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SYMMETRIC PRODUCTS OF ELLIPTIC CURVES AND SURFACES OF GENERAL TYPE WITH $p_g = q = 1$

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Surfaces with $p_g = q = 1$ have $2 \le K^2 \le 9$. The case $K^2 = 2$ was completely analysed in [Ca] (see also [CaCi, §5]), whereas in [CaCi] we completely classified the surfaces with $K^2 = 3$, leaving the proof of the existence of one of the two classes in which these surfaces fall, to the present paper. In fact the main theorem we prove here is the following:

Theorem. *Surfaces with $p_g = q = 1$, $K^2 = 3$ and genus of the Albanese fibres equal to 3 are exactly the minimal resolutions of hypersurfaces with at most rational double points as singularities in certain linear systems on three-fold symmetrics products of elliptic curves (these linear systems, whose general elements are smooth, are described in the statement of Theorem (3.1)). The moduli space is generically smooth, irreducible, uniruled of dimension 5.*

This theorem follows from results in §3, cf. Theorems (3.1) and (3.5). A rough outline of the contents of the paper is as follows. Section 1 is devoted to a detailed study of linear systems of divisors on r-fold symmetric products of elliptic curves, with special regards to properties of ampleness,

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cohomology, effectiveness, etc. It turns out that for all numerical equivalence classes the behavior does not depend on the linear equivalence class except for the divisors whose class is a multiple of the anticanonical divisor. For those the only effective classes are those obtained by adding to a plurianticanonical class a divisor class of r-torsion. Section 2 is devoted to determining the cohomology of these classes and the results we obtain are intimately related to the representation theory of the finite Heisenberg group of order r^3. Finally in §3 we apply this detailed analysis to the above-mentioned problems about existence and moduli of surfaces with $p_g = q = 1$, $K^2 = 3$. Our description of the moduli space is rather explicit and it is likely that this moduli space, which is clearly uniruled, is in fact rational. Concerning the status of classification of surfaces with $p_g = q = 1$, $2 \le K^2 \le 9$, and related open questions, we refer the reader to [CaCi].

1. Symmetric products of elliptic curves and their line bundles

(1) Symmetric products of elliptic curves. Let A be a curve of genus 1 and let $A^{(r)}$ be its r-fold symmetric product, namely the quotient of A^r by the obvious action of the symmetric group S_r. We will denote by $\pi_r: A^r \to A^{(r)}$ the natural map (we will drop the index r in π_r if no confusion arises). $A^{(r)}$ is the variety parametrizing all effective divisors of degree r on A, and one has the natural Abel-Jacobi map $\beta_r: A^{(r)} \to A$, realizing $A^{(r)}$ as a $\mathbf{P}^{r-1}$ bundle over A (again we will drop the index r in β_r if no confusion arises).

We briefly recall how to recover a vector bundle $\mathbf{E}_r$ of rank r over A such that $\mathbf{P}(\mathbf{E}_r) = A^{(r)}$. First one considers $A \times A$ and its projections $q_i: A \times A \to A$, $i = 1, 2$, onto the two factors. A Poincaré sheaf $\mathscr{P}_r$ for line bundles of degree r on $A \times A$ is effectively constructed as follows. One fixes a point $u \in A$ and defines $\mathscr{P}_r = [\Delta_A + (r-1)(A\times\{u\}) - (\{u\}\times A)]$, where Δ_A is the diagonal in $A \times A$, and it is clear that, setting $\mathbf{E}_r(u) = (q_1)_*(\mathscr{P}_r)$, one has $\mathbf{P}(\mathbf{E}_r) = A^{(r)}$.

In fact, we have an inductive exact sequence

$$0 \to \mathscr{P}_{r-1} \to \mathscr{P}_r \to \mathscr{P}_{r|A\times\{u\}} \to 0 \tag{1.1}$$

and we notice that $\mathscr{P}_{r|A\times\{u\}} \cong \mathscr{O}_A$, via the natural identification $A\times\{u\} = A$. Taking the direct image long exact sequence associated to (1.1) and to the map q_1, since $R^1q_{1*}(\mathscr{P}_r) = 0$, for $r \ge 1$, we obtain

$$0 \to \mathbf{E}_{r-1}(u) \to \mathbf{E}_r(u) \to O_A \to 0 \tag{1.2}$$

for $r > 1$, while for $r = 1$ we obtain

$$0 \to \mathbf{E}_1(u) \to O_A \to R^1q_{1*}(\mathscr{P}_0) \to 0.$$

Notice that $R^1q_{1*}(\mathscr{P}_0)$ is a skyscraper sheaf of length one supported at u. Whence $\mathbf{E}_1(u) \cong [-u]$. Furthermore the extension (1.2) does not split, since the divisors of $\mathscr{P}_r$ have zero intersection number with any divisor of the form $A\times\{v\}$ but are not algebraically equivalent to a sum of those; hence $h^0(\mathbf{E}_r(u)) = h^0(\mathscr{P}_r) = 0$.

We have thus given a proof of an assertion stated in [At, p. 451] (we observe here that our sheaves $\mathbf{E}_r(u)$ are the dual of the Atiyah's sheaves $E_r(u)$, also mentioned in [CaCi]).

We shall need in the sequel the following:

Lemma (1.3). *The bundles $\mathbf{E}_r(u)$ are stable.*

Proof. It is equivalent to prove that $E_r(u) = \mathbf{E}_r(u)^\vee$ is stable. Let W be a destablizing indecomposable subbundle of rank k of $E_r(u)$. Then $\deg(W) > 0$ and the Riemann-Roch theorem implies $0 < \deg(W) \le h^0(W) \le h^0(E_r(u)) = 1$; hence $\deg(W) = 1$. Atiyah's classification (see [At, Corollary to Theorem 7]) yields then that $W \cong E_k(v)$ for some $v \in A$. By (1.2) and since $h^0(\mathbf{E}_k(u)) = 0$, we have a nonzero homomorphism of $E_k(v)$ into $E_k(u)$ arising as the composition of the inclusion of $E_k(v)$ in $E_r(u)$ with the projection on $E_k(u)$. By [At, Lemma 22], we have

$$E_k(u) \otimes E_k(u)^\vee = \bigoplus_{\xi\in A_k} [\xi - 0] \tag{1.4}$$

where A_k is the subgroup of k-torsion points. Moreover there exists a line bundle $\mathscr{L}$ of degree 0 such that $\mathscr{L}^{\otimes k} \cong [v-u]$ and $E_k(v) \cong E_k(u)\otimes\mathscr{L}$. Hence the existence of a nontrivial homomorphism between $E_k(v)$ and $E_k(u)$ implies that $\mathscr{L}$ is a k-torsion bundle; hence $E_k(v) \cong E_k(u)$ and the above homomorphism is an isomorphism. But then $E_k(v)$ would be a direct summand of $E_r(u)$, a contradiction. q.e.d.

The geometrical meaning of (1.2) is the following. Dualizing (1.2) we see that there is a natural section of $O(1)$ whose locus of zeros consists of the set $D_u^{(r)}$ of all divisors of degree r on A containing u, which is isomorphic to $A^{(r-1)}$ (we shall write D_u instead of $D_u^{(r)}$ if no confusion arises).

For further use we will write D for the algebraic equivalence class of D_u. Moreover we will denote by F_v the fibre of β over $v \in A$, and simply by F its algebraic equivalence class. As for every projective bundle, the Picard group of $A^{(r)}$ is generated over the Picard group of the basis by the class of $O(1)$. Hence any divisor of $A^{(r)}$ is algebraically equivalent to a divisor of the form $mD + nF$ with n, m integers.

Definitions and Notations

$H^i(X, \mathscr{F})$ $H^i(\mathscr{F})$	For a coherent sheaf $\mathscr{F}$ on a variety X over $\mathbf{C}$, the field of complex numbers
$h^i(\mathscr{F})$	The $\mathbf{C}$-dimension of $H^i(\mathscr{F})$
$\chi(\mathscr{F})$	The Euler-Poincaré characteristic of $\mathscr{F}$
$[D]$	The invertible sheaf associated to a Cartier divisor D on a variety X
$\lvert D\rvert$ $\mathbf{P}(H^0([D]))$	The complete linear system of effective divisors linearly equivalent to a divisor D
A	A curve of genus 1
$\mathscr{P}_r$	A Poincaré sheaf for line bundles of degree r on $A \times A$. Explicitly $\mathscr{P}_r = [\Delta_A + (r-1)(A \times \{u\}) - (\{u\} \times A)]$, where Δ_A is the diagonal in $A \times A$
$\mathscr{P}_0$	Gives an isomorphism $A \to \mathrm{Pic}^0(A)$, which makes A an elliptic curve
$\oplus$	The sum on the elliptic curve A
$\ominus$	The difference on the elliptic curve A
A_k	The subgroup of k-torson points on A
A^r	The r-fold product of A
$A^{(r)}$	The r-fold symmetric product of A
$D_u^{(r)}$	The set of all divisors of degree r on A containing u, which is isomorphic to $A^{(r-1)}$
$\beta_r: A^{(r)} \to A$	The Abel-Jacobi map
$W^\vee$	The dual of a vector space or vector bundle
$\mathbf{P}(W^\vee)$	$\mathrm{Proj}(W_\sim)$, for a vector bundle W
Ω_X^i	The sheaf of holomorphic i-forms on the variety X
$\varepsilon_r = \exp(2\pi i/r)$	A primitive r-th root of the unity
$\mathbf{e}(z) = \exp(2\pi i z)$	For every complex number z
$\mu_r \cong \mathbf{Z}/r\mathbf{Z}$	The cyclic group of rth roots of unity
$\mathbf{G}_r$	The group $(\mathbf{Z}/r\mathbf{Z})^2$
$\mathbf{H}_r$	The Heisenberg group, the central extension of $\mathbf{G}_r$ with μ_r generated by two elements ρ and σ both of order r, satisfying the relation $\sigma^{-1}\rho^{-1}\sigma\rho = \varepsilon^{-1}$

