Topology Vol. 30, No. 4, pp. 641-651, 1991
Printed in Great Britain

0040-9383/91 $3.00 + .00

THE FUNDAMENTAL GROUP OF GENERIC POLYNOMIALS

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(Received in revised form 25 July 1990)

§0. INTRODUCTION

IN THIS short note we shall consider complex polynomials P in one variable as maps P: $\mathbb{C} \to \mathbb{C}$. In this framework a polynomial P of degree exactly $(n + 1)$ is said to be *generic* if the derivative P' has distinct roots $y_1, \ldots, y_n$, and the respective branch points $w_1 = P(y_1), \ldots, w_n = P(y_n)$, are also all distinct. Generic polynomials of degree $(n + 1)$ form an open set U_n in an affine space of dimension $(n + 2)$, and one can write down (cf. §1) an equation for the complement of U_n. The main object of this note is to establish the following.

MAIN THEOREM. *The fundamental group Γ_n of U_n is a direct product $\mathbb{Z} \times H_n$, and also occurs as a central extension*

$$(*) \qquad 1 \to \mathbb{Z} \to \Gamma_n \to G_n \to 1,$$

where $H_n \subset G_n$ are subgroups of the Artin braid group $\mathscr{B}_n$, such that

(i) *H_n is normal in G_n, $G_n/H_n \cong \mathbb{Z}/(n + 1)$ is generated by the coset of $(\Delta_n)^2$, the generator of the centre of $\mathscr{B}_n$*

(ii) *$(*)$ is induced by the epimorphism which sends $(m, h) \in \mathbb{Z} \times H_n$ to $(\Delta_n)^{2m} \cdot h$.*

More precisely, if $\sigma_1, \ldots, \sigma_{n-1}$ are the standard generators for $\mathscr{B}_n$, setting $\delta_s = \sigma_{s-1}\sigma_{s-2} \ldots \sigma_1$, then $(\Delta_n)^2 = (\delta_n)^n$, and H_n is generated by $(\delta_s)^{s+1}$ for $s = 2, 3, \ldots, n$, whereas G_n is generated by H_n and by $(\Delta_n)^2$. □

We shall see that the direct product $\Gamma_n = \mathbb{Z} \times H_n$ and the extension $(*)$ have an easy geometrical significance.

We remark that H_n is not only a subgroup of $\mathscr{B}_n$, but also (cf. §1 and Theorem 15) admits a natural epimorphism onto $\mathscr{B}_n$, and it is thus somehow a generalization of Artin's braid group.

In fact to H_n is associated the kernel P'_n of the canonical epimorphism onto the symmetric group $\mathscr{S}_n$ and we consider P'_n as a generalized pure braid group.

Using the concept of *simple* polynomials we also can define further generalizations of the pure braid group (cf. §1), but we postpone the study of those to a future paper.

From the point of view of combinatorial group theory, our results are not definite, since we have not yet found a presentation of H_n, G_n for each n: we hope to return to this topic in another paper. Nevertheless, we believe that, as far as geometry is concerned, our description is rather complete, and we would like to mention that the geometry of U_n was studied in a close fashion by Looijenga in [4], who proved that U_n is a $K(\pi, 1)$ space (he also treated other similar spaces).

Briefly, these are the contents of the paper. In §1 we show that the map assigning to $P \in U_n$ its ramification locus establishes an isomorphism $U_n \cong A(1, \mathbb{C}) \times M'_n$, where the ramification set is a point in M'_n and $A(1, \mathbb{C})$ is the group of affinities in one complex variable, acting on the target $\mathbb{C}$. Hence $\Gamma_n = \mathbb{Z} \times \pi_1(M'_n)$ (and in §3 we show that $\pi_1(M'_n) = H_n$).

Instead, still in §1, we show that the map associating to P its branch locus factors as a $A(1, \mathbb{C})$-fibre bundle (change of coordinates in the source $\mathbb{C}$) followed by a covering map identifying $G_n = \pi_1(U_n/A(1, \mathbb{C}))$ as a subgroup of Artin's braid group $\mathcal{B}_n$. G_n is explicitly determined in §1 as the stabilizer of a fixed edge labelled tree, and most of the technical work is done in §2 where, by a delicate inductive argument, it is shown that G_n and H_n are generated as indicated before. §3 essentially puts together the constructions in §1 and the algebraic manipulations in §2.

§1. THE BASIC GEOMETRICAL SET-UP

Let us consider the space V_n of polynomials of degree exactly $(n + 1)$ in $\mathbb{C}[z]$. If $P \in V_n$ we write $P = \sum_{i=0}^{n+1} a_i z^i$, where $a_{n+1} \neq 0$, hence $V_n = \mathbb{C}^{n+1} \times \mathbb{C}^*$. We view V_n as a space of holomorphic maps $P\colon \mathbb{C} \to \mathbb{C}$, hence (we set as usual $A(1, \mathbb{C}) = Aut(\mathbb{C})$) there is a natural action of $A(1, \mathbb{C}) \times A(1, \mathbb{C})$ on V_n given by

$$(\gamma', \gamma)(P) = \gamma' \circ P \circ \gamma^{-1}$$

and corresponding to changing coordinates in the source and in the target. The action of $A(1, \mathbb{C}) \times \{id\}$ is free, since if $\gamma'(w) = b'w + c'$, *then* $(a_0, \ldots, a_{n+1}) \to (b'a_0 + c', b'a_1, \ldots, b'a_{n+1})$ and in fact

$$V_n \cong A(1, \mathbb{C}) \times M_n,$$

where M_n is the space of monic polynomials of degree $(n + 1)$ with vanishing constant term.

The second action of $A(1, \mathbb{C})$, via $\{id\} \times A(1, \mathbb{C})$ is not everywhere free but there is a big open set where it is free.

