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d-VERY-AMPLE LINE BUNDLES AND EMBEDDINGS OF HILBERT SCHEMES OF 0-CYCLES

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The purpose of this short note is to sharpen one of the results of a recent paper by Beltrametti and Sommese (theorem 3.1. of [B-S]), concerning the notion of d-very ample line bundles.
In fact , our main theorem gives indeed another characterization of this notion.

§0 Notation and statement of the result

Definition 0.1 (cf. [B-S]) Let X be a complete algebraic variety over an algebraically closed field k (or a projective scheme over spec(k)).
i) by a 0-cycle on X we mean a purely 0-dimensional subscheme Z of X , defined by a sheaf of ideals J_Z of $\mathcal{O}_X$ (we set then $\mathcal{O}_Z = \mathcal{O}_X / J_Z$)
ii) the length d of Z is the dimension of the k- vector space $H^0(\mathcal{O}_Z)$
Let $\mathcal{L}$ be an invertible sheaf on X , then for any 0-cycle Z on X we can consider the restriction map r_Z to Z for the space of sections of $\mathcal{L}$, which fits into the exact sequence

$$(*)\ 0 \to H^0(X, J_Z\mathcal{L}) \to H^0(X,\mathcal{L}) \xrightarrow{r_Z} H^0(\mathcal{L}\otimes\mathcal{O}_Z) \longrightarrow$$
$$\longrightarrow H^1(X,J_Z\mathcal{L}) \longrightarrow H^1(X,\mathcal{L}) \longrightarrow 0 .$$

iii) $\mathcal{L}$ is said to be **d-very ample** if the restriction map r_Z is onto for every 0-cycle Z of length less than or equal to (d+1) .

Remark 0.2 i) The notion of 0-very ample corresponds to the classical notion of "spanned by global sections", while the classical notion of "very ample" is easily seen to correspond to the notion of 1-very ample.
ii) If $\mathcal{L}$ is 0-very ample , then $H^0(X,\mathcal{L})$ defines a morphism $\varphi_0 : X \longrightarrow \mathbb{P}(H^0(X,\mathcal{L})^*)$, which is an embedding precisely when $\mathcal{L}$ is 1-very ample .
iii) If $\mathcal{L}$ is d-very ample , then (*) associates to every 0-cycle Z of length = d+1 a subspace of $H^0(X,\mathcal{L})$ of codimension = d+1 (just $H^0(X,J_Z\mathcal{L})$!) , and this map yields indeed a morphism

(0.3) $\varphi_d : X^{[d+1]} \longrightarrow \mathrm{Grass}\,(d+1, H^0(X,\mathcal{L})^*)$,

where $X^{[d+1]}$ is the Hilbert scheme of 0-cycles on X of length equal to (d+1).
Then we claim that our previous remark ii) generalizes in a quite natural fashion, we have in fact the following :

MAIN THEOREM

The above morphism φ_d is an embedding if and only if $\mathcal{L}$ is d+1-very ample .

As we already mentioned , our main theorem improves upon theorem 3.1 of [B-S] , where it is proven that φ_d is 1-1 if $\mathcal{L}$ is (d + 1) - very ample and an embedding provided $\mathcal{L}$ is 3 d - very ample (the only reason there to assume that X is a smooth projective surface is just in order to let the Hilbert scheme of 0-cycles to be smooth).
We would finally like to call the reader' s attention to the quoted article by Beltrametti and Sommese ([B-S]), whose main theorem was an extension of Reider's criteria (cf. [Re]) in order to ensure d-very ampleness of the adjoint bundle of a nef line bundle on a smooth algebraic surface X.

