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POLYNOMIAL-LEMNISCATES, TREES AND BRAIDS

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§0. INTRODUCTION

LET $P \in \mathbb{C}[z]$ be a polynomial of degree $(n + 1)$. As in [2], we define P to be lemniscate generic if, $y_1, \ldots, y_n$ being the roots of P', then, setting $w_i = P(y_i)$, one has $w_i \neq 0$ for each $i = 1, \ldots, n$, and $|w_i| = |w_j| \Rightarrow i = j$. As we shall see in §1, P is lemniscate-generic if and only if $|P|^2$ has only non degenerate critical points, and has distinct critical values of index 1. In general, a big lemniscate of P is a singular level set $\Gamma_c = \{z \mid |P(z)| = c > 0\}$ of $|P|$, whereas a small lemniscate of P is a connected singular component of a big lemniscate.

In this context, P is lemniscate-generic if and only if P has $(n + 1)$ distinct roots, n big lemniscates, each one of which has only one small lemniscate, having only one singularity, which is an ordinary (real) double point, or *node* (cf. Fig. 1).

In the open set V_n of polynomials of degree exactly $(n + 1)$, lemniscate-generic polynomials form an open set $\mathscr{L}_n$ whose complement $V_n - \mathscr{L}_n$ is a union of real hypersurfaces.

To $P \in \mathscr{L}_n$ we associate the big lemniscate configuration, i.e. if Γ is the union of the singular level sets Γ_c of $|P|$ $(c \geq 0)$, the homeomorphism (or diffeomorphism) class of the pair $(\Gamma, \mathbb{C})$).

It is clear that the big lemniscate configuration does not change if P varies in a connected component of $\mathscr{L}_n$, and one of the main purposes of this article is to show that conversely, if P_1, P_2 have the same configuration, then they lie in the same connected component of $\mathscr{L}_n$.

The key point is that if E_n is the set of isomorphism classes of edge labelled trees with n edges, there is a natural mapping (cf. Fig. 1) from E_n to the set of big lemniscate configurations; moreover, as in [2], there is a natural action of the braid group $\mathscr{B}_n$ on E_n, and a natural isomorphism of $\mathbb{Z}^{n-1}$ with a subgroup Λ_n of $\mathscr{B}_n$.

Using these ideas, we can also give a very easy combinatorial description of all the connected components of $\mathscr{L}_n$.

We can now state the

MAIN-THEOREM. *There exists a natural bijection between*

(i) $\pi_0(\mathscr{L}_n)$

(ii) *The set of big lemniscate configurations*

(iii) *The set of* Λ_n*-orbits on* E_n

(iv) *The set* $\Sigma_{n-2} = \{(x_0, \ldots, x_{n-2}) \mid x_i \in \mathbb{N},\ 0 \leq x_i \leq i$, *no integer occurs three times in the sequence, i.e.,* $\forall m \in \mathbb{N}$ *the equation* $x_i = m$ *has at most two solutions* $x_i\}$.

Interest for this research stemmed from the second author's investigations [6] on ordinary differential equations integrable by rational functions (after a suitable change of coordinates).

In fact, the level curves $|P(z)| = c$ are the solutions of the O.D.E. $dz/dt = i\frac{P(z)}{P'(z)}$. In this other context, the small lemniscates are the singular solutions of the given O.D.E., and our main theorem allows also a combinatorial description of the small lemniscate configurations, and of how a sequence of numbers $(x_0, \ldots, x_{n-1}) \in \Sigma_{n-2}$ determines a small configuration.

In particular we see that all the a priori possible big and small configurations do in fact occur, and moreover (§4) that this can be achieved by using a polynomial with real coefficients.

The appendix is devoted to a counting problem: namely, if $a(n)$ is the number of big lemniscate configurations, and $b(n)$ is the number of small lemniscate configurations, we describe the generating functions

$$\sum b(n)t^n \quad \text{and} \quad \sum \frac{a(n)}{n!}t^n,$$

showing in particular that the latter function equals $(1 - \sin t)^{-1}$, whereas for the former we can give a sequence rapidly convergent to its radius of convergence, thereby at least estimating the asymptotic growth of the coefficients $b(n)$.

§1. THE BRANCH-POINTS FIBRATION

LEMMA 1.1. *$P \in \mathcal{L}_n \leftrightarrow |P|^2$ is a local Morse function (i.e., it has only non degenerate critical points) with distinct critical values of index 1.*

Proof. $|P|^2 = P\bar{P}$. Since $\partial(P\bar{P}) = (\partial P)\bar{P}$, $\bar{\partial}(P\bar{P}) = P(\overline{\partial P})$, the critical points are exactly the points where either P or $\partial P = P'$ vanishes. Clearly, when $P = 0$, the critical point z is non degenerate iff we have a simple root of P, i.e. $P(z) = 0 \Rightarrow P'(z) \neq 0$. On the other hand, if $P'(z) = 0$, since the Hessian matrix of $P\bar{P}$ at z is given by $\begin{pmatrix} \bar{P}\cdot\partial^2 P & 0 \\ 0 & P(\overline{\partial^2 P}) \end{pmatrix}$, we have a non degenerate critical point (of signature $(+1, -1)$) if and only if $\partial^2 P(z) \neq 0$.

Hence, if $P\bar{P}$ is a local Morse function, then P' has n distinct roots $y_1, \ldots, y_n$, while P has $(n+1)$ distinct roots $z_0, \ldots, z_n$, with $z_i \neq y_j$.

Finally, the critical values of index 1, if we set $P(y_i) = w_i$, are the n real numbers $|w_1|^2, \ldots, |w_n|^2$, and, since $w_i \neq 0$ (y_i is not a root of P), requiring moreover that they must be different is equivalent to require that $P \in \mathcal{L}_n$. Q.E.D.

Remark 1.2. If $P \in \mathcal{L}_n$, we can choose an ordering $y_1, \ldots, y_n$ of the roots of P' such that $0 < |w_1| < |w_2| < \ldots |w_n|$. Then Lemma 1.1 tells us that Γ, the big lemniscate configuration of P, equals the union of the finite set $P^{-1}(0)$ with the big lemniscates $\Gamma_{|w_1|}, \ldots, \Gamma_{|w_n|}$, where, as in the introduction, $\Gamma_c = \{z \mid (P(z)| = c\}$.

We notice that y_i is the unique singular point of $\Gamma_{|w_i|} \overset{\text{def}}{=} \Gamma_i$, and the singularity is a node (locally analytically isomorphic to $x^2 - y^2 = 0$, i.e. two smooth branches meeting transversally, cf. Fig. 1). We denote by Δ_i the connected component of Γ_i containing y_i.

Remark 1.3. Standard Morse theory (cf. [5]) shows that Γ_c and $\Gamma_{c'}$ are diffeomorphic by the gradient flow if $[c, c']$ contains no $|w_i|$, that Δ_i is an eight figure, and that Γ_c has exactly

$n+1-i$ connected components if (setting $|w_0| = 0$, $|w_{n+1}| = +\infty$) $|w_i| \leq c < |w_{i+1}|$.

Let's briefly recall the method used in [2] to study polynomial maps, and mention, without proof, some of the results stated.

One defines U_n to be $\{P \in \mathbb{C}[z] | \deg P = n+1,\ P$ has n distinct branch points $w_1, \ldots, w_n\}$.

Then, to $P \in U_n$ one can associate the branch set $B_P = \{w_1, \ldots, w_n\} \in W_n = \mathbb{C}^n - \Delta/\mathscr{S}_n$, where $\Delta = \{(w_1, \ldots, w_n) | w_i = w_j$ for some i, j with $i \neq j\}$ and $\mathscr{S}_n$ is the permutation group on n letters. W_n may also be viewed as the space of all monic polynomials $Q(z)$ of degree n with n distinct roots

$$Q(z) = \prod_{i=1}^{n} (z - w_i). \tag{1.4}$$

$$P \to B_P \text{ defines a holomorphic map } U_n \xrightarrow{\psi_n} W_n \tag{1.5}$$

which factors as $U_n \xrightarrow{\varphi_n} Z_n \xrightarrow{f_n} W_n$, and (cf. Prop. 5 of [2]) φ_n is a principal bundle with fibre the group $A(1, \mathbb{C})$ of coordinate transformations in the source $\mathbb{C}$, and $f_n : Z_n \to W_n$ is a (connected) covering space.