LEMMA 1. *Let $\gamma \in A(1, \mathbb{C})$, $P \in V_n$ and assume $P \circ \gamma^{-1} = P$, for some $\gamma \neq id$. Then there is an affine coordinate ζ s.t.*

(i) *$\gamma(\zeta) = \alpha\zeta$, with α a primitive* hth *root of* 1 *with $h|(n + 1)$,*
(ii) *setting $(n + 1) = h \cdot m$ there is a polynomial Q of degree m s.t. $P(\zeta) = Q(\zeta^h)$.*

Proof. Since $P(\gamma(z)) = P(z), \forall z$ the orbit $\{\gamma^m(z) | m \in \mathbb{Z}\}$ has at most $(n + 1)$ points, hence γ is of finite order h.

Taking a coordinate ζ centred at the fixed point of γ, we see that $\gamma(\zeta) = \alpha\zeta$ with α a primitive h^{th} root of unity. If $P(\zeta) = \sum_{i=0}^{n+1} b_i(\zeta)^i$, $P(\gamma(\zeta)) = \sum_{i=0}^{n+1} \alpha^i b_i(\zeta)^i$, an expression which equals $P \Leftrightarrow b_i = 0$ if $h \nmid i$. Since $b_{n+1} \neq 0$, we easily get the desired conclusion. □

LEMMA 2. *Let $\gamma, \gamma' \in A(1, \mathbb{C})$ be such that $\gamma' \circ P \circ \gamma^{-1} = P$ and $(\gamma, \gamma') \neq (id, id)$. Then there do exist integers m, h, t with $h \geq 2$, $mh + t = n + 1$, a coordinate ζ on the source $\mathbb{C}$, a coordinate w in the target $\mathbb{C}$, a primitive h^{th} root of unity α and a polynomial Q of degree m s.t., in the new coordinates,*

(i) *$\gamma(\zeta) = \alpha\zeta$, $\gamma'(w) = \alpha^{-t} w$,*
(ii) *$P(\zeta) = Q(\zeta^h) \cdot \zeta^t$, where $Q(0) = 0$ if $t = 0$.*

Proof: If $\gamma'(w) = \alpha' w + \beta'$, $\gamma^{-1}(z) = \hat{\alpha} z + \beta$, then from $P = \gamma' \circ P \circ \gamma^{-1}$ I get $P'(z) = \alpha' P'(\hat{\alpha} z + \beta) \cdot \hat{\alpha}$ by the chain rule. Hence the roots of P' are made of γ-orbits, thus γ has finite order.

Hence there is a coordinate ζ in the source s.t. $\gamma^{-1}(\zeta) = \alpha\zeta$, and α is a primitive h^{th} root of unity. Up to a translation in the target, we can assume $P(0) = 0$. Then if $P(\zeta) = \sum_{i=1}^{n+1} b_i(\zeta)^i = \alpha' \sum_{i=1}^{n+1} b_i \alpha^i(\zeta)^i + \beta'$, then $\beta' = 0$ and, whenever $b_i \neq 0$, $\alpha' \cdot \alpha^i = 1$, hence all these i's are congruent to a fixed integer t modulo h, whereas if $h = 1$, $\gamma = \gamma' = id$. □

Definition 1. Calling ramification (or critical) points the roots of P', branch points (or critical values) their images under P, $P \in V_n$ is said to be

(1) simple if two distinct ramification points map to distinct branch points
(2) generic if it has n distinct branch points $w_1, \ldots, w_n$
(3) lemniscate generic if there are n branch points different from zero and with $|w_i| \neq |w_j|$ if $i \neq j$.

Remark 1. Notions (1), (2) are $A(1, \mathbb{C}) \times A(1, \mathbb{C})$-invariant, (3) is only $SO(2, \mathbb{R}) \times A(1, \mathbb{C})$-invariant.

Let $S_n \supset U_n \supset L_n$ be the respective subsets of simple, generic, lemniscate generic polynomials.

PROPOSITION 3. *$V_n - U_n$ is a (closed) complex hypersurface, $V_n - L_n$ is a real hypersurface.*

Proof. Define $\int z^i = z^{i+1}/(i+1)$. Then the map $\psi\colon \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^n \to V_n$, such that

$$\psi(a_0, a_{n+1}, y_1, \ldots, y_n) = (n+1)a_{n+1}\left(\int \prod_{i=1}^{n} (z - y_i)\right) + a_0$$

is a finite holomorphic map such that $V_n \cong \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^n/\mathscr{S}_n$ (note thus that the polynomial maps modulo automorphisms in the target give a space $A(1, \mathbb{C}) \backslash V_n \cong M_n \cong \mathbb{C}^n/\mathscr{S}_n$, where the last homeomorphism is given by "ramification locus"). Hence it suffices to show that $\psi^{-1}(V_n - U_n), \ldots$ and so on are as in the statement; we can explicitly write their polynomial equations, which involve only the variables $(y_1, \ldots, y_n) = y$. Let

$$P_y = (n+1)\int \prod_{i=1}^{n} (z - y_i).$$

Then the equation of $\psi^{-1}(V_n - U_n)$ is given by $\prod_{i<j} P_y(y_i) - P_y(y_j) = F(y_1, \ldots, y_n) = 0$, and the equation of $\psi^{-1}(U_n - L_n)$ can also be given explicitly. We notice that F is a semi-symmetric (alternating) function of $(y_1, \ldots, y_n)$; if $\sigma_1(y), \ldots, \sigma_n(y)$ are the elementary symmetric functions, we have

$$P_y = (n+1) \sum_{j=0}^{n} (-1)^j \frac{z^{n-j+1}}{(n-j+1)} \sigma_j(y)$$

and thus

$$F = \Delta \cdot G$$

where the "discriminant" Δ is the classical semi-symmetric function

$$\Delta = \prod_{i<j} (y_i - y_j) \text{ and } G \text{ is the symmetric function}$$

$$G = \prod_{i<j} \sum_{k=0}^{n} (-1)^k \frac{n+1}{(n-k+1)} \sigma_k(y) \frac{y_i^{n-k+1} - y_j^{n-k+1}}{y_i - y_j}. \qquad \square$$

Remark 2. We let $M'_n = \mathbb{C}^n - \{F(y) = 0\}/\mathscr{S}_n$. Then $U_n \cong A(1, \mathbb{C}) \times M'_n$.

