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CHOW VARIETIES, HILBERT SCHEMES, AND MODULI SPACES OF SURFACES OF GENERAL TYPE

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Summary. The following are the main results of this paper.

Theorem A. *The number $\iota(y)$ of irreducible components of the moduli space of surfaces of general type with $K_S^2 = y$ satisfies for $y \geq 3$*

$$\iota(y) \leq 6^{(y+5/9)^{15}}.$$

If one restricts to regular surfaces ($q = 0$), then one has for $x, y \geq 3$ the better estimate $\iota^0(y) \leq y^3 \cdot (440y)^{76y^2}$.

Theorem B. *Let $\mathscr{H}^0$ (resp. $\mathscr{H}^*$) be the open subset of the Hilbert scheme parametrizing smooth (resp. normal) irreducible subvarieties of dimension k and degree d in $\mathbb{P}^n$, then the natural morphism of $\mathscr{H}^0_{\text{red}}$ (resp. $\mathscr{H}^*_{\text{red}}$) to the Chow variety is an isomorphism (resp. a homeomorphism).*

Theorem B is based on

Theorem (1.14). *Let V be an irreducible subvariety in $\mathbb{P}^n$, let $F = F_V$ be its Chow form, and let $W = W_F$ be the subscheme of $\mathbb{P}^n$ canonically associated to F. Then $V = W$ if V is a hypersurface, otherwise the equality $W = V$ holds exactly at the smooth points of V.*

Several other results are given concerning the complexity (cf. Introduction) of Chow varieties and Hilbert schemes.

0. Introduction

The motivation of this research was the desire to understand the complexity of the moduli space of the surfaces of general type (minimal, complete and smooth over $\mathbb{C}$) S such that their numerical invariants $\chi(\mathscr{O}_S)$, K_S^2 take some given integer values $\chi(\mathscr{O}_S) = x$, $K_S^2 = y$.

It is well known that these two invariants must take positive values and, conversely, that if these two numerical invariants are positive, then the surface (now not necessarily assumed to be minimal) is of general type except possibly if $\chi(\mathscr{O}_S) = 1$, $K_S^2 \leq 9$.

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Gieseker [Gie] proved that there exists a quasiprojective coarse moduli variety $\mathcal{M}_{x,y}$ for such surfaces, and there exists substantial literature (cf., e.g., [Per, Som, Xi, Chen1, Chen2] solving the "geographical problem" posed by Van de Ven [VdV] of determining for which pairs (x, y) $\mathcal{M}_{x,y}$ is nonempty.

The purpose of this paper is on the one hand to remark that one can give effective upper bounds for the number $\imath(x, y)$ of irreducible components of $\mathcal{M}_{x,y}$, and on the other hand to give, more generally, effective estimates for Chow and Hilbert varieties; it seems, though, that some deeper techniques are needed in order to attack the problem of determining the precise asymptotic growth of $\imath(x, y)$.

We should remark that, thanks to Ekedahl's extension [Ek] of Bombieri's work [Bo] in positive characteristic, our results are also valid over an algebraically closed field of arbitrary characteristic.

But, over $\mathbb{C}$, the question of finding a good bound for $\imath(x, y)$ is related to other interesting problems.

In fact, if S^{top} is the oriented topological 4-manifold underlying S, S^{diff} is the oriented $\mathscr{C}^\infty$ manifold underlying S, we can attach to S several integers in the following way.

Let $\mathcal{M}^{\text{top}}(S)$ be the subvariety of $\mathcal{M}_{x,y}$ corresponding to surfaces (orientedly) homeomorphic to S, and $\mathcal{M}^{\text{diff}}(S)$ be the subvariety of $\mathcal{M}^{\text{top}}(S)$ corresponding to surfaces (orientedly) diffeomorphic to S; we define then (0.1)

$\delta(S)$ = number of $\mathscr{C}^\infty$ inequivalent complex structures on S^{top}, i.e., more precisely, the cardinality of the elements of the partition given on $\mathcal{M}^{\text{top}}(S)$ by the subsets of the form $\mathcal{M}^{\text{diff}}(S'')$, with S'' homeomorphic to S;

$\gamma(S)$ = number of deformation types of complex structures on S^{top}, i.e., number of connected components of $\mathcal{M}^{\text{top}}(S)$;

$\gamma(x, y)$ = number of connected components of $\mathcal{M}_{x,y}$;

$\imath(x, y)$ = number of irreducible components of $\mathcal{M}_{x,y}$;

$\imath(S)$ = number of irreducible components of $\mathcal{M}^{\text{top}}(S)$;

$\gamma_\delta(S)$ = number of connected components of $\mathcal{M}^{\text{diff}}(S)$.

There are obvious inequalities holding among the above numbers, clearly for instance

$$\delta(S) \le \gamma(S) \le \iota(S) \le \iota(x, y),$$

and we know, as a consequence of important work of M. Freedman and S. Donaldson [Fr, Do1, Do2] that, as x, y tend to infinity, all the above four numbers can become arbitrarily large [Cat2, Cat5, F-M-M, Moi, O-VdV, Sal1, Sal2].

Only for the number $\gamma_\delta(S)$ of connected components of $\mathcal{M}^{\text{diff}}(S)$ do we lack a single example with $\gamma_\delta \ge 2$ (the problem here is to show that two surfaces are diffeomorphic without using the fact that they are deformations of each other).

Let also $\mathcal{M}^0_{x,y}$ be the subspace of the moduli space corresponding to surfaces with $q = h^1(\mathcal{O}_S) = 0$ (i.e., over $\mathbb{C}$, the first Betti number $b_1(S) = 0$), and accordingly let $\iota^0(x, y)$ be the number of its irreducible components.

Moreover, we set $\mathcal{M}_y$ to be equal to the union of the $\mathcal{M}_{x,y}$'s with y fixed

$$\mathcal{M}_y = \bigcup_x (\mathcal{M}_{x,y}), \tag{0.2}$$

and we define $\gamma(y)$, $\iota(y)$ accordingly.

We can now state again our main result on moduli spaces.

Theorem A. (1) $\iota(y) \le 6^{(y+5/9)^{15}}$.

(2) $\iota^0(x, y) \le 2y^2 \cdot (440y)^{76y^2}$.

As the reader can see, the bounds are rather unsatisfactory, but we would like to remark that even for the function $\delta(S)$ we have a lower bound which tends to infinity with y, since it is proven in [Sal2], that $\delta \ge c \cdot F(\log\log(y), \log\log\log(y))$.

It is of course an interesting *conjecture* whether there do exist polynomial bounds for $\iota^0(x, y)$, $\iota(x, y)$.

The first difficulty is that (cf. Remark 1.29) degrees tend to become of exponential type, and if the method of obtaining an upper bound for degrees is rather naive, one gets immediately a double exponential.

The situation for regular surfaces becomes better because we understand projections to a 3-dimensional projective space [Cat3], and we obtain families of low degree which dominate the moduli spaces (actually, for special values of the invariants one can even show in this way that the moduli space is unirational; cf. [Cat3, p. 104]).

For irregular surfaces one has to project to a 5-dimensional projective space.

In this framework of questions it is not so easy to obtain polynomial lower bounds, and, to my knowledge, this has not yet been done.

It could be useful in this respect to consider invariants of irreducible components of moduli spaces, such as dimension, generic tangent dimension, dimension of spaces of tensors for the general surface, rank of multilinear maps between these spaces, rank of contraction maps for the general surface, as well as inflectionary behaviour of canonical and pluricanonical maps as in [Cat2], or generic structure of Weierstrass loci (cf. [Cor]). More generally Ciliberto suggests to study the geometry of the surface over the generic point, whereas Moisezon looks at the braid monodromy of general pluricanonical projections [Moi1, Moi2].

As we already hinted at, since moduli spaces are dominated by Chow varieties and Hilbert schemes, we try to understand their complexity by studying the complexity of these other geometric objects.

Since the term "complexity" might sound unclear to some readers, we give a definition.

Definition. Given a locally closed algebraic set Z in projective space $\mathbb{P}^N$, we shall say that its unirational complexity is $\leq c$, if

(1) there is a finite decomposition $Z = \bigcup Z_i$, with Z_i locally closed;

(2) for each i, there is another locally closed algebraic set W_i in a projective space $\mathbb{P}^n$ and either a surjective morphism $\psi_i : W_i \to Z_i$, or a bijective morphism $\varphi_i : Z_i \to W_i$;

(3) $\sum_j \deg(W_j) \leq c$.

Finally, $c(Z)$ is defined to be the minimum of the c's as above.

Remark. If $\dim(Z) = 0$, clearly $c(Z) = \deg(Z) = \imath(Z)$, where $\imath(Z)$ is the number of irreducible components of Z.

If Z has higher dimension, then $\imath(Z) \leq c(Z) \leq \deg(Z)$, where $\deg(Z)$ is the sum of the degrees of its components.

If Z is irreducible and unirational, then $c(Z) = 1$, whence the name of unirational complexity.

Therefore, our simple-minded strategy will simply be to try to bound $c(Z)$ in order to bound $\imath(Z)$, and for this reason we shall try to give upper bounds for the degrees of Chow varieties, Hilbert schemes and other allied parameter varieties.

The first section starts with an elementary exposition of the theory of Chow (to be more precise, one should call them Cayley-Bertini-van der Waerden-Chow) forms, and of the natural subscheme of $\mathbb{P}^n$ associated to a hypersurface in a Grassmann manifold. We show then that, except for hypersurfaces, the subscheme associated to the Chow form of a variety V, although having the same support as V, equals V exactly at the smooth points of V.

Using this canonical subscheme, we prove the following result.

Theorem B. *Let $\mathscr{H}^0$ (resp. $\mathscr{H}^*$) be the open subset of the Hilbert scheme parametrizing smooth (resp. normal) irreducible subvarieties of dimension k and degree d in $\mathbb{P}^n$, then the natural morphism of $\mathscr{H}^0_{\text{red}}$ (resp. $\mathscr{H}^*_{\text{red}}$) to the Chow variety is an isomorphism (resp. a homeomorphism).*

We then end this section by giving some upper bound for the degrees of Chow varieties.

In the second section, in this spirit, we analyse firstly Hilbert resolutions of the coordinate rings of projectively normal varieties, whence we give upper bounds for some schemes which dominate the Hilbert schemes. In particular, we give some bounds for the number of irreducible components of Hilbert schemes.

Secondly, we look at the determinantal description of Hilbert schemes which was given by the work of Macaulay and Gotzmann ([Ma, Go], cf. also [Gr3]), and analyse its degree in terms of its equations.

In §3, finally, we firstly apply the results of the preceding sections to the moduli spaces of surfaces of general type; secondly, we analyse in more detail the case of regular surfaces, using the method of quasigeneric canonical projections introduced in [Cat3].

1. The method of Chow forms

Let $V = V_d^k \subset \mathbb{P}^n$ be an irreducible subvariety of dimension k and degree d.

Let us denote by H a point of the dual projective space $(\mathbb{P}^n)^\vee$, i.e. H corresponds to a hyperplane in $\mathbb{P}^n$.

Inside the product $(\mathbb{P}^n) \times ((\mathbb{P}^n)^\vee)^{k+1}$ we have the incidence correspondence I_V attached to V:

$$(1.1) \qquad \begin{gathered} I_V \subset V \times ((\mathbb{P}^n)^\vee)^{k+1}, \\ I_V = \{(x, H_1, \cdots, H_{k+1} | x \in H_j \quad \text{for all } j = 1, \cdots, k+1\}. \end{gathered}$$

I_V is a $(\mathbb{P}^{n-1})^{k+1}$ bundle over V, whence irreducible, and its projection to $((\mathbb{P}^n)^\vee)^{k+1}$ yields an irreducible hypersurface Γ_V which is then the locus of zeros of an (irreducible) polynomial

$$(1.2) \qquad F_V(H_1, \cdots, H_{k+1}),$$

the so called *Cayley-Bertini-van der Waerden-(Chow) form*, which is easily seen to be multi-homogeneous of degree d in each variable H_j.