Definition 1.6. Let E_n be the set of isomorphism classes of edge labelled trees with n edges (cf. Fig. 1).

Remark 1.7. E_n coincides with the equivalence classes of homomorphisms $\mu : \mathbb{F}_n \to \mathscr{S}_{n+1}$ such that

(i) $\mathbb{F}_n$ is a fixed free group with generators $\gamma_1, \ldots, \gamma_n$

(ii) $\mathscr{S}_{n+1}$ is the symmetric group in $(n+1)$ letters and $\mu(\gamma_i) = \tau_i$ is a transposition

(iii) $\mu(\mathbb{F}_n)$ is a transitive subgroup

(iv) μ and μ' are said to be equivalent (we shall write $[\mu] = [\mu']$) iff there exists $\pi \in \mathscr{S}_{n+1}$ such that

$$\mu'(\gamma) = \pi(\mu(\gamma))\pi^{-1} \quad \forall \gamma \in \mathbb{F}_n.$$

Indeed, such homomorphisms μ are in bijective correspondence with the set VE_n of edge and vertex labelled trees with n edges, in such a way that the transposition τ_i exchanges the 2 vertices joined by the i^{th}-edge of the tree.

In [2], Proposition 6, it is shown that, fixing in W_n the base point $\{1, \ldots, n\}$, then one can choose a fixed geometrical basis $\gamma_1, \ldots, \gamma_n$ of $\mathbb{F}_n = \pi_1(\mathbb{C} - \{1, \ldots, n\})$, and the monodromy of a polynomial P_0 with branch set $\{1, \ldots, n\}$ is represented by an element of E_n.

The reason why the vertices are not labelled is that, even if P_0 is fixed as a base point in U_n (resp: $\mathscr{L}_n$), the vertices of the tree correspond to the (unordered) set of roots of P_0.

Moreover, the monodromy of f_n is given by the following action of the braid group $\mathscr{B}_n = \pi_1(W_n)$ on E_n.

Since $\mathscr{B}_n$ acts as a group of automorphisms $\sigma : \mathbb{F}_n \to \mathbb{F}_n$ (cf. [1], §1, where, though, unlike in [2], the action is on the left), one can define, for $\varphi \in \mathscr{B}_n$, $[\mu] \in E_n$

$$\varphi([\mu]) = [\mu \circ \varphi^{-1}], \quad \text{and then} \tag{1.8}$$

$$(1.8) \text{ is the monodromy of the covering } Z_n \to W_n \tag{1.9}$$

(cf. [2], Prop. 7).

Definition 1.10. Let $Y_n \subset W_n'$ be the subset

$$\{\{w_1, \ldots, w_n\} | 0 < |w_1| < |w_2| < \ldots < |w_n|\},$$

and let Λ_n be the image of $\pi_1(Y_n, \{1, \ldots, n\}) \to \pi_1(W_n', \{1, \ldots, n\})$.

Remark 1.11. Writing $w_i = |w_i| \cdot w_i/|w_i|$, and $r_1 = |w_1|$, $r_2 = |w_2|/|w_1|, \ldots, r_n = |w_n|/|w_{n-1}|$, we see that Y_n is homeomorphic to $(S^1)^n \times (\mathbb{R}^+)^n$, hence $\pi_1(Y_n) \cong \mathbb{Z}^n$. The generators of $\pi_1(Y_n)$ give the braids T_j', which keep fixed the branch points $1, \ldots, n$ different from j, and move j in a circle around the origin $(j \to e^{2\pi i t} \cdot j)$. The basic observation is that

$$\mathscr{L}_n = \psi_n^{-1}(Y_n) \tag{1.12}$$

and the standard theory of covering spaces gives

PROPOSITION 1.13. *There is a bijection between the set $\pi_0(\mathscr{L}_n)$ of connected components of $\mathscr{L}_n$, and the set of Λ_n-orbits in E_n. Moreover, if a component $\mathscr{L}'$ of $\mathscr{L}_n$ corresponds to an orbit of cardinality r, then r equals the degree of the covering $\mathscr{L}'/A(1, \mathbb{C}) \to Y_n$.*

It remains therefore to study the action of Λ_n on E_n. It is easy to see that, in terms of the standard generators $\sigma_1, \ldots, \sigma_{n-1}$ of $\mathscr{B}_n$, the generators $T_1', \ldots, T_n'$ of $\pi_1(Y_n)$ map to the following braids $T_1, \ldots, T_n$.

$$T_1 = 1, \quad T_2 = \sigma_1^2, \ldots, \quad T_j = \sigma_{j-1}\sigma_{j-2} \ldots \sigma_2\sigma_1^2\sigma_2 \ldots \sigma_{j-1}. \tag{1.14}$$

Remark 1.15. Λ_n is a subgroup of the pure braid group $\mathscr{P}_n$, and one can in fact write

$$T_j = A_{1,j}A_{2,j} \ldots A_{j-1,j}.$$

It is also easy to see, by induction, that $\Lambda_n \cong \mathbb{Z}^{n-1}$.

Remark 1.16. The above formulae (1.14) hold for a given choice of a base point and of a basis of $\pi_1(\mathbb{C} - \{1, \ldots, n\})$ which is different from the one we shall adopt in §2.

§2. THE MAIN THEOREM AND ITS PROOF

In order to prove the main theorem, we need to make its statement stronger. We have defined in the introduction and in §1 the big lemniscate configuration of a polynomial $P \in \mathscr{L}_n$.

What in particular we are going to show is that any such figure occurs, up to isotopy, as the big lemniscate configuration of some polynomial.

To be more precise, we need to introduce the more general concept of a *possible big lemniscate n-configuration*: that is, as we shall more amply discuss in the proof of the theorem, an embedding ε in the complex plane $\mathbb{C}$ of the space X given by the disjoint union of $(n + 1)$ points, n figure eights, and $(n - 1)n/2$ circumferences, satisfying the same properties as the ones enjoyed by big lemniscate configurations of polynomials $P \in \mathscr{L}_n$.

MAIN THEOREM. *There exist natural bijections between the following sets*

(a) *The set $\pi_0(\mathscr{L}_n)$ of connected components of $\mathscr{L}_n$.*
(b) *The set of homeomorphism (isotopy) classes of big lemniscate configurations.*
(b′) *The set of homeomorphism (isotopy) classes of possible big lemniscate configurations.*
(c) *The set Σ_{n-2}.*
(d) *The set of Λ_n-orbits in E_n.*

Plan of the proof. (a) $\Leftrightarrow$ (d) is Proposition 1.13. The first step of proof shall consist in proving the equivalence (b′) $\Leftrightarrow$ (c). One may observe that (b) is a subset of (b′). Afterwards, the plan of proof proceeds as follows: show, as a second step, that the natural map from E_n to (possible big lemniscate configurations) is surjective (thereby proving the equality of (b), and (b′)). While showing this, it will be very convenient at the same time to prove that two edge labelled trees give homeomorphic big lemniscate configurations if and only if they lie in the same Λ_n-orbit. With this, the equivalence (b′) $\Leftrightarrow$ (d) shall be shown, and the proof of the theorem shall finally be through.

Step 1: Proof of (b′) $\Leftrightarrow$ (c) *of the Theorem.* Let P be a lemniscate generic polynomial. As in 1.2, we consider the big lemniscate configuration associated to P, i.e., if $w_1, \ldots, w_n$ are the branch points of P, $\Gamma = \Gamma_0 \cup \Gamma_{|w_1|} \cup \ldots \cup \Gamma_{|w_n|}$, where $\Gamma_c = \{z \mid |P(z)| = c\}$. We assume as in 1.2. that $0 < |w_1| < |w_2| < \ldots < |w_n|$.

We attach to Γ a tree $\mathscr{T}$, whose vertices represent connected components of Γ, and where an edge connects two vertices v, v' if there exists an i such that $v \subset \Gamma_{|w_i|}$, $v' \subset \Gamma_{|w_{i+1}|}$ (we set here $w_0 = 0$) and v is "inside" v', i.e., if you start from a smooth point $x \in v$ following the gradient of $P\bar{P}$ you get a curve meeting a point of v'. The tree $\mathscr{T}$ thus obtained has some remarkable properties To state them, we recall and state some combinatorial definitions. We apologize if some of the following definitions should not be standard.