$\mathrm{Aut}(G)$	The group of automorphisms of a group G
$\mathrm{Int}(G)$	The group of internal automorphisms of a group G
S	A minimal surface with $p_g = q = 1$
$A = A(S)$	The Albanese variety of S, a curve of genus 1
$\alpha: S \to A$	The Albanese morphism of S
$\mathscr{P}' = (\alpha \times \mathrm{id})^*(\mathscr{P})$	A Poincaré sheaf on $S \times A$
$g = g(S)$	The arithmetic genus of a fibre of α
$K = K_S$	A canonical divisor of S
K^2	The self-intersection of K $(2 \le K^2 \le 9)$
$\{K\} = \{C_t\}_{t \in A}$	The paracanonical system of S (see §1)
Y in $S \times A$	The paracanonical incidence correspondence, i.e., for every $t \in A$, $Y \cap (S \times \{t\}) = C_t \times \{t\}$
$\pi_S: S \times A \to S$	The projection onto the first factor
$\pi_A: S \times A \to A$	The projection onto the second factor
$\iota = \iota(S)$	The degree of $\pi_{S\mid Y}: Y \to S$, called the index of the paracanonical system. Roughly speaking this is the number of distinct curves of $\{K\}$ passing through the general point of S
$\mathscr{K} = \pi_S^*([K]) \otimes \mathscr{P}'$	The paracanonical sheaf on $S \times A$
$V = \alpha_*[K]$	The (locally free) direct image of the relative canonical sheaf for α on A
$\omega: S \to P(V^\vee)$	The (rational) relative canonical map
Λ^i	$R^i(\pi_A)_*\mathscr{K}$
$\lambda = \lambda(S)$	The length of Λ^1

We notice that every automorphism g of A naturally induces an automorphism $g^{(r)}$ of $A^{(r)}$. In particular the symmetric product $A^{(r)}$ has a 1-parameter group of automorphisms induced by the group of translations of A. If $u \in A \to u \oplus t \in A$ is a translation, the induced action on $A^{(r)}$ sends the divisor D_u to $D_{u \oplus t}$, resp. F_u to $F_{u \oplus rt}$.

Proposition (1.5). *The group of translations of A acts transitively on all the algebraic equivalence classes in $A^{(r)}$ except for the classes which are multiple of $rD - F$. Moreover the homomorphism ϕ which to $g \in \mathrm{Aut}(A)$ associates $g^{(r)}$ is an isomorphism onto* $\mathrm{Aut}(A^{(r)})$.

Proof. The first assertion is a straightforward consequence of the previous remark.

The injectivity of ϕ is obvious. Let us prove the surjectivity. First of all, any automorphism of $A^{(r)}$ induces an isomorphism of $\mathrm{Pic}^0(A^{(r)}) = \mathrm{Pic}^0(A)$ which is also induced by some automorphism of A. Hence it suffices to prove surjectivity onto the subgroup of automorphisms acting trivially on $\mathrm{Pic}^0(A^{(r)})$. Let γ be such an automorphism of $A^{(r)}$. As we shall see later on (cf. (1.6)), the canonical bundle K of $A^{(r)}$ is algebraically equivalent to $-rD + F$. Since F is the fibre of the Albanese map its algebraic equivalence class is preserved under any automorphism. The same happens for D since K is preserved and there is no torsion in $\mathrm{Num}(A^{(r)})$. Thus, multiplying γ by the automorphism induced by a suitable translation, we may assume that γ leaves every divisor D_u invariant. If $a_1, \cdots, a_r$ are distinct points on A, then the divisor $a_1 + \cdots + a_r$ is the only point of intersection of $D_{a_1}, \cdots, D_{a_r}$. Hence γ is the identity. q.e.d.

(2) Cohomology on projective bundles. Any line bundle $\mathscr{L}$ on a projective bundle $X = \mathbf{P}(V)$ of dimension r over a curve C is of the form $O(m) \otimes p^*(\mathscr{L}')$, where $\mathscr{L}'$ is a line bundle on C and $p: X \to C$ is the natural projection. The Leray spectral sequence for the map p degenerates at the first step; hence

$$\begin{aligned} H^i(\mathscr{L}) &= H^0(R^i p_*(\mathscr{L})) \oplus H^1(R^{i-1} p_*(\mathscr{L})) \\ &= H^0(R^i p_*(O(m)) \otimes \mathscr{L}') \oplus H^1(R^{i-1} p_*(O(m)) \otimes \mathscr{L}'). \end{aligned}$$

Now we have $R^i p_*(O(m)) = 0$ if either $i \neq 0, r-1$, or if $i = 0$, $m < 0$, or $i = r-1$, $-r < m$. Consequently all the cohomology is concentrated in degrees $i = 0, 1, r-1, r$, more precisely in degrees $i = 0, 1$ if $m \geq 0$ and in degrees $r-1, r$ if $m \leq -r$, whereas if $-r < m < 0$ all the cohomology groups vanish.

By relative duality for the map p, we have

$$R^{r-1} p_*(O(m)) \cong p_*(O(-m) \otimes \omega_{X|C})^\vee .$$

Using the Euler sequence

$$0 \to \Omega^1_{X|C} \to p^*(V^\vee \otimes O(-1)) \to O_X \to 0,$$

one deduces that

$$\omega_{X|C} \cong O(-r) \otimes p^*(\det(V)^\vee). \tag{1.6}$$

We observe for further use that

$$p_*(\omega^\vee_{X|C}) = \det(V) \otimes \mathrm{Sym}^r(V^\vee). \tag{1.7}$$

We obtain finally

$$\begin{aligned} &p_*(O(m)) = \mathrm{Sym}^m(V^\vee), \\ &R^{r-1} p_*(O(m)) \cong \det(V) \otimes \mathrm{Sym}^{-r-m}(V). \end{aligned} \tag{1.8}$$

One can therefore restrict oneself to determine the cohomology of the line bundles $\mathscr{L}$ for which $m \geq 0$. For those the only nonzero cohomology groups are H^0 and H^1. Of course, by using the Riemann-Roch theorem on the curve C, it suffices to compute the dimension of vector spaces of the type $H^0(\mathrm{Sym}^m(V) \otimes \mathscr{L}')$ and $H^0(\mathrm{Sym}^m(V^\vee) \otimes \mathscr{L}')$.

(3) Effective divisors on $A^{(r)}$. Let us go back to the particular case we were considering before, i.e. where $X = A^{(r)}$. We deal with $H^0(\mathrm{Sym}^m(\mathbf{E}_r(u)) \otimes \mathscr{L}')$.

Proposition (1.9). *$H^0(\mathrm{Sym}^m(\mathbf{E}_r(u)) \otimes \mathscr{L}') = (0)$ unless $m \geq 0$ and $m + nr \geq 0$, where $n = \deg \mathscr{L}'$.*

Proof. Let δ in A^r be the small diagonal, i.e. the set of all divisors of the type rv, $v \in A$. We notice that δ moves in an algebraic family of curves sweeping out the whole A^r. Assume that $mD+nF$ is (algebraically equivalent to) an effective divisor. Then $\pi^*(mD + nF) \cdot \delta \geq 0$. But now $\pi^*(D_u)$ is the sum of the r components $D_{i,u} = \{(u_1, \cdots, u_r): u_i = u\}$, each intersecting δ transversally at one point. Thus $\pi^*(D) \cdot \delta = r$. Instead the intersection number $\pi^*(F) \cdot \delta$ equals the degree of the map $\delta \to A$ sending $(v, \cdots, v)$ to rv; hence it equals r^2. q.e.d.

The relative dualizing sheaf on $A^{(r)}$ is isomorphic to the canonical sheaf $[K_r]$ (simply denoted by $[K]$ if no confusion arises). By (1.6) we have that $[-K] \cong O(r) \otimes p^*([-u])$; in particular the algebraic equivalence class of $[-K]$ is $rD - F$. By (1.7) we have, for any positive integer h,

$$p_*([-hK]) = [-u]^{\otimes h} \otimes \mathrm{Sym}^{rh}(E_r(u)). \tag{1.10}$$

Proposition (1.11). *$p_*([-hK])$ is a direct sum of r-torsion line bundles*

$$p_*([-hK]) = \bigoplus_{\xi \in A_r} [\xi - 0]^{n(h,\xi)}$$

with $0 < n(h, \xi) \leq r^{2(rh-1)}$.