One can define an analogous incidence relation in $V \times G(n-k-1, n)$, where $G(i, n)$ is the Grassmann manifold of i-dimensional projective subspaces π of $\mathbb{P}^n$,

$$(1.3) \qquad I'_V \subset V \times G(n-k-1, n), \qquad I'_V := \{(x, \pi) | x \in \pi\}.$$

Again the projection of I'_V to $G(n-k-1, n)$ is a hypersurface Γ'_V cut on $G = G(n-k-1, n)$, embedded by the dual Pluecker embedding, by a hypersurface of degree d.

In fact, if $x \in V$, there exists π meeting V only in x, hence $I'_V \to \Gamma'_V$ is generically 1-1 (actually one can verify that the above morphism is étale at the point (x, π) if x is a smooth point of V and π is transversal to V at x).

A polynomial equation F'_V for such a hypersurface is called a Chow form for V, and one has

$$(1.4) \qquad F_V(H_1, \cdots, H_{k+1}) = F'_V(H_1 \wedge \cdots \wedge H_{k+1}).$$

While F_V is unique up to multiplication by $a \in \mathbb{C}^*$, F'_V is not unique, an Chow forms should be thought of as elements in $\mathbb{P}(H^0(\mathcal{O}_G(d)))$.

It is well known (cf. [VdW1]), and we shall also see it later, that conversely F_V determines V set theoretically:

(1.5)

$$V = \{y \in \mathbb{P}^n | \text{ for all } (H_1, \cdots, H_{k+1}) \text{ s.t. } y \in H_j, F_V(H_1, \cdots, H_{k+1}) = 0\}.$$

In other terms, if one takes the bigger incidence correspondence

$$(1.6) \qquad I' \subset \mathbb{P}^n \times G, \qquad I' = \{(y, \pi) | y \in \pi\},$$

and lets $p_1 : I' \to \mathbb{P}^n$, $p_2 : I' \to G$ be the canonical projections, then we have that

$$(1.7) \qquad V = \{y | p_1^{-1}(y) \subset p_2^{-1}(\Gamma'_V)\}.$$

Another way of putting (1.7) is the following: let, for $y \in \mathbb{P}^n$, $G(y)$ be $= \{\pi | y \in \pi\}$. $G(y)$ is a Grassmann manifold $G(n-k-2, n-1)$, and (1.7) characterizes V as $V = \{y | G(y) \subset (\Gamma'_V)\}$. This geometrically means that if Γ'_V is a Chow hypersurface, then $p_2^{-1}(\Gamma'_V)$ is ruled by the algebraic foliation whose leaves are the subsets $\{y\} \times G(y)$.

The characterization (1.7) is of remarkable interest, because it allows in general to associate, to each hypersurface Γ' given by the vanishing of

$F' \in H^0(\mathcal{O}_G(d))$, not only a subvariety according to (1.7), but indeed a canonical subscheme $W_{F'}$ defined as follows. Let $H_j = (h_{0j}, \cdots, h_{nj})$, and take y to be a point in the principal open set $U_0 = \{y | y_0 \neq 0\}$: then $y = (1, y_1, \cdots, y_n)$, and if $y \in H_j$, we must have

$$h'_{0j} = h_{0j} + \sum_{i=1, \cdots, n} y_i h_{ij} = 0. \tag{1.8I}$$

We operate a change of coordinates in $(\mathbb{P}^n)^\vee$ by setting

$$\begin{aligned} h'_{ij} &= h_{ij} \quad \text{for } i \geq 1, \\ h'_{0j} &= h_{0j} + \sum_{i=1, \cdots, n} y_i h_{ij}, \end{aligned} \tag{1.8II}$$

(so that $h_{0j} = h'_{0j} - \sum_{i=1, \cdots, n} y_i h_{ij}$). Consider now

$$\begin{aligned} F'(H_1 \wedge \cdots \wedge H_{k+1}) &= F(H_1, \cdots, H_{k+1}) \\ &= F(h_{01}, \cdots, h_{n1}, \cdots, h_{0(k+1)}, \cdots, h_{n(k+1)}) \end{aligned}$$

and plug in the expression of the h_{ij}'s in terms of the h'_{ij}'s. Then we get

$$\begin{aligned} &\Psi(y, H'_1, \cdots, H'_{k+1}) \\ &\quad = F\left(h'_{01} - \sum_{i=1, \cdots, n} y_i h'_{i1}, h'_{11}, \cdots, h'_{n1}, \cdots, h'_{n(k+1)}\right) \end{aligned}$$

which can be written as an element of $\mathbb{C}[y_1, \cdots, y_n][h'_{ij}]$, multihomogeneous of degree d in each single variable H'_j, and of degree $\leq d$ in $y = (1, y_1, \cdots, y_n)$, i.e.

(1.9)

$$\begin{aligned} \Psi(y, H'_1, \cdots, H'_{k+1}) = F\Bigg(&h'_{01} - \sum_{i=1, \cdots, n} y_i h'_{i1}, h'_{11}, \cdots, h'_{n1}, \cdots, \\ &h'_{0(k+1)} - \sum_{i=1, \cdots, n} y_i h'_{i(k+1)}, h'_{1(k+1)}, \cdots, h'_{n(k+1)}\Bigg) \\ = &\sum_{\alpha = (\alpha_1, \cdots, \alpha_{k+1}),\ |\alpha_i| = d} \Psi_\alpha(y)(H'_1)^{\alpha_1} \cdots (H'_{k+1})^{\alpha_{k+1}} \end{aligned}$$

where $\alpha_1, \cdots, \alpha_{k+1}$ are multi-indices $\alpha_j = (\alpha_{0j}, \cdots, \alpha_{nj})$, with $|\alpha_j| = \sum_{i=0, \cdots, n} \alpha_{ij} = d$, and as usual $(H'_j)^{\alpha_j} = \prod_{i=0, \cdots, n} (h'_{ij})^{\alpha_{ij}}$.

Recall now that we are seeking the y's such that F vanishes whenever $y \in H_i$ for each i, i.e. Ψ vanishes whenever $h'_{0j} = 0$ for all $j \in \{1, \cdots, k+1\}$.

We define therefore a subscheme of U_0 as follows:

(1.10) $W_F \cap U_0$ is defined by the ideal generated by the $(\Psi_\alpha(y))$'s such that $\alpha_{0j} = 0$ for $j = 1, \cdots, k+1$.

In other words, $W_F \cap U_0$ is defined by the ideal generated by the coefficients of $F(-\sum_{i=1,\cdots,n} y_i h'_{i1}, h'_{11}, \ldots) \in \mathbb{C}[y_1, \cdots, y_n][h'_{11}, \cdots, h'_{n(k+1)}]$.

Remarks (1.11). (i) It is not difficult to check that the definition of W_F does not depend on the choice of the principal open set $U_i = \{y | y_i = 0\}$, and that it is independent of a particular choice of projective coordinates.

In fact, one can equivalently (cf. [VdW, p. 158], [A-N1, p. 43]) consider indeterminate antisymmetric matrices $S_1, S_2, \cdots, S_{k+1}$, and define W by the vanishing of the coefficients of the polynomial $F(S_1 x, S_2 x, \cdots, S_{k+1} x)$ in the above indeterminates S_i's. Moreover, if we perform a change of coordinates with generic coefficients, by setting, for $i \geq 1$,

$$h'_{ij} = \sum_{t=1,\cdots,n} a_{it} h''_{tj},$$

let us then consider, among the polynomials in $\mathbb{C}[y_1, \cdots, y_n][a_{it}]$, given by the new $(\Psi_\alpha(y, a))$'s, only those obtained by choosing precisely the α's for which

$$\alpha_{ij} = 0 \quad \text{for } j = 1, \cdots, k+1, \ i = 0, k+2, \cdots, n.$$

We shall call these polynomials special.

One can verify that the coefficients in the indeterminates a_{it} of the above new special $(\Psi_\alpha(y, a))$'s span the same old ideal generated by the $(\Psi_\alpha(y))$'s (here α is such that $\alpha_{0j} = 0$ for $j = 1, \cdots, k+1$).

In particular, an important consequence is that the above ideal is generated by the polynomials obtained from the $(\Psi_\alpha(y, a))$'s via taking a sufficient number of general specializations of the a_{it}'s. We shall see later that the geometrical meaning of this is that this ideal is spanned by a sufficient number of polynomial equations of conical hypersurfaces obtained by taking generic projections of V to a $\mathbb{P}^{k+1}$.

The following proposition characterizes Chow forms as being the irreducible polynomials for which $\dim(W_F)$ attains the maximal dimension.

Proposition (1.12). *Let*

$$F = F'(H_1 \wedge \cdots \wedge H_{k+1}),$$

with $F' \in H^0(G, \mathscr{O}_G(d))$*; then* $\dim W_F \leq k$. *Assume that* F *is irreducible and that* $\dim W_F = k$. *Then* F' *is a Chow form.*

Proof. We have that $\dim W_F \leq k$: otherwise, for all $\Pi \in G$, we would have $\Pi \cap W_F \neq \varnothing$ whence $p_2(p_1^{-1}(W_F)) = G$, contradicting the fact that $p_1^{-1}(W_F) \subset p_2^{-1}(\{F' = 0\})$. If $\dim W_F = k$, then there is an irreducible subvariety V of dimension k such that V is a subscheme of W_F. But, since F_V is an equation for $p_2(p_1^{-1}(V))$, we see that F_V divides F, hence $F_V = F$ by the irreducibility of F. q.e.d.

As we remarked in (1.5), (1.7), if we start from an irreducible subvariety V of dimension k, the subscheme $W = W_F$ associated to the Chow form of V is such that $\operatorname{supp}(W) = V$. It is therefore natural to ask when $W = V$, and one can give a complete answer to the above question.

We prefer, though, to give first an example which illustrates the situation.

Example (1.13). Consider the curve $V \subset \mathbb{P}^3$ given as the complete intersection $Q_1 \cap Q_2$ of the two quadric cones $\{x_0x_2 - x_1^2 = 0\} = Q_1$, $\{x_0x_3 - x_2^2 = 0\} = Q_2$. V is a rational curve with parametric equation $(t^4, t^3, t^2, 1)$ and has a singular point at the vertex $(0, 0, 0, 1)$ of Q_1, where the tangent dimension of V equals 2. We can easily find the Chow form of V, in fact if $\sum a_i x_i = 0$, $\sum b_i x_i = 0$ are two hyperplanes, $F_V(a, b)$ is given by the resultant of the two polynomials

$$a_0t^4 + a_1t^3 + a_2t^2 + a_3,$$
$$b_0t^4 + b_1t^3 + b_2t^2 + b_3,$$

$$F_V(a, b) = \det \begin{pmatrix} 0 & 0 & 0 & a_0 & a_1 & a_2 & 0 & a_3 \\ 0 & 0 & a_0 & a_1 & a_2 & 0 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 & a_3 & 0 & 0 \\ a_0 & a_1 & a_2 & 0 & a_3 & 0 & 0 & 0 \\ b_0 & b_1 & b_2 & 0 & b_3 & 0 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & b_3 & 0 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & 0 & b_3 \end{pmatrix}.$$

To obtain the equations of the subscheme W in the open set $\{x_3 \neq 0\}$, it suffices in the determinant to replace a_3 by $-\sum_{i=0,\cdots,2} a_i x_i$ and b_3 by $-\sum_{i=0,\cdots,2} b_i x_i$. One immediately sees that this determinant has order at least two in (x_0, x_1, x_2), whence $I_V \neq I_W$ (in the open set where $x_3 = 1$, $Q_2 = x_0 - x_2^2 = 0$ has order 1).