Definition 2.1.

(i) A *tree* T is a connected and simply connected graph.

(ii) An *end* of T is a point (necessarily a vertex) v such that $T - \{v\}$ is connected.

(iii) Given vertices v, v' of T there is a unique path from v to v', i.e. a minimal subtree $T' \subset T$ having v and v' as ends: the length of T' (i.e., the number of edges of T') is called the *geodesical distance of v and v'*.

(iv) The *height* $h(v)$ of a vertex v is the minimum of the geodesical distances from the ends of the tree.

(v) The *root radius* $\rho(v)$ of a vertex v is the maximum of the geodesical distances of v from the ends of the tree.

(vi) A *weak centre* of the tree T is a vertex v for which the root radius $\rho(v)$ attains a minimum.

(vii) A tree is called *central* if it has a unique weak centre v_0.

With the terminology introduced above, we can define a class of trees which are deeply related to big lemniscate configurations.

Definition 2.2. A tree T is called a *central balanced tree of length n* if

(1) T has $(n + 1)$ ends.

(2) T is central and the height $h(v_0)$, and the root radius $\rho(v_0)$ of the centre v_0, both equal n.

Definition 2.3. The weight w of a vertex v is the number of edges containing v. Hence ends are vertices of weight one, whereas vertices of weight three are called *nodes*.

Definition 2.4. A *simple central tree of length n* is a central balanced tree of length n such that

(i) the centre has weight 2

(ii) there are no vertices with weight strictly bigger than 3

(iii) there is, for each $i = 1, \ldots, (n - 1)$, exactly one node at distance i from the centre.

Remarks 2.5. (i), (ii), (iii) of 2.4 imply

(iv) there are exactly $(i + 1)$ vertices at distance i from v_0. In particular, 2.2.(1), follows from 2.2.(2) and 2.4.(i), (ii), (iii).

(v) There are exactly $(n + 1)(n + 2)/2$ vertices. One can notice moreover that (iii) could be replaced by

(iii′) there is at most one node at distance i from the centre.

PROPOSITION 2.6. *The tree $\mathcal{T}$ associated to the big lemniscate configuration Γ of a polynomial $P \in \mathcal{L}_n$ is a simple central tree of length n. Moreover, letting X be the topological space which is the disjoint union of $(n + 1)$ points, n figure eights, and $n(n - 1)/2$ circumferences, any simple central tree $\mathcal{T}$ of length n uniquely determines an isotopy class of embedding $\varepsilon_{\mathcal{T}}: X \subsetneq \mathbb{C}$, in such a way that $(\Gamma, \mathbb{C})$ is isotopic to $(\varepsilon_{\mathcal{T}}(X), \mathbb{C})$ if $\mathcal{T}$ is the tree associated to Γ.*

Finally, $\mathcal{T}$ and $\mathcal{T}'$ are isomorphic if and only if the associated isotopy classes are the same.

Proof. First of all we may set up a bijective correspondence between vertices of $\mathcal{T}$ and connected components of X, in such a way that ends correspond to isolated points, nodes to figure eights, and the centre to the remaining figure eight. We define the embedding $\varepsilon_{\mathcal{T}}$ by an inductive procedure, assuming that $\varepsilon_{\mathcal{T}}$ has been defined on $X^{(i)} \overset{\text{def}}{=}$ the union of the components of X corresponding to vertices v at distance at most i from the centre. The inductive procedure is based on the following definitions:

Definition 2.7. Given vertices v, v' of a simple central tree, or more generally, of a "rooted" tree (where a given vertex has been chosen as root), v is said to be a *direct successor* of v' if v' belongs to the path joining v and the centre v_0, and moreover $\operatorname{dist}(v, v_0) = \operatorname{dist}(v', v_0) + 1$.

Remark that a node and the centre have exactly two direct successors, ends have no successors, and all other vertices have exactly one direct successor. One can then inductively define the notion of successor.

Definition 2.8. A differentiable embedding of a figure eight E in $\mathbb{C}$, $\varepsilon: E \to \mathbb{C}$ is said to be *unreversed* if, B_1 and B_2 being the bounded connected components of $\mathbb{C} - \varepsilon(E)$, 0 being the singular point of E, then $\overline{B_1 \cup B_2} - \varepsilon(0)$ is disconnected.

Remark 2.9. There are only two isotopy classes of embeddings $\varepsilon: E \to \mathbb{C}$, the reversed and the unreversed one. In the reversed one, one can distinguish between B_1 and B_2 by imposing the condition $\partial B_1 \not\subset \partial B_2$, whereas in the unreversed case there is an orientation preserving diffeomorphism of $(\mathbb{C}, \varepsilon(E))$ exchanging B_1 and B_2. Moreover, both B_1 and B_2 are diffeomorphic to disks.

Remark 2.10. Any two embeddings of a circumference (resp.: of a point) in a disk are isotopic.

The initial part of the procedure consists in constructing an unreversed embedding in $\mathbb{C}$ of the figure eight E corresponding to the centre of the tree.

Assume now that one has constructed (inductively) an embedding $e^{(i)}: X^{(i)} \to \mathbb{C}$. For each vertex $v \in \mathcal{T}$ at distance $(i + 1)$ from the centre, there is a unique vertex v' such that v is a direct successor of v'. If $X_{v'}$ is the corresponding connected component of X, there are two possibilities:

(a) $X_{v'}$ is a simple closed curve and v is the only successor of v'
(b) $X_{v'}$ is a figure eight and v' has exactly two successors $v = v_1$, and v_2.

In case (a), $e^{(i)}(X_{v'})$ bounds only one bounded component B of $\mathbb{C} - e^{(i)}(X_{v'})$, and we may embed X_v in B (in an unreversed embedding if v is a node); in case (b) we choose an ordering of the two bounded components B_1, B_2 of $\mathbb{C} - e^{(i)}(X_{v'})$: then we may embed (X_{v_i}) in B_i for $i = 1, 2$, and in an unreversed way if v_i is a node. By induction we construct $e: X \to \mathbb{C}$ and the unicity follows from Remarks 2.9, 2.10. We omit to verify the other assertions, which follow immediately from our construction, and from the following

Remark 2.11. If Δ is a figure eight contained in the big lemniscate configuration associated to a polynomial $P \in \mathscr{L}_n$, then Δ is given by an unreversed embedding.

Proof. Otherwise, cf. 2.9., $|P|$ would have a local maximum in B_1, contradicting the maximum principle.

Q.E.D. for 2.11. and Prop. 2.6.

The proof of Step I shall be achieved via the following.

PROPOSITION 2.12. *There exists a bijective correspondence between the set of (isomorphism classes of) simple central trees of length n and the set Σ_{n-2}.*

Before proceeding to the proof of 2.12., we recall the well known (cf. [4]) Cayley-Prüfer correspondence for vertex labelled trees.

Definition 2.13. Let $\mathscr{T}'_n$ be the set of isomorphism classes of trees with n vertices and with a bijection of the set of vertices with the set $\{0, 1, \dots, (n-1)\}$. The Cayley-Prüfer correspondence is a bijective map

$$\mathscr{C}\mathscr{P}: \mathscr{T}'_n \to \{0, \dots, (n-1)\}^{\{0, \dots, (n-2)\}},$$

defined as follows: if $S \in \mathscr{T}'_n$, identifying $\{0, \dots, (n-1)\}$ with the set of vertices of S, we pick the end with biggest i, and if j is the neighbouring vertex, we set $x(n-2) = j$. The same can be done provided one has any strict total ordering of the vertices: therefore, if S' is the subtree of S obtained by deleting the vertex i and the edge $[i, j]$, we can assume by induction that $\mathscr{C}\mathscr{P}(S')$ is defined and equal to $(x(0), \dots, x(n-3))$, and thus there remains only to define $\mathscr{C}\mathscr{P}(S) = (x(0), \dots, x(n-3), x(n-2))$. We set now $X_{n-2} = \{0, \dots, (n-1)\}^{\{0, \dots, (n-2)\}}$, i.e.,

$$X_{n-2} = \{(x_0, \dots, x_{n-2}) | x_i \in \mathbb{N}, 0 \le x_i \le n-1\}. \tag{2.15}$$

Then $\Sigma_{n-2} \subset X_{n-2}$ corresponds to a subset of $\mathscr{T}'_n$, which we shall denote by $\hat{\mathscr{T}}'_n$: recall that $\Sigma_{n-2} = \{(x_0, \dots, x_{n-2}) \in X_{n-2} | 0 \le x_i \le i$, the equation $x_i = N$ has at most 2 solutions $\forall N\}$.