Proof. By (1.3) it is clear that $p_*([-hK])$ is semistable; hence, having degree 0, it is a direct sum of indecomposable bundles of degree 0. More precisely $p_*([-hK]) = (\Lambda^r E_r(u)^\vee)^{\otimes h} \otimes \mathrm{Sym}^{rh}(E_r(u))$ is a direct summand of $(E_r(u)^\vee \otimes E_r(u))^{\otimes rh}$, which, by (1.4), is isomorphic to $\bigoplus_{\xi \in A_r} [\xi - 0]^{\mu(h,\xi)}$, where $\mu(h, r) = r^{2(rh-1)}$. q.e.d.

Remark (1.12). The integer $n(h, \xi)$ depends only upon the order of torsion on the element ξ, because the monodromy of the universal family of elliptic curves with a level r-structure acts transitively on the set of elements of fixed order $r'|r$.

We can now prove the following:

Theorem (1.13). *Every divisor in the algebraic equivalence class of $mD + nF$ is linearly equivalent to an effective one if and only if $m \geq 0$ and $m + nr > 0$. There exist, for every positive integer h, a finite number of classes of effective divisors algebraically equivalent to $-hK$. For those, H^0 and H^1 have the same positive dimension.*

Proof. The second assertion follows by Proposition (1.11). To prove the first assertion it suffices to consider reducible divisors which are sums of the D_u's, the F_v's and effective divisors algebraically equivalent to $-K$. q.e.d.

(4) Linear systems on $A^{(r)}$. First we state the following:

Proposition (1.14). *An effective divisor is ample if and only if it is not algebraically equivalent to either a multiple of F or to a multiple of $-K$.*

Proof. It suffices to notice that D is ample (by Nakai-Moishezon) and F and $-K$ are numerically effective (nef) but not ample. This is clear for F. Moreover $\pi^*(-K) = \overline{\Delta}$, the big diagonal of A^r. Setting

$$\Delta_{i,j,t} = \{(u_1, \cdots, u_r) | u_i = u_j \oplus t\} \tag{1.15}$$

we have $\overline{\Delta} = \bigcup_{1 \leq i < j \leq r} \Delta_{i,j,0}$. Hence it is clear that $\overline{\Delta}$ moves in an algebraic family without base points and thus $-K$ is nef. The nonampleness of $-K$ follows from Proposition (1.11). q.e.d.

Remark (1.16). One of the two implications in the statement of Proposition (1.14) is valid more generally (cf. [Mi, Theorem 3.1]): in a projective bundle over a curve any effective divisor is nef if and only if the bundle is semistable.

We are able now to compute the cohomology of all divisors on $A^{(r)}$ which are not multiples of K.

Theorem (1.17). *If $\mathscr{D}$ is a line bundle associated to a divisor algebraically equivalent to $mD + nF$ then:*

(i) *if $m + nr \neq 0$, there is exactly one nonzero cohomology group of $\mathscr{D}$, in the following cases: $H^0(\mathscr{D})$ if $m \geq 0$, $m + nr > 0$, $H^1(\mathscr{D})$ if $m \geq 0$, $m + nr < 0$, $H^{n-1}(\mathscr{D})$ if $m \leq -r$, $m + nr > 0$, $H^n(\mathscr{D})$ if $m \leq -r$, $m + nr < 0$;*

(ii) *if $-r < m < 0$ all cohomology groups vanish;*

(iii) *if $m + nr = 0$ then the Euler-Poincaré characteristic of $\mathscr{D}$ is zero and there are at most two nonzero cohomology groups, namely*

$$H^0(\mathscr{D}) \text{ and } H^1(\mathscr{D}) \quad \text{if } m \geq 0, \qquad H^n(\mathscr{D}) \text{ and } H^{n-1}(\mathscr{D}) \quad \text{if } m \leq -r$$

and only for finitely many divisors $\mathscr{D}$ algebraically equivalent to $mD + nF$ are these cohomology groups nonzero.

Moreover the Euler-Poincaré characteristic of $\mathscr{D}$ is

$$\chi(\mathscr{D}) = (r!)^{-1}(m + rn) \cdot \prod_{i=1,\cdots,r-1} (m + i).$$

Proof. By the Kodaira vanishing theorem, if $\mathscr{D}$ is ample then $h^i(\mathscr{D}^\vee) = 0$ for all positive $i < n$. Moreover $[-K] \otimes \mathscr{D}$ is ample if $\mathscr{D}$ is such. Hence if $\mathscr{D}$ is ample all the cohomology vanishes except $H^0(\mathscr{D})$. A similar assertion is easily verified if $\mathscr{D}$ is a multiple of $[F]$. The assertions concerning the cohomology groups follow easily by Serre duality, both on $A^{(r)}$ and on A, Propositions (1.9) and (1.11) and the remarks we made in subsection 2. Finally we know that the Hilbert polynomial $\chi(\mathscr{D})$ is a polynomial of degree r in m and n, which vanishes on the lines $m = i$, for $i = -1, \cdots, -r+1$, and on the line $m + rn = 0$, whence the last assertion. Of course one could also have applied Riemann-Roch to compute $\chi(\mathscr{D})$. q.e.d.

Theorem (1.18). *If $\mathscr{D}$ is a line bundle associated to an effective divisor algebraically equivalent to $mD + nF$ then the complete linear system $|\mathscr{D}|$ associated to $\mathscr{D}$ is base point free if either $n \geq 0$ and $m + n \geq 2$ or $n < 0$ and $m + nr \geq 2$.*

Proof. The algebraic systems $\{D_u\}_{u \in A}$ and $\{F_v\}_{v \in A}$ are base point free. Hence the first assertion follows easily. The second assertion is proved by induction on r. The case $r = 1$ is clear. Restricting $\mathscr{D}$ to a divisor of the type D_u, which is isomorphic to $A^{(r-1)}$, we obtain a divisor $\mathscr{D}_u$ whose associated complete system is base point free by induction. On the other hand since $h^1([(m-1)D + nF]) = 0$ by Theorem (1.17), the system $|\mathscr{D}|$ cuts out the complete linear system on D_u, whence the assertion. q.e.d.

Proposition (1.19). *Let $\mathscr{D}$ be as above with $n < 0$ and $m + rn = 1$; there are only a finite number of base points.*

Proof. Arguing as before by restriction to a general D_u, we see that, by Theorem (1.71), (iii), $|\mathscr{D}|$ cuts out a complete linear system. Since $m + n(r-1) > 1 - n \geq 2$, we can then apply Theorem (1.18) and conclude that there are no base points on D_u. Hence the base locus has dimension at most 0. q.e.d.

Example (1.20). By Theorem (1.17), a linear system of the type $|3D - F|$ on $A^{(2)}$ is a pencil with base points, since $(3D - F)^2 = 3$. This shows that Proposition (1.19) cannot be improved.

Theorem (1.21). *If $\mathscr{D}$ is a line bundle associated to an effective divisor algebraically equivalent to $mD+nF$ then $|\mathscr{D}|$ is very ample if either $n \geq 0$, $m \geq 1$ and $m+n \geq 3$, or $n < 0$ and $m + nr \geq 3$.*

Proof. We start with a preliminary remark whose proof is trivial. Suppose we have a linear system $\mathscr{S}$ of divisors on a smooth variety V, passing through a point p. Suppose moreover that there is a divisor H of $\mathscr{S}$ smooth at p and such that $\mathscr{S}$ restricted to the irreducible component H' of H passing through p gives an embedding of H' at p: then $\mathscr{S}$ gives an embedding of V at p. We apply this remark taking for each point p of $A^{(r)}$ a divisor $H' = D_u$ for a suitable u, and a divisor in $|\mathscr{D} - H'|$ not passing through p (this is possible by Theorem (1.18)). By induction, since $h^1([\mathscr{D} - H']) = 0$, $|\mathscr{D}|$ restricted to H' gives an embedding. Therefore $|\mathscr{D}|$ gives at any point a local embedding. Since for any pair of distinct points of $A^{(r)}$ there is a divisor D_u containing one of the points but not the other, and since by Theorem (1.18), $|\mathscr{D} - D_u|$ has no base points, one has that $|\mathscr{D}|$ separates pairs of points in $A^{(r)}$. q.e.d.

2. The Heisenberg group and the anticanonical divisor of the symmetric product of an elliptic curve

In this section we will show how to compute the numbers $n(h, \xi)$ introduced in subsection 3 of §1. In order to do this we will use some representation theory for the Heisenberg group. We shall work here on the complex numbers.

(1) The Heisenberg group. Let r be a positive integer. We shall denote by ε_r, or simply by ε if no confusion arises, the primitive rth root of unity $\mathbf{e}(1/r) = \exp(2\pi i/r)$, which generates the cyclic group $\mu_r \cong \mathbf{Z}/r\mathbf{Z}$ of rth roots of unity. We also denote by $\mathbf{G}_r$ the group $(\mathbf{Z}/r\mathbf{Z})^2$.