§1 Proof of the main theorem

Lemma 1.1

Let A be a semilocal ring containing the field k and with residue fields isomorphic to k , let I be a finitely generated ideal of A of finite colength (i.e., B = A/I is Artinian) .
Then there does exist an ideal J of A with $I^2 \subset J \subset I$, and such that I / J is a 1-dimensional k-vector space.
Assume moreover that we are given a non zero homomorphism $f : I/I^2 \longrightarrow B = A/I$ of A- modules (equivalently,of B-modules) : then we can find such a J satisfying the further property that $J/I^2 \supset \ker f$.
Conversely, given J, such that C= A/J has a non trivial radical, we can find I with $I^2 \subset J \subset I$, and such that I / J is a 1-dimensional k-vector space.

Proof Let's first prove the first assertion.
Denote by N' the B-module I/I^2 and by K the B-submodule = ker(f) (set K=0 if we are not given f) . Set moreover N = N'/ K : by our assumption N is $\neq 0$,and a finite B-module.

Assume that M is a B-submodule of N such that N/M is a k-vector space of dimension 1 : then it suffices to take J as the inverse image of M under the surjection of I onto N.
To show the existence of such a submodule M ,let $\mathcal{M}$ be the radical of the Artinian ring B . By Nakayama's lemma, the submodule $\mathcal{M}N$ is a proper submodule of N ,and it suffices, replacing N by $N/\mathcal{M}N$, respectively B by $B/\mathcal{M}$, to reduce to the case where B is isomorphic to a direct sum ring k^r , and where N is a finite unitary B module (i.e., the identity of k^r acts as the identity on N).
Since N is finite, there is an epimorphism $(k^r)^n \longrightarrow N$, thus N has a filtration by B -submodules $0 \subset N_1 \subset N_2 \subset \ldots N_{nr-1} \subset N$ such that N_i/N_{i-1} is either 0 or isomorphic to k . Whence our claim .
To prove the second assertion, let C= A/J , consider the natural decreasing filtration on C given by the powers of the radical $\mathcal{M}$ of the Artinian ring C. Let C' be the last non zero ideal in the filtration. Choose then a 1-dimensional k- vector subspace L of C', and let I be its inverse image under the surjection of A onto A/J.
The inclusion $I^2 \subset J$ follows since $L^2 = 0$ in C.

qed.

We shall apply the previous lemma to the situation where A is the semilocal ring of functions which are regular at the points of supp(Z) (i.e.,given an affine open set U ⊃ supp(Z), we localize $H^0(U,\mathcal{O}_X)$ w.r.t the multiplicative set which is the complement of the union of the prime ideals corresponding to points of supp(Z)), and where I equals the ideal $I = H^0(U, J_Z\mathcal{O}_X)$.
Thus $B = \mathcal{O}_Z \cong H^0(\mathcal{O}_Z)$, and the first assertion of the lemma provides a 0-cycle Z' containing Z, and with length (Z') = length(Z) +1 .

Corollary 1.2

Let $f : J_Z / J_Z^2 \longrightarrow \mathcal{O}_Z$ be a non zero homomorphism of $\mathcal{O}_Z$ modules , and let $F : H^0(X, J_Z\mathcal{L}) \longrightarrow H^0(\mathcal{L}\otimes\mathcal{O}_Z) \cong \mathcal{O}_Z$ be the induced homomorphism of finite dimensional k-vector spaces. Then, if length(Z) = d+1, and $\mathcal{L}$ is (d+1) -very ample, then F is also nonzero.

Proof Pick a 0-cycle Z' of length (d+2) according to the assertion of lemma 1.1 .
By assumption, $H^0(X,\mathcal{L}) \xrightarrow{r_{Z'}} H^0(\mathcal{L}\otimes\mathcal{O}_{Z'})$ is surjective, hence $H^0(X,J_Z\mathcal{L})$ maps onto $J_Z / J_{Z'}$.
On the other hand,$J_{Z'} / J_Z^2$ contains ker f , hence the image of $H^0(X,J_Z\mathcal{L})$ into J_Z / J_Z^2 is not contained in ker f , and F is not zero.

qed.