Remark 3. For $n = 3$ we get $G = \text{constant} \prod_{i<j} (y_i - y_j)^2 (y_i + y_j - 2y_k)$ where $\{i, j, k\} = \{1, 2, 3\}$. In general $\Delta^2 | G$, but G/Δ^2 is not a product of linear forms. Notice finally that $\mathscr{S}_n$ is not open.

LEMMA 4. *$\{1\} \times A(1, \mathbb{C})$ acts freely on the open set U_n of generic polynomials.*

Proof. By Lemma 1, since otherwise there is a coordinate z in the source such that $P(z) = Q(z^h)$, with $h \geq 2$, and then either $P = (z^h)^m$ or P is not even simple: in fact $P'(z) = Q'(z^h) h z^{h-1}$ hence P is not simple if Q' has a non zero root. $\square$

Let W_n be $\mathbb{C}^n - \Delta/\mathscr{S}_n$, where Δ is the big diagonal $\{(w_1, \ldots, w_n) | \prod_{i<j} (w_i - w_j) = 0\}$, i.e. also W_n = monic polynomials of degree n with distinct roots.

If P is a generic polynomials, let $\{w_1, \ldots, w_n\} \in W_n$ be the set of its branch points: it is thus defined a polynomial map $\psi_n \colon U_n \to W_n$.

PROPOSITION 5. *ψ_n factors as $f_n \circ \varphi_n$, where $\varphi_n \colon U_n \to Z_n$ is an affine $A(1, \mathbb{C})$-principal bundle and f_n is a finite (unramified) connected covering space $f_n \colon Z_n \to W_n$.*

Proof. Take $\{w_1, \ldots, w_n\} = B \in W_n$ and consider the closed set $\psi_n^{-1}(\{B\})$, which is a union of $A(1, \mathbb{C})$ orbits (cf. Lemma 4). Each orbit is an isomorphism class of pairs $(\mathbb{P}^1, f \colon \mathbb{P}^1 \to \mathbb{P}^1)$ such that f has degree $n + 1$, $f(\infty) = \infty$ and f is totally ramified at ∞, and where $(\mathbb{P}^1, f) \cong (\mathbb{P}^1, f')$ if there exists an isomorphism g of $\mathbb{P}^1$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{P}^1 \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathbb{P}^1 \arrow[dl, "f'"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

Each pair $(\mathbb{P}^1, f)$ determines a connected covering $\mathbb{C} - f^{-1}(B)$ of $\mathbb{C} - B$ and in particular a monodromy homomorphism $\mu \colon \pi_1(\mathbb{C} - B, b_0) \to \mathscr{S}_{n+1}$, where the base point b_0 is chosen to be $b_0 = -2(\sqrt{-1}) \cdot \max_{w_i \in B} |w_i|$ and one has chosen a bijection $\{1, \ldots, n+1\} \cong f^{-1}(b_0)$. Clearly, if $\gamma_1, \ldots, \gamma_n$ are geometric generators of the free group $\pi_1(\mathbb{C} - B, b_0)$ (i.e., they are represented by simple loops around the respective points $w_1, \ldots, w_n$), then $\mu(\gamma_i)$ is a transposition τ_i. Conversely, given μ s.t.

($*$) $\mu \colon \pi_1(\mathbb{C} - B, b_0) \to \mathscr{S}_{n+1}$ maps a set of geometric generators to transpositions, and $Im(\mu)$ is a transitive subgroup, then, by Riemann's existence theorem, we have a connected compact Riemann surface (S, f) $S \xrightarrow{f} \mathbb{P}^1$ branched over B and, possibly, over ∞. By the

Riemann–Hurwitz formula, if g is the genus of S, we have, letting e_∞ be the branching index at ∞,

$$2g - 2 = -2(n+1) + n + e_\infty = -2 + e_\infty - n,$$

i.e., $2g = e_\infty - n$.

Since $g \geq 0$, $e_\infty \leq n$, we get $g = 0$ and $e_\infty = n$, in other words $S \cong \mathbb{P}^1$ and f is totally ramified at a point which we can assume to be ∞; hence we have $f(\infty) = \infty$, and f can be represented by a polynomial. Notice that f is unique up to an affine ($A(1, \mathbb{C})$ –) transformation in the source pointed Riemann surface $\cong \mathbb{P}^1$. Clearly, changing the bijection $\{1, \ldots, n+1\} \cong f^{-1}(b_0)$ alters μ only up to composing with an inner automorphism of $\mathscr{S}_{n+1}$, and we can thus define equivalence classes $[\mu]$ of such homomorphisms.

We let $Z_n = \{(B, [\mu]) | \mu\colon \pi_1(\mathbb{C} - B, b_0) \to \mathscr{S}_{n+1}$ satisfies $(*)\}$ which is a covering space of W_n since the fibration $T \subset \mathbb{C} \times W_n$, $T = \{(z, B) | z \notin B\}$ is locally trivial. We can conclude, since $\forall B\ \psi_n^{-1}(\{B\})$ consists then of a finite number of disjoint orbits, which are closed, hence $Z_n = U_n / A(1, \mathbb{C})$. Q.E.D.