LEMMA 2.16. *Under the Cayley-Prüfer correspondence, Σ_{n-2} corresponds to the set $\hat{\mathscr{T}}'_n$ of vertex labelled trees such that:*

(1) *taking 0 as a root (cf. 2.7.0), if j is a successor of i, then $j > i$*
(2) *the tree has no vertex of weight ≥ 4 and 0 is not a node (cf. 2.3.).*

Proof. Let's first prove that $\mathcal{CP}(\hat{\mathcal{V}}_n) \subset \Sigma_{n-2}$. In fact, we observe that, by (1), $(n-1)$ is an end, and removing it (together with the unique edge containing it), we obtain a tree S' in $\hat{\mathcal{V}}_{n-1}$.

The first assertion shows that $x_{n-2} \leq n-2$, and the second assertion implies, by induction, that we have $(x_0, \ldots, x_{n-3}) \in \Sigma_{n-3}$. To prove that $(x_0, \ldots, x_{n-2}) \in \Sigma_{n-2}$ it is enough to observe that if the equation $x_i = N$ had more than 3 solutions, then N would have at least 3 successors, contradicting (2).

In the other direction, it suffices to show that if $\mathcal{CP}(S) \in \Sigma_{n-2}$, then $S \in \hat{\mathcal{V}}_n$. Firstly, $(n-1)$ is an end: in fact, $(n-1)$ does not appear in the Cayley–Prüfer sequence, and if a vertex i does not appear in the sequence, then it must be an end. At some step of the Cayley–Prüfer procedure, i has become an end and one marks the neighbouring vertex x_j: now, if i were not an end, at some step another neighbouring vertex h would have become the biggest end and would have been deleted, but then i would appear in the sequence.

Secondly, we use again induction (the cases $n = 2, 3$ being trivial), letting S' be the subtree obtained by cancelling the end $(n-1)$: we may therefore $((x_0, \ldots, x_{n-3}) \in \Sigma_{n-3}!)$ assume that $S' \in \hat{\mathcal{V}}_{n-1}$. Moreover, S is obtained from S' by connecting $(n-1)$ (as an end) with the vertex x_{n-2} through an edge. Hence property (1) is clearly satisfied

It remains to prove property (2): if $x_{n-2} \neq 0$, then x_{n-2} is not a node of S', if $x_{n-2} = 0$, then 0 is an end of S'. But $S' \in \hat{\mathcal{V}}_{n-1}$, and it is easy to see then that a vertex with 2 successors appears twice in the Cayley–Prüfer sequence $(x_0, \ldots, x_{n-3})$. But since $(x_0, \ldots, x_{n-2}) \in \Sigma_{n-2}$, x_{n-2} cannot appear twice in the sequence $(x_0, \ldots, x_{n-3})$. Q.E.D.

The proof of Proposition 2.12 follows immediately from the following

LEMMA 2.17. *There is a natural bijection between the set of (isomorphism classes of) simple central trees of length n, and the set $\hat{\mathcal{V}}_n$ of vertex labelled trees* (cf. 2.16.).

Proof. Let $\mathcal{T}$ be a simple central tree of length n: to $\mathcal{T}$ we associate a vertex labelled tree $S \in \hat{\mathcal{V}}_n$.

First of all 0 corresponds to the centre of $\mathcal{T}$, and i to the unique node v_i of $\mathcal{T}$ at distance i from the centre. Moreover, an edge connects i and j if v_j is a successor of v_i, but v_j is not a successor of any v_h which is a successor of v_i. Notice that S is a tree clearly satisfying (1), and (2) holds since if j, k, l are direct successors of i, then the paths $[v_i, v_j]$, $[v_i, v_k]$, $[v_i, v_l]$ are disjoint, hence v_i has 3 distinct direct successors, contradicting 2.4 (ii).

We have just exhibited a map $\varphi : \mathcal{SCT}_n = \{\text{Simple central trees of length } n\} \to \hat{\mathcal{V}}_n$, and it suffices (the sets being finite) to provide a one side inverse $\psi : \hat{\mathcal{V}}_n \to \mathcal{SCT}_n$. Given $S \in \hat{\mathcal{V}}_n$, for each edge $[i, j]$ divide it into $(j-i)$ equal parts, thus creating $(j-i-1)$ new edges and vertices. If $i \neq 0$ is neither a node nor an end, or if $i = 0$ and 0 is an end, attach to i a string of $(n-i)$ consecutive edges (thus adding $(n-i)$ vertices).

Whereas, if i is an end of S, attach to i two such strings having i is an endpoint. The tree just obtained is $\psi(S) = \mathcal{T}'$. It is clear from the construction that $\mathcal{T}'$ has exactly n nodes, which are in such a natural bijection with the vertices of S (except 0), that if the node v_i corresponds to i, it has distance i from the vertex corresponding to 0. It is also clear that 0 has weight 2 and 2.4.(i), (ii), (iii) are verified.

Moreover, by construction, any path from v_0 to v_i continues until an end at distance $= n$, and 2.2.(2) holds. We conclude that $\mathcal{T}' \in \mathcal{SCT}_n$, since, by Remark 2.5, 2.2.(1) holds too. It is immediate now to verify that $\varphi(\psi(S)) = S$. Q.E.D.

We are now ready to begin

Step II: E_n surjects onto (b'), *and is a quotient map for the Λ_n-action.* Recall that to elements $[\mu]$ of E_n there correspond polynomials in $\mathcal{L}_n$ with monodromy $[\mu]$, and that by

Prop. 1.13 every connected component of $\mathscr{L}_n$ contains one such polynomial, and $[\mu]$ and $[\mu']$ belong to the same component if and only if they lie in the same Λ_n-orbit. First of all, then, let's describe how a tree in E_n determines a configuration of lemniscates. We fixed our base point $B_0 = \{1, \ldots, n\} \in W_n$ and we know that each element of E_n determines a covering space $P: \tilde{S} \to \mathbb{C}$, and $\Gamma = P^{-1}(\{0\} \cup \{|w| = 1\} \cup \ldots \{|w| = n\})$. Pictorially, the correspondence is as follows: for any midpoint of edge i draw an eight with centre there and $(n - i)$ circles, as in Fig. 1.

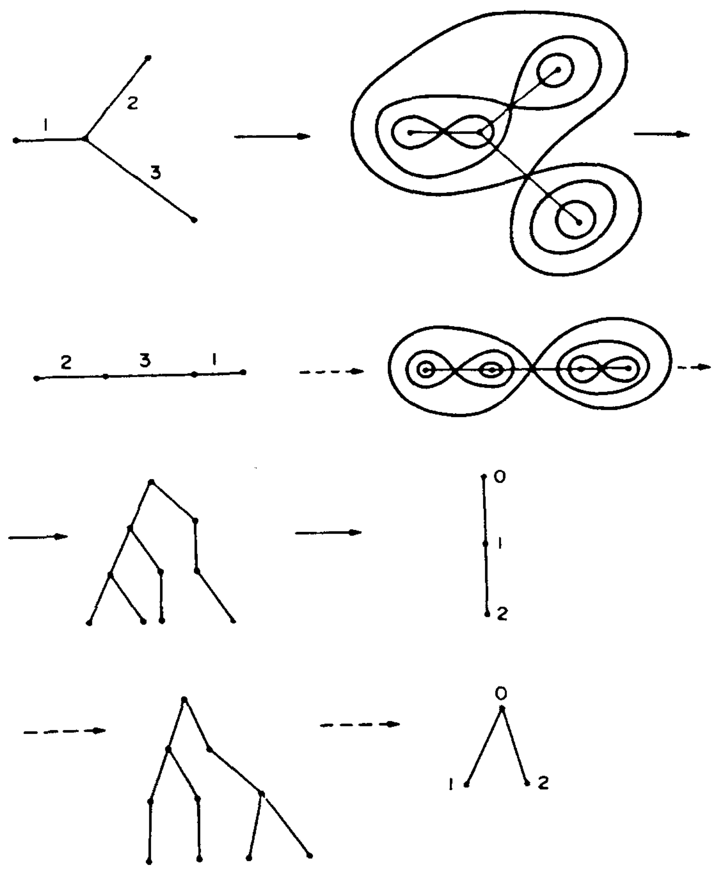

Fig. 1. Edge labelled trees → Big Lemniscates → Simple central trees → Vertex labelled trees