Theorem (1.14)(a). *Let V be an irreducible subvariety of dimension k in $\mathbb{P}^n$, let $F = F_V$ be its Cayley-Bertini-van der Waerden-(Chow) form, and let $W = W_F$ be the associated subscheme of $\mathbb{P}^n$, defined according to* (1.10). *Then* (*recall that* $\operatorname{supp}(W) = V$) *we have an equality of schemes $W = V$ at the smooth points of V.*

(b) *If moreover V is not a hypersurface, then the equality $W = V$ holds exactly at the smooth points of V.*

Remark (1.15). In fact, if V^* is a hypersurface of degree d in $\mathbb{P}^{k+1}$ with equation $f_d(x_0, \cdots, x_{k+1}) = 0$ then its Chow form $F(H_0, \cdots, H_k)$ equals $f((H_0 \wedge \cdots \wedge H_k)^\vee)$, where we have used, for a $(k+2)$-dimensional vector space X, the standard duality $\bigwedge^{k+1}(X) \times X \to \bigwedge^{k+2}(X)$ (in other words, if $H_0, \cdots, H_k$ are independent, $(H_0 \wedge \cdots \wedge H_k)^\vee$ is the unique point $x \in \mathbb{P}^{k+1}$ such that $H_0(x) = \cdots = H_k(x) = 0$).

In this case it is also easy to see that always one has $W_F = \{f = 0\}$, so that if V is a hypersurface, then $W = V$.

Proof of Theorem (1.14)(a). The main idea is to use the functoriality of Chow forms with respect to linear projections (cf. [Sa, pp. 47–48]). That is, let π' be a linear subspace of dimension $(n-k-2)$ which does not intersect V, and let $p : \mathbb{P}^n - \pi' \to \mathbb{P}^{k+1}$ be the projection with centre π'. Since $V \cap \pi' = \emptyset$, p is finite on V, and $p(V)$ is a hypersurface in $\mathbb{P}^{k+1}$. We furthermore assume

(1.16) $$p : V \to V^* = p(V) \subset \mathbb{P}^{k+1} \quad \text{is birational.}$$

Whence V^* is a hypersurface of degree d in $\mathbb{P}^{k+1}$. We can choose coordinates in $\mathbb{P}^n$ such that $\pi' = \{y_0 = \cdots = y_k = y_{k+1} = 0\}$. Then the hyperplanes in $\mathbb{P}^{k+1}$ correspond to the hyperplanes in $\mathbb{P}^n$ containing π', i.e., to the subspace $(\mathbb{P}^{k+1})^\vee$ of $(\mathbb{P}^n)^\vee$ given by $\{H | h_j = 0 \text{ for } j \geq k+2\}$.

The restriction F^* of the Bertini form $F(H_0, \cdots, H_{k+1})$ to $((\mathbb{P}^{k+1})^\vee)^{k+1}$ is just the Bertini form of V^*; in fact, more generally, the restriction of the Bertini form F to the hyperplanes belonging to a general linear subspace of $(\mathbb{P}^n)^\vee$ gives the Bertini form F' of the projected variety V', as soon as the degree of V' equals the one of V (since F' and the restriction of F have the same multidegree and have the same zero locus, they coincide up to a constant).

By Remark (1.15), F^* is essentially the equation of V^*. It is then easy to see (by Remark (1.15)) that the equation f^* of V^* is obtained as $\Psi_\alpha(y) = 0$ with α (cf. (1.10)) such that for all $j = 1, \cdots, k+1$, $\alpha_{0j} = \alpha_{k+2,j} = \cdots = \alpha_{nj} = 0$ (two different choices of such α's lead to the same equation, since we have that $F(H_1, \cdots, H_{k+1}) = F'(H_1 \wedge \cdots \wedge H_{k+1})$, therefore

$$\sum_{\alpha=(\alpha_1,\cdots,\alpha_{k+1})\alpha_{0,j}=\alpha_{k+2,j}=\alpha_{n,j}=0} \Psi_\alpha(y)(H_1')^{\alpha_1}\cdots(H_{k+1}')^{\alpha_{k+1}}$$

is proportional to the d^{th} power of the determinant of $|h'_{ij}|_{i,j=1,k+1})$.

Now let $x \in V$ be a smooth point: then we consider the open set in the Grassmannian consisting of the subspaces π' as above such that

(i) $\pi' \cap V = \varnothing$,

(ii) the projection p with centre π' gives a birational morphism $p|_V : V \to V^*$,

(iii) p gives a local isomorphism at x (i.e., $x = (p|_V)^{-1}(p(x))$, and $p|_V$ is of maximal rank at x).

Each such projection gives thus a hypersurface f^* of degree d (a cone over π') which is smooth at x, and has a tangent space at x equal to the join of π' and T_xV. It suffices therefore to take $(n-k)$ general hyperplanes H_i'' whose intersection is T_xV, and for each of them a generic $\pi_i' \subset H_i''$ satisfying the above three properties: then the corresponding hypersurfaces f_i^* define V locally at x, and the claimed result follows since the f_i^*'s belong to the ideal of the subscheme W. q.e.d.

Remark (1.17). As we already mentioned, what the previous proof shows is that, when the scheme W is obtained from the Chow form of a subvariety V^k, then W is the scheme theoretic intersection of the cones obtained by projecting V under a sufficient number of general projections to $\mathbb{P}^{k+1}$.

Proof of Theorem (1.14)(b). First of all, we claim that we can reduce to the case where V has codimension 2.

In fact, as we saw in the course of proving part (a), for a general projection of V to a variety V' of codimension 2, the Bertini form of V restricts to the Bertini form of V'. From the definition of W, W' it follows directly, as one can verify, that W' is the scheme theoretical image of W. Therefore, if V would equal W, the same would hold between V' and W'.

Now, to a singularity $x \in V$ in $\mathbb{P}^n$ we can associate an integer μ, which is the minimal order at x of a polynomial vanishing on V (alternatively, μ is 1 if the tangent dimension is smaller than n, otherwise it is the first degree in which the graded symmetric algebra over a minimal system of generators of the local ring R of x does not embed in the graded local ring associated to R). By virtue of Remark (1.17), it suffices to show that if V has codimension 2 and we project generically V to a hypersurface, then μ increases strictly. This is obvious if $\mu = 1$, whence we can assume $\mu \geq 2$. To this purpose, take affine coordinates (w, y) with $w = (w_1, \cdots, w_{k+1})$, such that $x = (0, 0)$, and such that our projection maps (w, y) to w.

Let $f_1, \cdots, f_h$ be generators of the ideal of V at x. By Noether's normalization lemma, we can assume that the f_i's are monic in the indeterminate y, therefore, the polynomials $g_1, \cdots, g_r$ obtained from $f_1, \cdots, f_h$ eliminating y through the method of the resultant system (cf. [vdW2]), have (by an easy lemma, cf. [Wa1]) order at least μ^2 at the origin.

It remains to observe that since the projection V^* of V is a hypersurface, V is the zero locus of any nonzero such a g_i, provided that such a g_i vanishes simply at a smooth point w' of V^*.

But, if x' is the corresponding smooth point of V, we can assume that locally at $x'V$ is the complete intersection of f_1 and f_2, whence follows immediately that some g_i vanishes simply at w'. q.e.d.

Remark (1.18). Similar results to Theorem 1.14 (including it, I believe) have been independently obtained by F. Amoroso in [Amo]. That paper deals also with the problem of giving estimates for a minimal power of the ideal of V which is contained in the ideal of W.

It is also worthwhile to mention how Bertini forms behave with respect to sections. That is, assume that we consider the section $Y = V \cap H_1^\wedge$ of V with a hyperplane $H_i^\wedge$. Then we can consider

$$\Phi(H_2, \cdots, H_{k+1}) := F_V(H_1^\wedge, H_2, \cdots, H_{k+1}) \tag{1.19}$$

and we remark that Φ (F_V being alternating) only depends upon $H_2 \cap H_1^\wedge, \cdots, H_{k+1} \cap H_1^\wedge$; which are hyperplanes in $H_1^\wedge$.

Proposition (1.20). (i) *Assume V is degenerate, i.e., contained in a hyperplane $H_1^\wedge$. Then, if F_V is the Bertini form of V in $\mathbb{P}^n$, and f_V is the Bertini form of V in $H_1^\wedge$, we have*

$$F_V(H_1, H_2, \cdots, H_{k+1}) = f_V(H_1 \cap H_1^\wedge, \cdots, H_{k+1} \cap H_1^\wedge).$$

(ii) *Assume, instead, V to be not necessarily degenerate and that $Y = V \cap H_1^\wedge$ is a proper algebraic subset of V, with Bertini form Φ_Y of the cycle associated to Y.*

We have then

$$F_V(H_1^\wedge, H_2, \cdots, H_{k+1}) = \Phi_Y(H_2 \cap H_1^\wedge, \cdots, H_{k+1} \cap H_1^\wedge).$$

Moreover, in case (i), *the subscheme W associated to F_V equals the subscheme associated to f_V, whereas in case* (ii) *the intersection of W with* *the hyperplane* $H_1^\wedge$ *is a subscheme of the scheme associated to* Φ_Y.

Proof. We can choose coordinates $(y_0, \cdots, y_n)$ such that $H_1^\wedge = \{y_n = 0\}$. If H_j is the hypersurface $\sum_{i=0,\cdots,n-1} h_{ij}y_i = 0$, then $H_j \cap H_1^\wedge = \{(y_0, \cdots, y_{n-1}, 0) | \sum_{i=0,\cdots,n-1} h_{ij}y_i = 0\}$.

Our first claim is simply that the Chow form F_V does not depend upon the variables $h_{n,1}, \cdots, h_{n,k+1}$ if $V \subset \{y_n=0\}$. But $F_V(H_1, \cdots, H_{k+1})=0$ if and only if $H_1 \cap \cdots \cap H_{k+1} \cap V \neq \emptyset$, which is equivalent to $(H_1 \cap H_1^\wedge) \cap \cdots \cap V \neq \emptyset$, which again is equivalent to $f_V(H_1 \cap H_1^\wedge, \cdots, H_{k+1} \cap H_1^\wedge) = 0$. We have two irreducible polynomials in $(H_1, \cdots, H_{k+1})$ with the same multidegrees and the same zero locus, whence they differ by a nonzero constant. An entirely similar argument shows the second equality.

In case (i), the scheme W associated to F_V equals the one associated to f_V simply because the variables h_{ni} do not appear in F_V.