PROPOSITION 6. *Let E_n be the set of (isomorphism classes of) edge labelled trees with n edges: then the choice of a geometric basis for $\pi_1(\mathbb{C} - B, b_0)$ determines a canonical bijection between $f_n^{-1}(B)$ and E_n.*

Proof. In the notation of the proof of Prop. 5, let $\tau_i = \mu(\gamma_i)$. Then we draw an edge for each τ_i and a vertex for each maximal subset R of $\{1, \ldots, n\}$ such that for $i, j \in R$ either $\tau_i = \tau_j$ or τ_i, τ_j don't commute. Then there is a bijection (μ is transitive) between the set of vertices and $\{1, \ldots, n+1\}$, and an edge τ_i has R as a vertex iff $i \in R$. □

We choose now as base point in W_n the set $B_0 = \{1, 2, \ldots, n\}$ and $b_{00} = -2n\sqrt{-1}$ as base point in $\mathbb{C} - B_0$. We recall (cf. [2]) that Artin's Braid group $\mathscr{B}_n$ can be defined as

(i) $\pi_1(W_n, B_0)$ and then the monodromy of the (tautological) fibration (defined in the proof of Prop. 5) $T \to W_n$ defines an isomorphism of $\mathscr{B}_n$ with

(ii) $\mathrm{Diff}^\infty(\mathbb{C} - B_0)/\mathrm{Diff}^{\infty,+}(\mathbb{C} - B_0)$ where ∞ means "equal to identity outside a circle of radius $2n$ around the origin", + means "isotopic to the identity". We can canonically fix geometrical generators $\gamma_1, \ldots, \gamma_n$ of $\pi_1(\mathbb{C} - B_0)$, then, via the action on $\pi_1(\mathbb{C} - B_0)$ induced by the monodromy, and the isomorphism of $\pi_1(\mathbb{C} - B_0)$ with the free group $\mathbb{F}_n$, we have an isomorphism

(iii) $\mathscr{B}_n = \{\varphi \in \mathrm{Aut}(\mathbb{F}_n) | (\gamma_1 \ldots \gamma_n)\varphi = \gamma_1 \ldots \gamma_n$ and $\exists \tau \in \mathscr{S}_n$ s.t. $(\gamma_i)\varphi$ is a conjugate of $\gamma_{\tau(i)}\}$.

Also, as an abstract group with generators and relations, $\mathscr{B}_n$ is the group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for} \quad |i - j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{cases}$$

and then the action of $\mathscr{B}_n$ on $\mathbb{F}_n$ is given by:

$$(\gamma_i)\sigma_i = \gamma_{i+1} \qquad (\gamma_{i+1})\sigma_i = \gamma_{i+1}^{-1}\gamma_i\gamma_{i+1}$$
$$(\gamma_j)\sigma_i = \gamma_j \quad \text{for} \quad j \neq i, i+1.$$

PROPOSITION 7. *The monodromy $\lambda_n\colon \mathscr{B}_n \to \mathscr{S}(E_n)$ of f_n is defined by $[\mu](\lambda_n(\varphi)) = [\varphi^{-1}\mu]$ (or, in more traditional notation, $\lambda_n(\varphi)([\mu]) = [\mu \circ \varphi^{-1}]$).*

Proof. The monodromy of φ gives a diffeomorphism of $\mathbb{C} - B_0$ bringing the canonical basis $\gamma_1, \ldots, \gamma_n$ to a new geometrical basis $\gamma'_1, \ldots, \gamma'_n$ in terms of which is now expressed μ, whereas $\mu \circ \varphi^{-1}$ expresses it in terms of the geometrical basis. □

§2. THE ALGEBRAIC COMPUTATION

Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be a fixed basis of a free group $\mathbb{F}_n$, and let $\sigma_1, \ldots, \sigma_{n-1}$ be the standard generators of a braid group $\mathscr{B}_n$, which act on $\mathbb{F}_n$ by the formulae:

$$(\gamma_i)\sigma_i = \gamma_{i+1}, \quad (\gamma_{i+1})\sigma_i = (\gamma_{i+1})^{-1}\gamma_i\gamma_{i+1}, \quad (\gamma_j)\sigma_i = \gamma_j \quad \text{for} \quad j \neq i, i+1.$$

Let $\tilde{E}_n$ be the set of the homomorphisms $\mu\colon \mathbb{F}_n \to \mathscr{S}_{n+1}$ into the symmetric group on $(n+1)$ letters such that μ is onto and $\tau_i = (\gamma_i)\mu$ is a transposition for every i. Let E_n be the set of the equivalence classes on $\tilde{E}_n$, where $[\mu] = [\nu]$ if and only if there exists $\pi \in \mathscr{S}_{n+1}$ such that $\pi((\gamma)\mu)\pi^{-1} = (\gamma)\nu$ for all $\gamma \in \mathbb{F}_n$. Then $\mathscr{B}_n$ acts on $\tilde{E}_n$, cf. Proposition 7, by composition: $(\gamma)(\mu\varphi) = (((\gamma)\varphi^{-1})\mu$ for $\mu \in \tilde{E}_n$, $\varphi \in \mathscr{B}_n$, $\gamma \in \mathbb{F}_n$. This action induces an action on E_n. The set E_n can be identified with a fiber of the covering $f_n\colon Z_n \to W_n$, and by Proposition 7 the action of $\mathbb{B}_n$ on E_n is the monodromy action $\lambda_n\colon \mathscr{B}_n \to \mathscr{S}(E_n)$ of the covering.

We want to find the fundamental group of Z_n. We fix a base point $[\mu_0]$ such that $(\gamma_i)\mu_0 = (i, i+1) \in \mathscr{S}_{n+1}$. Then $G_n = \pi_1(Z_n, [\mu_0])$ is isomorphic to the stabilizer of $[\mu_0]$ in $\mathscr{B}_n$: $G_n = \{\varphi \in \mathscr{B}_n | [\mu_0\varphi] = [\mu_0]\}$.

Let $\delta_s = \sigma_{s-1}\sigma_{s-2} \ldots \sigma_1 \in \mathscr{B}_n$ for $s = 2, 3, \ldots, n$. Then $(\Delta_n)^2 = (\delta_n)^n$ is a generator of the center of $\mathscr{B}_n$.

Let $H_n = \{\varphi \in \mathscr{B}_n | \mu_0\varphi = \mu_0\} \subset G_n$, and let $K_n = \langle(\delta_s)^{s+1}, s = 2, 3, \ldots, n\rangle \subset \mathscr{B}_n$. We want to prove the following result.

THEOREM 8. *$H_n = K_n$, while G_n is generated by $(\Delta_n)^2$ and by $(\delta_s)^{s+1}$, $s = 2, 3, \ldots, n-1, n$.*

We shall denote by π_0 the $(n+1)$-cycle $(n+1, n, \ldots, 2, 1) \in \mathscr{S}_{n+1}$ and by γ_{n+1} the element $(\gamma_n)\delta_n \in \mathbb{F}_n$. We shall first establish some simple properties of K_n, H_n, G_n and $\mathscr{B}_n$.

