 If Z is a cycle of length 2 supported at a smooth point p of X , then there do exist local coordinates (x, y) such that J_Z is generated by (x^2 , y) . The socle S coincides with the maximal ideal of R , and such a locally free extension exists if and only if S is not contained in the image of the restriction map r from H^0 ([K + L]) . I.e. , either p is a base point and Im (r) = 0 , or p is not a base point and r is not onto .

Example 1.7. If Z consists of m distinct smooth points , p_1 , .. p_m , then E is locally free iff α_p is non zero for each p = p_1 , .. p_m .
In this case we have a non trivial extension (by which we mean , not obtained from an extension $0 \longrightarrow \mathcal{O}_X \longrightarrow E' \longrightarrow [L] \longrightarrow 0$) if and only if the points p_i are projectively dependent via the rational map associated to the linear system |K + L| , or ,more precisely , if the linear functionals e_i , for i= 1, ..m , given by evaluation at p_i (and in fact only defined up to a scalar multiple) are linearly dependent ; this is in fact the condition that r be not surjective .
We obtain a locally free sheaf if no p_i is a base point of | K + L | and if , q_i being the image point of p_i , there does exist among the q_i's a relation of linear dependence with all the coefficients different from zero.
To understand what this geometrical condition means, we may assume that q_1 , .. q_h is a maximal set of linearly independent elements among the q_i's : then , since the given field k is infinite , such a relation of linear dependence exists if and only if h < m and the remaining q_j's do not all lie in one of the coordinate hyperplanes of the projective space of dimension (h-1) spanned by the points q_1, ..q_h .

Remark 1.8. The following observation came out in a conversation I had with Mauro Beltrametti . Assume that X is smooth and that Z is a 0-cycle for which the restriction map r is not onto , whereas for each subscheme Z' of Z the restriction map r' is onto . Then the image of r is a hyperplane in R(Z) , hence there is a unique nonzero linear form α vanishing on Im (r), and a corresponding extension E is locally free (implying that Z must be a l.c.i.). In fact , otherwise E is contained in its double dual E' which is locally free , and gives an extension $0 \longrightarrow \mathcal{O}_X \longrightarrow E' \longrightarrow J_{Z'}[L] \longrightarrow 0$ where now Z' is a proper subscheme of Z . By assumption this sequence is split locally at Z , hence also the extension giving E is locally split, a contradiction.

The following lemma is essential in order to be able to prove that the adjoint linear systems | K + L | give embeddings of X .

Lemma 1.9. If p is a smooth point of X and $H^0([L])$ surjects onto $\mathcal{O}_Z$ for each l.c.i. 0-cycle Z of length 2 supported at p , then | L | gives an embedding at p .

Proof. Let M_p be the maximal ideal of the local ring $\mathcal{O}_{X,p}$: if $H^0([L])$ does not surject onto $\mathcal{O}_{X,p} / \mathcal{M}_p^2$, by our assumption , the image is 2-dimensional and intersects $\mathcal{M}_p/\mathcal{M}_p^2$ in a 1 -dimensional subspace W. Thus we obtain a contradiction by considering the length 2 cycle Z defined by $\mathcal{M}_p^2$ and by W .

Q.E.D.

Remark 1.10 The lemma does not hold already for a A_1 singularity . In fact , if $H^0(\mathcal{M}_p[L])$ does not surject onto $\mathcal{M}_p/\mathcal{M}_p^2$, then the image is contained in a 2- plane W in $\mathcal{M}_p/\mathcal{M}_p^2$ .Unfortunately W and $\mathcal{M}_p^2$ generate a length 2 , but not a l.c.i. cycle , because if the line W^v in the Zariski tangent space is tangent to X , then J_Z is not locally generated by two elements .

§ 2 PLURICANONICAL EMBEDDINGS OF SURFACES OF GENERAL TYPE

In this section k is an algebraically closed field of characteristic 0 and X is the canonical model of a surface of general type : thus X is a normal Gorenstein projective surface with ω_X ample , and if S is a minimal resolution of singularities of X , S is a minimal surface of general type.

To a singular point p of X there corresponds a divisor E on S , called a fundamental cycle , and consisting, with suitable multiplicities , of all the curves mapping down to p (hence these are all curves which have 0 intersection number with K) . The main property we want to mention here (cf. [Ar] for more details) is that there is a natural isomorphism (given by pull -back) between $\mathcal{O}_{X,p} / \mathcal{M}_p^{\ 2}$ and $H^0 (\mathcal{O}_{2E}) \equiv H^0 (\mathcal{O}_{2E} (mK))$, and therefore a pluricanonical system $| \omega_X^{\ m} |$ gives an embedding at p if and only if the sequence

(2.1.) $0 \longrightarrow H^0 ([mK -2E]) \longrightarrow H^0 ([mK]) \longrightarrow H^0 (\mathcal{O}_{2E} (mK)) \longrightarrow 0$
is exact .