We recall that the Heisenberg group $\mathbf{H}_r$ is the unique central extension

$$0 \to \mu_r \to \mathbf{H}_r \to \mathbf{G}_r \to 0$$

associated to the standard symplectic form $\lambda: ((a_1, a_2), (a_1', a_2')) \in \mathbf{G}_r \to a_1 a_2' - a_2 a_1' \in \mathbf{Z}/r\mathbf{Z}$. In concrete terms, $\mathbf{H}_r$ is generated by a unique central element ε such that $\varepsilon^r = 1$ and by two elements ρ and σ both of order r, satisfying the relation $\sigma^{-1}\rho^{-1}\sigma\rho = \varepsilon^{-1}$.

$\mathbf{H}_r$ admits a standard faithful irreducible representation V and a classical theorem by Stone-von Neuman asserts that, up to isomorphism, V is the unique irreducible representation of $\mathbf{H}_r$ where μ_r acts by scalar multiplication. An explicit description of V is as follows. Let V be the vector space $\mathbf{C}^r$. For reasons that will become clear in a while, we shall denote by $\mathscr{B} = (\theta^{(i)})_{i=0,\cdots,r-1}$ the natural basis of V. Then V becomes a representation of $\mathbf{H}_r$ if one lets μ_r act by scalar multiplication and lets ρ and σ act on $\mathscr{B}$ as follows:

$$\rho: \theta^{(i)} \to \varepsilon^i \theta^{(i)}, \qquad \sigma: \theta^{(i)} \to \theta^{(i+1)},$$

where i has to be understood as a representative of a class in $\mathbf{Z}/r\mathbf{Z}$.

If $\mathscr{B}^\vee = (\eta^{(i)})_{i=0,\cdots,r-1}$ is the dual basis to $\mathscr{B}$ in $V^\vee$, the induced dual representation of $\mathbf{H}_r$ on $V^\vee$ is, of course, defined by

$$\varepsilon: \eta^{(i)} \to \varepsilon^{-1}\eta^{(i)}, \quad \rho: \eta^{(i)} \to \varepsilon^{-i}\eta^{(i)}, \quad \sigma: \eta^{(i)} \to \eta^{(i+1)}.$$

We notice now that $\mathrm{Aut}(\mathbf{H}_r)$ acts on the center μ_r and on $\mathbf{H}_r/\mu_r \cong \mathbf{G}_r$. Let $\mathrm{Aut}^0(\mathbf{H}_r)$ be the subgroup of $\mathrm{Aut}(\mathbf{H}_r)$ of the automorphisms of $\mathbf{H}_r$ acting as the identity on the center. We now prove the following analogue of Theorem 6, p. 30 of [Ig]:

Proposition (2.1). $\mathrm{Aut}^0(\mathbf{H}_r)/\mathrm{Int}(\mathbf{H}_r) \cong \mathrm{Sp}(2, \mathbf{Z}/r\mathbf{Z}) \cong \mathrm{SL}(2, \mathbf{Z}/r\mathbf{Z})$.

Proof. The symplectic form λ on $\mathbf{G}_r$ is obtained by choosing representatives α and β of two lifts of $a, b \in \mathbf{G}_r$ and setting $\lambda(a, b) = [\alpha, \beta] \in \mu_r \cong \mathbf{Z}/r\mathbf{Z}$. On the other hand any symplectic transformation g leaves invariant the cocycle in $H^2(\mathbf{G}_r, \mu_r)$ which classifies the given extension. Hence there is an automorphism of $\mathbf{H}_r$ in $\mathrm{Aut}^0(\mathbf{H}_r)$ inducing the given transformation g.

To finish the proof it suffices to show that any automorphism g which is the identity on μ_r and induces the identity on $\mathbf{G}_r$ is in $\mathrm{Int}(\mathbf{H}_r)$. It is easy to verify that all those isomorphisms g are of the form $g(\varepsilon) = \varepsilon$, $g(\rho) = \varepsilon^h \rho$, $g(\sigma) = \varepsilon^k \sigma$ for suitable integers h and k, and they are therefore the inner automorphisms associated to $\rho^h \sigma^k$. q.e.d.

Finally we observe that if $g \in \mathrm{Aut}^0(\mathbf{H}_r)$, then composing the standard representation V with g we obtain an isomorphic representation by the quoted theorem of Stone-von Neumann. In particular if $\mathrm{Gl}(V) \to \mathrm{Gl}(W)$ is a homomorphism, the decomposition of W into irreducible summands is $\mathrm{Aut}^0(\mathbf{H}_r)$-equivariant.

(2) Representation theory for the Heisenberg group. Let now h be a positive integer, and let $W_h = \mathrm{Sym}^{hr}(V^\vee)$ be the associated representation. Since μ_r acts trivially on W_h, W_h actually lifts from a representation γ of $\mathbf{G}_r = (\mathbf{Z}/r\mathbf{Z})^2$, which we are now going to describe.

We choose a basis of W_h to be given by the monomials $\eta^{(I)} = \eta^{(i_1)}\eta^{(i_2)} \cdots \eta^{(i_{hr})}$, where $I = (i_1, \cdots, i_{hr})$ is a multi-index such that $0 \le i_1 \le i_2 \le \cdots \le i_{hr} \le r-1$. We shall find it convenient to consider I as an element of a suitable subset $\mathcal{I}$ of $(\mathbf{Z}/r\mathbf{Z})^{hr}$. If we let R and S be the two generators of $\mathbf{G}_r$ induced by ρ and σ respectively, the monomials $\eta^{(I)}$ are eigenvectors for R, since R acts by sending $\eta^{(I)}$ to $\varepsilon^{(i_1+i_2+\cdots+i_{hr})}$, whereas the action of S is as follows:

$$S: \eta^{(I)} \to \eta^{(I+\mathbf{1})}$$

where $\mathbf{1}$ denotes the multi-index $(1, \cdots, 1)$ and the sum takes place in $(\mathbf{Z}/r\mathbf{Z})^{hr}$, up to reordering of the entries of the multi-index $I+\mathbf{1}$.

Of course W_h splits into irreducible representations of degree 1 of $\mathbf{G}_r$ and we will describe this splitting. More precisely we notice that W_h splits according to the S-orbits in the set $\mathcal{A} = (\eta^{(I)})_{I\in\mathcal{I}}$, and we are going to see how each of the summands corresponding to an orbit splits into irreducible representations of degree 1.

Recall that we may identify $\mathbf{G}_r^\vee$ with $\mathbf{G}_r$ in the following way: to $b = (b_1, b_2) \in \mathbf{G}_r$ we associate the character

$$\chi_b: (c_1, c_2) \in \mathbf{G}_r \to \varepsilon^{c_1b_1+c_2b_2} \in \mathbf{C}^*.$$

With this identification we have

$$\chi_b(R) = \varepsilon^{b_1}, \qquad \chi_b(S) = \varepsilon^{b_2}.$$

Let $\mathcal{D} = \mathcal{D}(I)$ be the S-orbit of $\mathcal{A}$ containing $\eta^{(I)}$, and suppose $\mathcal{D}$ consists of $k = k(\mathcal{D})$ elements. This means that $I + k\mathbf{1} = I$, where k is minimal under this condition, and $\mathcal{D} = \{\eta^{(I)}, \eta^{(I+\mathbf{1})}, \cdots, \eta^{(I+(k-1)\mathbf{1})}\}$. Moreover we notice that, since S has period r, k divides r; hence there is a $d = d(\mathcal{D})$ such that $r = kd$.

Let us consider the invariant subspace $W_h(\mathcal{D})$ of W_h corresponding to $\mathcal{D}$ and let $\gamma_{\mathcal{D}}: \mathbf{G}_r \to \mathrm{GL}(W_h(\mathcal{D}))$ be the induced representation. If $\chi_b \in \mathbf{G}_r^\vee$, we let $\mu(\mathcal{D}, \chi_b)$ be the integer such that

$$W_h(\mathcal{D}) = \bigoplus_{\chi_b\in\mathbf{G}_r^\vee} \chi_b^{\mu(\mathcal{D}, \chi_b)}.$$

Lemma (2.2). *One has $\mu(\mathcal{D}, \chi_b) \le 1$ and equality holds if and only if*

$$i_1 + \cdots + i_{hr} + b_1 \equiv 0 \pmod r, \qquad b_2 \equiv 0 \pmod{d(\mathcal{D})}.$$

Proof. By character theory we have

$$\mu(\mathcal{D}, \chi_b) = (1/r^2)\cdot \sum_{(c_1, c_2)\in\mathbf{G}_r} \mathrm{tr}(\gamma_{\mathcal{D}}(c_1, c_2))\cdot\chi_b(c_1, c_2)^{-1}$$
$$= (1/r^2)\cdot \sum_{(c_1, c_2)\in\mathbf{G}_r} \mathrm{tr}(\gamma_{\mathcal{D}}(c_1, c_2))\cdot\varepsilon^{-(c_1b_1+c_2b_2)}.$$