LEMMA 9.

(a) $K_n \subset H_n$.

(b) $(\gamma_i)\delta_n = \gamma_{i+1}$, $i = 1, \ldots, n$, $(\gamma_{n+1})\mu_0 = (1, n+1)$, *thus for every* $\gamma \in \mathbb{F}_n$ $(\gamma\delta_n)\mu_0 = \pi_0((\gamma)\mu_0)\pi_0^{-1}$ *and* $\delta_n \in G_n$.

(c) $(\gamma_1\gamma_2 \ldots \gamma_n)\varphi\mu_0 = \pi_0$ *for every* $\varphi \in B_n$.

(d) $G_n = \langle H_n, (\Delta_n)^2\rangle$, *hence H_n is a normal subgroup of G_n of index $n+1$.*

Proof. Notice first of all that δ_s only affects $\gamma_1, \ldots, \gamma_s$, and, setting $\gamma_{(s)} = \gamma_1 \ldots \gamma_s$, $\mathscr{B}_{s-1}$ acts trivially on $\gamma_{(s)}$. In particular $(\gamma_i)\delta_s = \gamma_{i+1}$ for $i \leq s-1$, whereas $(\gamma_s)\delta_s = (\gamma_{(s)})^{-1}\gamma_1\gamma_{(s)}$. Hence follows easily $(\gamma_i)\delta_s^s = (\gamma_{(s)})^{-1}\gamma_i\gamma_{(s)}$ for each $i = 1, \ldots, s$. Therefore

$$\begin{cases} (\gamma_i)\delta_s^{s+1} = (\gamma_{(s)})^{-1}\gamma_{i+1}\gamma_{(s)} & \text{for } i \leq s-1 \\ (\gamma_s)\delta_s^{s+1} = (\gamma_{(s)})^{-2}\gamma_1\gamma_{(s)}^2 & \\ (\gamma_i)\delta_s^{s+1} = \gamma_i & \text{for } i \geq s+1. \end{cases}$$

Part (a) follows immediately, because all we have to verify is that $(\delta_s)^{s+1} \in H_n$, i.e.,

$$\gamma_i(\delta_s)^{s+1}\mu_0 = (i, i+1).$$

But indeed $\gamma_{(s)}\mu_0 = (s+1, \ldots, 1)$ and the desired equalities are then verified. In particular $\gamma_{n+1} = (\gamma_n)\delta_n = (\gamma_{(n)})^{-1}\gamma_1\gamma_{(n)}$, hence $(\gamma_{n+1})\mu_0 = (1, n+1)$ and thus we have checked also part (b). (c) follows since, as we noticed, $(\gamma_1 \ldots \gamma_n)\varphi = \gamma_{(n)}$, and $\gamma_{(n)}\mu_0 = \pi_0 = (n+1, \ldots, 1)$.

Let's prove now part (d). By our previous computations $(\Delta_n)^2$ acts on $\mathbb{F}_n$ by $\gamma \to (\gamma_{(n)})^{-1}\gamma\gamma_{(n)}$, hence $(\Delta_n)^2 \in G_n$. Let $\varphi \in G_n$. There exists (by definition of G_n) $\pi \in \mathcal{S}_n$ such that $\pi((\gamma\varphi)\mu_0)\pi^{-1} = \gamma\mu_0$ for all $\gamma \in \mathbb{F}_n$. In particular $\pi((\gamma_1 \ldots \gamma_n)\varphi\mu_0)\pi^{-1} = (\gamma_1 \ldots \gamma_n)\mu_0$. Thus $\pi\pi_0\pi^{-1} = \pi_0$ and $\pi = (\pi_0)^k$ for some k. Therefore $(\gamma\varphi)(\delta_n)^{nk}\mu_0 = \gamma\mu_0$ $\forall\gamma$ and $\varphi(\delta_n^n)^k \in H_n$. Hence, $\varphi(\Delta_n)^{-2k} \in H_n$. We finally observe that $(\Delta_n^2)^k \in H_n$ if and only if k is a multiple of $(n+1)$. Q.E.D.

In order to prove Theorem 8 it is enough to prove that $H_n = K_n$. We shall prove the following stronger statement by induction on n.

PROPOSITION 10. *$H_n = K_n$. If $\varphi \in \mathcal{B}_n$ and $\alpha = (\gamma_i)\varphi$ and $(\alpha)\mu_0 = (j, j+1)$ for some $1 \le i$, $j \le n$ then there exists $\psi \in K_n$ such that $(\alpha)\psi = \gamma_j$.*

INDUCTION HYPOTHESIS. *Proposition 10 is true for $n-1$.*

For $s < n$ we can identify H_s and K_s with subgroups of G_n. Indeed we can restrict elements of $\tilde{E}_n$ to $\mathbb{F}_s = \langle\gamma_1, \ldots, \gamma_s\rangle \subset \mathbb{F}_n$. Also $\mathcal{B}_s$ is naturally isomorphic to $\langle\sigma_1, \ldots, \sigma_{s-1}\rangle \subset \mathcal{B}_n$ and the action of $\mathcal{B}_s$ on $\mathbb{F}_n$ is completely determined by its restriction to $\mathbb{F}_s$, since it acts trivially on $\gamma_{s+1}, \ldots, \gamma_n$. Then $H_s = \{\varphi \in \mathcal{B}_s | (\gamma\varphi)\mu_0 = (\gamma)\mu_0 \text{ for } \gamma \in \mathbb{F}_s\} \subset H_n$, and $K_s = \langle(\delta_j)^{j+1}, j = 2, \ldots, s\rangle$. By the induction hypothesis $H_{n-1} = K_{n-1}$.