In Fig. 1 is also depicted the associated simple central tree $\mathscr{T}$. It is somehow easier to understand the map $v: E_n \to \mathscr{SCT}_n$, the set of simple central trees of length n constructed via Proposition 2.6. This map can formally be described as follows: ends of $\mathscr{T} = v(A)$ correspond to vertices of A, and vertices of $\mathscr{T}$ at distance k from the centre correspond to connected components of the $(n - k)$ skeleton of A, which is defined as the union of the vertices of A with the first $(n - k)$ edges of A. Edges in $\mathscr{T}$ are defined according to inclusions of components of the $(n - k)$ skeleton in components of the $(n - k + 1)$ skeleton.

In fact, it is clear that passing through the critical value i makes the number of connected components of Γ_c drop by one. On the other hand, if the edge i connects two vertices v, v' but v, v' are not in the interior parts of the figure eight $\Delta_i \subset \Gamma_i$, then, choosing 0 as base point of $\pi_1(\mathbb{C} - \{1, \ldots, n\})$, we see that a loop around i cannot have a monodromy exchanging v and v' (which are roots of P). There remains to prove the following

Proposition 2.18. *$v: E_n \to \mathscr{SCT}_n$ is surjective and if $v(A) = v(A') = \mathscr{T}$, then A and A' are in the same Λ_n-orbit.*

Proof. Let $A_{(i)}$ be the ith-skeleton of A: the proof will be based upon induction on i. I.e., we assume inductively that the following assertions hold true:

(a_i) if $\nu(A) = \nu(A') = \mathscr{T}$, then there exists a transformation $\sigma \in \langle T_1, \ldots, T_i \rangle$ such that $\sigma(A')_{(i)} \cong A_{(i)}$

(b_i) for each $\mathscr{T} \in \mathscr{S}\mathscr{C}\mathscr{T}_n$ there exists A s.t. $A_{(i)}$ maps onto the subtree of $\mathscr{T}$ at distance at least $(n - i)$ from the centre.

Let's prove the inductive step (b_{i+1}): we need to add an edge $(i + 1)$ to $A_{(i)}$ in such a way that it joins two vertices belonging to the 2 connected components C, D of $A_{(i)}$ which correspond to the 2 direct successors of the unique node of $\mathscr{T}$ at distance $(n - i - 1)$ from the centre; then we can add *ad libitum* (i.e., to our taste) other edges to complete $A_{(i+1)}$ to a tree A.

To prove (a_{i+1}), we may assume that, by the inductive assumption, $A_{(i)} \cong A'_{(i)}$, and it suffices to show that we can act with a $\sigma \in \langle T_1, \ldots, T_{i+1} \rangle$ to have $A_{(i+1)} \cong A'_{(i+1)}$. As before, call C, D the connected components of $A_{(i)}$ containing the two vertices of $(i + 1)$. Let $T_C = \prod_{j \in C} T_j$, $T_D = \prod_{j \in D} T_j$, where the indexes in the products stand of course for edges of $A_{(i)}$. For this proof, again we choose 0 as base point in $\mathbb{C} - \{1, \ldots, n\}$ and a geometric basis $\Gamma_1, \ldots, \Gamma_n$ as in Fig. 2. The action of T_i is given by a full twist around the circle of radius i, hence

(a) $T_i(\Gamma_j) = \Gamma_j$ if $i > j$

(b) $T_i(\Gamma_i) = \Gamma_{i-1}\Gamma_{i-2} \ldots \Gamma_1 \Gamma_i \Gamma_1^{-1} \ldots \Gamma_{i-2}^{-1}\Gamma_{i-1}^{-1}$

(c) $T_i(\Gamma_h) = \Gamma_i^{-1}\Gamma_h\Gamma_i$ for $i < h$.

In fact $T_i(\Gamma_i)$ is as in Fig. 3, $T_i(\Gamma_h)$ as in Fig. 4. We remark again that the present action differs from 1.14 because of the change of base point for the generators $\Gamma_1, \ldots, \Gamma_n$. It is then

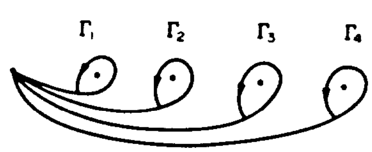

Fig. 2.

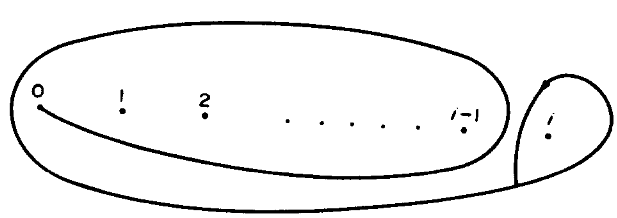

Fig. 3.

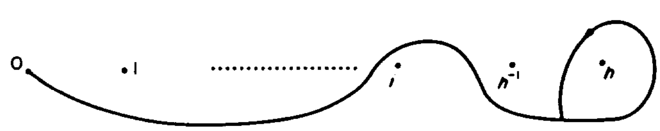

Fig. 4.

easy to see that, for $j \in C$, $T_C(\Gamma_j) = T_1 \ldots T_i(\Gamma_j)$, but we observe that in general $(i \geq j)$.

$$\begin{aligned} T_1 \ldots T_i(\Gamma_j) &= T_i \ldots T_1(\Gamma_j) = T_i \ldots T_2(\Gamma_1^{-1}\Gamma_j\Gamma_1) \\ &= T_i \ldots T_j(\Gamma_1^{-1}\Gamma_2^{-1} \ldots \Gamma_{j-1}^{-1}\Gamma_j\Gamma_{j-1} \ldots \Gamma_1) \\ &= T_i \ldots T_{j+1}(\Gamma_1^{-1}\Gamma_2^{-1} \ldots \Gamma_{j-1}^{-1}\Gamma_{j-1} \ldots \Gamma_1\Gamma_j\Gamma_1^{-1} \ldots \Gamma_{j-1}^{-1}\Gamma_{j-1} \ldots \Gamma_1) \\ &= T_i \ldots T_{j+1}(\Gamma_j) = \Gamma_j. \end{aligned}$$

Hence

$$\begin{cases} T_C(\Gamma_j) = \Gamma_j & \forall j \leq i \\ T_D(\Gamma_j) = \Gamma_j \end{cases}$$

But, if we choose Γ_{i+1}, $T_C(\Gamma_{i+1})$, if $j_1, \ldots, j_r$ are the ordered $(j_1 < j_2 < \ldots < j_r)$ edges of C, equals (by the above computation)

$$\Gamma_{j_1}^{-1} \ldots \Gamma_{j_r}^{-1}\Gamma_{i+1}\Gamma_{j_r} \ldots \Gamma_{j_1}.$$

Hence the action of T_C^{-1} on the transposition $\mu(\Gamma_{i+1})$ is to conjugate it with the product $\Gamma_{j_r} \ldots \Gamma_{j_1}$, a cyclical permutation on the vertices of the component C. Similarly for T_D^{-1}, hence, using two appropriate powers of T_C^{-1}, respectively T_D^{-1}, we can choose at our wish the respective points of C, D, which shall be the two vertices of the edge $(i + 1)$. Q.E.D.