Notice that

$$\gamma_{\mathcal{D}}(c_1, c_2)(\eta^{(I)}) = \varepsilon^{-c_1(i_1+\cdots+i_{hr})}\cdot\eta^{(I+c_2\mathbf{1})}$$

and therefore $\mathrm{tr}(\gamma_{\mathcal{D}}(c_1, c_2))$ is zero unless c_2 is a multiple of k, in which case $\mathrm{tr}(\gamma_{\mathcal{D}}(c_1, c_2)) = k\varepsilon^{-c_1(i_1+\cdots+i_{hr})}$. Hence

$$\mu(\mathcal{D}, \chi_b) = (k/r^2)\cdot \sum_{c_1\in\mathbf{Z}/r\mathbf{Z}}\ \sum_{j=0,\cdots,d-1} \varepsilon^{-(c_1b_1+c_2b_2)c_1(i_1+\cdots+i_{hr})}$$
$$= (k/r^2)\cdot \sum_{j=0,\cdots,d-1}\ \sum_{c_1\in\mathbf{Z}/r\mathbf{Z}} \varepsilon^{-(c_1b_1+c_2b_2)-c_1(i_1+\cdots+i_{hr})}$$
$$= (k/r^2)\cdot \sum_{j=0,\cdots,d-1} \varepsilon^{-kjb_2}\cdot\delta$$

where $\delta = 0$ unless $i_1 + \cdots + i_{hr} + b_1 \equiv 0 \pmod r$, in which case $\delta = r$. In fact one has $\sum_{c_1\in\mathbf{Z}/r\mathbf{Z}} \varepsilon^{xc_1} = 0$, unless $x \equiv 0 \pmod r$. By the same reason we finally have $\mu(\mathcal{D}, \chi_b) = 0$ unless $b_2 \equiv 0 \pmod d$, in which case $\mu(\mathcal{D}, \chi_b) = kdr/r^2 = 1$. q.e.d.

Let us remark that the condition $b_2 \equiv 0 \pmod d$ in Lemma (2.2) means that there is an integer u such that $b_2 = ud = ur/k$, namely $ur = kb_2$, which one the other side is the same as saying that k is a multiple of the period $\beta(b_2)$ of b_2 in $\mathbf{Z}/r\mathbf{Z}$. Another remark is that, given $\mathcal{D}$, the quantity $\sigma(\mathcal{D}) = i_1 + \cdots + i_{hr} \in \mathbf{Z}/r\mathbf{Z}$ computed by a given $\eta^{(I)}$ in $\mathcal{D}$ is independent of $\eta^{(I)}$ and depends only on $\mathcal{D}$.

Now, if $\chi_b \in \mathbf{G}_r^\vee$, we denote by $m(h, \chi_b)$ the integer such that

$$W_h = \bigoplus_{\chi_b\in G_r^\vee} \chi_b^{m(h, \chi_b)}.$$

Proposition (2.3). *Let $b = (b_1, b_2) \in \mathbf{G}_r$, let $\beta = \beta(b_2)$ be the period of b_2 in $\mathbf{Z}/r\mathbf{Z}$, and let $K = r/\beta$. Moreover, let $\nu(b, k)$ be the number of orbits $\mathcal{D}$ of $\mathcal{A}$ of cardinality k such that $\sigma(\mathcal{D}) + b_1 = 0$. Then*

$$m(h, \chi_b) = \sum_{k'|K} \nu(b, \beta k').$$

Proof. One clearly has $m(h, \chi_b) = \sum_{\mathcal{D}} \mu(\mathcal{D}, \chi_b)$ where the sum is taken over all the orbits. The assertion follows by Lemma (2.2). q.e.d.

For further use we shall also describe the associated representation on $W'_h = (\Lambda^r V)^{\otimes d}$. Again we notice that μ_r acts trivially on W'_h, which is therefore an irreducible representation of degree 1, hence a character, of $\mathbf{G}_r$. More precisely we have the following proposition, whose proof is clear:

Proposition (2.4). *W'_h is the character $\chi^{(h)}$ such that*

$$\chi^{(h)}(R) = \varepsilon^{hr(r-1)/2}, \qquad \chi^{(h)}(S) = (-1)^{h(r-1)}.$$

Hence if either h is even or r is odd, then $\chi^{(h)}$ is trivial; if r is even, and h is odd then $\chi^{(h)} = \chi_{(r/2, r/2)}$.

Let us finally consider the representation $\mathscr{W}^{(h)} = W_h \otimes W'_h$ of $\mathbf{G}_r$. Let

$$\mathscr{W}^{(h)} = \bigoplus_{\chi_b \in \mathbf{G}_r^\vee} \chi_b^{\mathscr{L}(h, \chi_b)}.$$

As a consequence of Proposition (2.4) we have

Corollary (2.5). *Let $b \in \mathbf{G}_r$. One has $\mathscr{L}(h, \chi_b) = m(h, \chi_b)$ if either h is even or r is odd. Otherwise $\mathscr{L}(h, \chi_b) = m(h, \chi_{b'})$, where $b' = b + (r/2, r/2)$.*

(3) An application of the above formulas. We shall apply the above formulas to the case where r is a prime number and to the description of the representation $\mathscr{W}^{(h)}$.

First we consider the case $r = 2$:

Proposition (2.6). *Let $r = 2$. If h is even, then $\mathscr{L}(h, \chi_b) = h/2$ unless $b = 0$, in which case $\mathscr{L}(h, \chi_b) = (h/2) + 1$. If h is odd, then $\mathscr{L}(h, \chi_b) = (h+1)/2$ unless $b = 0$, in which case $\mathscr{L}(h, \chi_b) = (h-1)/2$.*

Proof. A multi-index in $\mathscr{I}$ can be written in the form $I_j = (0^j, 1^{2h-j})$ meaning that the first j entries are 0, the latter are 1. Let us denote by $\eta^{(j)}$ the corresponding memorial in A. There is therefore only one S-orbit of cardinality 1, and precisely $\mathscr{D}_h = \{\eta^h\}$, whereas all other orbits $\mathscr{D}_j = \{\eta^j, \eta^{2h-j}\}$, $j = 0, \cdots, h-1$, have cardinality 2. The assertion follows as an easy application of Proposition (2.3) and Corollary (2.5). q.e.d.

Next we turn to the case r is an odd prime number. We have:

Proposition (2.7). *Let r be an odd prime number. Then*

$$\mathscr{L}(h, \chi_b) = m(h, \chi_b) = \left[\binom{r+rh-1}{rh} - 1\right] \Big/ r^2$$

unless $b = 0$, in which case

$$\mathscr{L}(h, \chi_b) = m(h, \chi_b) = \left\{\left[\binom{r+rh-1}{rh} - 1\right] \Big/ r^2\right\} + 1.$$

Proof. By Corollary (2.5) we have $\mathscr{L}(h, \chi_b) = m(h, \chi_b)$ for every $b \in \mathbf{G}_r$. For every multi-index $I \in \mathscr{I}$, the orbit $\mathscr{D}$ of $\eta^{(I)}$ has either cardinality r or cardinality 1. The latter case only happens if $I = (0^h, 1^h, \cdots, (r-1)^h)$, where we use a notation similar to the one introduced in the case $r = 2$. Therefore, since $\dim W_h = \binom{r+rh-1}{rh}$, there are exactly $[\binom{r+rh-1}{rh} - 1]/r$ orbits of cardinality r and only one orbit of cardinality 1.

The orbit $\mathscr{D}$ of cardinality 1 corresponds, by Lemma (2.2), to the trivial character χ_0. In fact for $b = (b_1, b_2) \in \mathbf{G}_r$, we have $\mu(\mathscr{D}, \chi_b) = 1$ if and only if $b_2 = 0$ and $b_1 - h[0 + 1 + \cdots + (r-1)] = -hr(r-1)/2 = 0$ in $\mathbf{Z}/r\mathbf{Z}$.

An orbit $\mathscr{D}$ of cardinality r, again by Lemma (2.2), contributes 1 to each character of the type $(\sigma(\mathscr{D}), b_2)$ where b_2 can be arbitrary. Consider the set $\mathrm{Orb}^{(r)}$ of such orbits of cardinality r of $\mathscr{A}$. The map $\sigma: \mathscr{D} \in \mathrm{Orb}^{(r)} \to \sigma(\mathscr{D}) \in \mathbf{Z}/r\mathbf{Z}$ is surjective. It is then clear from the above discussion that the proposition follows from the fact that the fibres of σ have all the same cardinality. This in turn can be proved as follows. By Proposition (2.1) and the following remark, we deduce that $\mathscr{L}(h, \chi_b)$ is independent of b, for $b = (b_1, b_2) \neq 0$. Therefore the cardinality of $\sigma^{-1}(b_1)$ is equal to $\mathscr{L}(h, \chi_b)$ for any $b = (b_1, b_2)$ with $b_2 \neq 0$, and these are equal. q.e.d.

In particular we see that for $r = 3$ we have $\mathscr{L}(h, \chi_b) = h(h+1)/2$ for all $b \in \mathbf{G}_3$ with $b \neq 0$, whereas $\mathscr{L}(h, \chi_0) = [h(h+1)/2] + 1$.