LEMMA 11. *If $\varphi \in \mathcal{B}_n$ and $(\gamma_n)\varphi = \gamma_n$ then $\varphi \in \mathcal{B}_{n-1}$. If also $\varphi \in H_n$ then $\varphi \in K_n$. If $\varphi \in H_n$ and $(\gamma_1)\varphi = \gamma_1$ then $\varphi \in \langle\sigma_2, \ldots, \sigma_{n-1}\rangle$.*

Proof. The first and the last part follow from the description of $\mathcal{B}_n$ as $\mathrm{Diff}^\infty(\mathbb{C} - B_0)$. The second part follows from the induction hypothesis. □

By a *geometric basis* of $\mathbb{F}_n$ we shall mean (as in § 1) a basis of the form $(\gamma_1)\varphi, \ldots, (\gamma_n)\varphi$ for some $\varphi \in \mathcal{B}_n$. An element α of $\mathbb{F}_n$ is *simple* if it belongs to some geometric basis (i.e., recalling characterization (iii) above of $\mathcal{B}_n$, α is simple iff it is represented by a simple loop around a branch point (the conjugate by a path of a small circle around the branch point): hence if $\alpha \in \mathbb{F}_{n-1}$ is simple in $\mathbb{F}_n$, it is also simple in $\mathbb{F}_{n-1}$).

LEMMA 12. *Let α and γ_i belong to the same geometric basis and let $(\alpha)\mu_0 = (\gamma_j)\mu_0$. Then there exists $\varphi \in K_n$ such that $(\alpha)\varphi = \gamma_j$.*

Proof. Let $\alpha = (\gamma_p)\tau$, $\gamma_i = (\gamma_q)\tau$, $1 \le p$, $q \le n$. If $q < p$ let $s = -i+1$ and let $\psi = \sigma_1 \ldots \sigma_{q-1}\tau(\delta_n)^s$. If $p < q$ let $s = n - i$ and let $\psi = (\sigma_q \ldots \sigma_{n-1})^{-1}\tau(\delta_n)^s$. If $p < q$ then $(\gamma_n)\psi = \gamma_n$ so $\psi \in B_{n-1}$, by Lemma 11, and therefore $(\gamma_p)\psi = (\alpha)(\delta_n)^s \in \mathbb{F}_{n-1}$. If $q < p$ then $(\gamma_1)\psi\delta_n = \gamma_1$ so $\psi\delta_n \in \langle\sigma_2, \ldots, \sigma_{n-1}\rangle$ and thus $(\gamma_p)\psi\delta_n \in \langle\gamma_2, \ldots, \gamma_n\rangle$ hence $(\gamma_p)\psi = (\alpha)(\delta_n)^s \in \mathbb{F}_{n-1}$ by Lemma 9(b). In both cases, by Lemma 9(b) there exists an integer k, $1 \le k \le n-1$, such that $(\gamma_p\psi)\mu_0 = (\gamma_k)\mu_0$ and either $\gamma_k = (\gamma_j)(\delta_n)^s$ or $\gamma_k = (\gamma_j)(\delta_n)^s(\delta_n)^{\pm(n+1)}$. In fact, $(\gamma_p\psi)\mu_0 = (\alpha)(\delta_n)^s\mu_0 = \pi_0^s(\alpha\mu_0)\pi_0^{-s}$. Thus, by the remark made previously about simple elements, we can apply the induction hypothesis and there exists therefore $\varphi \in K_n$ such that $(\alpha)(\delta_n)^s\varphi = \gamma_k$. Since $(\delta_n)^n$ is in the center of B_n and $(\delta_n)^{n+1} \in K_n$, δ_n normalizes K_n, therefore $(\delta_n)^s\varphi(\delta_n)^{-s} \in K_n$. It follows that there exists $\varphi_1 \in K_n$ such that $(\alpha)\varphi_1 = \gamma_j$ as required. Q.E.D.

For $\tau \in \mathcal{B}_n$ the *length* of τ is the length of the shortest word in the letters σ_i and $(\sigma_i)^{-1}$ representing τ. For a simple element α the *index* $I(\alpha)$ of α is the length of the shortest element $\tau \in \mathcal{B}_n$ such that $\alpha = (\gamma_i)\tau$ for some i, $1 \le i \le n$.

LEMMA 13. *Let α be a simple element such that $(\alpha)\mu_0 = (\gamma_p)\mu_0$ for some p. Then there exists $\varphi \in K_n$ such that $(\alpha)\varphi = \gamma_p$.*

Proof. (By induction on $l(\alpha)$). If $l(\alpha) = 0$ then $\alpha = \gamma_p$. Let $l(\alpha) = k$ and let τ be an element of length k such that $(\gamma_q)\tau = \alpha$, where $1 \le q \le n$. We want to find a geometric basis which contains α and an element β such that $l(\beta) < k$ and $(\beta)\mu_0 = (\gamma_j)\mu_0$ for some j. Let

$$\tau_1 = \sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)\dots(\sigma_{q-2}\sigma_{q-3}\dots\sigma_1)(\sigma_{q+1}(\sigma_{q+2}\sigma_{q+1}) \dots(\sigma_{n-1}\sigma_{n-2}\dots\sigma_{q+1}))^{-1}$$