Assume that $m > 1$: then $H^1 ([mK]) = 0$ (cf . [Bom]) , and the exactness of (2.1.) amounts to the vanishing $H^1 ([mK -2E]) = 0$.

Lemma 2.2. If E is a fundamental cycle on a minimal surface of general type S , then $H^1 ([mK -2E]) = 0$, provided $m > 3$, or $m=3$, $K^2 > 2$.

Proof. At page 188 of [Bom] (proof of theorem 3 , where E is though denoted Z) , it is shown that the desired vanishing holds if $H^0 ([(m -1) K -2E])$ is not zero , and one has moreover $m^2 K^2 > 9$, $m + K^2 > 4$.

We can therefore assume that $H^0 ([(m -1) K -2E]) = 0$.
Since also $H^2 ([(m -1) K -2E]) = 0$ (in fact the dual space is $H^0 ([(2-m)K + 2E])$, which is zero for $m > 2$, otherwise we would have an effective divisor with negative intersection number with K) , the conclusion is that , by the Riemann-Roch formula , $1/2 (m-1)(m-2) K^2 -4 + \chi$ is non-positive.
Since $K^2 > 2$, $m > 2$, $\chi > 0$, the only possibility is that $m = K^2 =3$, $\chi = 1$.
If $H^1 ([mK -2E])$ is non zero , recalling that $m = 3$, we have a non-trivial extension

(@) $0 \longrightarrow \mathcal{O}_S \longrightarrow E \longrightarrow \mathcal{O}_S (2K - 2E) \longrightarrow 0.$

We obtain immediately that $H^0 (E)$ has dimension 1 , whereas $c_1^{\ 2} (E) = 4$ and $c_2 (E) = 0$, hence E is numerically unstable (cf. [Bog] , [Rei]) and we have a Bogomolov destabilizing extension

(#) $0 \longrightarrow \mathcal{O}_S (M) \longrightarrow E \longrightarrow J_Z (D) \longrightarrow 0$,
where Z is a 0-cycle , and the divisor M - D is in the positive cone .

Recall also that M + D is linearly equivalent to 2K - 2E .
Therefore $K (M - D) > 0$,and $K M + K D = 2 K^2 = 6$, hence $K M > 3$,

while $K E < 3$.
As a consequence we get $H^0([-M]) = 0$: tensoring both exact sequences (@) and (#) by $\mathcal{O}_S(-M)$, we obtain that $H^0(E(-M))$ is at least 1-dimensional and is a subspace of $H^0([D])$, so that we may assume D is an effective divisor.

Recall though that by our assumption $H^0([M + D]) = 0$,hence $H^0([M]) = 0$ too. We noticed that $3 = K^2 < K M$, hence $H^0([K - M]) = 0$, and dually $H^2([M]) = 0$, so that the Riemann-Roch formula gives us that $1/2(M^2 - MK) + 1$ is a nonpositive number , i.e. $M^2 < KM - 1$.
We have $(M + D)^2 = 4(K - E)^2 = 4(K^2 + E^2) = 4 = D^2 + 2MD + M^2 < D^2 + 2MD + KM - 1$; since $KM + KD = 6$, we obtain

$$(2.3) \quad KD < 2MD + D^2 + 1 .$$

<u>Claim 2.4.</u> $D = 0$.
<u>Proof of the claim.</u> By the Index theorem $3D^2 = K^2 D^2$ is less than or equal to $(KD)^2$ which is in turn at most 4 , by a previously obtained inequality , thus D^2 is at most 1 .
Observe now that $c_2(E) = 0$, thus $\deg(Z) + MD = 0$, and MD is non positive. Looking at (2.3) , since KD is non negative , we immediately obtain that $MD = 0 = \deg(Z)$. Again , (2.3) gives that KD is at most 1 , hence $3D^2$ is bounded by $(KD)^2$ which is at most 1 , hence D^2 is nonpositive . Once again (2.3) gives $KD = D^2 = 0$, and we conclude that $D = 0$ since the selfintersection form is strictly negative definite for the effective divisors orthogonal to K .

<u>Q.E.D.for the claim</u>

By the claim (we noticed also that deg Z = 0) , (#) reduces to an extension of the following form

$$0 \longrightarrow \mathcal{O}_S(2K - 2E) \longrightarrow E \longrightarrow \mathcal{O}_S \longrightarrow 0 .$$

Since the first term of the above sequence has no global sections , by our assumption , the above sequence is easily seen to give a splitting of (@) , a contradiction .

<u>Q.E.D. for Lemma2.2</u>

<u>Corollary 2.5.</u> If X is the canonical model of a surface of general type , then the m^{th} pluricanonical system $|\omega_X{}^m|$ gives an embedding of X whenever $m > 4$, or $m=4$, $K^2 > 1$, $m=3$ and $K^2 > 2$.

Proof. The proof follows theorem 1 of [Rei] , lemma 1.9. , and lemma 2.2..

Q.E.D.

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