Finally, if $\Phi(h_{02}, \cdots, h_{n-1,2}, \cdots) = F(0, \ldots, 0, 1, h_{02}, \cdots, h_{n2}, \cdots)$, then the subscheme $W_\Phi \subset \{y_n = 0\}$, restricted to the open set $y_0 \neq 0$ is obtained by expanding

$$A = F\left(0, \cdots, 0, 1, -\sum_{i=1,\cdots,n-1} y_i h'_{i2}, h'_{1,2}, \cdots, h'_{n,2}, \cdots\right)$$

and looking at its coefficients viewing it as an element of $\mathbb{C}[y_1, \cdots, y_{n-1}] \cdot [h'_{ij}]$ (here $j \geq 2$). Whereas, the intersection of the scheme W with the hyperplane $\{y_n = 0\}$ is obtained by applying the same procedure to

$$B = F\left(-\sum_{i=1,\cdots,n-1} y_i h'_{i1}, h'_{1,1}, \cdots, h'_{n,1}, \cdots\right).$$

It suffices to observe that the specialization $h'_{i1} = 0$ for $i \leq n-1$, $h'_{n1} = 1$ yields A from B, whence the desired result. q.e.d.

The main direction in which we are going to apply Theorem (1.14) is the comparison between the Hilbert scheme and the Chow variety.

Observe now that equations (1.10) define a canonical subscheme $\mathscr{W}''$ inside the product $\mathbb{P}(H^0(G, \mathscr{O}_G(d))) \times \mathbb{P}^n$ ($\mathscr{W}''$ is defined by equations which are linear in the coefficients of F).

Proposition (1.12) tells us that, if $\mathscr{U}$ is the open set $\{F | F$ is irreducible$\}$, then $F \in \mathscr{U}$ is a Chow form iff the dimension of the fibre $W_F = (\mathscr{W}'' \cap p_1^{-1}(\{F\}))$ equals k.

Since k is the maximal dimension of a fibre of the proper map $p_1 : \mathscr{W}'' \cap p_1^{-1}(\mathscr{U}) \to \mathscr{U}$, we find that there is an algebraic subset $\mathscr{C}$ ($\mathscr{C}$ stands for

"Chow") of $\mathscr{U}$ consisting exactly of the F's which are Chow forms.

It is important to remark that by Proposition (1.12) and Theorem (1.14), if $F \in \mathscr{C}$, then W_F equals $(W_F)_{\text{red}} = V$ at the smooth points of V.

Defintiion (1.21). One defines $\mathscr{C} = \mathscr{C}_{k,d}(\mathbb{P}^n)$ with its reduced structure to be the Chow variety parametrizing irreducible k-dimensional subvarieties of degree d in $\mathbb{P}^n$. Letting $p: \mathscr{W} \to \mathscr{C}$ be the projection induced by p_1, we define the Chow family $\mathscr{W}$ to be the restriction to $\mathscr{C}$ of the family $\mathscr{W}''$ defined by (1.10).

$\mathscr{C}$ contains the open subset $\mathscr{C}^0$ consisting of the F's in $\mathscr{C}$ for which $(W_F)_{\text{red}} = V$ is smooth, hence equal to W_F.

Over $\mathscr{C}^0$ we thus get a smooth, hence flat family $p^0 : \mathscr{W}^0 \to \mathscr{C}^0$.

(1.22). This family induces a morphism $\varphi: \mathscr{C}^0 \to \mathscr{H}^0$, where $\mathscr{H}^0$ is the open subset of the Hilbert scheme parametrizing smooth irreducible subvarieties of dimension k and degree d in $\mathbb{P}^n$.

To be more precise, we have a flat family $p^0: \mathscr{W}^0 \to \mathscr{C}^0$, and one knows (cf., e.g., [Mu]) that the Hilbert polynomial of the fibres is constant on each connected component of $\mathscr{C}^0$.

Thus we have a finite number of Hilbert polynomials, and we let $\mathscr{H}''$ (cf. [Gro, Mu, Go]) be the union of the Hilbert schemes corresponding to these Hilbert polynomials.

By the universal property of the Hilbert scheme we have a universal family $\chi'' \to \mathscr{H}''$, $\chi'' \subset \mathscr{H}'' \times \mathbb{P}^n$, such that any other flat family with one of the given Hilbert polynomials is a pull back of it by a suitable morphism of the base of the family to $\mathscr{H}''$.

$\mathscr{H}''$ contains an open subset $\mathscr{H}^0$, consisting of the points whose fibres are smooth, and clearly our morphism $\varphi: \mathscr{C}^0 \to \mathscr{H}''$ maps to $\mathscr{H}^0$.

We can similarly consider (by a slight abuse of language) the Hilbert scheme $\mathscr{H}$ parametrizing irreducible reduced subvarieties of dimension k and degree d in $\mathbb{P}^n$: actually $\mathscr{H}$ is a union of open sets of Hilbert schemes and $\mathscr{H}$ is known (cf. [Gro]) to have a finite numbers of components because there exists a surjective and injective morphism $\psi: \mathscr{H}_{\text{red}} \to \mathscr{C}$.

(1.23). The morphism $\psi: \mathscr{H}_{\text{red}} \to \mathscr{C}$ is obtained as follows: Let $\chi \to \mathscr{H}$, $\chi \subset \mathscr{H} \times \mathbb{P}^n$ be the universal family, and consider the fibre product $\mathscr{X} \times_{\mathbb{P}^n} I'$, $I' \subset \mathbb{P}^n \times G$ being as in (1.6). Then the scheme theoretic image $\mathscr{F}$ to $\mathscr{H} \times G$ defines a hypersurface, whose fibres over the points of $\mathscr{H}$ are the Chow hypersurfaces of the fibres of $\chi \to \mathscr{H}$. In [Mu1], page 111, is proven that $\mathscr{F} \to \mathscr{H}$ is flat, then $\mathscr{F}$ defines a relative Cartier divisor in $\mathscr{H} \times G \to \mathscr{H}$, whence one gets a morphism $\psi': \mathscr{H} \to \mathbb{P}(H^0(\mathscr{O}_G(d)))$.

Whence we get a morphism $\psi: \mathscr{H}_{\text{red}} \to \mathbb{P}(H^0(\mathscr{O}_G(d)))$ which clearly factors through $\mathscr{C}$.

Theorem (1.24). *Let $\mathscr{H}^0$ be the open subset of the Hilbert scheme parametrizing smooth irreducible subvarieties of dimension k and degree d in $\mathbb{P}^n$. Then the restriction to $\mathscr{H}^0_{\text{red}}$ of the natural morphism $\psi: \mathscr{H}_{\text{red}} \to \mathscr{C}$, is an isomorphism onto the open subset $\mathscr{C}^0$ consisting of the F's in $\mathscr{C}$ for which $(W_F)_{\text{red}} = V$ is smooth.*

Proof. It suffices to verify that φ and the restriction of ψ to $\mathscr{H}^0_{\text{red}}$ are inverse morphisms to each other. q.e.d.

Remark (1.25). A more general result than above, concerning an isomorphism between nonreduced schemes, is stated in [An, p. 128, Remark 7.3.1].

With our elementary approach we can give several scheme structures to Chow, but we do not yet understand for which one there is a natural morphism from Hilbert to Chow.

We continue now to analyse the natural morphism $\psi: \mathscr{H}_{\text{red}} \to \mathscr{C}$, at the points which correspond to normal varieties.

Theorem (1.26). *Let $\mathscr{H}^*$ be the open subset of the reduced Hilbert scheme parametrizing normal irreducible subvarieties of dimension k and degree d in $\mathbb{P}^n$. Then ψ^*, the restriction to $\mathscr{H}^*$ of the natural morphism $\psi: \mathscr{H}_{\text{red}} \to \mathscr{C}$, yields a homeomorphism onto the open subset $\mathscr{C}^*$ consisting of the F's in $\mathscr{C}$ for which $(W_F)_{\text{red}} = V$ is normal.*

Proof. First of all, we remark that, since we have chosen $\mathscr{H}$ to parametrize irreducible reduced subvarieties of $\mathbb{P}^n$, ψ is injective: in fact F determines (W_F) which in turn determines $(W_F)_{\text{red}} = V$. The surjectivity is clear by definition, and it suffices to show that ψ is open.

To show that ψ is open, let V^* be a normal irreducible subvariety of degree d in $\mathbb{P}^n$, and let F^* be its Chow form. View F^* as a point of $\mathscr{C}$, and let $[V^*]$ be the point of $\mathscr{H}_{\text{red}}$ corresponding to V^*. Let, moreover, $\mathscr{D}$ be an irreducible component of $\mathscr{C}$ at F^*, C be an irreducible curve of $\mathscr{D}$ passing through F^*.

Let T be the normalization of C, and consider the fibre product $T \times_C \mathscr{W} = \mathscr{Y}$. $Y = \mathscr{Y}_{\text{red}} \to T$ is an algebraic family of normal varieties in the sense of [Ha, p. 263]: Y has all fibres irreducible of dimension k, and generically reduced fibres by Theorem 1.14. By Theorem 9.11 of [Ha] (based on Hironaka's Lemma 9.12), $Y = \mathscr{Y}_{\text{red}} \to T$ is a flat family of schemes, with central fibre reduced and normal by 9.12 of [Ha]. We can shrink T to T' and achieve that all fibres are reduced and normal. By the universal property of the Hilbert scheme, the normalization map $T' \to C \subset \mathscr{C}$ factors through the composition of a suitable morphism $f: T' \to$

$\mathscr{H}_{\text{red}}$ with the canonical morphism $\psi: \mathscr{H}_{\text{red}} \to \mathscr{C}$. In particular, firstly a neighbourhood of F^* in C lies in the image under ψ of a neighbourhood of $[V^*]$; secondly $\mathscr{C}^*$ is open. q.e.d.

I conjectured the above theorem in 1988, but for long time what I was missing was exactly Hironaka's lemma: I am extremely grateful to J. Kollar for pointing it out to me.

In fact, we also have a corollary related to work of Andreotti and Norguet (cf. [A-N1]), who prove that the weak normalization of the Chow variety of a projective variety is independent of the projective embedding (before Barlet, in [Ba], proved independence of the projective embedding).

Corollary (1.27). *$\psi: \mathscr{H}_{\text{red}} \to \mathscr{C}$ establishes an injective correspondence between the set of irreducible components of $\mathscr{H}^*_{\text{red}}$ and the set of irreducible components of $\mathscr{C}$. ψ also yields an isomorphism between the respective weak normalizations of $\mathscr{H}^*_{\text{red}}$ and $\mathscr{C}^*$.*

Proof. The first assertion is clear, since $\mathscr{C}^*$ is open, so its irreducible components have as closure distinct irreducible components of $\mathscr{C}$. For the second assertion, $\psi: \mathscr{H}^*_{\text{red}} \to \mathscr{C}^*$ is a homeomorphism, in particular the respective weak normalizations are homeomorphic, whence isomorphic. q.e.d.

As mentioned in the introduction, in order to give an upper bound for the number of components of $\mathscr{C} = \mathscr{C}_{k,d}$ we shall give an upper bound for its degree.

We recall the idea that Andreotti [An] used in order to bound from above the cardinality of a finite algebraic set.

Lemma (1.28) (Andreotti-Bezout inequality). *Let $Z = U \cap V$ be a quasiprojective scheme, with U an open set in $\mathbb{P}^N$, and V a closed subscheme in $\mathbb{P}^N$, defined by a homogenous ideal $I = (f_1, f_2, \cdots, f_h)$ where f_i is a homogeneous polynomial of degree d_i, and $d_1 \geq d_2 \geq \cdots d_h$. Let $Z_1, \cdots, Z_t$, be the irreducible components of Z_{red}, and r the maximum, for $i = 1, \cdots, t$, of the codimension of Z_i in $\mathbb{P}^N$. Then the degree of Z, i.e., the sum of the degrees of the Z_i's, is at most $d_1 d_2 \cdots d_r$.*

Proof. In general, we can assume, by adding to each f_i a suitable element of the ideal $(f_{i+1}, \cdots, f_h)$, that in the local ring of the generic point of each Z_i, the ideal spanned by $(f_1, \cdots, f_j)$ is either primary or it has codimension j. This shows that we can reduce to the case where the number of equations h equals the codimension r.