§3. SMALL LEMNISCATE CONFIGURATIONS

As in the introduction, if $P \in \mathscr{L}_n$, the *small lemniscate configuration* associated to P is the homeomorphism class of the pair $(Y_P, \mathbb{C})$ where $Y_P = P^{-1}(0) \cup \Delta_{|w_1|} \cup \ldots \cup \Delta_{|w_n|}$, where $\Delta_{|w_i|}$ is the component of $\Gamma_{|w_i|}$ homeomorphic to a figure eight. We have thus $(n + 1)$-points and n figure eights, embedded in $\mathbb{C}$ by an unreversed embedding, and again we want to give first an easier combinatorial description of the possible configurations, then to show that all the a priori possible configurations do in fact occur.

Definition 3.1. The rooted tree associated to a small lemniscate configuration is a rooted tree R with n unlabelled vertices corresponding to the components $\Delta_{|w_1|}, \ldots, \Delta_{|w_n|}$, with $\Delta_{|w_n|}$ as a root vertex, and an edge connecting two vertices corresponding to $\Delta_{|w_i|}$, $\Delta_{|w_j|}$ if $\Delta_{|w_i|}$ lies in the interior of $\Delta_{|w_j|}$, and there is no $\Delta_{|w_k|}$ lying in the interior of $\Delta_{|w_j|}$, and having $\Delta_{|w_i|}$ in its interior.

Remark 3.2. The rooted tree R is simply obtained from the vertex labelled tree $S \in \hat{\mathscr{T}}_n$ introduced in 2.16, 2.17, simply by forgetting the vertex labelling except for the "root" 0.

We obtain immediately from the results of §2, and by the previous remark

THEOREM 3.3. *The map which to a connected component $U \in \pi_0(\mathscr{L}_n)$ associates the small lemniscate configuration of a polynomial P, is isomorphic, via the bijection $\pi_0(\mathscr{L}_n) \Leftrightarrow \hat{\mathscr{T}}_n$, to the forgetful mapping $\xi : \hat{\mathscr{T}}_n \to \hat{\mathscr{R}}_n$ (where $\hat{\mathscr{R}}_n$ is the set of isomorphism classes of rooted trees with n vertices, such that every vertex has weight $1 \leq w \leq 3$, and the root has weight ≤ 2), given by forgetting the labellings $1, \ldots, (n - 1)$.*

COROLLARY 3.4. *All a priori possible small lemniscate configurations do in fact occur.*

Proof. Given a rooted tree $R \in \hat{\mathscr{R}}_n$, it suffices to choose a labelling of the other vertices such that if j is a successor of i, then $j > i$. That this is possible can easily be shown by induction, in fact it suffices to label $(n - 1)$ any vertex without successor. Q.E.D.

Remark 3.5. Notice that the number of allowed labellings for a tree $R \in \hat{\mathscr{R}}_n$ tells the number of connected components of $\mathscr{L}_n$ which realize the small lemniscate configuration associated to R. This number is 1 only if R is a string, otherwise it can be very big, as it will follow from the results in the appendix.

§4. REAL POLYNOMIALS

The main purpose of this section is to show that every big lemniscate configuration occurs as the lemniscate configuration associated to a suitable real polynomial $P \in \mathbb{R}[x]$ of degree $(n + 1)$ with $(n + 1)$ distinct roots. To this purpose, we give the following.

Definition 4.1. An edge labelled tree $A \in E_n$ is said to be *linear* if it does not have nodes (i.e., it is homeomorphic to $[0, 1]$).

PROPOSITION 4.2. *Every Λ_n-orbit in E_n contains a linear tree A.*

Proof. By Proposition 2.18, every orbit corresponds uniquely to a simple central tree $\mathscr{T}$ of length n. First we shall establish a bijection between the set of ends of $\mathscr{T}$ and a subset of $[0, 1]$ of the form $\{a_0, \ldots, a_n\}$, where $a_0 = 0 < a_1 < \ldots < a_n = 1$. Then we construct A simply by considering $\{a_0, \ldots, a_n\}$ as the set of vertices of A, and letting the intervals $[a_i, a_{i+1}]$ be the edges of A. There remains to give a labelling to each edge $[a_i, a_{i+1}]$. We can do this in a unique fashion by giving to $[a_i, a_{i+1}]$ the label k iff $2k$ equals the geodesic distance of the two ends corresponding to a_i, a_{i+1}. It is easy to verify that $A \in E_n$, and that $v(A) = \mathscr{T}$. Q.E.D.

(4.3) Given a monic lemniscate generic polynomial $P(x) = \prod_{i=1}^{n} (x - x_i)$ of degree $n + 1$ with real roots $x_0 < x_1 < \ldots < x_n$, we consider the following linear tree A' such that

(i) the vertices of A' are the points $x_0, \ldots, x_n$
(ii) the edges of A' are the intervals $[x_i, x_{i+1}]$
(iii) $[x_i, x_{i+1}]$ has the label j if and only if, letting y_j be the only root of $P'(x)$ in the interval (x_i, x_{i+1}) then, setting as usual $w_j = P(y_j)$, we have $|w_1| < |w_2| < \ldots < |w_n|$.

The following lemma is a particular case of a theorem of C. Davis ([3]).

LEMMA 4.4. *Given a linear tree A, there do exist real numbers $x_0 < x_1 < \ldots < x_n$ such that A is isomorphic to the tree A' obtained as in* (4.3).

Proof. Notice that $|P(y_i)| = \max\{|P(y)| \mid y \in [x_i, x_{i+1}]\}$ and that if $|x_i - x_{i+1}| \to 0$, this maximum also tends to 0. Thus, fixing i, we can achieve that $|P(y_i)|$ be minimum. The rest of the proof follows easily. Q.E.D.

THEOREM 4.5. *For each big lemniscate configuration, there exists a (real) monic lemniscate generic polynomial with $n + 1$ (distinct) real roots $P(x) = \prod_{i=1}^{n} (x - x_i)$, whose big lemniscate configuration Γ is isotopic to the given one.*

Proof. By the main theorem, Prop. 2.18, Prop. 4.2 and Lemma 4.4 we can take a polynomial $P(x)$ as above whose associated linear tree A' (cf. 4.3) represents the Λ_n-orbit associated to the given lemniscate configuration. To finish the proof it suffices to verify that the tree A' gives the monodromy of P. But the n paths connecting x_i with x_{i+1} and obtained by replacing, in the segment $[x_i, x_{i+1}]$, a small segment with center y_j with the lower semicircumference with same center and ends, are such that their images under P form a geometric basis of $\pi_1(\mathbb{C} - \{w_1, \ldots, w_n\})$. Q.E.D.

Acknowledgements Both authors are grateful to the University of California, San Diego, for providing, in the spring of 1986, hospitality and neighbouring offices. In this way, the methods developed by the first author for different purposes, found a beautiful and interesting (at least so we believe) application. The first author would also like to thank the Institute Mittag-Leffler in Djursholm for providing, in September 1986, excellent environment for research, and to express his deep gratitude to Torsten Ekedahl for trying to convince him that there should be a bijection between the set of big lemniscate configurations and the set of Z^{n-1}-orbits in E_n. The hospitality of the S.F.B. of Göttingen in June 1989 allowed the first author to write down the final version.

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APPENDIX

COUNTING LEMNISCATE CONFIGURATIONS

By Fabrizio Catanese (with the cooperation of Rick Miranda, Don Zagier, Enrico Bombieri).

§A1. BIG CONFIGURATIONS

Our first purpose is to give a formula for the number of big lemniscate configurations of lemniscate generic polynomials of degree $(n+1)$. By the main theorem, this number equals the cardinality of Σ_{n-2}, where

$$\Sigma_n = \{(x_0, \ldots, x_n) | 0 \le x_i \le i, \quad \text{and, } \forall\, m \in \mathbb{N} \text{ there are at most 2 integers } 0 \le i \le n \text{ such that } x_i = m\}. \quad (0)$$

Theorem 1. *Set* $a_0 = 0$, $a_n = $ *cardinality of* Σ_{n-1}. *Then* $\sum_{n=0}^{\infty} \frac{a_n}{n!} t^n = (1 - \sin t)^{-1}$.

We shall give the proof of Theorem 1 by a sequence of steps. The first one, suggested by Rick Miranda, is to study the generating function corresponding to the a_n's through a generating function in two variables, for whose coefficients it is possible to write a recursion relation. So we define

$$\Sigma_{n,k} = \{(x_0, \ldots, x_n) \in \Sigma_n | \operatorname{card}\{x_0, \ldots, x_n\} = n + 1 - k\}.$$

In other words, $\Sigma_{n,k}$ consists of the sequences where k integers occur twice.