Let $\psi = \tau_1\tau$. Consider the basis $(\gamma_s)\psi$, $s = 1, \dots, n$. Then $(\gamma_q)\psi = \alpha$. Among the letters $1, 2, \dots, n+1$ there is at least one, say j, which appears only in one transposition $(\gamma_s\psi)\mu_0$ and does not appear in $(\gamma_q\psi)\mu_0$, since a tree with n edges, $n \ge 2$, has at least one vertex not lying on a given edge. We may assume by symmetry that $s < q$. Consider $\psi_1 = \delta_s^{-1}\psi = (\sigma_{s-1}\sigma_{s-2}\dots\sigma_1)^{-1}\psi$. We have $(\gamma_i)\psi_1 = (\gamma_{i-1})\psi$ for $2 \le i \le s$, $(\gamma_i)\psi_1 = (\gamma_i)\psi$ for $s+1 \le i \le n$, and one can verify that $(\gamma_1)\psi_1 = (\gamma_{q-s})\tau$. Clearly the letter j does not appear in $(\gamma_i\psi_1)\mu_0$ for $i \ne 1$ so it appears only in $(\gamma_1\psi_1)\mu_0$. Since $((\gamma_n\dots\gamma_1)\psi_1)\mu_0 = \pi_0^{-1}$, and j does not appear in $(\gamma_i\psi_1)\mu_0$ for $i \ge 2$, we must have $(\gamma_1\psi_1)\mu_0 = (\gamma_j)\mu_0$. Since $\alpha = (\gamma_q)\tau$ has index k, the first factor of τ is σ_{q-1} or $(\sigma_q)^{-1}$. It follows that $l(\gamma_1\psi_1) < k$ (so $\gamma_1\psi_1$ is the sought for β!). By the induction hypothesis there exists $\varphi \in K_n$ such that $(\gamma_1\psi_1)\varphi = (\gamma_j)$. Then $(\gamma_q\psi_1)\varphi = (\alpha)\varphi$ and $(\alpha\varphi)\mu_0 = (\gamma_p)\mu_0$, because $\varphi \in K_n$, and therefore, by Lemma 12, there exists $\varphi_1 \in K_n$ such that $(\alpha\varphi)\varphi_1 = \gamma_p$ as required. Q.E.D.

Proof of Proposition 10. The proof follows by induction. In fact, Lemma 13 provides the proof for the second assertion. Let now $\varphi \in H_n$. Then $(\gamma_n\varphi)\mu_0 = (\gamma_n)\mu_0$ and by Lemma 13 there exists $\varphi_1 \in K_n$ such that $(\gamma_n\varphi)\varphi_1 = \gamma_n$. Now $\varphi\varphi_1 \in H_n$ and by Lemma 11 $\varphi\varphi_1 \in K_n$. It follows that $\varphi \in K_n$. Q.E.D.

§3. PROOF OF THE MAIN THEOREM

We summarize the geometric set up of §1 in the following diagram:

$$A(1,\mathbb{C}) \times M'_n \cong U_n \xrightarrow{\varphi_n} U_n/A(1,\mathbb{C}) = Z_n \xrightarrow{f_n} W_n. \tag{3.1}$$

Thus, setting $H'_n = \pi_1(M'_n)$, since $\Gamma_n = \pi_1(U_n)$, $G_n = \pi_1(Z_n)$, and φ_n is an affine $A(1,\mathbb{C})$-bundle, we have the following diagram of group homomorphisms

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z} \arrow[r, "i"] \arrow[dr, shift left, "i_1"] & \Gamma_n \arrow[r, "\varphi_*"] \arrow[d, equal] \arrow[dl, shift left] \arrow[dr, shift left] & G_n \arrow[r] & 1 \\
& \mathbb{Z} & \mathbb{Z} \times H'_n \arrow[l, "p_1"] \arrow[r, "p_2"] \arrow[ur, shift left, "i_2"] & H'_n &
\end{tikzcd} \tag{3.2}$$

THEOREM 14.

(a) *$p_1 \circ i$ is given by multiplication by $(n+1)$.*

(b) $\varphi_*(i_1(1)) = (\Delta_n)^2$

(c) *$\varphi_* \circ i_2$ gives an isomorphism of H'_n with H_n.*

(d) *using the isomorphism $\varphi_* \circ i_2$ to identify H'_n and H_n, $i(\mathbb{Z}) = \ker\varphi_*$ is generated by $i_1(n+1)\cdot i_2(\Delta_n^{-2(n+1)})$.*

Proof. A path representing $i(1)$ is given by an orbit of the 1-parameter subgroup $(P,t) \to P(e^{2\pi i t}z)$. On the other hand, p_1 is associated to the map carrying a polynomial P to its leading coefficient a_{n+1}, hence, since the leading coefficient of $P(e^{2\pi it}z)$ is $a_{n+1}\cdot(e^{2\pi(n+1)t})$, we have shown (a).

(b): $\varphi_*(i_1(1))$ is represented by the braid moving B_0 to $e^{2\pi it}B_0$, and it is well-known that this braid is $(\Delta_n)^2$ (cf. [5]).

(c) Let's first show that $\varphi_*(i_2(H'_n)) \subset H_n$. In fact $\varphi_*(i_2(H'_n))$ consists of the braids coming from paths β in the subspace M'_n of monic polynomials with vanishing constant term. We have seen algebraically in Lemma 9 that H_n is the kernel of the epimorphism of G_n to the cyclic group of order $(n + 1)$ generated by π_0 in $\mathscr{S}_n$. Geometrically, this homomorphism is easily seen to be given as follows: you take a path β in U_n, and you choose the base point in $\mathbb{C} - B_0$ in a sufficiently big circle such that all polynomials in β are uniformly well approximated by the map $z' \to [1/(a_{n+1})](z')^{n+1}$, where z' is a local coordinate in the Reimann sphere around the point ∞.

The inverse images of the base point are then permuted cyclically, according to how a_{n+1} moves in the path β. But if β is in M'_n, then $a_{n+1} = 1$ constantly on β, hence the inverse images are not permuted, and the braid is in H_n.

Now, $i(1)$ is a generator of the kernel of φ_* and, by (a), $i(1) \in \mathbb{Z} \times H'_n$ is a pair $(n + 1, h')$ with $\varphi_* i_2(h') = (\Delta_n^2)^{-(n+1)}$, and also in particular $i_2(H'_n)$ has trivial intersection with $\ker(\varphi_*)$, hence H'_n is isomorphic to a subgroup of H_n, which contains $(\Delta_n^2)^{n+1}$. Since φ_* is surjective, the index of H'_n in G_n is at most $n + 1$: hence H'_n is isomorphic to H_n via $\varphi_* \circ i_2$, and (d) follows immediately. Q.E.D.

As the reader will have noticed, Theorem 14 is a more precise formulation of the main theorem.

We consider now the epimorphism $H_n \xrightarrow{\tau} \mathscr{B}_n$ induced by the inclusion $M'_n \subset M_n$ (cf. Remark 2, §1).