We can now apply induction on r, the case $r = 1$ being obvious (notice that the upper bound would be better if one knew that f_h is not identically zero). It suffices to show that if Z' is an irreducible variety of degree m,

f is a polynomial of degree d not in the ideal of Z', then the algebraic set $Z' \cap \{f = 0\}$ decomposes as a union of varieties whose degrees add up to a number not exceeding md. This follows since, if we intersect Z' with a general linear subspace L such that $L \cap Z'$ is a smooth curve of degree m, then the length of $L \cap Z' \cap \{f = 0\}$ does not exceed md. q.e.d.

The above bound is sharp, but it is rather crude if we want to use it only to bound the number t of irreducible components of Z.

As a matter of fact, in most applications, we shall use the following trivial but useful remark (which we already used in the introduction, when we mentioned that the unirational complexity is greater or equal to the number of irreducible components).

Remark (1.29). Let $Z = U \cap V$ be a locally closed algebraic set. Then $\imath(Z) \leq c(Z)$. I.e., assume that Z is the union of locally closed subsets $W_1, \cdots, W_r$, for each of which we have a surjective morphism $f_i\colon X_i \to W_i$, where X_i is a locally closed projective algebraic set. Then the number $\imath(Z)$ of the irreducible components of Z is at most the sum of the degrees of the X_i's (defined as in Theorem (1.26)).

The same result holds if we have bijective morphisms $f_i\colon W_i \to X_i$.

We end this section by giving applications to Chow varieties and Hilbert schemes, using a result of Green and Morrison [G-M, Theorem 4], which generalizes an old result of Cayley for space curves [Cay1, Cay2].

Theorem [G-M]. *An irreducible F is a Chow form if and only if its coefficients satisfies certain equations* ((2.5), *resp.* (2.16) *in loc. cit.*) *of respective degrees* 2, 3.

We would like only to comment on the beautiful idea behind this theorem: we saw that (1.7) characterizes Chow hypersurfaces as the hypersurfaces Γ'_V such that $p_2^{-1}(\Gamma'_V)$ is ruled by the algebraic foliation whose leaves are the subsets $\{y\} \times G(y)$. Now, the quadratic equations simply imply that the foliation is tangent to the hypersurface, whereas the cubic equations are just the Frobenius conditions of integrability of the distribution of tangential subspaces on the hypersurface: if both are verified, we have an analytic foliation on $p_2^{-1}(\Gamma'_V)$ which must coincide with the algebraic foliation whose leaves are the subsets $\{y\} \times G(y)$, whence Γ'_V is a Chow hypersurface.

From the theorem of Green and Morrison follows

Theorem (1.30). *Let $\mathscr{C} = \mathscr{C}_{k,d}(\mathbb{P}^n)$ be the open subset of the Chow variety parametrizing Chow forms of irreducible k-dimensional subvarieties of degree d in $\mathbb{P}^n$. Let, for $b \leq a$, $C(a, b)$ be the binomial coefficient. Then the number $\imath(\mathscr{C})$ of irreducible components of $\mathscr{C}$ satisfies $\imath(\mathscr{C}) \leq$*

$\deg(\mathscr{C}) \le 3^{\lambda}$, *where* $\lambda = C(C(n+d, d), k+1) - 1$.

Proof. We are in the position to apply the Andreotti-Bezout inequality (1.26) to $\mathscr{C} = \mathscr{C}_{k,d}(\mathbb{P}^n) \subset \mathbb{P}(H^0(\mathscr{O}_G(d)))$. The dimension of this last projective space is at most λ, since F is multihomogeneous of degree d in each variable $H_1, \cdots, H_{k+1}$. Now, $\mathscr{C}$ is defined by equations of degree at most 3 by the theorem of Green and Morrison , so we are done. q.e.d.

Remarks (1.31). (a) Let $\mathscr{C}^- \supset \mathscr{C}_{k,d}(\mathbb{P}^n)$ be the closed subset formed by the Chow forms of k-dimensional cycles of degree d (to a cycle $\sum n_i V_i$ one associates the Chow form $\prod F_{V_i}^{n_i}$). It is not clear to me whether the equations of Green and Morrison define $\mathscr{C}^-$ or something different.

(b) Another interesting question is: what is the most natural scheme structure with which to endow the set $\mathscr{C}$? Of course, one would not want to give up the embedding $\to \mathbb{P}(H^0(\mathscr{O}_G(d)))$. To this purpose, it is clear that one has to keep track of the natural morphism $\psi' : \mathscr{H} \to \mathbb{P}(H^0(\mathscr{O}_G(d)))$ extending the morphism $\psi' : \mathscr{H}_{\text{red}} \to \mathbb{P}(H^0(\mathscr{O}_G(d)))$ we considered above (cf. [Mu1]).

(c) Another question is whether in (1.20)ii) equality holds for a general section.

(d) One may ask how sharp is the bound of Theorem 1.30.

In the case where $d = 1$, we have the Grassmann manifold $\mathscr{C}_{k,1}(\mathbb{P}^n) = G(k, n)$ whose degree equals (cf. [K1, p. 463])

$$((k+1)(n-k))!1!2!\cdots k!/(n-k)!\cdots n!.$$

We want to compare this asymptotically with 3^{λ}, where ($d = 1$) $\lambda = C(n+1, k+1) - 1$.

Keeping k fixed and letting n go to infinity, we can use Stirling's formula and see that the degree is asymptotic to a constant times $(k+1)$ to the power $(k+1)n$, whereas our bound yields for λ a constant times $(n+1)^{k+1}$: i.e., instead of having the exponential of a linear function of n, we replace it by the exponential of a power of n.

If instead for n fixed we choose the highest degree, e.g. we set $n = 2k+1$, we get for the degree $((k+1)^2)!1!2!\cdots k!/(k+1)!\cdots(2k+1)!$. Taking logarithms, we get (since $\int x \log x \approx (1/2)x^2 \log x$) something asymptotic to $k^2 \log k$. Whereas the logarithm of our bound is a constant times 2^{2k}. So this case is worse, because we replace an exponential in a power of n by a double exponential. It seems thus that exponentials are unavoidable, whereas double exponentials are due to the poor method we use for the estimation from above.

2. The method of Hilbert resolutions and Hilbert schemes

In this section we consider the following situation: given a smooth manifold X and a birational morphism $\varphi: X \to Y \subset \mathbb{P}^n$, we consider the invertible sheaf $\mathcal{L}$ on X given as $\mathcal{L} = \varphi^*(\mathcal{O}_{Pn}(1))$, and we consider the graded ring

$$(2.1) \qquad \mathcal{R} = \mathcal{R}(X, \mathcal{L}) = \bigoplus_{m\in\mathbb{N}}(H^0(X, \mathcal{L}^{\otimes m})).$$

$\mathcal{R}$ is a graded finite module over the graded ring

$$(2.2) \qquad \mathcal{A} = \bigoplus_{m\in\mathbb{N}}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}n}(m))),$$

and in fact the Serre correspondence associated to $\mathcal{R}$ the coherent sheaf $\mathcal{F} = \varphi_*(\mathcal{O}_X) = \varphi_*(\mathcal{L})(-1)$.

Let k be the dimension of X and let $\delta \le k+1$ be the depth of $\mathcal{R}$ as an $\mathcal{A}$-module. By Hilbert's syzygy theorem, there exists a minimal free resolution of $\mathcal{R}$ of length $n-\delta$:

$$(2.4) \qquad 0 \to L_{n-\delta} \to L_{n-\delta-1} \to \cdots \to L_2 \to L_1 \to L_0 \to \mathcal{R} \to 0.$$

Here L_i is free and if we denote its rank by r_i, then we can write

$$(2.5) \qquad L_i = \bigoplus_{j=1,\cdots,r_i} \mathcal{A}(-m_{i,j}).$$

We have, (cf., eg., [Gr2, Theorem 1.2]) that $L_i = \bigoplus_m (B_{i,m}) \otimes \mathcal{A}(-m)$, where the $\mathbb{C}$-vector space $B_{i,m}$ equals

$$(2.6) \qquad B_{i,m} = \operatorname{Tor}_i^{\mathcal{A}}(\mathcal{R}, \mathbb{C})_m.$$

An important notion in this context is Castelnuovo's notion of regularity (cf. loc. cit. and [Mu2, Lecture 14]).

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$; then $\mathcal{F}$ is said to have regularity equal to h if $H^i(\mathbb{P}^n, \mathcal{F}(h-i)) = 0$ for all $i > 0$, and h is minimal with this property. An $\mathcal{A}$-module is said to have regularity h if the associated sheaf has regularity h. Then [Gr2, Theorem 2.3]

$$h = \max\{m - i | B_{i,m} \neq 0\}.$$

In other words, if $\mathcal{R}$ is h-regular, then $m_{i,j} \le i + h$.

In our particular situation, since $\mathscr{R}$ has no terms of negative degree and we have a minimal resolution, one can easily show by induction that

(2.6) $$i \le m_{i,j} \le i + h.$$

We can now split (2.4) into a series of short exact sequences:

(2.7)
$$\begin{gathered} 0 \to I_1 \to L_0 \to \mathscr{R} \to 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \to I_{i+1} \to L_i \to I_i \to 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \to L_{n-\delta} \to L_{n-\delta-1} \to I_{n-\delta-1} \to 0. \end{gathered}$$

First of all, we want to give an upper bound for the ranks r_i of the free modules L_i appearing in a minimal free resolution. We can do this in terms of the regularity h of $\mathscr{R}$ and its mode of generation (i.e., L_0). In fact, rather crudely, we can use (2.6) to infer that $r_i \le h^0(\mathbb{P}^n, \mathscr{F}_i(h+i))$, where $\mathscr{F}_i$ is the locally free sheaf associated to the free module L_i. Using the minimality of the resolution we can actually say more, i.e.,

(2.8) $$r_i \le h^0(\mathbb{P}^n, \mathscr{I}_i(h+i)) \quad \text{where } \mathscr{I}_i \text{ is the sheaf associated to } I_i.$$

Thus $r_i \le h^0(\mathbb{P}^n, \mathscr{I}_i(h+i)) \le h^0(\mathbb{P}^n, \mathscr{F}_{i-1}(h+i)) \le C(n+h+1, h+1)r_{i-1}$, whence $r_i \le C(n+h+1, h+1)^{i-1} r_1$. Finally, it remains to bound r_1: but here we can simply write $r_1 \le h^0(\mathbb{P}^n, \mathscr{I}_1(h+1)) \le h^0(\mathbb{P}^n, \mathscr{F}_0(h+1)) - h^0(\mathbb{P}^n, \mathscr{F}(h+1)) \le h^0(\mathbb{P}^n, \mathscr{F}_0(h+1))$. Thus

(2.9) $$r_i \le C(n+h+1, h+1)^{i-1} h^0(\mathbb{P}^n, \mathscr{F}_0(h+1)).$$

The above inequality is most meaningful in the case when $L_0 = \mathscr{A}$, where it reads out as

(2.9′) $$r_i \le C(n+h+1, h+1)^i.$$

Remark that $L_0 = \mathscr{A}$ means that the image Y of X is projectively normal. Whereas, if we moreover assume that Y is nondegenerate (not contained in a hyperplane), then necessarily, (since $2 \le m_{1,j}$) $i+1 \le m_{i,j} \le i+h$. It follows that

(2.10) if Y is nondegenerate and projectively normal we have $r_i \le C(n+h, h)^{i-1} C(n+h+1, h+1)$.