(4) The plurianticanonical systems of the symmetric product of an elliptic curve. Let A be an elliptic curve with $A \cong \mathbf{C}/\mathbf{Z}+\mathbf{Z}\Omega$, where $\Omega \in \mathbf{C}$ is in the upper half-plane. We shall identify the group $A_r \cong (\mathbf{Z}+\mathbf{Z}\Omega)/r(\mathbf{Z}+\mathbf{Z}\Omega)$ of r-torsion points of A to $\mathbf{G}_r$. With this identification we have

Theorem (2.8). *For each point $\xi \in A_r$ one has that the integer $n(h, \xi)$ introduced in (1.11) equals the above $\mathscr{L}(h, \chi_\xi)$.*

Proof. The theta functions $\theta[{}^{i/r}_{0}](rz, r\Omega)$, $i = 0, \cdots, r-1$, form a basis for $H^0(A, [r0])$. Thus a basis for $H^0(A, [r0 + w])$ is given by the family of theta functions $\theta^{(i)}(w) = \theta[{}^{i/r}_{0}](rz + w, r\Omega)$, $i = 0, \cdots, r-1$ (abusing notation we denote by the same letter w a point in $\mathbf{C}$ and the corresponding point in A). If we vary w in $\mathbf{C}$, the family $\mathscr{B} = (\theta^{(i)})_{i=0,\cdots,r-1}$ gives a trivializing frame of a vector bundle V of rank r which is the pull back to $\mathbf{C}$ of a vector bundle E on A such that $\mathbf{P}(E) \cong A^{(r)}$.

The group $\mathbf{Z}^2 \sim \mathbf{Z} + \mathbf{Z}\Omega$ acts on V and by comparing $\theta^{(i)}(w + \omega)$ with

$\theta^{(i)}(w)$ with $\omega \in \mathbf{Z} + \mathbf{Z}\Omega$ one finds cocycle relations for E. It is well known (see [Mu, pp. 123–124]) that these relations are given by

$$\theta^{(i)}(w+1) = \varepsilon^i \theta^{(i)}(w), \qquad \theta^{(i)}(w+\Omega) = \mathbf{e}(-z-(w/r)-(\Omega/2r^2))\theta^{(i+1)}(w)$$

with $i = 0, \cdots, r-1$.

Consider now the vector bundle $\mathrm{Sym}^{rh}(V^\vee) \otimes \det(V)^{\otimes h}$, which is the pull back to $\mathbf{C}$ of the vector bundle $p_*([-hK])$ considered in subsection 3 of §1. It is not difficult to verify that $\mathrm{Sym}^{rh}(V^\vee) \otimes \det(V)^{\otimes h}$ is defined by a constant cocycle, and hence it is associated with a representation of $\mathbf{G}_r$ which can easily be identified with $\mathscr{W}^{(h)}$. Comparing then the splitting of $\mathscr{W}^{(h)}$ into characters of $\mathbf{G}_r$ described above and the splitting of $p_*([-hK])$ into torsion line bundles proved in Proposition (1.11), we deduce the assertion. q.e.d.

Remark (2.9). Taking into account Remark (1.12) one obtains another indirect proof of the claim given at the end of the proof of Proposition (2.7) that all the fibres of σ have the same cardinality if r is odd.

(5) The plurianticanonical systems on $A^{(2)}$. According to Proposition (2.6), on the twofold symmetric product $A^{(2)}$ of an elliptic curve A there are three isolated curves which are algebraically, but not linearly, equivalent to $-K$.

It is not difficult to locate these curves on $A^{(2)}$. In fact, let ξ be a nonzero 2-torsion point of A, and let us consider the curve T_ξ on $A^{(2)}$ which is the image of the morphism $\phi_\xi : t \in A \to t + (t \oplus \xi) \in A^{(2)}$. Notice that ϕ_ξ realizes A as an unramified double cover of T_ξ, and indeed $T_\xi = A/\langle\xi\rangle$. It is easy to see that $T_\xi \cdot D = 1$ and $T_\xi \cdot F = 2$, and thus T_ξ is algebraically equivalent to $-K$. Hence the curves T_ξ, $\xi \in A_2 - \{0\}$, are the three required curves. To be slightly more precise, we have that

$$[T_\xi] \cong [-K + p^*(\xi - 0)]. \tag{2.10}$$

This is an easy consequence of the following:

Proposition (2.11). *Let $\mathscr{L}$ and $\mathscr{L}'$ be line bundles on $A^{(r)}$ such that for a given $u \in A$ one has $\mathscr{L}_{|D_u} \cong \mathscr{L}'_{|D_u}$. Then $\mathscr{L} \cong \mathscr{L}'$.*

Proof. It suffices to prove the proposition in the case $\mathscr{L}'$ is the trivial bundle. The hypothesis is then that $\mathscr{L}_{|D_u} \cong O_{D_u}$. Since any fibre F of $p: A^{(r)} \to A$ is isomorphic to $\mathbf{P}^{r-1}$ and $[D]_{|F} \cong O_{\mathbf{P}^{r-1}}(1)$, we have that $\mathscr{L}_{|F} \cong O_{|F}$ for any fibre F. Hence $\mathscr{L} \cong p^*(\mathscr{N})$, with $\mathscr{N}$ a line bundle on A. Moreover $D_u \cong A^{(r-1)}$ and $p_{|D_u}: D_u \to A$ can be interpreted as the composition of the natural projection $A^{(r-1)} \to A$ with the translation by u on A. Since $\mathscr{L}_{|D_u} \cong p^*(\mathscr{N})_{|D_u} \cong p^*_{|D_u}(\mathscr{N})$ is trivial and since $p^*_{|D_u}$ clearly induces an injection of $\mathrm{Pic}(A)$ into $\mathrm{Pic}(D_u) \cong \mathrm{Pic}(A^{(r-1)})$, we have that $\mathscr{N}$ hence $\mathscr{L}$, is trivial. q.e.d.

To prove formula (2.10), take now a divisor $D = D_u$ on $A^{(2)}$. The intersection of D with T_ξ is the point of D corresponding to the divisor $u + (u \oplus \xi)$. If we choose the canonical identification of $D = D_u$ with A via the projection p, we have $[T_\xi]_{|D} \cong [u \oplus \xi]$. On the other hand, the adjunction formula tells us that $[-K + p^*(\xi - 0)]_{|D} \cong [D + p^*(\xi - 0)]_{|D} \cong [D_{u\oplus\xi}]_{|D}$, which after identification of D with A is again $[u\oplus\xi]$. Formula (2.10) follows then by Proposition (2.11).

Consider now the map $\pi: A^2 \to A^{(2)}$. For each $u \in A$ we have a curve Δ_u on A^2, namely the image of the morphism $\psi_u : t \in A \to (t, t\oplus u) \in A^2$. Of course Δ_0 is the diagonal of A^2 and $\mathscr{P} = \{\Delta_u\}_{u\in A}$ is an elliptic pencil of curves on A^2 parametrized by A. Moreover $\pi(\Delta_u) = \pi(\Delta_v)$ if and only if $v = \pm u$. Hence we have a base point free, rational pencil $\mathscr{Q}$ of curves on $A^{(2)}$ such that the projection map π induces a two-to-one cover $\Pi: \mathscr{P} \to \mathscr{Q}$ which is ramified at four points corresponding to four curves of $A^{(2)}$. These are precisely the diagonal Δ, where π ramifies, and the three curves T_ξ, each appearing with multiplicity two in the pencil $\mathscr{Q}$. This shows that $\mathscr{Q} = |-2K|$, and this agrees with Proposition (2.6) which tells us that $h^0(A^{(2)}, [-2K]) = 2$. According to Proposition (2.6) we also have three isolated curves algebraically, but not linearly, equivalent to $-2K$. These clearly are the three curves of the type $T_\xi + T_\eta$ where ξ and η are two distinct nonzero 2-torsion points of A.

In general if $h > 1$ is odd, the linear systems of curves of $A^{(2)}$ algebraically equivalent to $-hK$ are, with obvious meaning of the notation, $T_\xi + T_\eta + T_{\xi\oplus\eta} + [(h-3)/2]\mathscr{Q}$, $T_\xi + [(h-1)/2]\mathscr{Q}$, $T_\eta + [(h-1)/2]\mathscr{Q}$, $T_{\xi\oplus\eta} + [(h-1)/2)\mathscr{Q}$; if h is even we have instead $(h/2)\mathscr{Q}$, $T_\xi + T_\eta + [(h-2)/2)]\mathscr{Q}$, $T_\xi + T_{\xi\oplus\eta} + [(h-2)/2)]\mathscr{Q}$, $T_\eta + T_{\xi\oplus\eta} + [(h-2)/2)]\mathscr{Q}$.

(6) The anticanonical pencil of $A^{(3)}$. Let us now turn to $A^{(3)}$. In view of the results of subsection 2, we have here $\dim|-K| = 1$; hence we have a pencil of anticanonical surfaces on $A^{(3)}$. We shall now study some geometrical properties of this anticanonical pencil.

Again let ξ be a nonzero 2-torsion point of A and let us consider the morphism $\tau_\xi : (u, v) \in A \times A \to u + v + (u \oplus \xi) \in A^{(3)}$, whose image is a surface which we shall denote by Σ_ξ. Notice that τ_ξ factorizes through an obvious bijective map $(A/\langle\xi\rangle) \times A \to \Sigma_\xi$ which is in fact an isomorphism;

hence Σ_ξ is a smooth surface in $A^{(3)}$. We have

Proposition (2.12). *Σ_ξ is an anticanonical divisor on $A^{(3)}$.*

Proof. Let $F = F_v$ be a general fibre of $p: A^{(3)} \to A$. The surfaces F and Σ_ξ intersect transversally along a curve of which we want to compute the degree inside the projective plane F. This is the same as counting the number of divisors of degree 3 which Σ_ξ has in common with a general g_3^1 on A. We claim that this number is 3.