Lemma 3. *If* $A_{n,i} = $ *card* $\Sigma_{n,i}$, *then*

$$A_{n,i} = (i+1)A_{n-1,i} + (n + 2 - 2i)A_{n-1,i-1}.$$

Proof. Let $\pi:\Sigma_n \to \Sigma_{n-1}$ be the map such that $\pi(x_0, \ldots, x_n) = (x_0, \ldots, x_{n-1})$. Then $\pi(\Sigma_{n,i}) \subset \Sigma_{n-1,i} \cup \Sigma_{n-1,i-1}$. Moreover if $y = (y_0, \ldots, y_{n-1})$ belongs to $\Sigma_{n-1,i}$, $\pi^{-1}(\{y\}) \cap \Sigma_{n,i}$ has cardinality $(i+1)$, since x_n must be different from all the y_j's. Whereas if $y \in \Sigma_{n-1,i-1}$, x_n has to be chosen among the $n - 2(i-1)$ numbers which occur exactly once in the Prüfer sequence y. Q.E.D.

We set, for convenience (cf. Thm. 1).

$$a(n, i) = \text{cardinality of } \Sigma_{n-1,i}, \text{ so that by Lemma 3} \tag{4}$$

$$a(n, i) = (i+1)a(n-1, i) + (n+1-2i)a(n-1, i-1). \tag{4'}$$

Remark 5. $a(n, i) = 0$ for $2i > n$. The first values of $a(n, i)$ are given by the following table

	$i =$	0	1	2	3
$n =$	0	1			
	1	1			
	2	1	1		
	3	1	4		
	4	1	11	4	
	5	1	26	34	
	6	1	57	180	34
	7	1	120	568	496

It is obvious that $a_n < n!$, on the other hand, by (4'), since

$$a_n = \sum_{i=0}^{n/2} a(n, i),$$

$$a_n = \sum_{i=0}^{(n-1)/2} a(n-1, i)[(i+1) + (n-1-2i)]$$

$$= \sum_{i \le (n-1)/2} a(n-1, i)(n-i) \ge \left(\frac{n+1}{2}\right) a_{n-1},$$

hence $a_n \ge (n+1)!2^{-n}$. Hence, in order to avoid to have a diverging series as generating function, we follow Zagier's advice and we set

$$A(x, t) = \sum_{n,i=0}^{\infty} \frac{a(n, i)}{n!} x^i t^n = 1 + t + \frac{1+x}{2} t^2 + \frac{1+4x}{6} t^3 + \frac{1+11x+4x^2}{24} t^4 + \ldots$$

where $a(n, i) = 0$ for $n < 2i$. Then the recursion formula (4') translates into the following P.D.E.

$$(1 - xt)\frac{\partial A}{\partial t} + (2x^2 - x)\frac{\partial A}{\partial x} = A. \tag{8}$$

In fact, we have:

$$\frac{\partial A}{\partial t} = \sum_{m,j \ge 0} \frac{a(m,j)}{(m-1)!} x^j t^{m-1} = \sum_{\substack{i \ge 0 \\ n \ge 0}} x^i t^n \frac{a(n+1, i)}{n!}$$

$$xt\frac{\partial A}{\partial t} = \sum_{m,j \ge 0} \frac{a(m,j)}{(m-1)!} x^{j+1} t^m = \sum_{\substack{i \ge 1 \\ n \ge 0}} x^i t^n \frac{a(n, i-1)}{(n-1)!}$$

$$x^2\frac{\partial A}{\partial x} = \sum_{m,j \ge 0} \frac{a(m,j)}{m!} j x^{j+1} t^m = \sum_{\substack{i \ge 0 \\ n \ge 0}} x^i t^n (i-1) \frac{a(n, i-1)}{n!}$$

$$x\frac{\partial A}{\partial x} = \sum_{n,i \ge 0} \frac{a(n, i)i}{n!} x^i t^n.$$

Looking at the coefficient of $x^i t^n$, multiplied by $n!$, we have for $i \ge 1$

$$a(n+1, i) - na(n, i-1) + 2(i-1)a(n, i-1) - ia(n, i) = a(n, i)$$

since

$$a(n+1, i) = (i+1)a(n, i) + [n - 2(i-1)]a(n, i-1) \quad \text{by (4')}.$$

Moreover, clearly $A(x, t)$ satisfies the Boundary values

$$\begin{cases} A(0, t) = e^t \\ A(x, 0) = 1 \end{cases} \tag{9}$$

We seek for a change of coordinates

$$(x, c) \to (x, f(x, c)) = (x, t) \tag{10}$$

where the P.D.E. reduces to an O.D.E. in the x variable. Since $\frac{d}{dx}A(x, f(x, c)) = \frac{\partial A}{\partial x} + f'\frac{\partial A}{\partial t}$, we want to find $f(x)$ such that $\frac{\partial A}{\partial x} + f'\frac{\partial A}{\partial t} = (2x^2 - x)^{-1}A$, i.e., in view of (8), and since $f' = \frac{dt}{dx}$, we need to solve the O.D.E.

$$dt(2x^2 - x) = (1 - xt)dx \tag{11}$$

Dividing by $-x(1 - 2x)^{1/2}$, we get

$$dt(1 - 2x)^{1/2} - t(1 - 2x)^{-1/2}dx = \frac{-dx}{x(1 - 2x)^{1/2}}$$

which yields, by integration,

$$t(1 - 2x)^{1/2} = \int \frac{dx}{x(1 - 2x)^{1/2}} = -\log\frac{1 - (1 - 2x)^{1/2}}{1 + (1 - 2x)^{1/2}} + c \tag{12}$$

(the last equality holds up to a constant of integration which can be set equal to c). In the (x, c) variables the P.D.E. for A becomes

$$\frac{d\log A}{dx} = \frac{1}{2x^2 - x} \Rightarrow \tag{13}$$

there is a function $F(c)$ such that

$$A(x, c) = \left(\frac{1 - 2x}{x}\right) \cdot F(c). \tag{14}$$

We set for convenience

$$u = \log\frac{1 - \sqrt{1 - 2x}}{1 + \sqrt{1 - 2x}} \tag{15}$$

Thus

$$\begin{cases} c = t(1 - 2x)^{1/2} + u \\ A(x, t) = \left(\frac{-2x + 1}{x}\right)F(t\sqrt{1 - 2x} + u). \end{cases}$$

The boundary condition (cf. (9)) $A(x, 0) \equiv 1$ implies

$$F(u) = \frac{x}{1 - 2x} \text{ which equals, by (15), and an easy calculation,} \tag{16}$$

$$F(u) = \frac{2}{e^u + e^{-u} - 2}. \tag{17}$$

Whence

$$A(x, t) = 2\frac{(1 - 2x)}{x}\left[e^{t\sqrt{1-2x}} \cdot \frac{1 - \sqrt{1 - 2x}}{1 + \sqrt{1 - 2x}} + e^{-t\sqrt{1-2x}}\frac{1 + \sqrt{1 - 2x}}{1 - \sqrt{1 - 2x}} - 2\right]^{-1}$$

$$= \left[\cos h\left(\frac{t}{2}\sqrt{1 - 2x}\right) - \frac{\sin h\left(\frac{t}{2}\sqrt{1 - 2x}\right)}{\sqrt{1 - 2x}}\right]^{-2}$$

The last formula clearly shows that $A(0, t) = e^t$. Plugging in $x = 1$, we get

$$A(1, t) = -4[e^{it}(1-i)^2 + e^{-it}(1+i)^2 - 4]^{-1}$$
$$= \left[1 + \frac{i}{2}(e^{it} - e^{-it})\right]^{-1} = (1 - \sin t)^{-1}. \qquad \text{Q.E.D. for Theorem 1.}$$

§A2. SMALL CONFIGURATIONS

By Theorem 3.3, and Corollary 3.4, we know that there is a natural bijection between the set of isotopy classes of small lemniscate configurations and the set $\hat{\mathscr{R}}_n$ of isomorphism classes of rooted trees with n vertices such that each vertex has weight $w \leq 3$, and the root has weight $w \leq 2$. We set

$$\begin{cases} b_n = \#(\hat{\mathscr{R}}_n) \\ f(t) = \sum_{n=0}^{\infty} b_n t^n, \quad \text{where we have set } b_0 (= b_1 = b_2) = 1. \end{cases} \tag{19}$$

It is convenient to have set $b_0 = 1$, since we get thus the following recursion formulae.