The above result can be used to prove explicit boundedness results for projectively normal (nondegenerate) varieties with Castelnuovo regularity smaller than h. In fact, we have at most N choices for the ranks

$r_1, r_2, \cdots, r_{n-\delta}$ of the free modules appearing in a minimal free resolution, where

(2.11) $$N = C(n+h+1, h+1)^{n-\delta} C(n+h, h)^{(n-\delta)(n-\delta-1)/2}.$$

For each choice of the ranks r_i we have at most $C(r_i + h, h)$ choices for the integers $m_{i,j}$. All together, the possible types of Hilbert resolutions are at most

(2.12)
$$\begin{aligned} N' &= N \prod_{j=1,\cdots,n-\delta} C\{C(n+h, h)^{j-1} C(n+h+1, h+1) + h, h\} \\ &\leq N(h!)^{-(n-\delta)} C(n+h+1, h+1)^{h(n-\delta)} C(n+h, h)^{h(n-\delta)(n-\delta-1)/2} \\ &= (h!)^{-(n-\delta)} C(n+h+1, h+1)^{(h+1)(n-\delta)} C(n+h, h)^{(h+1)(n-\delta)(n-\delta-1)/2}. \end{aligned}$$

Moreover, after the type of the Hilbert resolution is chosen, the matrices α_i (whose entries have degree at most h) yielding the homomorphisms of the exact sequence (2.4) belong to an affine space of dimension D equal to (2.13)

$$\begin{aligned} D &\leq C(n+h, h) \sum_{j=1,\cdots,n-\delta} r_j r_{j-1} \\ &\leq C(n+h, h) C(n+h+1, h+1)^2 C(n+h, h)^{2(n-\delta)-1} \\ &\leq C(n+h+1, h+1)^2 C(n+h, h)^{2(n-\delta)}. \end{aligned}$$

Theorem (2.14). *Let $\mathscr{H} = \mathscr{H}(n, \delta, h)$ be the open subset of the Hilbert scheme parametrizing nondegenerate, projectively normal subschemes whose projective coordinate ring $\mathscr{R}$ has depth at least δ and Castelnuovo regularity less than h in $\mathbb{P}^n$. Then $\mathscr{H}$ is dominated by a scheme $\mathscr{H}'$, a disjoint union of quasiprojective schemes such that the degree of $\mathscr{H}'$ (sum of the degree of its components) is bounded from above by the expression (N' and D being as in (2.12) and (2.13)) $\nu = N' 2^D$. In particular ν is an upper bound for the number of irreducible components of $\mathscr{H}$.*

Proof. The result follows mutatis mutandis from the above discussion and the Andreotti-Bezout inequality (1.28), since $\mathscr{H}'$ is the family of matrices $\alpha_1, \alpha_2, \cdots, \alpha_{n-\delta}$ yielding all the possible Hilbert resolutions. We saw that these $(n-\delta)$-tuples of matrices belong to the disjoint union of at most N' affine spaces of dimension at most D. These matrices must satisfy the quadratic equations $\alpha_{i+1}\alpha_i = 0$.

Conversely, inside the closed set of matrices such that $\alpha_{i+1}\alpha_i = 0$ we have the open set $\mathscr{H}''$ such that the complex (2.4) (where the homomorphisms are given by the α_i's) is indeed exact. Since $L_0 = \mathscr{A}$, $\mathscr{R}$ is indeed the coordinate ring of a subscheme Y of $\mathbb{P}^n$.

The condition of projective normality of Y yields a smaller open subset $\mathscr{H}'$, whereas the conditions concerning depth and Castelnuovo regularity follow now automatically. q.e.d.

Remarks (2.15). (i) For further use, we try to have a closer look at the expression $\nu = N'2^D$. Since $D = C(n+h+1, h+1)^2 C(n+h, h)^{2(n-\delta)}$, $N' = (h!)^{-n-\delta} C(n+h+1, h+1)^{(h+1)(n-\delta)} C(n+h, h)^{(h+1)(n-\delta)(n-\delta-1)/2}$, and for our applications h, δ will be small compared to n, we see that the dominant term is $\exp(\log 2\{C(n+h, h)^{2(n-\delta)}\} \approx \exp(\text{const}. n^{2hn})$, so that we have a double exponential here too.

(ii) It is well known that the number of solutions of bilinear equations is smaller than the number of solutions of quadratic equations. For instance, in $\mathbb{P}^a \times \mathbb{P}^b$, if we set $n = a+b$, then the self intersection of a divisor of type $(1, 1)$ equals $C(a+b, a)$, which is clearly smaller than $2^n = 2^{a+b}$. But the biggest binomial, when $a = b$, is by Stirling's formula asymptotic to $(\pi/2n)^{-1/2} 2^n$. Therefore we have not lost much in the estimate of Theorem (2.14).

(iii) If Y is projectively normal, it is in particular normal. If we restrict to the open subset where Y is reduced and irreducible, we do not need to worry about the dimension of Y, since this is the degree of the Hilbert polynomial of Y, which is completely determined by the numerical datum of the integers $m_{i,j}$.

Whence follows

Corollary (2.16). *The same bound as in Theorem* (2.14) *holds for the number of irreducible components of the open set of the Hilbert scheme parametrizing nondegenerate, projectively normal subvarieties of dimension* k, *depth at least* δ *and Castelnuovo regularity at most* h.

A small improvement of the upper bound can instead be obtained under the assumption that the subvariety Y is projectively Gorenstein (such would be, for example, a canonically embedded variety with all the pluri-irregularities equal to zero).

Definition (2.17). Let Y be a projective variety, let $\mathscr{R}$ be its projective coordinate ring, and consider a minimal Hilbert resolution of $\mathscr{R}$ as in (2.4).

Y is said to be projectively Gorenstein (of level μ) if $\mathscr{R}$ is Gorenstein of level μ, i.e., if we transpose (that is, we apply the functor $\text{Hom}(-, \mathscr{A})$ to) the exact sequence

$$0 \to L_{n-k} \to L_{n-k-1} \to \cdots \to L_2 \to L_1 \to L_0$$

we again obtain an exact sequence yielding a minimal free resolution of a module isomorphic to $\mathscr{R}(n+1+\mu)$.

If $\mathscr{R}$ is Gorenstein, then (since any two minimal free resolutions are isomorphic)

(2.18). One has $L_{n-k-i} \cong \mathrm{Hom}(L_i, \mathscr{A})(-n-1-\mu)$, and moreover it is easy to show that one can assume, for $i \leq (n-k-2)/2$, that α_i is the transpose of α_{n-k-i}.

If Y is projectively Gorenstein, then by sheafifying (2.4) and computing the $\mathscr{E}xt$ groups, we obtain that Y is Gorenstein and its invertible dualizing sheaf ω_Y is induced by an invertible sheaf $\mathscr{O}_{\mathbb{P}^n}(\mu)$ on $\mathbb{P}^n$. Moreover, by the local to global spectral sequence of Ext, it follows in particular that all the higher irregularities of Y, $H^i(\mathscr{O}_Y)$ vanish $(i = 1, \cdots, k-1)$. Therefore this class of varieties is rather restricted, and one must go to weighted projective spaces (cf. [Ca4]) to enlarge it to a wider class.

In general it has been proven by Buchsbaum and Eisenbud [B-E]) that if Y is projectively Gorenstein, then one can take (2.4) to be self-dual (i.e., α_i is plus or minus the transpose of α_{n-k-i} for all i). In any case, this is not so crucial for asymptotic estimates. What improves is that the number N of choices for the ranks of the terms in the Hilbert resolution goes down (notice that now the depth equals $k+1$) to

$$N = C(n+h+1, h+1)^{(n-k)/2} C(n+h, h)^{(n-k)(n-k-2)/8},$$

whereas for the type of Hilbert resolutions N' goes down to

$$N' = (h!)^{-(n-\delta)/2} C(n+h+1, h+1)^{(h+1)(n-k)/2} C(n+h, h)^{(h+1)(n-k)(n-k-2)/8},$$

accordingly also D goes down to $C(n+h+1, h+1)^2 C(n+h, h)^{n-k}$.

Our last type of general results concern the complexity of the Hilbert schemes.

In order to do this, we recall Macaulay's results [Ma] and Gotzmann's determinantal description of the Hilbert scheme (cf. [Go], also [Gr3]). Let $\mathscr{I}_Y$ be a coherent sheaf of ideals on $\mathbb{P}^n$, Y the associated subscheme, and let $P(m)$ be the Hilbert polynomial of Y, namely $P(m) = \chi\ (\mathscr{O}_Y(m))$.

If we let $Q(m)$ be the complementary Hilbert polynomial, so that $P(m) + Q(m) = C(n+m, m)$, then $Q(m) = \chi(\mathscr{I}_Y(m))$, and for m large $H^0(\mathbb{P}^n, \mathscr{I}_Y(m))$ is a subspace of dimension $Q(m)$ inside $\mathscr{A}_m = H^0(\mathbb{P}^n, \mathscr{O}_{Pn}(m))$.

In fact, m large means exactly that m should be at least the Castelnuovo regularity h of the sheaf $\mathscr{O}_Y$.

Mumford [Mu2] proved that, fixing the Hilbert polynomial $P(m)$, then every such sheaf $\mathscr{O}_Y$ has regularity bounded by a polynomial in the coefficients of $P(m)$.

Let h_P be the maximum of the Castelnuovo regularities of the sheaves $\mathscr{O}_Y$ with fixed Hilbert polynomial $P(m)$.

Then it is known [Gr3] that there exist integers $a_1 \geq a_2 \geq \cdots \geq a_d \geq 0$, such that

(2.19)

$$P(m) = C(m+a_1, a_1) + C(m+a_2-1, a_2) + \cdots + C(m+a_d-(d-1), a_d)$$

and such that $h_P \leq d$.

With this choice of d, in fact one lets W be the set of the first $Q(d)$ monomials of degree d in the lexicographic order, I the graded ideal of $\mathscr{A}$ generated by W, and then, for $m \geq d$, $Q(m)$ is the dimension of I_m.

(2.20). For all subspaces V of $\mathscr{A}_d$ of dimension $Q(d)$, let $I(V)$ be the graded ideal generated by V. Then, the theorem of Macaulay says that $I(V)_{d+1}$ has always dimension at least $Q(d+1)$.

(2.21). Gotzmann's persistence theorem says that, if V is such that equality holds, $\dim I(V)_{d+1} = Q(d+1)$, then $\dim I(V)_m = Q(m)$ for all $m \geq d$, whence $I(V)$ defines a subscheme Y with Hilbert polynomial $P(m)$.

Whence one reaches a simple determinantal description of the Hilbert scheme Hilb_P parametrizing subschemes Y with a fixed Hilbert polynomial $P(m)$:

$\mathrm{Hilb}_P \subset \mathrm{Grass}(Q(d), \mathscr{A}_d)$, where Grass denotes the Grassmannian of vector subspaces of fixed dimension $Q(d)$, is the subscheme

(2.22)

$$\begin{aligned}\mathrm{Hilb}_P &= \{V \in \mathrm{Grass}(Q(d), \mathscr{A}_d) | \dim I(V)_{d+1} = Q(d+1)\} \\ &= \{V \in \mathrm{Grass}(Q(d), \mathscr{A}_d) | \mu\colon V \otimes \mathscr{A}_1 \to \mathscr{A}_{d+1} \text{ has rank } \leq Q(d+1)\}.\end{aligned}$$

It follows directly from this description that Hilb_P is complete and connected.