Consider in fact the curve G of $A^{(2)}$ formed by all divisors of degree 2 contained in a divisor of the g_3^1. Since $D \cdot G = 2$ and $F \cdot G = 1$, G is a general curve of a linear system on $A^{(2)}$ of the form $|D + F|$, which is base point free by Theorem (1.18). Since $(D+F)\cdot(-K) = 3$, G cuts out 3 distinct points on the curve T_ξ. Each of these points is a divisor of the form $t + (t \oplus \xi)$ contained in a divisor of the g_3^1, which therefore gives rise to an intersection of the g_3^1 with Σ_ξ and conversely any such intersection arises in this way.

The above discussion proves that Σ_ξ is algebraically equivalent to a divisor of the form $3D + nF$.

Consider now a divisor $D = D_u$ which we may identify with $A^{(2)}$. With this identification $\Sigma_\xi \cap D$ is easily seen to consist of the union of the two components T_ξ and $D_{u\oplus\xi}$. We claim that the scheme-theoretical intersection of Σ_ξ and D is reduced. Suppose in fact that T_ξ appears in the intersection with multiplicity α and $D_{u\oplus\xi}$ with multiplicity β. Since T_ξ is algebraically equivalent to $2D - F$ on $A^{(2)}$, we should then have $\alpha + 2\beta = 3$ which yields $\alpha = \beta = 1$.

We may thus conclude that, after the identification of D with $A^{(2)}$, one has $[\Sigma_\xi]_{|D} \cong [T_\xi + D_{u\oplus\xi}]$. Hence by (2.10) and by adjunction, we deduce $[\Sigma_\xi]_{|D} \cong [-K_D + p^*(\xi - 0) + D_{u\oplus\xi}] \cong [-K_D + D_u] \cong [-K]_{|D}$. But then Proposition (2.11) yields the assertion. q.e.d.

Now take the three anticanonical divisors Σ_ξ with ξ a nonzero 2-torsion point of A. Since $|-K|$ is a pencil on $A^{(2)}$, any two of these divisors generate the anticanonical pencil. We shall need for future reference some information concerning the base locus of $|-K|$.

Consider any pair of divisors Σ_ξ and Σ_η. The intersection Γ of these divisors is the curve of $A^{(3)}$ which is the image of the map $\kappa: t \in A \to t + (t\oplus\xi) + (t \oplus \eta) \in A^{(3)}$. Notice that $D \cdot \Gamma = 3$. More precisely, if $D = D_u \cong A^{(2)}$, we saw that $\Sigma_\xi \cdot D = T_\xi + D_{u\oplus\xi}$ and $\Sigma_\eta \cdot D = T_\eta + D_{u\oplus\eta}$. Since T_ξ and T_η do not intersect, $D \cap \Gamma$ consists of the three distinct points $D_{u\oplus\xi} \cap T_\eta = \{(u\oplus\xi) + (u\oplus\xi\oplus\eta)\}$, $D_{u\oplus\eta} \cap T_\xi = \{(u\oplus\eta) + (u\oplus\xi\oplus\eta)\}$, $D_{u\oplus\xi} \cap T_{u\oplus\eta} = \{(u\oplus\xi) + (u\oplus\eta)\}$. This shows that Σ_ξ and Σ_η intersect transversally along Γ. Since Σ_ξ and Σ_η generate the anticanonical pencil, Γ is the base locus of $|-K|$.

Remark (2.13). As we know from subsection 3, $n(3, 1, \xi) = 1$ if $\xi \in A_3 - \{0\}$ and $n(3, 1, 0) = 2$. This can be proved in a purely geometric way, without the computations of subsections 1–3, in the following way.

The above discussion yields that on $A^{(3)}$ one has $\dim|-K| \geq 1$. On the other hand by Theorem (1.17) and by restricting to a general divisor D, one sees that $\dim|-K| = 1$; hence $n(3, 1, 0) = 2$. Since $p_*(-K)$ is a rank 10 vector bundle on A, and by taking into account Remark (1.12), one finds $n(3, 1, \xi) = 1$ for all $\xi \in A_3 - \{0\}$.

We have not been able to geometrically describe the eight divisors of the form $|-K + p^*(\xi - 0)|$, with $\xi \in A_3 - \{0\}$.

3. On surfaces of general type with $p_g = q = 1$ and $g = K^2 = 3$

(1) Generalities on surfaces with $p_g = q = 1$. We briefly recall some results from [CaCi]. Let S be a minimal surface of general type with $p_g = q = 1$. Its Albanese variety A is a curve of genus 1. Let $\alpha: S \to A$ be the Albanese morphism. We shall denote by g the genus of the fibres of α.

On S there is the so-called *paracanonical system* (see [CaCi, §1]) which is a 1-dimensional system of curves $\{K\} = \{C_t\}_{t\in A}$ parametrized by A, whose general element is algebraically, but not linearly, equivalent to K. For $t \in A$ general, one has $h^0(S, [C_t]) = 1$, but for particular values of t one might have $h^0(S, [C_t]) > 1$. A relevant invariant of S is the number λ (defined in [CaCi, §1]; see also notation and conventions at the beginning of this paper) which, roughly speaking, is the number of points $t \in A$, counted with multiplicity, such that $h^0(S, [C_t]) > 1$. Another relevant invariant of S together with g, λ and K^2, is the so-called *index* of the paracanonical system ι, namely the number of distinct curves of the paracanonical system passing through the general point of S (see again notation and conventions). We recall the relation $\iota = g - \lambda$ (see [CaCi, Theorem (2.1)]).

Consider the locally free sheaf $V = \alpha_*[K]$ of rank g and degree 1 on A. Then one has the projective bundle $p: \mathbf{P}(V^\vee) \to A$ and a rational map $\omega: S \to \mathbf{P}(V^\vee)$, the so called *relative canonical map*, such that $\alpha = p \circ \omega$.

One of the results from [CaCi] (see Theorem (2.3)) is that if $\lambda = 0$ then $\mathbf{P}(V^\vee) \cong A^{(\iota)}$.

The possible values for K^2 are $2 \leq K^2 \leq 9$ for these surfaces (see [Bo]). The case $K^2 = 2$ has been extensively studied in [Ca] (see also [CaCi, §5]): one has then also $\iota = g = 2$, and hence $\lambda = 0$.

In [CaCi, §5] we began the study of the case $K^2 = 3$. Here we have either $\lambda = 1$, and therefore $g = 3$ and $\iota = 2$, or $\lambda = 0$ and $g = \iota = 3$. The existence of surfaces of the first type was proved in [CaCi]. About surfaces of the second type we recall the following theorem stated in [CaCi] as Theorem (5.8):

Theorem (3.1). *If $g = K^2 = 3$, then $\omega: S \to A^{(3)}$ is a morphism which is birational onto its image. Moreover ω is an isomorphism of the canonical model of S onto $\omega(S)$, which is a divisor in a linear system homologous to $|4D - F|$ having at most rational double points as singularities. Such surfaces do in fact exist.*

Proof. The first assertion was proved in [CaCi] where it was also shown that no Albanese fibre is hyperelliptic. Consider then the exact sequence

$$0 \to \mathscr{L} \to \mathrm{Sym}^4(V) \to \alpha * [K^{\otimes 4}] \to \mathscr{T} \to 0$$

where $\mathscr{L}$ is a line bundle and $\mathscr{T}$ is a torsion sheaf, since the middle map is generically surjective. Hence $\deg(\mathscr{L}) = \deg(\mathscr{T}) + 1 \geq 1$. This shows that $\omega(S)$ is a divisor homologous to $4D - mF$, with $m \geq 1$, on $A^{(3)}$. But Proposition (1.9) implies $m = 1$. Using the adjunction formula on $A^{(3)}$, and the fact that the system $\{D_t\}_{t \in A}$ pulls back via ω to the paracanonical system $\{K\}$ on S (see [CaCi, proof of Theorem (2.3) and Lemma (4.10)]) we see that $\omega(S)$ has at most rational double points as singularities. The existence will be proved separately in Theorem (3.2). q.e.d.