First of all take now in M'_n as base point $P_0 = z^{n+1} - (n + 1)z = (n + 1)\int(z^n - 1)$, and consider the path in M'_n based on P_0

$$\gamma(t) = e^{2\pi it(n+1)/n} P_0(z \cdot e^{-2\pi it/n}). \tag{3.3}$$

Clearly $\varphi_*(\gamma)$ is an element of $H_n(\gamma(t) \in M'_n)$ and indeed, since φ is not affected by the action of $A(1, \mathbb{C})$ in the source $\mathbb{C}$, $\varphi_*(\gamma)$ is the braid given by the 1-parameter group $(w, t) \to e^{2\pi it(n+1)/n}w$. This 1-parameter group gives a diffeomorphism of $\mathbb{C} - B'_0$, where B'_0 is the branch set of P_0, i.e. $B'_0 = \{P_0(\zeta) | \zeta^n = 1\} = \{-n\zeta | \zeta^n = 1\}$. It follows then that

$$\varphi_*(\gamma) \text{ is the braid } (\delta_n)^{n+1}. \tag{3.4}$$

On the other hand, if we only look at the ramification points of P, the effect of $\gamma(t)$ on the roots $y_1, \ldots, y_n$ of P' is given by the action of the 1-parameter group $(y, t) \to ye^{2\pi it/n}$ on them. We have therefore

$$\tau(\gamma_n^{n+1}) = \gamma_n. \tag{3.5}$$

THEOREM 15. *Under the isomorphism $H_n \cong \pi_1(M'_n)$ (cf. Theorem 14), the epimorphism $\tau: H_n = \pi_1(M'_n) \to \pi_1(M_n) = \mathscr{B}_n$ induced by the inclusion $M'_n \subset M_n$ is such that*

$$\tau(\gamma_s^{s+1}) = \gamma_s.$$

Proof. We shall prove the result by induction on n. Recall that $H_{n-1} \subset \mathscr{B}_{n-1} \subset \mathscr{B}_n$. In view of (3.5) it will suffice to show that there is a homomorphism $j_*: \pi_1(M'_{n-1}) \to \pi_1(M'_n)$ such that

(i) the composition of $j_*: H'_{n-1} = \pi_1(M'_{n-1}) \to \pi_1(M'_n) = H'_n$ with $\varphi_* \circ i_2$ coincides with the previously described isomorphism of H'_{n-1} with the given subgroup H_{n-1} of $\mathscr{B}_{n-1}$ composed with a fixed embedding of $\mathscr{B}_{n-1}$ into $\mathscr{B}_n$.

(ii) under the natural epimorphism of $H'_n \twoheadrightarrow \mathscr{B}_n$, $j_*(H'_{n-1})$ lands into $\mathscr{B}_{n-1}$, by a homomorphism equal to the natural epimorphism $H'_{n-1} \twoheadrightarrow \mathscr{B}_{n-1}$.

To define j_*, notice that $\pi_1(M'_{n-1})$ is generated by a finite number of loops in M'_{n-1} which therefore lie on a compact set K of M'_{n-1}.

Now, if $Q \in K$, there do exist $(y_1, \ldots, y_{n-1})$ such that

$$Q = n \int \prod_{i=1}^{n-1} (z - y_i)$$

Let $u \in \mathbb{R}$ be a negative number with $|u| \gg 0$: then we define

$$j_u(Q) = P = (n+1) \int \prod_{i=1}^{n-1} (z - y_i) \cdot (z - u) \tag{3.6}$$

Clearly j_u sends $K \subset M_{n-1}$ into M_n if

$$|u| > \max_{Q \in K} \max_i |y_i|$$

On the other hand, integration by parts gives

$$P = \left(\frac{n+1}{n}\right)\left[Q \cdot (z-u) - \int Q\right], \tag{3.7}$$

hence

$$P(u) = -\frac{n+1}{n}\left(\int Q\right)(u) = -\frac{1}{n} u^{n+1} + O(|u|^n), \tag{3.8}$$

where, if $Q \in K$, the remainder is in absolute value $\leq \mathrm{Cost}\,|u|^n$. Whereas,

$$P(y_i) = -\frac{n+1}{n} \cdot uQ(y_i) + O(1), \tag{3.9.}$$

(again, the remainder is bounded by a constant on K). Since there is $\varepsilon > 0$ such that $|Q(y_j) - Q(y_i)| > \varepsilon$ (for $i \neq j$) uniformly on K, it follows that for $|u| >$ cost, then $P \in M'_n$.

Let us consider the paths $y_i(t)$ ($t \in [0, 1]$) which induce generators of $\pi_1(M'_{n-1})$. Then the $Q_t(y_i(t))$ describe a finite number of braids and it is rather clear that there is a constant $\delta > 0$ such that, if $|f_i(t)| < \delta$, then the braids given by the roots $Q_t(y_i(t))$ and by the perturbed roots $Q_t(y_i(t)) + f_i(t)$ are isotopic. By (3.9), choosing u such that $|u| > [O(1)n]/[(n+1)\delta]$, we see that the braid associated to the roots $P_t(y_i(t))$ is the same as the one we started with.

On the other hand, if we fix u with $|u| \gg 0$, the $P_t(y_i(t))$ move in a circle with center the origin and of radius cost $|u|$, while $P_t(u)$ moves in a disjoint circle (with center on $-(1/n)\,u^{n+1}$). Hence the braid of P_t belongs to a fixed embedding of $\mathscr{B}_{n-1}$ and coincides, in terms of this embedding, with the braid of Q_t. Q.E.D.

Acknowledgements.—Both authors would like to thank Columbia University and the organizers of the special year in Algebraic Geometry 1986–1987 for providing a stimulating environment where their cooperation begun. The first author would like also to thank the University of Amsterdam and the Max Planck Institute in Bonn; this research was motivated by some unpublished results obtained during his stay in Amsterdam in April 1985, and in Bonn the final version of the paper was begun. The second author acknowledges the hospitality of the University of Pisa, and support from the (Italian) M.P.I.

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