Remark (2.23). Let C be a canonical curve in $\mathbb{P}^n$, so $n = g-1$. Here, the Hilbert polynomial $P(m) = 2nm - n$. The polynomial can also be written as $P(m) = (m+1) + m + \cdots + (m+2-2n) + (n-1)(2n-2) - 2$, whence d is $\geq 2n(n-1)$, which grows like the square of the degree. But, on the other hand, the Castelnuovo regularity of a canonical curve is 3.

Instead, for a surface of general type which is 3-canonically embedded, then $P(m) = 1/2(3m-1)3my + x$. Therefore d grows less than $9^3 y^4$, asymptotically. We shall comment later on this point.

We want now to use this description to bound the complexity of the Hilbert scheme. Since the dimension of Y, i.e., the degree of $P(m)$, is smaller than the dimension n of the ambient space, it is convenient to consider also the annihilator V^* of V inside the dual space $\mathscr{A}_d^*$ of $\mathscr{A}_d$.

Moreover, we shall work on the Stiefel manifold instead of on the Grassmannian. That is, we pick a basis $f_1, f_2, \cdots, f_{Q(d)}$ for V, or alternatively a basis $\varphi_1, \varphi_2, \cdots, \varphi_{P(d)}$ for V^*.

Let $\mathscr{S} = \{f_1, f_2, \cdots, f_{Q(d)} \in \mathscr{A}_d | \operatorname{rank}\{f_i \cdot x_j\}_{i=1,\cdots,Q(d),\ j=0,\cdots,n} \leq Q(d+1)\}$.

Then clearly the degree of $\mathscr{S}$ is $\leq Q(d+1)^{Q(d)C(n+d,d)}$.

An easy argument in linear algebra shows that, if μ^* is the dual map of μ, $\xi_0, \cdots, \xi_n$ is the dual basis of the x_j's, and $\eta_1, \cdots, \eta_r$ is a basis of $\operatorname{Im}(\mu^*)$, then the Hilbert scheme is dominated by an open set of

$$\begin{aligned}\mathscr{S}^* = \{\varphi_1, \varphi_2, \cdots, \varphi_{P(d)} | \operatorname{rank}\{\varphi_i \cdot \xi_j, \eta_t\} \\ \leq C(n+1+d, n) + (n+1)P(d) - P(d+1)\}.\end{aligned}$$

The degree of $\mathscr{S}^*$ is $\leq [(n+1)P(d)]^{C(n+d,d)P(d)}$.

We thus obtain the following

Theorem (2.2.4). *Let* Hilb_P *be the Hilbert scheme parametrizing subschemes* Y *of* $\mathbb{P}^n$ *with a fixed Hilbert polynomial* $P(m)$. *Then the complexity of* Hilb_P *is* $\leq [(n+1)P(d)]^{C(n+d,d)P(d)}$, d *being as in* (2.19).

Remark (2.25). For a surface of general type which is 3-canonically embedded, we have $P(m) = 1/2(3m-1)3my + x$, $n+1 \leq 4y$, $d \leq 9^3y^4$, thus we obtain the estimate $[9^8y^{10}]^{\{9^2y^3\}^{4y}}$, which is a double exponential estimate rather worse than the one provided by the method of Chow forms (see also Theorem A in the Introduction).

One can ask whether we can improve the previous bounds if we do not want to estimate the complexity of the whole Hilbert scheme, but only of some components where the Castelnuovo regularity is generically much lower.

Notice in fact that, by the existence of a universal family on Hilb_P, and by the definition of Castelnuovo regularity, we have a stratification of Hilb_P by open sets $\operatorname{Hilb}_{P,h}$ where the regularity is at most h.

We can use the result of Macaulay-Gotzmann in order to give a lower degree realization of $\operatorname{Hilb}_{P,h}$:

Proposition (2.26). $\operatorname{Hilb}_{P,h}$ *is an open set in the locally closed subscheme* $H_{P,h} = \{U \in \operatorname{Grass}(Q(h), \mathscr{A}_h) | U$ *is such that* $\dim I(U)_t = Q(t)$ *for each* t *with* $h \leq t \leq d+1\}$.

Proof. On $\operatorname{Hilb}_{P,h}$ is defined, by the base change theorem, a natural morphism to $\operatorname{Grass}(Q(h), \mathscr{A}_h)$ which associates to $V \in \operatorname{Grass}(Q(d), \mathscr{A}_d)$

the subspace $H^0(\mathbb{P}^n, \mathcal{I}_V(h))$, $\mathcal{I}_V$ being the sheaf of ideals associated to the graded ideal $I(V)$ generated by V. Clearly, the image of $\mathrm{Hilb}_{P,h}$ is contained in $H_{P,h}$. Conversely, the map associating to U the subspace $V = I(U)_d$ defines, by (2.22), a morphism of $H_{P,h}$ into Hilb_P. Firstly, it is clear that the composition of the two above morphisms is the identity of $\mathrm{Hilb}_{P,h}$, secondly, again by semicontinuity, the image of $\mathrm{Hilb}_{P,h}$ is open in $H_{P,h}$. q.e.d.

Remark (2.27). It is not clear to us that $\mathrm{Hilb}_{P,h}$ should be dense in $H_{P,h}$. But, certainly, we can bound the complexity of $\mathrm{Hilb}_{P,h}$ by the one of $H_{P,h}$.

With an entirely similar proof to the one of Theorem (2.24) we obtain, in view of the foregoing Remark (2.27), the following

Theorem (2.28). *The complexity of* $\mathrm{Hilb}_{P,h}$ *is bounded from above by* $\min\{[C(d+n+1-h, n)P(h)]^{C(n+h,h)P(h)}, Q(d+1)^{Q(h)C(n+h,h)}\}$.

Remark (2.29). For a surface of general type which is 3-canonically embedded, we have (cf. Remark (2.25)) $n+1 \leq 4y$, $h = 3$, $d \leq 9^3 y^4$, $P(3) \leq 23y$, $P(d) \leq 9^7 y^8$, thus we obtain an asymptotic estimate choosing the minimum between $[9^3 y^3]^{\{4y\}^{(4y)^4}}$, which is again a double exponential estimate, and $[9^3 y^3]^{\{4y\}^{4^4 y^6}}$, which is worse.

It is striking that in this case the estimate (2.25) is better.

The next section will be entirely devoted to applying the above general estimates to the case of surfaces of general type, and to comparing this case with the estimates obtained by the method of Chow forms.

3. Degree of moduli spaces and the method of quasigeneric projections

Throughout this section, S will be the minimal model of a surface of general type, and X its canonical model.

(3.1) $X = \mathrm{Proj}(\mathcal{R})$ where $\mathcal{R}$ is the canonical ring,

$$\mathcal{R} = \mathcal{R}(S, \mathcal{O}_S(K_S))) = \bigoplus_{m \in \mathbb{N}} (H^0(S, \mathcal{O}_S(mK_S))),$$

K_S being a canonical divisor on S.

As in the Introduction, we set $\chi(\mathcal{O}_S) = x$, $K_S^2 = y$. We observe that, since $9x \geq y \geq 2x - 6$ by the inequality of Bogomolov-Miyaoka-Yau and the inequality of Noether, if we are not too much concerned about constants, we can asymptotically regard y as the main numerical invariant.

We have (cf. [Bo]) that $P(m) = h^0(S, \mathscr{O}_S(mK_S)) = x + ym(m-1)/2$ and that, if $\varphi_m : S \to Y_m \subset \mathbb{P}^{P(m)-1}$ is the rational map associated to $H^0(S, \mathscr{O}_S(mK_S))$, then (cf. also [Cat6, Theorem 1.11]).

(3.2). Except a finite number of families of surfaces, for $m \geq 3$ φ_m is a birational morphism and induces an isomorphism $\psi_m : X \to Y_m$ (for the exceptions, one has $K^2 \leq 2$, and, but for one irreducible family, $9 = 0$). Whence the image Y_3 is a normal surface of degree $9y$ in a projective space of dimension $n \leq 4y$, having at most Rational Double Points (RDPs) as singularities.

Let $\mathscr{C} = \mathscr{C}_{2,9y}(\mathbb{P}^n)$ be the open subset of the Chow variety parametrizing Chow forms of irreducible surfaces of degree $9y$ in $\mathbb{P}^n$. By Theorem 1.28 the number $\imath(\mathscr{C})$ of irreducible components of $\mathscr{C}$ satisfies $\imath(\mathscr{C}) \leq 3^{\lambda'}$, where $\lambda' = 1/6C(13y, 9y)^3$. To get a better feeling, we use Stirling's formula which tells us that $C(13y, 9y)$ is asymptotic to c^y, where the constant c is of the order of magnitude of $C(13, 9)$ which is less than 1000. We finally get $\imath(\mathscr{C}) \leq (3^{1/6})^{1000^y}$. We do not want to use this result to obtain an estimate for $\imath(y)$. Instead, we observe that any such surface Y_3 can be isomorphically projected to the projective space $\mathbb{P}^5$.

Theorem (3.3). *For* $y \geq 3$, $\imath(y) \leq 6^{(y+5/9)^{15}}$.

Proof. Let as before $\mathscr{M}_y = \bigcup_x (\mathscr{M}_{x,y})$.

Consider the Hilbert scheme $\mathscr{H}'$ parametrizing normal surfaces Y_3 of degree $9y$ in the projective space of dimension 5.

By what has been said previously, every surface in $\mathscr{M}_{x,y}$ is such that a projection of its tricanonical model Y_3 yields a surface with RDPs occurring in the universal family over $\mathscr{H}'$. Restrict $\mathscr{H}'$ to the open set $\mathscr{H}$ such that all the fibres Y over $\mathscr{H}$ have RDPs, the minimal resolution S of Y is minimal (this condition is open by the main result of [Kod]), and its numerical invariant $K_S^2 = y$. Then we have a surjective morphism of $\mathscr{H}$ onto $\mathscr{M}_y$, whence $\imath(y) \leq \imath(\mathscr{H}_{\text{red}})$. Finally $\imath(\mathscr{H}_{\text{red}}) \leq \imath(\mathscr{C}_{2,9y}(\mathbb{P}^5))$ by Corollary (1.27), whence the desired result. q.e.d.

We proceed now to analyse the results that can be asymptotically obtained by the method of Hilbert resolutions. We recall that (cf. [Bo, Gas]) with our previous notation (3.2).

(3.4). Except a finite number of families of surfaces, for $m \geq 4$ φ_m is a birational morphism and induces an isomorphism $\psi_m : X \to Y_m$ where Y_m is projectively normal.

We apply to $Y = Y_5$ the results of the second section, observing that

Y_5 is a surface with RDPs, of degree $25y$ in a projective space $\mathbb{P}^n$ with $n = x + 10y \le 11y$. It is easy to see that the Castelnuovo regularity of $\mathcal{O}_Y$ is at most 3, since $h^2(\mathbb{P}^n, \mathcal{O}_Y(h-2)) = h^0(S, \mathcal{O}_S((5(2-h)+1)K_S)) = 0$ if $h \ge 3$, and then $h^1(\mathbb{P}^n, \mathcal{O}_Y(2)) = h^1(S, \mathcal{O}_S(-9K_S)) = 0$ by the Kodaira vanishing theorem (cf., e.g., [Bo]). We have

$$\begin{aligned} N' &\le 6^{-(n-1)} C(n+4, 4)^{4(n-1)} C(n+3, 3)^{4(n-1)(n-2)/2} \\ &\le 6^{-n}(n+3)^{6n(n-1)}, \end{aligned}$$

while $D \le C(n+4, 4)^2 C(n+3, 3)^{2(n-1)} \le (n+3)^{6n+2}$.