(2) Existence of surfaces with $p_g = q = 1$ and $g = K^2 = 3$. Let us prove the following:

Theorem (3.2). *If A is an elliptic curve and $\mathscr{D}$ is a line bundle of $A^{(3)}$ associated to an effective divisor algebraically equivalent to $4D - F$ then the general element of $|\mathscr{D}|$ is a smooth surface with $p_g = q = 1$ and $g = K^2 = 3$.*

In order to prove the theorem it suffices to show the following:

Lemma (3.3). *$|\mathscr{D}|$ has at most simple base points.*

Proof. If a point p does not belong to some effective divisor algebraically equivalent to $3D - F$, we are done since the D_u's are all smooth. Therefore it suffices to prove that the intersection of all the above divisors

is empty. As we saw in subsection 6 of §2, the base locus of $|-K|$ is the smooth curve $\Gamma = \{x + (x \oplus \xi) + (x \oplus \eta)\}_{x \in A}$ where ξ and η are distinct points of order two of A. Let us recall that Γ is counted with multiplicity 1 in the base locus, as one sees by intersection with a general D_u. If a divisor H algebraically equivalent to $3D - F$ intersects Γ, then it contains Γ, since Γ is an orbit of A acting by translations on $A^{(3)}$. Remark now that H is irreducible by (1.17); hence each divisor of $|-K|$ different from H intersects H in a curve which is algebraically equivalent to Γ and then equals Γ since it contains Γ. But then, since $|-K|$ is a linear pencil, we derive $H \in |-K|$. This shows that Γ has empty intersection with each of the eight effective divisors algebraically equivalent to $3D - F$. q.e.d.

Remark (3.4). We suspect that indeed any system of the type $|\mathscr{D}_u| = |-K + D_u|$ is base point free. In fact, by Proposition (1.19), we already know that for each divisor $\mathscr{D}_u$ algebraically equivalent to $\mathscr{D}$ the linear system $|\mathscr{D}_u|$ has at most finitely many base points. Moreover, since A (as contained in $\mathrm{Aut}(A^{(r)})$) acts transitively on the linear equivalence classes of the $\mathscr{D}_u$'s, either every or no $|\mathscr{D}_u|$ has base points. In the latter case we are done; in the former case, these points sweep out a finite number of smooth curves Δ which are indeed A-orbits. By restricting to any divisor D_v passing through a point $p \in \Delta$ we see that p is not a base point unless $\mathscr{D}_u - D_v + K$ is a divisor of 3-torsion. This immediately implies that Δ is of the form $\{x + (x \oplus a) + (x \oplus b)\}_{x \in A}$, where a and b are points of 3-torsion. Moreover there exists a point of 3-torsion c such that Δ is contained in the intersection of all divisors in $|-K + d|$ where d is a 3-torsion point distinct from $c, c \oplus a, c \oplus b$. Therefore such a point d cannot be 0 and we may assume $c = 0$. If the set $\{0, a, b\}$ consists of three elements, the intersection of Δ with a general D_u consists of three distinct points, which are the complete intersections in D_u of two of the above divisors $|-K+d|$. This gives rise to a contradiction by an argument similar to one used in the proof of the above lemma. The case in which $\{0, a, b\}$ consists of less than three points should be finally analyzed.

(3) Moduli of surfaces with $p_g = q = 1$ and $g = K^2 = 3$. Theorems (3.1) and (3.2) yield the existence of a family containing all the surfaces with $p_g = q = 1$ and $g = K^2 = 3$. We briefly sketch the existence of such a family. Take $\mathscr{A} \to \mathscr{H}$ the universal family of elliptic curves with some level structure, and form its symmetric fibre product $f: \mathscr{A}^{(3)} \to \mathscr{H}$. The invertible sheaves $[4D_0 - F_0]$ fit together to give an invertible sheaf $\mathscr{L}$ on $\mathscr{A}^{(3)}$. Since by Theorem (1.17) one has $\dim |4D - F| = 4$ on

$A^{(3)}$, the sheaf $f_*(\mathcal{L})$ is locally free of rank 5. Let $\mathcal{X}$ be $\mathbf{P}(f_*(\mathcal{L}))$ with its projection $g: \mathcal{X} \to \mathcal{H}$. We can then consider the fibred product $\mathcal{X} \times_{\mathcal{H}} \mathcal{A}^{(3)} \to \mathcal{H}$ and inside the total space we can consider the scheme $\mathcal{S}$ defined by the obvious equation $(\sigma, x) \in \mathcal{S} \Leftrightarrow \sigma(x) = 0$. Finally we have a projection $\mathcal{S} \to \mathcal{X}$ and there is a dense open subset of $\mathcal{X}$ such that for any point $x \in \mathcal{X}$, the fibre S_x of $\mathcal{S} \to \mathcal{X}$ over x is a normal surface with rational double points, whose desingularization is a surface with $p_g = q = 1$ and $g = K^2 = 3$, and any such a surface appears in this way. Notice that $\mathcal{X}$ is a smooth variety of dimension 5.

We shall now prove:

Theorem (3.5). *If $x \in \mathcal{X}$ is such that the corresponding surface S_x is smooth, then the Kodaira-Spencer map for $\mathcal{S} \to \mathcal{X}$ at x is an isomorphism and $\mathcal{S} \to \mathcal{X}$ is, around x, the Kuranishi family for S_x.*

Proof. Let $S \in |4D_0 - F_0|$ in $A^{(3)}$ be a smooth surface with $p_g = q = 1$ and $g = K^2 = 3$ and let T_S be its tangent sheaf. Since $\chi(T_S) = 2K^2 - 10\chi(O_S) = -4$, if we show that $h^2(T_S) = h^0(\Omega^1_S \otimes \omega_S) = 1$, we have that $h^1(T_S) = 5$.

To this purpose we notice that $h^0([-S] \otimes \omega_S) = h^1([-S] \otimes \omega_S) = 0$. Consider in the fact the exact sequence

$$0 \to O_{A^{(3)}} \to [S] \to [S]_{|S} \to 0.$$

By Theorem (1.17) one has $h^1([S]) = h^2(O_{A^{(3)}}) = h^2([S]) = h^3(O_{A^{(3)}}) = 0$. Thus $h^1([S]_{|S}) = h^1([-S] \otimes \omega_S) = 0$ and $h^2([S]_{|S}) = h^0([-S] \otimes \omega_S) = 0$.

Moreover we have $h^0(\Omega^1_S \otimes \omega_S) = h^0(\Omega^1_{A^{(3)}} \otimes \omega_S)$. In fact this follows by the exact sequence

$$0 \to [-S] \otimes \omega_S \to \Omega^1_{A^{(3)}} \otimes \omega_S \to \Omega^1_S \otimes \omega_S \to 0$$

and by what we proved above.

By the exact sequence

$$0 \to \omega_S \to \Omega^1_{A^{(3)}} \otimes \omega_S \to \Omega^1_{A^{(3)}|A} \otimes \omega_S \to 0$$

we are done if we prove that $h^0(\Omega^2_{A^{(3)}|A} \otimes \omega_S) = 0$. We finally use the Euler sequence, which, since $O_{A^{(3)}}(-D_0) \otimes \omega_S \cong O_S$, reads as

$$0 \to \Omega^1_{A^{(3)}|A} \otimes \omega_S \to p^*(E_3(0)) \otimes \mathcal{O}_S \to \omega_S \to 0.$$

Now we finish the proof that $h^1(T_S) = 5$ by observing that the restriction map $H^0(p^*(E_3(0)) \otimes \mathcal{O}_S) \cong H^0([D_0]_{|S}) \to H^0(\omega_S)$ is clearly an isomorphism.

After proving that the tangent dimension of the base $\mathcal{B}$ of the Kuranishi family is equal to 5, we notice that two surfaces S_1 and S_2 in the family $\mathcal{S} \to \mathcal{X}$ are isomorphic if and only if:

(i) the corresponding elliptic curves A_1 and A_2 are isomorphic to a third elliptic curve A;

(ii) identifying A_1 and A_2 with A and observing that the Albanese morphism is defined up to translation, we have that S_1, S_2 are isomorphic by an automorphism g of $A^{(3)}$ induced by a translation in A. But then $g(S_1) = S_2$ implies $g(D_0) = D_0$; hence (cf. (1.5)) g is the identity.

Notice that if the isomorphism between A_1 and A_2 does not preserve the given level structure, and S_1 is sufficiently close to S_2 (thus there is a fixed diffeomorphism between S_1 and S_2), then the isomorphism between S_1 and S_2 does not act trivially on cohomology. The above argument shows that the natural map $\mathcal{X} \to \mathcal{B}$ is a locally injective and surjective morphism of a smooth manifold of dimension 5 to a subspace of a 5-dimensional manifold; hence it is a local biholomorphism.

Alternatively, consider the normal bundle exact sequence

$$0 \to H^0(T_{A^{(3)}}) \to H^0(N_{S|A^{(3)}}) \to H^1(T_S) \to H^1(T_{A^{(3)}}) \to 0$$

and observe that $p: A^{(3)} \to A$ induces isomorphisms $H^i(T_{A^{(3)}}) \cong H^i(T_A)$ for $i = 0, 1$. The Kodaira-Spencer map $\rho: T_{\mathcal{X}} \to H^1(T_S)$ induces a surjection σ onto $H^1(T_{A^{(3)}})$ since $T_{\mathcal{X}}$ surjects onto $H^1(T_A)$. One easily verifies that the ker σ is a supplementary space to $H^0(T_{A^{(3)}})$. Hence ρ is an isomorphism. q.e.d.

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(F. Catanese) Dipartimento di Matematica, Università di Pisa, Via Buonarroti 2, 56100 Pisa (Italy)

E-mail address: CATANESE@ICNUCEVM.BITNET

(C. Ciliberto) Dipartimento di Matematica, II Università di Roma, Via O. Raimondo, 00173 Roma (Italy)

E-mail address: CILIBERTO@MAT.UTOVRM.IT