PROPOSITION 20. $b_{2n} = b_{2n-1}b_0 + \dots + b_n b_{n-1}$, $b_{2n+1} = b_{2n}b_0 + b_{2n-1}b_1 + \dots + (b_n^2 + b_n)/2$.

Proof. Erasing the root of a tree in $\hat{\mathscr{R}}_m$, one obtains a pair of trees (one of them is possibly empty), in $\hat{\mathscr{R}}_i \cup \hat{\mathscr{R}}_{m-1-i}$, with $2i \leq m - 1$. Conversely, two trees as above combine to yield a tree in $\hat{\mathscr{R}}_m$: one has to take special care of the case when $i = m - 1 - i$, that is, $m = 2i + 1$, where the (unordered) pair may consist of equal trees. Q.E.D.

To simplify the recursion we set

$$b_j = 0 \quad \text{for} \quad j \in 1/2\mathbb{Z} - \mathbb{Z} \tag{21}$$

thus we have, for $j \geq 1$

$$b_j = \frac{1}{2}\sum_{i=0}^{j-1} b_i b_{j-i-1} + \frac{1}{2} b_{(j-1)/2}. \tag{22}$$

Multiplying by t^j and summing over j we get

$$f(t) = 1 + \sum_{j \geq 1} b_j t^j = 1 + \frac{1}{2} t \left[\sum_{j \geq 1} \left(\sum_{i=0}^{j-1} b_i t^i b_{j-1-i} t^{j-1-i} \right) + b_{(j-1)/2} (t^2)^{(j-1)/2} \right],$$

that is,

$$f(t) = 1 + \tfrac{1}{2} t (f^2(t) + f(t^2)). \tag{23}$$

Before proceeding to discuss the functional equation (23) for the generating function f, we would like to give some lower and upper bounds for the b_n's, which in particular imply that the power series $f(t)$ has a strictly positive radius of convergence t_0 (t_0 is also the first singular point of f on the positive half-axis, since f has positive coefficients).

For this purpose, we define functions $\gamma(t)$, $\psi(t)$ by the following recursions

$$\begin{cases} c_0 = 1 \quad c_{n+1} = \frac{1}{2}\left(\sum_{i=0}^{n} c_i c_{n-i}\right) \\ d_0 = 1 \quad d_{n+1} = \left(\sum_{i=0}^{n} d_i d_{n-i}\right) \\ \gamma(t) = 1 = \sum_{n \geq 0} c_n t^n \\ \psi(t) = 1 = \sum_{n \geq 0} d_n t^n \end{cases} \tag{24}$$

By virtue of the recursion formulae (22), (24) we obtain

$$\gamma(t) \leq f(t) \leq \psi(t). \tag{25}$$

The auxiliary functions γ, ψ are algebraic, since they clearly satisfy

$$\begin{cases} \gamma = 1 + \frac{1}{2}t\gamma^2 \Leftrightarrow \gamma = \dfrac{1 - \sqrt{1-2t}}{t} \\ \psi = 1 + t\psi^2 \Leftrightarrow \psi = \dfrac{1 - \sqrt{1-4t}}{2t} \end{cases} \tag{26}$$

Expanding the power series γ

$$\gamma(t) = -t^{-1}\left[\sum_{i=1}^{\infty}(-2t)^i\binom{1/2}{i}\right] = \sum_{i=1}^{\infty}\left(\prod_{j=2}^{i}\frac{2j-3}{j}\right)t^{i-1}$$

we get the rather crude estimates

$$c_n = \prod_{j=2}^{n+1}(2 - 3/j) \leq b_n \leq 2^n c_n = d_n \tag{27}$$

implying that $\log b_n$ grows linearly with n. We follow now Bombieri's beautiful idea to relate the function $f(t)$ to the iteration of complex quadratic polynomials studied by Douady and Hubbard (cf. [1A]). We set

$$\begin{cases} g(t) = 1 - tf(t) = 1 - \sum_{i=0}^{\infty} b_i t^{i+1} \\ v(z) = g(z^2)/z = \dfrac{1}{z} - \sum_{i=0}^{\infty} b_i z^{2i+1} \end{cases} \tag{28}$$

Multiply the functional equation by $2t$ and replace $tf(t)$ by $(1 - g(t))$: then $g(t)$ satisfies

$$(\bullet) \qquad g(t^2) = g^2(t) + 2t.$$

Letting $t = z^2$, and using $g(z^2) = zv(z)$, finally we characterize v as the solution of the following problem

$$\begin{cases} v(z^2) = v(z)^2 + 2 \\ v(z) = \dfrac{1}{z} + \ldots \text{(power series in } z). \end{cases} \tag{29}$$

One can rephrase (29) as follows, setting

$$w = 1/z \tag{30}$$

Then $\xi(w) = v(1/w)$ is a local biholomorphism sending ∞ on the Riemann sphere to itelf and with differential equal to 1 at ∞: we denote by $\varphi_2 = \varphi$ the local inverse to this biholomorphism, so that setting

$$F_c(x) = x^2 + c \quad \text{for} \quad c \in \mathbb{C}, \quad \text{we have} \tag{31}$$

$$\begin{cases} F_2 \circ \xi = \xi \circ F_0 \quad \text{or, equivalently,} \\ \varphi_2 \circ F_2 \circ \varphi_2^{-1} = F_0. \end{cases} \tag{32}$$

We refer to [1A] for the following properties of $\varphi_2 = \varphi$. If Σ denotes the Riemann sphere, $\varphi = \varphi_2$ extends to an analytic homeomorphism of

$$\Sigma - L_2 \to \Sigma - B(0, R), \quad \text{where} \quad R = (t_0)^{-1/2}$$

(t_0 being the radius of convergence of $f(t), g(t)$), $B(0, R)$ is the ball with centre 0 and radius R, whereas, if, F_2^n stands for the nth iterate of F_2, and

$$\eta = \eta_2 = \log|\varphi_2(z)| = \lim_{n\to\infty} 2^{-n}\log|F_2^n(z)|, \tag{33}$$

L_2 is the figure eight $L_2 = \{z \mid \eta_2(z) \leq \eta_2(0)\}$,
and its interior contains the Cantor set (34)
$K_2 = \{z \mid F_2^n(z) \text{ does not tend to } \infty\}$.

We can now draw the following conclusions: since R^{-1} is the radius of convergence of

$$\frac{1}{v(z)} = z\left(1 - \sum_{i=0}^{\infty} b_i z^{2i+2}\right)^{-1},$$

which is a series with positive coefficients, we get that

$$R = \varphi_2(0) = \exp\left(\lim_{n\to\infty} 2^{-n}\log F_2^n(0)\right). \tag{35}$$

Setting

$$z_1 = 2, \qquad z_n = F_2(z_{n-1}) = F_2^n(0), \tag{36}$$

and observing that

$$\log(z_{n+1}) = 2\log(z_n) + \log\left(1 + \frac{2}{z_n^2}\right), \tag{37}$$

we obtain

$$R = \exp\left(\frac{1}{2}\log 2 + \sum_{n=1}^{\infty} 2^{-n-1}\log\left(1 + \frac{2}{z_n^2}\right)\right) = 2^{1/2}\left(1 + \frac{1}{2}\right)^{1/4}\left(1 + \frac{1}{18}\right)^{1/8}\left(1 + \frac{2}{z_3^2}\right)^{1/16}\cdots \tag{38}$$

Since $R = (t_0)^{-1/2}$, we get

THEOREM 2. $\limsup_{n\to\infty} \frac{1}{n}\log b_n = -2\log R$, *where* R *is given by* (38).

I would like to thank also Adrien Douady for reminding me about (35).

REFERENCES

1A. A. Douady and J. H. Hubbard: Itérationes des polynomes quadratiques complexes, *Comptes Rendues A.S. Paris*, **294** (1982), 123–126.