MAIN THEOREM

If $\mathcal{L}$ is d-very ample, then the morphism
$\varphi_d : X^{[d+1]} \longrightarrow$ Grass (d+1, H^0 $(X,\mathcal{L})^*$) is an embedding if and only if $\mathcal{L}$ is also (d+1) -very ample.

Proof Given two 0-cycles of length d+1, Z and Z', assume that φ_d (Z) = φ_d (Z').
Then $H^0(X,J_Z\mathcal{L}) = H^0(X,J_{Z'}\mathcal{L}) = H^0(X,(J_Z+J_{Z'})\mathcal{L})$, hence if Z" is defined by the ideal $(J_Z+J_{Z'})$, then
$H^0(X,\mathcal{L}) \xrightarrow{r_{Z''}} H^0(\mathcal{L}\otimes\mathcal{O}_{Z''})$ has a (d+1)-dimensional image.
If length(Z") $\geq$ d+2, we can find, by lemma 1.1, a 0-cycle W of length =(d+2), such that W is contained in Z"; but then r_W cannot be surjective, which is impossible if $\mathcal{L}$ is (d+1) -very ample.
We have shown the injectivity of φ_d under the assumption that $\mathcal{L}$ be (d+1) -very ample ; with the same assumption, the assertion that φ_d is an embedding follows directly from corollary 1.2 .
In fact, the tangent space to $X^{[d+1]}$ at the point Z, by standard deformation theory, coincides with the sections of the normal sheaf to Z,which is exactly $\mathrm{Hom}_{\mathcal{O}_Z}(J_Z/J_Z^2, \mathcal{O}_Z)$.
On the other hand, the tangent space to the Grassmanian at φ_d (Z) coincides with $\mathrm{Hom}_k(H^0(X,J_Z\mathcal{L}), H^0(\mathcal{L}\otimes\mathcal{O}_Z))$, and it is not difficult to verify (see e.g. [B-S]) that the differential of φ_d carries ,notation being as in corollary 1.2, f to F . Thus corollary 1.2 simply says that the differential of φ_d is injective.
In the other direction, if there exists a 0-cycle Z' of length (d+2) with $r_{Z'}$ not surjective, there are two possibilities .
If $\mathcal{O}_{Z'}$ is reduced , then we have (d+2) distinct points , and any choice of (d+1) of them yields ,by our assumptions, distinct 0-cycles with the same image under φ_d .
Otherwise, if $\mathcal{O}_{Z'}$ is not reduced, we can pick-up, by lemma 1.1,a 0-cycle Z of length (d+1) contained in Z' ,and such that $J_Z \supset J_{Z'} \supset J_Z^2$
We then let f in $\mathrm{Hom}_{\mathcal{O}_Z}(J_Z / J_Z^2, \mathcal{O}_Z)$ be the composition of the natural surjection of J_Z / J_Z^2 onto
$J_Z / J_{Z'} \cong k$ with the natural embedding of k into $\mathcal{O}_Z$.
By construction, the associated homomorphism F in the space
$\mathrm{Hom}_k(H^0(X,J_Z\mathcal{L}), H^0(\mathcal{L}\otimes\mathcal{O}_Z))$ is zero, and we succeeded in showing that the differential of φ_d is not injective.

qed.

REFERENCES

[B-S] - Beltrametti,M.-Sommese,A.J.: "Zero cycles and k-th order embeddings of smooth projective surfaces", in 'Problems on surfaces and their classification', Proc. Cortona 1988, Symposia Mathematica, INdAM, Academic Press

[B-F-S] - Beltrametti,M.-Francia,P. -Sommese,A.J.: "On Reider's method and higher order embeddings", Duke Math. Jour. **58**,2,(1989),425-439

[Rei] - Reider,I. : "Vector bundles of rank 2 and linear systems on algebraic surfaces" , Ann. of Math. , **127** , (1988), 309-316

[Tju] - Tjurin,A.: "Cycles,curves and vector bundles on an algebraic surface" , Duke Math. Jour. , **54** ,(1987), 1-26

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