Then

$$\begin{aligned} \nu = N' 2^D &\le (n+3)^{6n(n-1)} 2^{-2n+(n+3)^{6n+2}} \\ &\le (11y)^{66y(11y)} 2^{-20y+(11y)^{66y}}, \end{aligned}$$

which is asymptotic to $2^{(11y)^{66y}}$. Thus, by Theorem (2.14) and by the usual argument that an open set of the Hilbert scheme dominates the moduli space we obtain

$$\iota(x, y) \le (11y)^{66y(11y)} 2^{-20y+(11y)^{66y}}, \tag{3.5}$$

which is asymptotic to $2^{(11y)^{66y}}$.

This estimate is definitely worse than the one gotten by the method of Chow forms (Theorem (3.3)), and also the one gotten (cf. (2.25)) by the method of the Hilbert scheme. The final slogan will be that Enriques wins Chow who wins Hilbert, except that the first match only takes place for regular surfaces ($q = 0$). In fact, let me recall a method, inspired by Enriques' treatment of pluricanonical surfaces [En] which I introduced in [Cat3] (cf. also [Cat4] for improvements and [Cat6] for a related survey), namely *The method of quasigeneric projections*:

(3.6). A quasigeneric birational canonical projection is a morphism to a weighted projective space $\varphi: S \to \mathbb{P} = \mathbb{P}(e_0, e_1, e_2, e_3)$ such that

(i) the four integers e_0, e_1, e_2, e_3 are normalized, that is, any three of them are relatively prime,

(ii) the morphism φ is given by four sections y_0, y_1, y_2, y_3 where y_i is a section of $H^0(S, \mathcal{O}_S(e_i K_S))$, and the four sections have no common zeros,

(iii) φ is birational onto the image Y of S.

For simplicity, excluding the finite number of families with $p_g \le 1$, $K^2 \le 2$, we can assume (this follows, e.g., from [Cat3, p. 81] and (3.2)) that

(3.7). If S is a surface of general type with $q = 0$, $p_g \geq 2$, $K^2 \geq 3$ there exists a quasigeneric birational canonical projection $\varphi: S \to Y \subset \mathbb{P} = \mathbb{P}(1, 1, 2, 3)$.

As in [Cat3], we follow an idea of Petri, revisited by Arbarello and Sernesi [A-S], and we view the ring

$$\mathcal{R} = \mathcal{R}(S, \mathcal{O}_S(K_S))) = \bigoplus_{m\in\mathbb{N}} (H^0(S, \mathcal{O}_S(mK_S)))$$

as a module over the polynomial ring $\mathcal{A} = \mathbb{C}[y_0, y_1, y_2, y_3]$, and since the assumption $q = 0$ ensures that $\mathcal{R}$ is Cohen-Macaulay, one can take a minimal resolution of length 1

$$0 \to L_1 \to L_0 \to \mathcal{R} \to 0 \tag{3.8}$$

where $L_0 = \bigoplus_{j=1,\cdots,h} \mathcal{A}(-m_j)$, and by duality, setting

$$s = e_0 + e_1 + e_2 + e_3, \tag{3.9}$$

$L_1 = \bigoplus_{j=1,\cdots,h} \mathcal{A}(s+1-m_j)$. For further use, we set $1 = v_1, v_2, \cdots, v_h$ the corresponding minimal system of generators of $\mathcal{R}$ as an $\mathcal{A}$-module.

One can further assume (loc. cit. Theorem 3.8) that the matrix α giving the homomorphism (3.8) is symmetric and has an irreducible determinant yielding the equation of Y; in particular,

$$K^2 e_0 \cdot e_1 \cdot e_2 \cdot e_3 = \sum_{j=1,\cdots,h} (s + 1 - 2m_j), \tag{3.10}$$

where we should recall that $m_1 = 0$. Thus

$$6y(= 6K^2) = \sum_{j=1,\cdots,h} (8 - 2m_j). \tag{3.11}$$

Since φ is birational, the determinant of α is irreducible, whence, if we assume $0 = m_1 < m_2 \cdots \leq m_h$, then

$$m_j < 8 - m_{h-j+2}, \quad \text{in particular, } 0 = m_1 < m_2 \leq \cdots \leq m_h \leq 6. \tag{3.12}$$

Before we proceed, we need to quote the main result of [Cat3], asserting not only that the matrix α satisfies the following rank condition (or Rouche' Capelli, or Ring condition)

(R.C.) if β is the adjoint matrix $= \bigwedge^{h-1}(\alpha)$, then $\beta_{i,j}$ belongs to the ideal generated by $\beta_{1,1}, \beta_{1,2}, \cdots, \beta_{1,h}$, (in particular there exist polynomials $\lambda^k_{i,j}$ with

$$\beta_{i,j} = \sum_{k=1,\cdots,h} (\lambda^k_{i,j} \beta_{1,k}) \tag{3.13}$$

but conversely, ([Cat3, Theorem 4.22], [Cat4, Theorem 23]).

(3.14). Let there be given positive integers $0 = m_1 < m_2 \le \cdots \le m_h$, a symmetric $h \times h$ matrix with entries $\alpha_{i,j}$ homogeneous of degree $(s + 1 - m_i - m_j)$ with irreducible determinant, and moreover satisfying R.C. (i.e., (3.13) holds); then if the subscheme $X \subset \mathbb{P}(e_0, e_1, e_2, e_3, m_2, m_3, \cdots, m_h)$ defined by the equations

$$\sum_{j=1,\cdots,h} \alpha_{i,j} y_j = 0,$$

$$v_i \cdot v_j = \sum_{k=1,\cdots,h} (\lambda^k_{i,j} v_k)$$

satisfies the open condition of having at most RDP's as singularities, then X is the canonical model of a surface of general type.

We can go back now to the inequality (3.12): observe first of all that $m_j = 1$ for $1 \le j \le x-2$ $(p_g = x-1)$. By (3.11) it follows immediately, since $0 < m_j$ for $j \ge 2$, that $h \ge y - 1/3$, but since $m_j + m_{h-j+2} < 8$, and therefore

$$\begin{aligned} 12y &= 16 + \sum_{j=2,\cdots,h} (16 - 2(m_j + m_{h-j+2})) \\ &\ge 16 + 2(h-1), \end{aligned}$$

then

$$h \le 6y - 7. \tag{3.15}$$

At this stage we let the calculations, especially regarding constants, be less coarse than the ones we dealt with previously. Recall that $0 = m_1 < m_2 \le \cdots \le m_h \le 6$ and that $m_j = 1$ for $j \le x-2$. Let $m_{x-1} = \cdots = m_{x-2+a} = 2$, $m_{x-1+a} = 3$. Let us consider the integer a: since the bigenus $P(2) = h^\circ(S, \mathcal{O}_S(2K_S)) = x + y$, $a = P(2) - \dim W$, where W is the subspace of $H^0(S, \mathcal{O}_S(2K_S))$ spanned by the monomials $y_0^2, y_0y_1, y_1^2, y_0v_2, y_1v_2, \cdots, y_0v_{x-2}, y_1v_{x-2}$; now, we can assume that the minimum of the divisors $\operatorname{div}(y_0)$, $\operatorname{div}(y_1)$, equals the fixed part of the canonical system, whence it follows easily that for any pair of sections s, σ of $H^0(S, \mathcal{O}_S(K_S))$, the equation $sy_0 = \sigma y_1$ implies that s is a scalar multiple of y_1, σ a multiple of y_0, and thus $a = y - x + 3$. The same argument implies that there are no relations in degree 2, whence indeed $0 = m_1 < m_2 \le \cdots \le m_h \le 5$.

Let $m_{x-1+a} = \cdots = m_{x-2+a+b} = 3$, $m_{x-1+a+b} = \cdots = m_{x-2+a+b+c} = 4$, $m_{x-1+a+b+c} = \cdots + m_h = 5$. Set also, for convenience, $h = x - 1 + a +$

$b+c+d=y+1+b+c+d$. By (3.12), $d \leq y+1$, $c+d \leq y+1+b$, by (3.11), $b=y-x-2+d$. Therefore c, d determine the other integer, and the second inequality can be written as $c \leq 2y-1-x$. Therefore, $c \cdot d \leq 2y^2$, and

(3.16). The possible types for the resolution (3.8) are at most $2y^2$.

A boring calculation (keeping in mind that the number of monomials of degree $1, 2, \ldots$ in $\mathscr{A}$ is $2, 4, 7, 10, 16, 22, 31, 40, \cdots$) which we omit here shows that, once we have fixed the type of resolution (i.e., the integers c, d), then the matrix α is a point in an affine space of dimension

$$N \leq 58y^2. \tag{3.17}$$

For the points α of this affine space we shall inspect the equations given by the Rank Conditions. We want the entries $\beta_{i,j}$ of $\bigwedge^{h-1}(\alpha)$ such that there exist polynomials $\lambda^k_{i,j}$ with $\beta_{i,j}=\sum_{k=1,\cdots,h}(\lambda^k_{i,j}\beta_{i,k})$. Since α_{ij} has degree $8-m_i-m_j$, then $\beta_{i,j}$ has degree $6y-8-m_i-m_j \leq 6y+2$, whereas the polynomials $\lambda^k_{i,j}$ have degree at most 10. Since, given a degree $d \leq 10$, there are at most 73 monomials of degree d, it follows that the R.C. are, for each choice of i, j, conditions that the vector $\beta_{i,j}$ is linearly dependent upon $r_{i,j}h$ vectors, where $r_{i,j} \leq 73$.

It is worthwhile to mention that all the coefficients of the above vectors are homogeneous polynomials of degree $(h-1)$ in the coefficients of the polynomials $\alpha_{i,j}$ which are the entries of α.

The locus $V=\{\alpha|\alpha$ satisfies R.C.$\}$ is stratified as the union of at most

$$\prod_{i\leq j=1,\cdots,h} r_{i,j}h \leq (73h)^{h(h+1)/2} \leq (440y)^{18y^2} \tag{3.18}$$

subloci, distinguished according to the ranks $\rho_{i,j}$ of the $r_{i,j}h$ vectors, upon which we want $\beta_{i,j}$ to be dependent. Each such sublocus, in turn, is defined, inside a suitable open subset, by the vanishing of minors of order $\rho_{i,j}$ of the matrix obtained by bordering the previous $r_{i,j}h$ vectors with the vector $\beta_{i,j}$, and clearly these are equations of degree less than $440y$ in the coefficients of the polynomials $\alpha_{i,j}$. By the Andreotti-Bezout lemma (1.26) the degree of these subloci is

$$\leq (440y)^{58y^2}. \tag{3.19}$$

Summing up, we have at most $2y^2(440y)^{18y^2}$ subloci of degree $\leq (440y)^{58y^2}$ which dominate the moduli space $\mathscr{M}^0_{x,y}$ of regular surfaces with invariants x, y.

Theorem (3.20). *The number* $\iota^0(x, y)$ *of irreducible components of the moduli space* $\mathcal{M}^0_{x,y}$ *of regular surfaces of general type* S *with invariants* $\chi(\mathcal{O}_S) = x$, $K_S^2 = y$ *satisfies for* $x, y \geq 3$ *the inequality*

$$\iota^0(x, y) \leq 2y^2 \cdot (440y)^{76y^2}.$$

Whence also the asymptotic inequality

$$\iota^0(x, y) \leq y^{77y^2}.$$

Remark (3.21). Using the degree of the determinantal varieties one could slightly improve the above upper bound, but without changing its exponential nature.

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